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Integration of the Primer Vector in a Central Force Field[1]

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Abstract. This paper examines the primer vector which governs optimal solutions for orbital transfer when the central force field has a more general form than the usual inverse-square-force law. Along a null-thrust arc that connects two successive impulses, the two sets of state and adjoint equations are decoupled. This allows the reduction of the problem to the integration of a linear first-order differential equation, and hence the solution of the optimal coasting arc in the most general central force field can be obtained by simple quadratures. Immediate applications of the results can be seen in solving problems of escape in the equatorial plane of an oblate planet, satellite swing by, or station keeping around Lagrangian points in the three-body problem.

1. Equations of Optimal Trajectories

A rocket M, considered as a mass point, is moving in a central force field with center of attraction at 0 (Fig. 1). At time t, the state of the rocket is defined by the position vector $\mathbf{r}(t)$, the velocity vector $\mathbf{V}(t)$, and the characteristic velocity C. Note that C gives a measure of the fuel expenditure:

$$C = \int_0^t |\, \boldsymbol{\Gamma} \,|\, dt, \tag{1}$$

where $\boldsymbol{\Gamma}$, the acceleration due to the propulsive force $\mathbf{F}$, is given by

$$\boldsymbol{\Gamma} = \mathbf{F}(t)/m(t), \tag{2}$$

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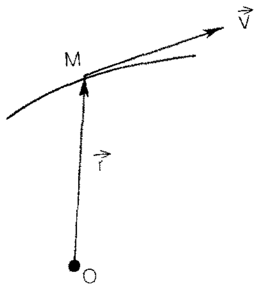

Fig. 1. State vectors.

m being the instantaneous mass of the rocket. The control at any instant t is the vector acceleration $\mathbf{\Gamma}$, subject to the constraint

$$0 \leqq \Gamma \leqq \Gamma_{\max}. \tag{3}$$

The optimal transfer problem, known as the Lawden problem, is as follows.

At the initial time $t = 0$, $\mathbf{r} = \mathbf{r}_0$, $\mathbf{V} = \mathbf{V}_0$, $C = 0$. The vectors $\mathbf{r}_0$, $\mathbf{V}_0$ are prescribed. At the final time $t = t_f$, $\mathbf{r} = \mathbf{r}_f$, $\mathbf{V} = \mathbf{V}_f$. The vectors $\mathbf{r}_f$ and $\mathbf{V}_f$ are also prescribed. The final time t_f may be fixed in advance, or subject to the condition $t_f \leqq t_{\max}$, or may be completely free. The problem is to find the time history of $\mathbf{\Gamma}(t)$ such that the final characteristic velocity C_f is a minimum.

First, we have the state equations

$$d\mathbf{r}/dt = \mathbf{V}, \tag{4}$$

$$d\mathbf{V}/dt = \mathbf{\Gamma} - g(r)\,\mathbf{r}/r. \tag{5}$$

Here, $g(r)$ is the gravitational force per unit mass, with $g > 0$ for an attracting force field. Using the maximum principle, we introduce the adjoint elements $\mathbf{p}_r$, $\mathbf{p}_v$, p_c to form the Hamiltonian

$$H = \mathbf{p}_r \cdot \mathbf{V} + \mathbf{p}_v \cdot (\mathbf{\Gamma} - g(\mathbf{r}/r)) + p_c \Gamma. \tag{6}$$

To maximize H, $\mathbf{\Gamma}$ must be collinear to $\mathbf{p}_v$ and

$$H = \mathbf{p}_r \cdot \mathbf{V} - g(\mathbf{p}_v \cdot \mathbf{r})/r + (p_v + p_c)\Gamma. \tag{7}$$

From this reduced Hamiltonian, we have the Lawden's condition for the selection of Γ (see Ref. 1)

$$\begin{aligned} \Gamma &= 0, && \text{if} \quad (p_v + p_c) < 0, \\ \Gamma &= \Gamma_{\max}, && \text{if} \quad (p_v + p_c) > 0, \\ \Gamma &= \text{intermediate}, && \text{if} \quad (p_v + p_c) = 0. \end{aligned} \tag{8}$$

Furthermore, if $\Gamma_{\max}$ is constant,

$$dp_c/dt = -\partial H/\partial C = 0.$$

Hence,

$$p_c = p_{c_f} = -1. \tag{9}$$

We have three types of optimal trajectories according to the value of the switching function k defined as

$$k = p_v - 1. \tag{10}$$

For $k < 0$, we have a ballistic or null thrust (NT) arc. For $k > 0$, we have a maximum thrust (MT) arc. For $k = 0$, we have a singular or intermediate (IT) arc.

It is seen that the adjoint vector $\mathbf{p}_v$, known as the primer vector, governs the optimal thrust control. The primer vector along the MT-arcs is not known, but its solution along the NT-arcs has been obtained by Lawden for a Newtonian force field (Ref. 1). The application to optimal switching for impulsive transfers is given in Ref. 2. Following the procedure of Lawden, Brooks and Smith have derived a set of three linear differential equations of the second order for the components of the primer vector along the NT-arcs in a general central force field (Ref. 3). Lawden also obtained the solution for the primer vector along the planar IT-arcs for the free-time case (Ref. 1), and Marchal has integrated the planar IT-arcs for the fixed-time case (Ref. 4). Recently, Vinh and Marchal obtained the solution for the primer along the IT-arcs for the fixed-time case in a general central force field (Ref. 5).

With the integrals of motion obtained by Pines (Ref. 6), it is now possible to integrate completely the equations of the optimal trajectories for the NT-arcs in a general central force field. We shall use a rotating coordinate system $OXYZ$ such that O is at the center of the force field, the X-axis along the position vector, positive outwards, the Y-axis parallel to the circumferential direction, positive toward the direction of motion, and the Z-axis completing a right-handed system (Fig. 2). For this system, we have the components

$$\mathbf{r} = (r, 0, 0), \qquad \mathbf{V} = (X, Y, 0), \qquad \mathbf{p}_r = (a, b, c), \qquad \mathbf{p}_v = (\alpha, \beta, \gamma). \tag{11}$$

The adjoint $\mathbf{p}_v$ and $\mathbf{p}_r$ must satisfy the adjoint equations

$$d\mathbf{p}_v/dt = -\mathbf{p}_r, \tag{12}$$

$$d\mathbf{p}_r/dt = (1/r)[g\mathbf{p}_v + (\mathbf{p}_v \cdot \mathbf{r})(g^1 - g/r)\,\mathbf{r}/r], \tag{13}$$

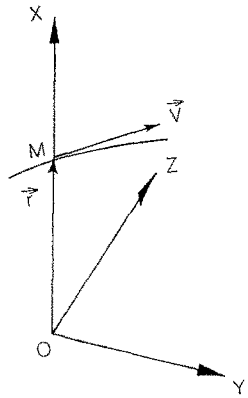

Fig. 2. Rotating axes.

where g^1 denotes the first derivative of g with respect to r. If the gravitational field is time invariant, we have the integral $H = \text{const}$, with

$$aX + bY - \alpha g + k\Gamma = H. \tag{14}$$

For a central force field, a vector integral exists, and we have (Refs. 6–7)

$$(d/dt)[\mathbf{p}_v \times \mathbf{V} + \mathbf{p}_r \times \mathbf{r}] \\ = (d\mathbf{p}_v/dt) \times \mathbf{V} + \mathbf{p}_v \times (d\mathbf{V}/dt) + (d\mathbf{p}_r/dt) \times \mathbf{r} + \mathbf{p}_r \times (d\mathbf{r}/dt). \tag{15}$$

Using Eqs. (4), (5), (12), (13), and noticing that along an optimal trajectory $\mathbf{p}_v$ is parallel to $\mathbf{\Gamma}$, we have

$$(d/dt)[\mathbf{p}_v \times \mathbf{V} + \mathbf{p}_r \times \mathbf{r}] = 0. \tag{16}$$

This gives

$$\mathbf{p}_v \times \mathbf{V} + \mathbf{p}_r \times \mathbf{r} = \mathbf{D} = \text{const}. \tag{17}$$

2. Primer Vector Along Null Thrust Arcs

Consider a central force field given as

$$\mathbf{g} = -g(r)\,\mathbf{r}/r, \tag{18}$$

with $g(r) > 0$ for an attracting field. Along the null thrust arcs, Eq. (5) gives an integral for the angular momentum

$$\mathbf{r} \times \mathbf{V} = \mathbf{h} = \text{const}. \tag{19}$$

The trajectory is planar. We take the fixed coordinate system $Oxyz$ such that O is at the center of attraction and the z-axis along the fixed vector $\mathbf{h}$. The rotating system $OXYZ$ is determined by the polar angle $\theta = (O\mathbf{x}, O\mathbf{X})$. Any vector transformation between the two systems follows the rule for rotation of axes

$$\mathbf{F}_{xyz} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{F}_{XYZ}, \tag{20}$$

and any time-derivative transformation is obtained from

$$(d\mathbf{F}/dt)_{xyz} = (d\mathbf{F}/dt)_{XYZ} + \dot{\theta}\mathbf{k} \times \mathbf{F}, \tag{21}$$

where $\mathbf{k}$ denotes the unit vector along the z-axis.

By projecting the vector integral (17) onto the fixed axes, using the components on the rotating axes, we have

$$\begin{aligned} \gamma Y \cos\theta + \gamma X \sin\theta + cr \sin\theta &= hC_1, \\ \gamma Y \sin\theta - \gamma X \cos\theta - cr \cos\theta &= hC_2, \\ -\alpha Y + \beta X + br &= hC_3, \end{aligned} \tag{22}$$

where the C_i are constants. Along the arcs of null thrust, $\Gamma = 0$, and we rewrite (14) with a new Hamiltonian constant as

$$aX + bY - \alpha g = h^2 H. \tag{23}$$

To express the components X, Y of the velocity in terms of r, we use the relations

$$r^2\dot{\theta} = h, \qquad X = \dot{r} = hr'/r^2, \qquad Y = r\dot{\theta} = h/r, \tag{24}$$

where the prime denotes derivation with respect to θ.

Within the rotating frame, we rewrite (12) as

$$(d\mathbf{p}_v/dt)_{xyz} = (d\mathbf{p}_v/dt)_{XYZ} + \dot{\theta}\mathbf{k} \times \mathbf{p}_v = -\mathbf{p}_r,$$

or

$$(h/r^2)(\mathbf{p}_v' + \mathbf{k} \times \mathbf{p}_v) = -\mathbf{p}_r, \tag{25}$$

where in this last equation the differentiation is taken with respect to the rotating frame. By projecting the equation into the rotating axes, we have a set of three first-order differential equations for the components of the primer vector

$$(h/r^2)(\alpha' - \beta) = -a, \qquad (h/r^2)(\beta' + \alpha) = -b, \qquad (h/r^2)\gamma' = -c. \tag{26}$$

Upon substitution of (24) into (22) and solving for the first two equations, we immediately have for γ without integration

$$\gamma = r(C_1 \cos \theta + C_2 \sin \theta). \tag{27}$$

Next, using (22-3) and the Hamiltonian integral (23) to solve for a and b, and upon substituting into (26-1) and (26-2), we have the following linear first-order differential equations for the other two components α and β of the primer vector:

$$\alpha' + ((gr^4/h^2) - r)(\alpha/r') = (C_3 - Hr^2)(r^2/r'), \tag{28}$$

$$\beta' - (r'/r)\,\beta = -2\alpha - rC_3 . \tag{29}$$

Along the NT-arcs, the equation of the trajectory can be integrated once the functional form $g(r)$ is specified, and the radial distance r can be expressed in terms of θ. Upon substituting the value of r and r' into (28), we see that the linear equation in α can be integrated by two quadratures, and subsequently Eq. (29) in β can also be integrated. Finally, the adjoint vector $\mathbf{p}_r$ is obtained from (26) without integration.

3. Remarks

(a) By the change of variable from θ to u such that

$$u = 1/r, \qquad g(r) = g(1/u), \tag{30}$$

we can write (28) in the form

$$d\alpha/du + (u - (g/h^2u^2))(\alpha/u'^2) = (H/u^2 - C_3)(1/u'^2). \tag{31}$$

For a central force, we have the integral

$$u'^2 = -u^2 + f(u) + a_1 , \tag{32}$$

where

$$f(u) = (2/h^2) \int (g/u^2)\, du. \tag{33}$$

Upon substitution of (32) into (31), the solution for the component α of the primer can be expressed in terms of u, that is, of r alone.

(b) In general, the component β is expressed as a function of θ by the expression

$$\beta = r[C_5 - \int (C_3 + 2\alpha/r)\, d\theta]. \tag{34}$$

The constant of integration C_4 enters linearly into the expression of α by the integration of the linear equation (31).

(c) The constant of integration a_1 in (32) and another constant a_2 obtained by the integration of that equation are calculated by specifying the initial condition of the null thrust arc. Besides these orbital constants, the components α, β, of the primer depend linearly on six constants of integration, the C_i, and the Hamiltonian constant H.

(d) The reason for which we have only a first-order differential equation for the primer vector $\mathbf{p}_v$ can be seen by inspecting Eqs. (12)–(13) and the first integral (17). Along a ballistic arc, the state and adjoint equations are decoupled, and hence, after these three equations are solved, they give the adjoint vectors $\mathbf{p}_v$ and $\mathbf{p}_r$. We have two first-order vector differential equations, and one vector relation, all of them linear in $\mathbf{p}_v$ and $\mathbf{p}_r$. Hence, using the vector integral, we can delete one differential equation (in our case the $\mathbf{p}_r$-equation) and, upon elimination of $\mathbf{p}_r$, we have a first-order differential equation in $\mathbf{p}_v$. For this reason, if we choose to delete the $\mathbf{p}_v$-equation, we will come up with a linear differential equation in $\mathbf{p}_r$, a detail which the reader can easily verify.

4. Conclusions

In this paper, we have shown that, in the problem of orbital transfer, the solution of the optimal coasting arc in the most general central force field can be obtained by simple quadratures. The complete solution not only can be seen as a generalization of the results obtained by Lawden, but it may have far reaching applications. For example, in problems of orbital transfer using a large planet to assist with gravity pull, the man-made satellite arrives in hyperbolic path along the equatioral plane of the perturbing planet. If the planet has an acute oblateness, then the gravitational force, although still central, has a more general form than the usual inverse-square-force law, and applications of the results in this paper will give the necessary modifications for optimal transfer as compared to the unperturbed force field. Of course, the inverse problem is the problem of escape in the equatorial plane of an oblate planet. Another application is the problem of station keeping around the Lagrangian points in the three-body problem. In this problem, the resulting force acting on the satellite is also central, and the equation of the coasting arc can also be obtained. Hence, the form of the primer vector also can be obtained by simple quadratures.

References

1. Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths Publishers, London, England, 1963.
2. Vinh, N. X., *Exact Relations of Optimum Switching in the Problem of Impulsive Transfer*, Journal of the Astronautical Sciences, Vol. 17, No. 6, 1970.
3. Brookes, C. J., and Smith, J., *Optimum Rocket Trajectories in a General Force-Field*, Astronautica Acta, Vol. 15, No. 3, 1970.
4. Marchal, C., *Généralisation Tridimensionnelle et Étude de l'Optimalité des Arcs à Poussée Intermédiaire de Lawden (Dans un Champ Newtonien)*, La Recherche Aérospatiale, No. 123, 1968.
5. Vinh, N. X., and Marchal, C., *The Lawden's Singular Arcs in a Newtonian Force Field*, Paper presented at the 21st International Astronautical Congress, Constance, Germany, 1970.
6. Pines, S., *Constants of the Motion for Optimum Thrust Trajectories in a Central Force Field*, AIAA Journal, Vol. 2, No. 11, 1964.
7. Marec, J. P., *Transferts Optimaux Entre Orbites Elliptiques Proches*, *ONERA*, Report No. 121, 1967.