INTEGRO-DIFFERENTIAL OPERATORS ON VECTOR BUNDLES

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Introduction. This article considers a fairly general class of operators on sections of a vector bundle over a compact manifold, including the "smooth" differential operators and singular integral operators. The members of this class share many of the properties of differential operators, particularly the elliptic ones. Two general advantages have motivated the development. First, it leads to transparent proofs of the familiar results for elliptic equations, on regularity, the Fredholm alternative, and eigenfunction expansions; and for a larger class than the differential operators. These proofs are not new; rather some of the techniques used in the case of differential operators appear here as general properties of the class of integro-differential operators considered. A second advantage of the larger system, not extensively exploited in this article, is topological. Homotopies (in the class of smooth functions) of the characteristic polynomial of a differential operator can be "lifted" to homotopies of the operator itself in the class of integro-differential operators considered, but not (generally) in the class of differential operators. This is an important help in treating some questions raised by Gelfand [6]; some of the questions concerning the index have now been answered by Atiyah and Singer [1].

To find the notation and main results, one can read §1–§3 (except for proofs), §6, and the definitions and statements of theorems and corollaries from the remaining sections.

The paper is organized as follows. §1 describes the well-known function spaces on $\boldsymbol{R}^n$ that are involved, as well as certain operators on them. §2 describes the singular integral operators and their symbols. §3 extends this collection to one that contains the differential operators on $\boldsymbol{R}^n$, as well as the inverses of the invertible elliptic operators. The symbol $\sigma(A)$ of an operator A is defined, and the behavior of σ under composition of operators is discussed. §4 considers the behavior of σ under coordinate changes. §5 gives some necessary lemmas from functional analysis. §6 establishes the notation for vector bundles, and the analogs for bundles of the function spaces of §1. §7 defines the singular integral operators on sections of a vector bundle E over a compact manifold X, and their symbols. If A is a singular integral operator from sections of one bundle E

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to sections of another bundle F over X, $\sigma(A)$ is defined as a fibre-preserving map of the cotangent space of X into the bundle of homomorphisms of E into F. Theorem 7.1 characterizes the class of symbols, and Theorem 7.2 relates manipulations of the operators (sum, product, adjoint) to manipulations of symbols. §8 applies these results to obtain the theorems on solvability of elliptic equations (Theorem 8.3) and regularity of solutions (Theorem 8.1, corollary). §9 gives some further results in the same direction. §10 gives some theorems on expansions in eigenfunctions of self-adjoint operators, characterizing the expansion of C^∞ sections.

The remaining sections consider various closures of the class of operators considered here, in order to give a satisfactory account of the effect of tensoring, and the homotopy invariance of the index(2).

Some of the results given here have been announced by Dynin [5]; his announcement has affected the structure of this article. The author is also indebted to Professors Atiyah, Bott, and particularly I. M. Singer (who read a somewhat preliminary version) for questions, discussion, and suggestions.

1. $H^r(\boldsymbol{R}^n)$. $\boldsymbol{R}^n$ denotes real n-dimensional Euclidean space. Its points are denoted by $x=(x_1,\cdots,x_n), y$, etc.; the inner product is $(x,y)=\sum_1^n x_jy_j$, the norm is $|x|=(x,x)^{1/2}$. S^{n-1} is the unit sphere $\{|x|=1\}$. If $\alpha=(\alpha_1,\cdots,\alpha_n)$ is an n-tuple of non-negative integers, then $|\alpha|=\sum\alpha_j$, and $x^\alpha=\prod_1^n(x_j)^{\alpha_j}$. D^α and $(\partial/\partial x)^\alpha$ are notations for the partial derivative $\prod_1^n(\partial/\partial x_j)^{\alpha_j}$. $f\hat{}$ is the Fourier transform of f, $f\hat{}(\xi)=(2\pi)^{-n/2}\int e^{i(x,\xi)}f(x)dx$. Then

$$(D^\alpha f)\hat{}(\xi) = (-i\xi)^\alpha f\hat{}(\xi). \tag{1}$$

$L^2(\boldsymbol{R}^n)$ is the usual Hilbert space, with inner product $(f,g) = \int fg^* = \int f\hat{}\,g\hat{}^*$, both integrals being over $\boldsymbol{R}^n$. Here $*$ denotes complex conjugate.

For integer $r\geqq 0$, $H^r(\boldsymbol{R}^n)=\{f: f\in L^2$, and $\|f\|_r<\infty\}$, where

$$\|f\|_r = \sqrt{\int |f\hat{}(\xi)|^2(1+|\xi|^2)^r d\xi}. \tag{2}$$

Thus $H^r(\boldsymbol{R}^n)\subset H^{r-1}(\boldsymbol{R}^n)$ for $r>0$, $H^0(\boldsymbol{R}^n)=L^2(\boldsymbol{R}^n)$, and $\|f\|_0=\sqrt{(f,f)}$. The norm $\|\ \|_r$ defines the topology of $H^r(\boldsymbol{R}^n)$, which is a Banach space. $\|\ \|$ will generally be used in place of $\|\ \|_0$.

Note that, in view of (1) and the fact that every polynomial of degree $\leqq r$ is dominated by a constant times $(1+|\xi|^2)^{r/2}$, $H^r(\boldsymbol{R}^n)$ (r a positive integer) consists of all functions in L^2 whose derivatives (in the sense of distributions) of order $\leqq r$ are also in L^2.

For $r<0$, $H^r(\boldsymbol{R}^n)$ is the anti-dual of $H^{-r}(\boldsymbol{R}^n)$, i.e., the space of all continuous mappings $\lambda: H^{-r}(\boldsymbol{R}^n)\to \boldsymbol{C}$ such that $\lambda(af+g)=(a^*)(\lambda(f))+\lambda(g)=(a\lambda)(f)+\lambda(g)$.

(2) *Added in proof.* An appendix derives formulas for the index of elliptic operators on sections of trivial bundles over Euclidean space, and over two-dimensional manifolds.

(Here $a \in C$, f and g are in $H^{-r}(R^n)$.) As usual, $\|\lambda\|_r = \sup|\lambda(f)|$ for $\|f\|_{-r} = 1$. It is very well known that L^2 is isomorphic to its anti-dual by the correspondence $g \leftrightarrow \lambda_g$, $\lambda_g(f) = (g,f)$. An even more obvious identification allows us to write $H^r(R^n) \subset H^{r-1}(R^n)$ for all $r \leqq 0$. Thus, for all $r = 0, \pm 1, \cdots, H^r(R^n)$ is embedded in a norm-decreasing manner into $H^{r-1}(R^n)$.

For each $r = 0, \pm 1, \cdots$, $H^{-r}(R^n)$ is isomorphic to the anti-dual of $H^r(R^n)$.

One could also make the following equivalent definition: $H^r(R^n) = \{f : f$ is a tempered distribution, $f^\wedge$ is a locally integrable function, and $\int |f^\wedge(\xi)|^2(1 + |\xi|^2)^r d\xi < \infty\}$, for $r = 0, \pm 1, \cdots$. Then formula (2) defines $\| \ \|_r$ for all r.

$B^\infty(R^n)$ is the set of all functions $f: R^n \to C$ such that f and all its partial derivatives are bounded. (It follows that all these derivatives are continuous.) A base of neighborhoods at 0 is given by $U_m = \{f : f \in B^\infty(R^n), |D^\alpha f(x)| < 1/m$ for all $x \in R^n, |\alpha| \leqq m\}$. Since f is continuous and bounded if $f^\wedge \in L^1(R^n)$, it follows from formulas (1) and (2) that if $f \in H^r(R^n)$ for some $r > |\alpha| + n/2$, then $D^\alpha f$ is continuous and bounded by some constant times $\|f\|_r$. (This is a primitive form of Soboleff's inequality.) Thus the inverse limit $H^\infty(R^n) = \bigcap H^r(R^n)$ is included continuously in $B^\infty(R^n)$.

The direct limit $H^{-\infty}(R^n) = \bigcup H^r(R^n)$ is isomorphic in an obvious way to the anti-dual of $H^\infty(R^n)$. Each $H^r(R^n)$ $(r = 0, \pm 1, \cdots, \pm \infty)$ is closed under multiplication by functions in $B^\infty(R^n)$. $H^\infty(R^n)$ is dense in $H^r(R^n)$ for $r = 0, \pm 1, \cdots, \pm \infty$. (The last two observations have close analogs in the usual theory of distributions.)

We will be concerned with linear maps $A: H^\infty \to H^\infty$. We say that such an A is continuous: $H^r \to H^s$ if $\sup \|Af\|_s < \infty$, the sup being over all f in H^∞ with $\|f\|_r = 1$. If A is continuous: $H^k \to H^{k-r}$, for all $k = 0, \pm 1, \cdots$, we say that A is *of order* $\leqq r$. In particular, $D^\alpha = (\partial/\partial x)^\alpha$ is of order $|\alpha|$, i.e., of order $\leqq |\alpha|$ and not of order $\leqq |\alpha| - 1$. If $\Delta = -\sum_1^n (\partial/\partial x_j)^2$, then $I + \Delta$ is a linear isometry: $H^r(R^n) \to H^{r-2}(R^n)$ for $|r| < \infty$.

If A and A^* are both linear maps: $H^\infty(R^n) \to H^\infty(R^n)$, and $(Af, g) = (f, A^*g)$ for all $f, g \in H^\infty(R^n)$, then A^* is called the *formal adjoint* of A. Since H^∞ is dense in $H^{-\infty}$, A^* is unique, and it is clear that A is the formal adjoint of A^*. Not every map A has a formal adjoint; but if A is of order $\leqq r$ for some $r < \infty$, then its extension $A_k: H^k \to H^{k-r}$ is bounded and has an adjoint $A_k^*: H^{r-k} \to H^{-k}$. Thus any operator of order $\leqq r < \infty$ has a formal adjoint A^* of order $\leqq r$, the restriction of the A_k^* to $H^\infty(R^n)$.

The operators of order zero carry a natural topology, which can be described by the neighborhoods of zero $U_m = \{A : A$ is of order zero and $\sup \|Af\|_k / \|f\|_k \leqq 1/m$ for $|k| \leqq m\}$. We call this the order zero topology; it makes the operators of order zero into a Frechet space.

2. **Singular integral operators.** We define the Riesz operator R^α by $(R^\alpha f)^\wedge(\xi) = (\xi/|\xi|)^\alpha f^\wedge(\xi)$. If the a_α are functions in B^∞, then any finite sum

$\sum a_\alpha R^\alpha$ defines an operator of order zero. The B^∞ singular integral operators are the closure of the space spanned by these finite sums, in a certain topology to be described.

These operators arise in connection with partial differential equations as follows. Define Λ_0 by $(\Lambda_0 f)\hat{}(\xi) = |\xi| f\hat{}(\xi)$. Then any B^∞ differential operator $L = \sum_{|\alpha|=m} a_\alpha D^\alpha$, homogeneous of order m, has a representation $L = (\sum_{|\alpha|=m} a_\alpha R^\alpha)(-i\Lambda_0)^m$. In this factorization the second term $(-i\Lambda_0)^m$ is easily analyzed, and the first term is an operator of order zero, which allows one to apply some analytic techniques not directly available for differential operators.

These operators are called singular integral operators because of their representation by means of singular convolutions. For instance, if $\alpha = (1,0,\cdots,0)$, then

$$(a_\alpha R^\alpha f)(x) = (\text{const}) \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} a_\alpha(x)(x_1 - y_1)|x-y|^{-n-1} f(y)dy,$$

for f in $H^\infty(\boldsymbol{R}^n)$. (If f is in $H^0(\boldsymbol{R}^n)$, the above formula holds almost everywhere, as shown in [2].) Important results on composition and change of variable are proved by using this representation. We refer to other articles for these results, and prove here only those that are easily established by using Fourier transforms.

The symbol of $\sum a_\alpha R^\alpha$ is obtained by taking the Fourier transform of the convolution terms: $\sigma(\sum a_\alpha R^\alpha)(x,\xi) = \sum a_\alpha(x)(\xi/|\xi|)^\alpha$. This is analogous to the characteristic polynomial of a differential operator. The symbol $\sigma(\sum a_\alpha R^\alpha)$ has certain properties which we formalize in defining $B^\infty(\boldsymbol{R}^n \times S^{n-1})$: this is the set of all functions $h(x,\xi)$ on $\boldsymbol{R}^n \times \{\xi : |\xi| > 1\}$, such that $h(x,t\xi) = h(x,\xi)$ for $t > 1$, and all of whose derivatives are bounded on $\boldsymbol{R}^n \times \{|\xi| > 1\}$. $B^\infty(\boldsymbol{R}^n \times S^{n-1})$ carries a natural topology with neighborhoods of zero given by

$$U_m = \{h : |(\partial/\partial x)^\alpha (\partial/\partial \xi)^\beta h(x,\xi)| < 1/m \text{ for } |\alpha| + |\beta| < m,\ |\xi| > 1,\ x \text{ in } \boldsymbol{R}^n\}.$$

THEOREM 2.1. (i) *If $\sum a_\alpha R^\alpha$ and $\sum b_\beta R^\beta$ define the same operator on $H^0(\boldsymbol{R}^n)$, then $\sigma(\sum a_\alpha R^\alpha) = \sigma(\sum b_\beta R^\beta)$.* (ii) *$\sigma$ is a 1-1 map of the space of operators of the form $\sum a_\alpha R^\alpha$, into $B^\infty(\boldsymbol{R}^n \times S^{n-1})$.* (iii) *$\sigma^{-1}$ extends by continuity to a one-one map σ^{-1} of $B^\infty(\boldsymbol{R}^n \times S^{n-1})$ into the operators of order zero on $H^\infty(\boldsymbol{R}^n)$, in the order zero topology.*

This is proved in [3], and in [10]. Part (i) also follows from Theorem 2.2 below. For part (iii), see also [11, pp. 665–666].

On the basis of Theorem 2.1, we make the following definition.

DEFINITION 2.1. The B^∞ singular integral operators on $H^\infty(\boldsymbol{R}^n)$ are the operators in $\sigma^{-1}(B^\infty(\boldsymbol{R}^n \times S^{n-1}))$. If $A = \sigma^{-1}(h)$, then $\sigma(A) = h$.

In particular, this class contains the multipliers $M_\phi : M_\phi f = \phi f$, for ϕ in

$B^\infty(\boldsymbol{R}^n)$; and $\sigma(M_\phi)(x,\xi) = \phi(x)$. We shall generally use ϕ for both the function and the multiplier, since it is clear from the context which is intended.

The following result, which sheds some light on the connection between symbols and operators, is essentially due to Gohberg [7].

THEOREM 2.2. *Given x_0 in $\boldsymbol{R}^n$, ξ_0 in $\boldsymbol{R}^n$, $\xi_0 \neq 0$, there is a sequence $\{\phi_m\}$ of functions in $B^\infty(\boldsymbol{R}^n)$ such that* (i) *$\phi_m(y) = 0$ for $|x_0 - y| > 1/m$,* (ii) *$\int |\phi_m(y)|^2 dy = 1$, and* (iii) *for each B^∞ singular integral operator A, $\| A\phi_m - \sigma(A)(x_0,\xi_0)\phi_m \| \to 0$.*

Proof. By density, it suffices to consider $A = \sum a_\alpha R^\alpha$. Since properties (i) and (ii) imply that $\| a_\alpha \phi_m - a_\alpha(x_0)\phi_m \| \to 0$, it is enough to exhibit a sequence ϕ_m with properties (i) and (ii) and such that $\| R^\alpha \phi_m - (\xi_0/|\xi_0|)^\alpha \phi_m \| \to 0$. In [12, Lemma 15], are exhibited functions ψ_m in B^∞ such that (a): $\int |\psi_m|^2 = 1$, (b): $\int_{|y-x_0|>1/2m} |\psi_m(y)|^2 dy < 1/2m$, and (c): the support of $(\psi_m)^\wedge$ is contained in $\{|\xi_0 - (\xi/|\xi|)| < 1/m\}$. By (a) and (c), $\| R^\alpha \psi_m - (\xi_0/|\xi_0|)^\alpha \psi_m \| \to 0$. To obtain ϕ_m it suffices to multiply ψ_m by a B^∞ function θ_m such that $|\theta_m| \leqq 1$, $\theta_m = 0$ in $|y - x_0| > 1/m$, $\theta_m = 1$ in $|y - x_0| < 1/2m$; and let $\phi_m = \theta_m \psi_m / \| \theta_m \psi_m \|$.

3. **B^∞ operators of order r.** Following the example of Dynin [5], we fit the singular integral operators and differential operators into a common framework. In order to do this briefly, we introduce the isomorphism $\Lambda: H^k(\boldsymbol{R}^n) \to H^{k-1}(\boldsymbol{R}^n)$ given by $(\Lambda f)^\wedge(\xi) = (1 + |\xi|^2)^{1/2} f^\wedge(\xi)$.

DEFINITION 3.1. A B^∞ operator of order r is any of the form $A = B\Lambda^r$, where B is a B^∞ singular integral operator. We set $\sigma(A)(x,\xi) = |\xi|^r \sigma(B)(x,\xi)$.

Given any function h in $B^\infty(\boldsymbol{R}^n \times S^{n-1})$, there is a unique A, a B^∞ operator of order r, such that $\sigma(A)(x,\xi) = |\xi|^r h(x,\xi)$.

The result of composition of two such operators, or change of variable in one of them, leads to certain remainder terms of lower order. We use S_r as a generic symbol for these remainders: S_r denotes any operator which is continuous: $H^k(\boldsymbol{R}^n) \to H^{k-r}(\boldsymbol{R}^n)$ for all k.

Recall that $(\Lambda_0 f)^\wedge(\xi) = |\xi| f^\wedge(\xi)$. Since $[(1 + |\xi|^2)^{r/2} - |\xi|^r](1 + |\xi|^2)^{1-r/2}$ is a bounded function of ξ for each $r > 0$, it follows that $\Lambda^r = \Lambda_0^r + S_{r-2}$. Thus since $D^\alpha = R^\alpha(-i\Lambda_0)^{|\alpha|}$, the B^∞ operators of order r include the differential operators of order r, modulo operators of lower order.

The following result is an easy extension of Theorem 5 in [3].

THEOREM 3.1. *Let B_j $(j = 1,2,3,4)$ be B^∞ singular integral operators (of order zero) on $H^\infty(\boldsymbol{R}^n)$ and $\sigma(B_1)\sigma(B_2) = \sigma(B_3), \sigma(B_4) = \sigma(B_1)^*$. Then*

(i) *$B_1\Lambda^r + B_2\Lambda^r$ is a B^∞ operator of order r, and $\sigma(B_1\Lambda^r + B_2\Lambda^r) = \sigma(B_1\Lambda^r) + \sigma(B_2\Lambda^r)$.*

(ii) *$B_1\Lambda^r B_2 \Lambda^p = B_3 \Lambda^{r+p} + S_{r+p-1}$.*

(iii) *$(B_1\Lambda^r)^* = B_4\Lambda^r + S_{r-1}$.*

Proof. (i) is clear. (ii) is proved in [3, Theorem 5], and in [10, p. 7–05], for $r=p=0$. The case for $r\neq 0$ thus reduces to showing $\Lambda^r B_2 = B_2\Lambda^r + S_{r-1}$. If $r>0$, this reduces in turn to showing that $\Lambda_0 B_2 = B_2\Lambda_0 + S_0$, which is proved in [3, Theorem 5]. If $r<0$, then $\Lambda^r B_2 = \Lambda^r B_2 \Lambda^{-r}\Lambda^r = \Lambda^r\Lambda^{-r}B_2\Lambda^r + \Lambda^r S_{-r-1}\Lambda^r$ (by the case just proved), so $\Lambda^r B_2 = B_2\Lambda^r + S_{r-1}$.

(iii) is also shown in [3], for $r=0$. In general we have $(B_1\Lambda^r)^* = \Lambda^r B_1^* = B_1^*\Lambda^r + S_{r-1}$, and from the case $r=0$, $B_1^*\Lambda^r = B_4\Lambda^r + S_{-1}\Lambda^r$. This completes the proof.

4. **Coordinate changes.** If U is open in $\boldsymbol{R}^n$, $B^\infty(U)$ denotes those functions in $B^\infty(\boldsymbol{R}^n)$ which vanish except in some compact subset of U. If V is also open in $\boldsymbol{R}^n$ and $h: V\to U$ is a C^∞ diffeomorphism of V onto U, then there is an associated map $h^*: B^\infty(U)\to B^\infty(V)$ given by $(h^*f)(y) = f(h(y))$ for y in V, $h^*f(y)=0$ for y not in V.

If L is any B^∞ differential operator of order r, then L maps $B^\infty(U)\to B^\infty(U)$, for each U. We have $h^*L = L'h^* + S$, where S is a differential operator of order $<r$, and $\sigma(L')(y,\eta) = \sigma(L)(h(y),\xi)$ (wherein $\eta_k = \sum \xi_j(\partial h_j/\partial y_k)(y)$, i.e., $\eta = (\partial h/\partial y)\xi)$). Thus for differential operators the symbol $\sigma(L)$ is appropriately viewed as a function on the cotangent bundle of $\boldsymbol{R}^n$. This is easy to interpret in the case of first order operators, i.e., vector fields. Then L is a section of the tangent bundle of $\boldsymbol{R}^n$, and hence can be viewed as a function on the dual tangent bundle, i.e., on the cotangent bundle. This function is precisely $\sigma(L)$.

A similar result holds for B^∞ operators.

THEOREM 4.1. *Let A be a B^∞ operator of order r on $\boldsymbol{R}^n$, and let $\sigma(A)(x,\xi)=0$ if $x\notin C$, where C is some compact subset of $\boldsymbol{R}^n$. Let U be open, $C\subset U$, and $h: V\to U$ be a diffeomorphism. Then $h^*A = A'h^* + S_{r-1}$, where S_{r-1} maps $B^\infty(U)$ into $B^\infty(V)$, and $\sigma(A')(y,(\partial h/\partial y)\xi) = \sigma(A)(h(y),\xi)$.*

Proof. The condition on $\sigma(A)$ guarantees that A maps $B^\infty(U)$ into $B^\infty(U)$, so that h^*A makes sense.

In the case $r=0$, the result is established in [11, pp. 674–677].

In case $r\geqq 1$, $A = B\Lambda^r = B\Lambda_0^r + S_{r-2}$, so we may consider the transformation of $B\Lambda_0^r$. Now $\Lambda_0 = \sum_{|\alpha|=1} iR^\alpha D^\alpha$, and $R^\alpha D^\alpha = D^\alpha R^\alpha$, so $\Lambda_0^r = \sum_{|\alpha+\beta|=r}\gamma_\alpha R^\beta D^\alpha$, where the γ_α are constants. Thus $B\Lambda_0^r = \sum_{|\alpha|=r} B_\alpha D^\alpha$, where the B_α are B^∞ singular integral operators of order zero whose symbols vanish off C. Thus it suffices to check the transformation of BD, where B is a B^∞ singular integral operator and D a differential operator of order r. In this case we have $h^*BD = B'h^*D + S_{-1}D = B'D'h^* + S_{-1}D + B'S_{r-1}$. Now the symbol of B' is known from the case $r=0$, and we checked the symbol of D' before stating the present theorem. Thus the result for $r>0$ follows from Theorem 3.1, part (ii).

The case $r<0$ can be referred to $r\geqq 0$ by liberal use of Theorem 3.1. Let $B, B_1, \cdots, B_4$ denote B^∞ singular integral operators of order zero; let $\sigma(B)$ vanish off some compact subset of U; let $\psi \in B^{\infty}(U)$ and $\psi\sigma(B) = \sigma(B)$; let $\phi = h^*\psi$; and finally let the T^j denote maps that are bounded : $H^k \to H^{k-r+1}$ for all k. Then since $\sigma(B)\sigma(\psi) = \sigma(B)$ we have, from Theorem 3.1, that $B = B\psi + S_{-1}$. Thus $h^*B\Lambda^r = h^*B\psi\Lambda^r + T^1 = B_1 h^*\psi\Lambda^r + T^2$, where $\sigma(B_1)$ is known from the case $r = 0$. Continuing in this vein, we get

$$\begin{aligned} h^*B\Lambda^r &= B_1 h^*\psi\Lambda^r + T^2 = \phi B_1 \Lambda^r \Lambda^{-r} h^*\psi\Lambda^r + T^2 \\ &= B_1\Lambda^r(\phi\Lambda^{-r})h^*\psi\Lambda^r + T^3 = B_1\Lambda^r h^* B_2 \Lambda^{-r}\psi\Lambda^r + T^4 \\ &= B_1\Lambda^r h^* B_3 + T^5 = B_1\Lambda^r B_4 h^* + T^6 = B'\Lambda^r h^* + T^7. \end{aligned}$$

The symbols of the B_j, and finally of B', may be calculated from the case $r \geqq 0$ of the present theorem, or from Theorem 3.1, to complete the proof.

5. **Results related to compactness.**

LEMMA 5.1 (RELLICH'S LEMMA). *Let A be a bounded operator: $H^{k-1}(\mathbf{R}^n) \to H^{r+k}(\mathbf{R}^n)$, let $\psi \in B^{\infty}(\mathbf{R}^n)$, and let ψ have compact support. Then $A\psi$ and ψA are compact operators: $H^k(\mathbf{R}^n) \to H^{r+k}(\mathbf{R}^n)$.*

Proof. $A = (A\Lambda)\Lambda^{-1}$, and $A\Lambda$ is bounded: $H^k(\mathbf{R}^n) \to H^{r+k}(\mathbf{R}^n)$. Thus it suffices to show that $\Lambda^{-1}\psi$ is compact on $H^k(\mathbf{R}^n)$. Let J_S be defined by $(J_S f)\hat{}(\xi) = (1 + |\xi|^2)^{-1/2} f\hat{}(\xi)$ if $|\xi| \leqq S$, $(J_S f)\hat{}(\xi) = 0$ if $|\xi| > S$. Then $\| J_S - \Lambda^{-1} \|_k = (1 + S^2)^{-1/2} \to 0$ as $S \to \infty$, so it suffices to show $J_S\psi$ is compact on $H^k(\mathbf{R}^n)$. Suppose then that $\int |f_n\hat{}(\xi)|^2 (1 + |\xi|^2)^k d\xi \leqq 1$. We will show that the convolutions $\psi\hat{} * f_n\hat{}$ are uniformly equicontinuous on $|\xi| \leqq S$, from which it follows that some subsequence of $J_S\psi f_n$ converges in H^k, by Arzela's theorem. For the equicontinuity, $|\psi\hat{} * f_n\hat{}(\xi) - \psi\hat{} * f_n\hat{}(\eta)| = |\int f_n\hat{}(Z)[\psi\hat{}(\xi - Z) - \psi\hat{}(\eta - Z)]dZ| \leqq \| e_\xi\psi - e_\eta\psi \|_{-k}$, where $e_\xi(y) = \exp(i(\xi, y))$. If $k \geqq 0$,

$$\| e_\xi\psi - e_\eta\psi \|_{-k} \leqq \| e_\xi\psi - e_\eta\psi \|_0 \leqq \max_{|y| \leqq R} |e_\xi(y) - e_\eta(y)| \, \| \psi \|_0 \leqq R|\xi - \eta| \, \| \psi \|_0,$$

where R is such that $\psi(y) = 0$ for $|y| \geqq R$. If $k = -m < 0$, then $\| e_\xi\psi - e_\eta\psi \|_{-k}$ can be bounded by a combination of the H^0 norms of $e_\xi\psi - e_\eta\psi$ and its derivatives of order $\leqq m$, which leads again to $\| e_\xi\psi - e_\eta\psi \|_m \leqq C|\xi - \eta|$, where C depends on R, S, and the derivatives of ψ of order $\leqq m$.

The result for $A\psi$ follows by considering adjoints.

LEMMA 5.2. *If A is a B^{∞} operator of order r, ϕ and ψ are in $B^{\infty}(\mathbf{R}^n)$, and $\phi(x_0)\psi(x_0)\sigma(A)(x_0, \xi_0) \neq 0$ for some x_0 and ξ_0, then $\phi A\psi$ is neither compact: $H^k(\mathbf{R}^n) \to H^{k-r}(\mathbf{R}^n)$, nor bounded: $H^{k-1}(\mathbf{R}^n) \to H^{k-r}(\mathbf{R}^n)$.*

Proof. By Lemma 5.1, it suffices to show $\phi A\psi$ is not compact: $H^k(\mathbf{R}^n) \to H^{k-r}(\mathbf{R}^n)$. Since Λ is an isomorphism: $H^m(\mathbf{R}^n) \to H^{m-1}(\mathbf{R}^n)$, this is equivalent to showing

$\Lambda^{k-r}\phi A\psi\Lambda^{-k}$ is not compact on $H^0(\boldsymbol{R}^n)$. But by Theorem 3.1, $\Lambda^{k-r}\phi A\psi\Lambda^{-k} = B + S_{-1}$, where $\sigma(B)(x_0,\xi_0) = |\xi_0|^{-r}\phi(x_0)\psi(x_0)\sigma(A)(x_0,\xi_0) \neq 0$, and B is a B^∞ singular integral operator of order zero. Considering the sequence $\{\phi_m\}$ of Theorem 2.2, we have on the one hand that $\|B\phi_m\| \to |\sigma(B)(x_0,\xi_0)| \neq 0$. On the other hand, if $B + S_{-1}$ were compact we would have $\|B\phi_n\| \to 0$, as follows. The supports of the ϕ_m shrink to a point, so ϕ_m converges weakly to zero in $H^0(\boldsymbol{R}^n)$. Further if $\psi(y) = 1$ for $|y - x_0| \leqq 1$ and ψ is in $B^\infty(\boldsymbol{R}^n)$ with compact support, then $S_{-1}\psi$ is compact (from Lemma 5.1), and $S_{-1}\phi_m = S_{-1}\psi\phi_m \to 0$. Finally, if $B + S_{-1}$ is compact, it follows that $\|B\phi_m\| \to 0$.

LEMMA 5.3. *If $r < -n/2$, and ψ is in $H^0(\boldsymbol{R}^n)$, then $\psi\Lambda^r$ is a Hilbert-Schmidt operator on $H^0(\boldsymbol{R}^n)$; and $\Lambda^r\psi$ extends by continuity from a map: $H^\infty(\boldsymbol{R}^n) \to H^0(\boldsymbol{R}^n)$ to a Hilbert-Schmidt operator on $H^0(\boldsymbol{R}^n)$.*

Proof. First, if $f \in H^0(\boldsymbol{R}^n)$ then $(\Lambda^r f)^\wedge$ is in L^1,

$$\int |(\Lambda^r f)^\wedge(\xi)|\,d\xi \leqq \left[\int (1 + |\xi|^2)^r d\xi\right]^{1/2} \|f\|,$$

so $\|\psi\Lambda^r f\| \leqq C\|\psi\|\,\|f\|$, and $\psi\Lambda^r$ is bounded on $H^0(\boldsymbol{R}^n)$. Moreover, $\Lambda^r f(x) = \int K_r(x-y)f(y)dy$, where $K_r \in H^0(\boldsymbol{R}^n)$. Thus the kernel k of $\psi\Lambda^r$ satisfies $\int\int |k(x,y)|^2 dydx = \int\int |\psi(x)K_r(x-y)|^2 dydx < \infty$.

The case for $\Lambda^r\psi$ is referred to $\psi^*\Lambda^r$ by taking adjoints.

6. **Vector bundles.** For a brief discussion of vector bundles, see Lang [9, Chapter III].

Let X be a C^∞ compact manifold of dimension n, without boundary. E and F denote C^∞ vector bundles over X, of complex dimension p. A *section* is a (not necessarily continuous) map $f: X \to E$ such that $\pi f =$ identity, where π is the projection of E onto X. The fibre $\pi^{-1}(x)$ is denoted by E_x. Each point x_0 in X has a neighborhood U such that there are a C^∞ diffeomorphism χ of U onto an open subset V of $\boldsymbol{R}^n$, and p C^∞ sections $\{\alpha_m\}$ such that $\{\alpha_m(x)\}$ is a basis of E_x for each x in U. A collection such as $\{\alpha_m\}$ is called a C^∞ basis of sections over U, or simply a basis of E_U.

If Y is another C^∞ manifold and F a q-dimensional complex vector bundle over Y, then HOM(E,F) is a bundle over the product manifold $X \times Y$, whose fibre $H_{(x,y)}$ at (x,y) is the space of all linear transformations of E_x into F_y. The topology of HOM(E,F) is the compact-open topology of maps of E into F. The manifold structure of HOM(E,F) is determined in an obvious way by local bases of sections of E and F. If $Y = X$, then Hom(E,F) is defined as the restriction of HOM(E,F) to the diagonal of $X \times Y$.

We will assume a C^∞ volume element on the manifold X, and a C^∞ Hermitian inner product in the bundles E, etc. (but no inner product in the tangent bundle). The inner product in E is denoted by $((\ ,\ ))$, and its integral over X by $(\ ,\ )$.

Thus for any C^∞ sections f and g of E, $((f,g))$ is a C^∞ function on X, and (f,g) is a complex number. A bundle with such a structure will be called a Hermitian bundle. In such a bundle the Gram-Schmidt process provides orthonormal local bases, i.e., bases such that $((\alpha_j,\alpha_k)) = \delta_{jk}$. If E and F are Hermitian, then $\mathrm{HOM}(E,F)$ is Hermitian, with $((f_z,g_z)) = \mathrm{trace} f_z g_z^*$. Here $z=(x,y)$ is in $X \times Y$, f_z and g_z are in the fibre at (x,y), and g_z^* is the adjoint of g_z.

$T(X)$ is the tangent bundle of X. If $U \subset X$ and χ is a coordinate map of U onto $V \subset \boldsymbol{R}^n$, then χ gives a basis of sections over U which is denoted by $\{\partial/\partial\chi_j\}$. The dual basis in the cotangent bundle $T^*(X)$ is $\{d\chi_j\}$. $T'(X)$ is the subbundle of $T^*(X)$ obtained by deleting the origin from each fibre. The projection of $T'(X)$ onto X is denoted by τ.

If ϕ is a C^∞ function with support in $U \subset X$, and χ a coordinate map of U onto V in $\boldsymbol{R}^n$, then ϕ_χ is the map of functions on X into functions on $\boldsymbol{R}^n$ given by $\phi_\chi g(y) = \phi(\chi^{-1}(y))g(\chi^{-1}(y))$ for $y \in V$, and $\phi_\chi g(y) = 0$ otherwise. Similarly ϕ^χ maps functions on $\boldsymbol{R}^n$ into functions on X by $\phi^\chi f(x) = \phi(x)f(\chi(x))$ for x in U, $\phi^\chi f(x) = 0$ otherwise.

For $r \geqq 0$, $H^r(E)$ is the collection of (equivalence classes of) sections of E such that $\phi_\chi((f,\beta_j))$ is in $H^r(\boldsymbol{R}^n)$ for each coordinate map χ, each local basis $\{\beta_j\}$ over the domain of χ, and each C^∞ function ϕ with support in the domain of χ. $H^r(E)$ inherits a topology from $H^r(\boldsymbol{R}^n)$; neighborhoods of zero in $H^r(E)$ require the Euclidean H^r norms of finitely many $\phi_\chi((f,\beta_j))$ to be small. $H^r(E)$ can be made into a Banach space by using a C^∞ partition of unity $\sum \phi^j = 1$, with coordinate maps χ^j and orthonormal bases $\{\beta_k^j\}$ in the support of each ϕ^j: $(\|f\|_r)^2 = \sum_{j,k}(\|\phi^j_{\chi^j}((f,\beta^j_k))\|_r)^2$.

$H^0(E)$ is a Hilbert space with inner product $(f,g) = \int_X((f,g))$. $H^r(E)$, for $r<0$, is the anti-dual of $H^{-r}(E)$. As in the Euclidean case, we use the inner product to identify $H^0(E)$ with its anti-dual, and consider $H^r(E)$ embedded in $H^{r-1}(E)$ for all r. $H^\infty(E)$ is the inverse limit $\bigcap H^r(E)$, and is dense in $H^r(E)$ for each r. Since X is compact, $H^\infty(E)$ is the space of all C^∞ sections of E, and its anti-dual $H^{-\infty}(E) = \bigcup H^r(E)$ is essentially the space of distributional sections of E.

The inner product (f,g), defined for f and g in $H^0(E)$, extends to a pairing between $H^k(E)$ and $H^{-k}(E)$ for all k: if $k \geqq 0$, and f is (an anti-linear functional) in $H^{-k}(E)$, then $(f,g) = f(g)$ for g in $H^k(E)$. If f is in $H^k(E)$ $(k \geqq 0)$, then $(f,g) = g(f)^*$ for g in $H^{-k}(E)$.

There is an anti-isomorphism α_m between $H^{-m}(E)$ and the dual $H^m(E)$, given by $\alpha_m f(g) = (g,f)$ for g in $H^m(E)$.

The various spaces $H^k(E)$ can be defined directly in terms of distributional sections, without assuming a Hermitian structure. All the results below that can be stated without such a structure would still hold, since such a structure can always be imposed.

Let F be another Hermitian bundle over X. A linear map $A: H^\infty(E) \to H^\infty(F)$ isan operator of order $\leqq r$ if and only if for each $k = 0, \pm 1, \cdots$, A extends to a continuous map $A_k: H^k(E) \to H^{k-r}(F)$. An operator of order $\leqq r$ has a *formal adjoint* $A^*: H^\infty(F) \to H^\infty(E)$, such that $(Af, g) = (f, A^*g)$ for f in $H^\infty(E)$, g in $H^\infty(F)$. This is obtained by considering the adjoints $A_k^*: H^{r-k}(F) \to H^{-k}(E)$, of the maps $A_k: H^k(E) \to H^{r-k}(F)$.

7. **Singular integral operators on bundles.** In the following, S_r denotes an operator of order $\leqq r$, a "lower order terms" remainder. E and F are Hermitian bundles over X.

DEFINITION 7.1. A linear map $A: H^\infty(E) \to H^\infty(F)$ is a B^∞ operator of order r if and only if

(i) for ϕ and ψ in $C^\infty(X)$ with disjoint support, $\phi A \psi$ is an operator of order $\leqq r-1$, and

(ii) if $U \subset X$, χ is a coordinate map of U into $\boldsymbol{R}^n$, and $\{\beta_j\}$ and $\{\gamma_k\}$ are bases of E and F, respectively, over U, and ϕ and ψ are in $C^\infty(X)$ with support in U, then

$$\phi A \psi\left(\sum a_j \beta_j\right) = \sum_{j,k} (\phi^\chi A_{kj} \psi_\chi a_j)\gamma_k + S_{r-1},$$

where A_{kj} is a B^∞ operator of order r on $\boldsymbol{R}^n$.

The *symbol* $\sigma(A)$ is the map of $T'(X)$ into $\mathrm{Hom}(E,F)$ given by

$$\sigma(A)(\xi)\left[\sum a_j(x)\beta_j(x)\right] = \sum_{j,k} \left[\sigma(A_{kj})(\chi(x), z)\right] a_j(x)\gamma_k(x), \tag{3}$$

where $x = \tau(\xi)$, $z = (z_1, \cdots, z_n)$, $\xi = \sum z_j d\chi_j$, and x is any point such that $\phi(x)\psi(x) \neq 0$. τ is the projection of $T'(X)$ onto X.

It is not immediately clear that $\sigma(A)$ is well defined. However, by Lemma 5.2, $\sigma(A_{kj})(y,z)$ is uniquely determined, for y such that $\phi(\chi^{-1}(y))\psi(\chi^{-1}(y)) \neq 0$. Then the invariance of $\sigma(A)$ under changes of the coordinate map χ follows from Theorem 4.1. Finally, the invariance of $\sigma(A)$ for changes in the bases $\{\beta_j\}$ and $\{\gamma_k\}$ follows from Theorem 3.1, bearing in mind that multiplication M_ϕ by a C^∞ function ϕ with compact support is a B^∞ operator of order zero, with $\sigma(M_\phi)(\xi) = \phi(x)$, where ξ lies over x.

A B^∞ operator of order r is in particular an operator of order $\leqq r$, i.e., continuous from $H^k(E)$ to $H^{k-r}(F)$. Any operator A of order $\leqq r-1$ is a B^∞ operator of order r, and $\sigma(A) = 0$.

Note that if τ is the projection of $T'(X)$ onto X, and π that of $\mathrm{Hom}(E,F)$ onto X, then $\sigma(A)$ maps $\tau^{-1}(x)$ into $\pi^{-1}(x)$ for each x in X. It is also clear that if A is a B^∞ operator of order r, then $\sigma(A)$ is C^∞; and if $t > 0$, then $\sigma(A)(t\xi) = t^r\sigma(A)(\xi)$. We express this by saying that $\sigma(A)$ is a fibre-preserving C^∞ map, homogeneous of degree r, from $T'(X)$ into $\mathrm{Hom}(E,F)$.

In the next proof we require, for a given covering of X by open sets $\{U_j\}$, a collection $\{\phi_j\}$ of C^∞ functions with the support of ϕ_j in U_j, $\phi_j \geqq 0$, and $\sum \phi_j^2 = 1$. These are obtained from any C^∞ set $\{\psi_j\}$ with the support of ψ_j in U_j, $\psi_j \geqq 0$, and $\sum \psi_j > 0$, by letting $\phi_j = \psi_j\left[\sum (\psi_j)^2\right]^{-1/2}$.

THEOREM 7.1. *σ maps the set of B^∞ operators of order r linearly onto the fibre-preserving C^∞ maps, homogeneous of degree r, from $T'(X)$ into* $\mathrm{Hom}(E,F)$. *The kernel of σ is the set of operators of order $\leqq r-1$.*

Proof. That σ is linear into is clear. To show it is onto, let $\{U_k\}$ be an open covering of X such that E and F have bases $\{\beta_i^k\}$ and $\{\gamma_j^k\}$ over U_k, and such that there is a coordinate map χ^k of U_k into $\boldsymbol{R}^n$. Let ϕ_k have support in U_k, $\sum\phi_k^2 = 1$, and ϕ_k be C^∞. Let ψ be a map: $T'(X) \to \mathrm{Hom}(E,F)$ of the type considered in the statement of this theorem. Then there are C^∞ functions ψ_{ij}^k on $\tau^{-1}(U_k)$ such that $\psi(\xi)(\sum_i a_i(x)\beta_i^k(x)) = \sum_{ij}\psi_{ij}^k(\xi)a_i(x)\gamma_j^k(x)$, where $x = \tau(\xi) \in U_k$. Then there are B^∞ operators of order r on $\boldsymbol{R}^n$, A_{ij}^k, such that $\sigma(A_{ij}^k)(y,z) = \psi_{ij}^k(\xi)$, where $\xi = \sum z_j d\chi_j(x)$, $x = \tau(\xi)$, and $y = \chi^k(x)$. Now let $A^k(\sum a_i\beta_i^k) = \sum_{ij}(\phi\ A_{ij}^k\phi_\chi a_i)\gamma_j$, where $\phi = \phi_k$ and $\chi = \chi^k$; and set $A = \sum A^k$. Then A satisfies Definition 7.1, part (i), and for the given neighborhoods, coordinates, and bases it satisfies part (ii) of Definition 7.1. The same argument that shows the symbol is invariant now shows that A satisfies Part (ii) of Definition 7.1 for all neighborhoods, coordinates, and bases. Since $\sigma(A^k)(\xi) = \phi_k(\tau(\xi))^2\psi(\xi)$, it follows that $\sigma(A) = \psi$.

It has already been noted that all operators of order $\leqq r-1$ are in the kernel of σ. Conversely, if $\sigma(A) = 0$ then for the A_{kj} of Definition 7.1 we have $(\phi\circ\chi^{-1})(\psi\circ\chi^{-1})\sigma(A_{kj}) = 0$, so $\phi^\chi A_{kj}\psi_\chi$ is of order $\leqq r-1$, and $\phi^\chi A\psi_\chi$ is of order $\leqq r-1$ for each ϕ and ψ.

The main features of the B^∞ operators introduced above are given in Theorems 7.1 and 7.2. Theorem 7.2 refers to the adjoint of a symbol $\sigma(A):T'(X) \to \mathrm{Hom}(E,F)$. This is the map $\sigma(A)^*:T'(X) \to \mathrm{Hom}(F,E)$ such that $((\sigma(A)(\xi)f,g)) = ((f,\sigma(A)^*(\xi)g))$ for each f in E_x and g in F_x, where $x = \tau(\xi)$.

THEOREM 7.2. *Let E_1, F_1, E_2, and F_2 be Hermitian bundles over X. Let $A:H^\infty(E_1)\to H^\infty(F_1)$ and $B:H^\infty(E_2)\to H^\infty(F_2)$ be B^∞ operators of orders r and p respectively. Then the following results hold.*

(a) *If $r = p$, $E_1 = E_2$, and $F_1 = F_2$, then $A + B$ is a B^∞ operator of order r, and $\sigma(A+B) = \sigma(A) + \sigma(B)$.*

(b) *If $E_1 = F_2$, then AB is a B^∞ operator of order rp, and $\sigma(AB) = \sigma(A)\sigma(B)$.*

(c) *The formal adjoint A^* of A is a B^∞ operator of order r, and $\sigma(A^*) = \sigma(A)^*$.*

Proof. (a) is clear. For (b), we check first condition (i) of Definition 7.1. Let ϕ and ψ be in $C^\infty(X)$ and have disjoint supports. Choose θ so that $\theta = 1$ in a neighborhood of the support of ϕ, and so that the support of θ is disjoint from that of ψ. Then $\phi AB\psi = \phi A\theta B\psi + \phi A(1-\theta)B\psi$ is of order $\leqq r+p-1$, since $\theta B\psi$ is of order $\leqq p-1$ and $\phi A(1-\theta)$ of order $\leqq r-1$.

To check part (ii) of Definition 7.1, suppose ϕ, ψ, and χ given, and choose θ so that $\theta = 1$ on a neighborhood of the support of ϕ, while the support of θ lies in the domain of the coordinate map χ. Then write $\phi AB\psi = \phi A\theta^2 B\psi$

$+\phi A(1-\theta^2)B\psi$, and observe that the second term is of order $\leqq r+p-1$. Now let $\{\beta_j\}$ be a local basis in E_2, $\{\gamma_i\}$ a local basis in $F_2=E_1$, and $\{\delta_k\}$ a local basis in F_1. Then by Theorem 3.1,

$$\phi A\theta^2 B\psi(\textstyle\sum a_j\beta_j)=\sum_{ijk}(\phi^\chi A_{ki}\theta_\chi\theta^\chi B_{ij}\psi_\chi a_j)\delta_k=\sum_{jk}(\phi^\chi C_{kj}\psi_\chi a_j)\delta_k+S_{r+p-1},$$

where

$$\sigma(C_{kj})(y,z)=\theta^2(\chi^{-1}(y))\sum_i \sigma(A_{ki})(y,z)\sigma(B_{ij})(y,z).$$

Considering A^*, let ϕ and ψ have disjoint support. Then $\phi A^*\psi=(\psi^*A\phi^*)^*$ is an operator of order $\leqq r-1$. To obtain a representation in local coordinates, let ϕ and ψ have support in $U\subset X$, and χ be a coordinate map of U into $\mathbf{R}^n$. Let $\{\beta_i\}$ and $\{\gamma_j\}$ be orthonormal bases (over U) for E_1 and F_1 respectively. Then we have for $f=\sum a_i\beta_i$ and $g=\sum b_j\gamma_j$ that $(f,\phi A^*\psi g)=(A\phi^*f,\psi g)$ $=\sum_{ij}\int_{R^n}(\psi_\chi b_j)^*(A_{ij}\Lambda^r\phi_\chi^* a_i)v=\sum_{ij}\int_{R^n}\phi_\chi^* a_i(\Lambda^r A_{ij}^* v\psi_\chi b_j)^*$, where A_{ij} is a B^∞ singular integral operator of order zero plus an operator of order $\leqq -1$, and v gives the volume element on X in the local coordinates χ. From Theorem 3.1, the last expression is $\sum_{ij}\int\phi_\chi^* a_i(A'_{ij}\Lambda^r\psi_\chi b_j)^*v+(f,S_{r-1}g)$, where A'_{ij} is a B^∞ singular integral operator and $\sigma(A'_{ij})$ is the complex conjugate of $\sigma(A_{ij})$. This yields the representation of $\phi A\psi$, and the symbol $\sigma(A^*)$, thus completing Theorem 7.2.

Each C^∞ section γ of Hom(E,F) gives rise to a B^∞ operator of order zero, C, such that $\sigma(C)(\xi)=\gamma(\tau(\xi))$, as follows: $(Cf)(x)=\gamma(x)(f(x))$. The following analog of Theorem 2.2 describes a related local behavior for arbitrary B^∞ operators of order zero.

THEOREM 7.3. *Given $x_0\in X$, and sections α of $T^*(X)$ and β of E such that $\alpha(x_0)\neq 0$ and $((\beta,\beta))=1$ in a neighborhood of x_0, then there exists a sequence $\{\theta^m\}$ in $C^\infty(X)$ such that $\|\theta^m\beta\|=1$, the support of θ^m converges to x_0, and for each B^∞ operator A of order* 0, $\|A\theta^m\beta-[\sigma(A)\circ\alpha]\theta^m\beta\|\to 0$.

Proof. Let χ be a coordinate map in a neighborhood U of x_0, let $\{\gamma_i\}$ be a basis of F over U, and let $\{\beta_j\}$ be a basis of E over U with $\beta_1=\beta$. Let ψ equal 1 in a neighborhood of x_0, and the support of ψ lie in U. Then let $\psi A\psi(\sum a_j\beta_j)=\sum_{ij}(\psi^\chi A_{ij}\psi_\chi a_j)\gamma_i+S_{-1}(\sum a_j\beta_j)$. Let $\alpha(x)=\sum z_j(x)d\chi_j(x)$, and $z(x)=(z_1(x),\cdots,z_n(x))$, $z^0=z(x_0)$, and $\phi^m_{\chi(x_0),z^0}$ be as in Theorem 2.2. Then define $\{f^m\}$ in $C^\infty(X)$ so that $f^m=0$ where $\psi=0$, and $\psi_\chi f^m=\phi^m_{\chi(x_0),z^0}$ for m sufficiently large. Then

$$\|A_{i1}\psi_\chi f^m-\sigma(A_{i1})(\chi(\cdot),z(\cdot))\psi_\chi f^m\|\leqq$$

$$\|A_{i1}\psi_\chi f^m-\sigma(A_{i1})(\chi(x_0),z^0)\psi_\chi f^m\|+\|[\sigma(A_{i1})(\chi(\cdot),z(\cdot))-\sigma(A_{i1})(\chi(x_0),z^0)]\psi_\chi f^m\|\to 0.$$

Now for m sufficiently large, $\psi^2 f^m=f^m$, so $Af^m\beta=\psi A\psi f^m\beta+S_{-1}f^m\beta$.

Since $f^m\beta$ converges weakly to zero, we have $\|S_{-1}f^m\beta\|\to 0$ (Lemma 5.1), and consequently $\| Af^m\beta - [\sigma(A)\circ\alpha]f^m\beta \| \to 0$. Since $\|f^m\beta\|^2 = \int_X |f^m|^2 \geqq C > 0$, we can set $\theta^m = f^m \|f^m\|^{-1}$.

COROLLARY. *$\|A\theta^m\beta\|$ converges to $\|\sigma(A)(\alpha(x_0))\beta(x_0)\|$ (where $\|\phi\|^2 = ((\phi,\phi))$ for ϕ in F_{x_0}).*

Proof. $\|[\sigma(A)\circ\alpha]\theta^m\beta\|^2 = \int_X |\theta^m|^2 \|[\sigma(A)\circ\alpha]\beta\|^2$. Since $\int_X |\theta^m|^2 = 1$ and the support of θ^m converges to x_0, the result follows.

Note that if $E = F = X \times C$, then $\sigma(A)$ is essentially a map of $T'(X)$ into C, and the sections α and β need not appear explicitly in the statement of Theorem 7.3, only $\alpha(x_0)$ is needed.

DEFINITION 7.2. $\|\sigma(A)(\xi)\|$ is the norm of $\sigma(A)(\xi)$ as a linear transformation from $E_x \to F_x$, where $x = \tau(\xi)$. If A is of order zero, $|\sigma(A)| = \sup_\xi \|\sigma(A)(\xi)\|$.

Theorem 7.3 has the following corollary.

LEMMA 7.1. *For some section β of E, the sequence $\theta^m\beta$ of Theorem 7.3 satisfies $\|A\theta^m\beta\| \to \|\sigma(A)(\alpha(x_0))\|$ while $\|\theta^m\beta\| = 1$.*

Proof. We already have

$$\lim \|A\theta^m\beta\|^2 = \lim \|[\sigma(A)\circ\alpha]\theta^m\beta\|^2 = \lim \int_X \sum |\sigma(A_{i1})(\chi(\cdot), z(\cdot))\theta^m(\cdot)|^2$$

$$= \sum |\sigma(A_{i1})(\chi(x_0), z^0)|^2 = \|\sigma(A)(\alpha(x_0))\beta(x_0)\|^2.$$

Thus it suffices to choose the section β so that

$$\|[\sigma(A)(\alpha(x_0))]\beta(x_0)\|^2 = \|\sigma(A)(\alpha(x_0))\|^2.$$

Since the $\theta^m\beta$ converge weakly to zero, we find immediately

LEMMA 7.2. *If A is a B^∞ operator of order 0, and K is any compact operator from $H^0(E)$ to $H^0(F)$, then $|\sigma(A)| \leqq \|A + K\|$.*

8. Elliptic operators: the basic results.

DEFINITION 8.1. A B^∞ operator $A: H^\infty(E) \to H^\infty(F)$ is *elliptic* if and only if, for each ξ in $T'(X)$, $\sigma(A)(\xi)$ is an isomorphism of E_x onto F_x, where ξ lies over x.

The collection of elliptic B^∞ operators of order r from $H^\infty(E)$ to $H^\infty(F)$ is denoted $\mathscr{E}_r(E,F)$.

Thus if A is elliptic, $\sigma(A)(\xi)$ has an inverse $[\sigma(A)(\xi)]^{-1}$; the function $\sigma(A)^{-1}: \sigma(A)^{-1}(\xi) = [\sigma(A)(\xi)]^{-1}$ is the symbol of an elliptic operator A' which serves more or less as an inverse of A. The existence of A' is the basis of the results on existence and smoothness of solutions of elliptic equations. The regularity follows almost immediately from Theorem 7.2, so we treat that first. In Theorem 8.1, I_E is the identity: $H^\infty(E) \to H^\infty(E)$.

THEOREM 8.1. *If A is in $\mathscr{E}_r(E,F)$, then there is an A' in $\mathscr{E}_{-r}(F,E)$ such that $A'A = I_E + S'_{-1}$ and $AA' = I_F + S''_{-1}$. If B is any operator of order $\leqq -r$*

such that either $BA = I_E + S_{-1}$ *or* $AB = I_F + S_{-1}$, *then* B *is in* $\mathscr{E}_{-r}(F,E)$ *and* $\sigma(B) = \sigma(A')$.

Proof. From Theorem 7.1, there is an A' in $\mathscr{E}_{-r}(F,E)$ such that $\sigma(A')\sigma(A) = \sigma(I_E)$, and $\sigma(A)\sigma(A') = \sigma(I_F)$. From Theorem 7.2, $\sigma(AA' - I_F) = 0$ and $\sigma(A'A - I_E) = 0$. From Theorem 7.1, $AA' - I_F$ and $A'A - I_E$ are of order -1.

If $BA = I_E + S_{-1}$, then $B + BS''_{-1} = BAA' = A' + S_{-1}A'$, so $B - A'$ is of order $\leqq -1 - r$, and B is a B^∞ operator of order $-r$ with $\sigma(B) = \sigma(A')$. An identical argument holds if $AB = I_E + S_{-1}$, and the proof is complete.

COROLLARY. *If* A *is in* $\mathscr{E}_r(E,F)$, f *is in* $H^{-\infty}(E)$, *and* Af *is in* $H^k(F)$, *then* f *is in* $H^{k+r}(E)$.

Proof. Let A' be as in Theorem 8.1, and suppose f is in $H^m(E)$ with $m < k + r$. Since Af is in $H^k(F)$ and A' is of order $-r$, $A'Af = f + S_{-1}f$ is in $H^{k+r}(E)$. Thus $f = A'Af - S_{-1}f$ is in $H^{m+1}(E)$. The corollary follows by induction.

We turn now to the question of solvability of $Af = g$. From the point of view of functional analysis, the characteristic property of elliptic operators is that they satisfy a weakened form of Fredholm's alternative. We continue this section by defining this property, and stating the main general results in connection with it.

DEFINITION 8.2. Let X and Y be Banach spaces. A bounded operator $A: X \to Y$ is an F-operator if and only if $A(X)$ is closed, $A^{-1}(0)$ is finite dimensional, and $A^{*-1}(0)$ is finite dimensional.

The index of such an operator is $\operatorname{ind}(A) = \dim(A^{-1}(0)) - \dim(A^{*-1}(0))$.

A rather complete discussion of bounded and unbounded F-operators is given in [8]. Proofs of the results quoted here are also given, for the case $X = Y$, in [12]; those for $X \neq Y$ are nearly identical.

The requirement that the range $A(X)$ be closed has the following consequences. (See [4, pp. 487–488].) First, $A(X) = \{y : A^*y^* = 0 \Rightarrow y^*(y) = 0\}$. Second, $A^*(Y^*) = \{x^* : Ax = 0 \Rightarrow x^*(x) = 0\}$. Finally, $A^*(Y^*)$ is closed.

THEOREM 8.2. (i) *If* $A: X \to Y$ *and* $B: Y \to Z$ *are F-operators, then so is* BA, *and* $\operatorname{ind}(BA) = \operatorname{ind}(B) + \operatorname{ind}(A)$.

(ii) *A is an F-operator if and only if there are operators* A' *and* A'' *such that* $A'A = I_X + K'$, $AA'' = I_Y + K''$, *where* K' *and* K'' *are compact operators, and* I_X *and* I_Y *are identity operators.*

(iii) *A is an F-operator with* $\operatorname{ind}(A) \leqq 0$ *if and only if there is an operator* K *with finite range such that* $A + K$ *maps* X *isomorphically onto a closed subspace of* Y, *of finite codimension; such an* $A + K$ *has a bounded left inverse.*

(iv) *If* A *is an F-operator and* K *compact, then* $A + K$ *is an F-operator and* $\operatorname{ind}(A + K) = \operatorname{ind}(A)$.

(v) *If* A *is an F-operator, then there is an* $\varepsilon > 0$ *such that, if* $\|A - B\| < \varepsilon$, *then* B *is an F-operator and* $\operatorname{ind}(B) = \operatorname{ind}(A)$.

The next lemma connects the compact operators of the above theorem with the "lower order" operators of Theorem 8.1.

LEMMA 8.1. *If S is continuous*: $H^{k-1}(E) \to H^{k-r}(F)$, *then S is compact*: $H^k(E) \to H^{k-r}(F)$.

Proof. If $\sum \phi_i = 1$ is a C^∞ resolution of the identity then $S = \sum \phi_i S \phi_j$. Thus it suffices to show that $\phi S \psi$ is compact: $H^k(E) \to H^{k-r}(F)$ whenever the support of ϕ lies in the domain of a coordinate map over which E is trivial, and ψ has a similar support. Then the result can be transferred to a system of operators on $H^k(\mathbf{R}^n)$, to which Lemma 5.1 applies.

The proof of the fundamental result on solvability is now immediate.

THEOREM 8.3. *Let A be in $\mathscr{E}_r(E,F)$. Then* (i) *for each k, A extends to an F-operator $A_k: H^k(E) \to H^{k-r}(F)$.* (ii) *The null space of A_k is the same as the null space of $A: H^\infty(E) \to H^\infty(F)$.* (iii) *If A^* is the formal adjoint of A, then* $\operatorname{ind}(A) = \dim(A^{-1}(0)) - \dim(A^{*-1}(0))$. (iv) *The range of A_k is the set* $\{f \text{ in } H^{k-r}(F)$: $A^*g = 0$ *implies* $(f,g) = 0\}$. (v) *If A' is in $\mathscr{E}_r(E,F)$ and $\sigma(A') = \sigma(A)$, then* $\operatorname{ind}(A') = \operatorname{ind}(A)$.

Proof. The first statement follows from Theorem 8.1, Lemma 8.1, and Theorem 8.2(ii). The second statement follows from the corollary of Theorem 8.1. For statement (iii), choose $k \geqq 0$ so that $k - r \geqq 0$. Then the adjoint $A_k^*$of A_k is the extension A^*_{r-k} of the formal adjoint A^* to $H^{r-k}(F)$.

Considering the statement (iv), since the range of A_k is closed, it equals $\{f$ in $H^{k-r}(F): A_k^*\lambda = 0$ implies $\lambda(f) = 0\}$. Because of the connection between the formal adjoint and A_k^*, this set equals $\{f$ in $H^{k-r}(F): A^*_{r-k}g = 0$ implies $(f,g) = 0\}$. Finally since $\sigma(A^*) = \sigma(A)^*$, A^* is also elliptic, and the solutions g in $H^{r-k}(F)$ of $A^*_{r-k}g = 0$ are in $H^\infty(F)$, hence are the same as the solutions of $A^*g = 0$.

Finally, if $\sigma(A') = \sigma(A)$ then $A' - A$ is an operator of order $\leqq r - 1$, hence compact from $H^k(E)$ to $H^{k-r}(F)$ (by Lemma 8.1). Then from Theorem 8.2(iv), $\operatorname{ind}(A') = \operatorname{ind}(A)$.

Theorem 8.3 has a converse, given by Theorem 9.2 below.

9. Applications of the basic elliptic results.

LEMMA 9.1. *If A is in $\mathscr{E}_r(E,E)$ and $\sigma(A)^* = \sigma(A)$, then* $\operatorname{ind}(A) = 0$.

Proof. If $A^* = A$, then $\operatorname{ind}(A) = 0$ by Theorem 8.3(iii). If $\sigma(A)^* = \sigma(A)$ then $\sigma(A) = \sigma((1/2)(A + A^*))$, so by Theorem 7.1 and Lemma 8.1, A_k differs from $(1/2)(A + A^*)_k$ by a compact operator from $H^k(E)$ to $H^{k-r}(F)$. Then from Theorem 8.2(iv), $\operatorname{ind}(A) = \operatorname{ind}((1/2)(A + A^*)) = 0$.

THEOREM 9.1. *Let ψ be a fibre-preserving C^∞ map, homogeneous of degree one, from $T'(X)$ into* $\operatorname{Hom}(E,E)$, *and suppose $\psi(\xi)$ is positive definite for each ξ*

in $T'(X)$. Then there is an operator P in $\mathscr{E}_1(E,E)$ such that $(Pf,f)\geqq(f,f)$ for f in $H^0(E)$, $P^*=P$, and $\sigma(P)=\psi$. For each integer k, P^k is in $\mathscr{E}_k(E,E)$ and extends to an isomorphism of $H^s(E)$ onto $H^{s-k}(E)$ for each integer s.

Proof. Given a P in $\mathscr{E}_1(E,E)$ such that $(Pf,f)\geqq(f,f)$ and $P=P^*$, the statement about P^k follows easily. For if $f\in H^s(E)$ and $Pf=0$, then by Theorem 8.1, corollary, f is in $H^\infty(E)$, so that $0\geqq(f,f)$; thus P is 1-1 on $H^s(E)$. Since $P=P^*$, it follows from 8.3(iv) that P is onto, and then by the closed graph theorem that P is an isomorphism of $H^s(E)$ onto $H^{s-1}(E)$. Consequently P^k is an isomorphism of $H^s(E)$ onto $H^{s-k}(E)$. To show P^k is in $\mathscr{E}_k(E,E)$ for each k, we show P^{-1} is in $\mathscr{E}_{-1}(E,E)$ and refer to Theorem 7.2. But P^{-1} is of order -1, and is a regularizer for P, so that $P^{-1}\in\mathscr{E}_{-1}(E,E)$ by Theorem 8.1.

To construct a self-adjoint P with $\sigma(P)=\psi$ and $(Pf,f)\geqq(f,f)$, consider a function $\phi\geqq 0$ with support in $U\subset X$, an orthonormal basis $\{\beta_j\}$ of E_U, and a coordinate map $\chi: U\to \boldsymbol{R}^n$. Let $P^\sim$ be the matrix representing ψ in the basis $\{\beta_j\}$, and let B be a matrix of Euclidean B^∞ singular integral operators with $\sigma(B)(y,z)=[|z|^{-1}P^\sim(\sum z_j d\chi_j(x))]^{1/2}$, where $|z|^2=\sum z_j^2$ and $y=\chi(x)$. Let Λ be the operator defined in §2 $(\Lambda^2=I+\Delta)$, and set $P_\phi(\sum a_j\beta_j)=\sum_{mkj}(\phi^\chi B^*_{km}\Lambda B_{kj}\phi_\chi a_j)\beta_m$, where B_{kj} is the entry in row k and column j of the matrix of operators B, and B^*_{kj} is its formal adjoint. We now have P_ϕ in $\mathscr{E}_1(E,E)$, $\sigma(P_\phi)(\xi)=\phi^2(\tau(\xi))\psi(\xi)$, $P_\phi=P^*_\phi$, and $(P_\phi f,f)\geqq 0$.

Finally, take a collection $\{\phi_j\}$ of functions like the ϕ above, such that $\sum\phi_j^2=1$, and set $P=I+\sum P_{\phi_j}$.

Remark. The point of Theorem 9.1 is that a positive symbol ψ exists, and so an isomorphism P exists. This can be stated without reference to the Hermitian structure, by requiring that $\psi(\xi)$ be a positive multiple of the identity for each ξ.

Theorem 9.2. *If a B^∞ operator A of order r extends to an F-operator A_k from $H^k(E)$ to $H^{k-r}(F)$ for some k, then A is elliptic, and* $\dim E=\dim F$.

Proof. Let P in $\mathscr{E}_1(E,E)$ and Q in $\mathscr{E}_1(F,F)$ be isomorphisms as in Theorem 9.1. Let $B=Q^{k-r}AP^{-k}$; we will show B is elliptic, so that A is elliptic and $\dim E=\dim F$. Now $B_0=Q^{k-r}A_k P^{-k}$ is an F-operator from $H^0(E)$ to $H^0(F)$, and $(B_0^*)B_0$ is an F-operator from $H^0(E)$ to $H^0(E)$ with index zero. By Theorem 8.2(iii) there is a compact K such that $B_0^*B_0+K$ has a bounded left inverse $C: H^0(E)\to H^0(E)$. Hence $\|(B_0^*B_0+K)f\|\geqq\|C\|^{-1}\|f\|$. Consider now the sections $\theta^m\beta$ of Theorem 7.3. Since they converge weakly to zero in $H^0(E)$, we have $\|K\theta^m\beta\|\to 0$, and consequently by the corollary of Theorem 7.3, $\|(B_0^*B_0+K)\theta^m\beta\|\to\|[\sigma(B^*B)(\alpha(x_0))]\beta(x_0)\|$ for the given sections α of $T^*(X)$ and β of E. Since $\|\theta^m\beta\|=1$, we have $\|[\sigma(B^*B)(\alpha(x_0))]\beta(x_0)\|\geqq\|C\|^{-1}$ for $\alpha(x_0)$ in T'_{x_0} and $\beta(x_0)$ of length one in E_{x_0}. Thus $\sigma(B^*B)(\xi)$ is one-one and onto for each ξ. Since $\sigma(B^*B)=\sigma(B^*)\sigma(B)$, $\sigma(B)$ is one-one.

An identical argument shows $\sigma(B)^* = \sigma(B^*)$ is one-one, so $\sigma(B)$ is an isomorphism, and B is elliptic. Finally, $A = Q^{r-k}BP^k$ is elliptic.

We now consider a local form of the assertion of Theorem 8.3 about the range of an elliptic operator. First, two conventions: If f and g are in $H^{-\infty}(E)$ and U is a subset of X, then "$f = g$ in U" means that $(f,\phi) = (g,\phi)$ for all ϕ in $H^{\infty}(E)$ such that $\phi = 0$ in $X - U$. An operator A is "surjective in U" if and only if, given g in $H^{-\infty}(F)$, there is an f in $H^{-\infty}(E)$ with $Af = g$ in U. The null space of A is denoted $N(A)$.

THEOREM 9.3. *Let U be closed and A be elliptic. Then A is surjective in U if and only if no nonzero function in $N(A^*)$ has support in U.*

Proof. Suppose h is in $N(A^*)$, $Af = h$ in U, and h has support in U. Then $(h,h) = (f,A^*h) = 0$. Thus A is surjective in U only if no nonzero function in $N(A^*)$ has support in U.

Assume now the condition on $N(A^*)$, and let L be the set of linear functionals on $N(A^*)$ of the form $\lambda(h) = (h,\phi)$ for some C^{∞} section ϕ vanishing in U. Then L is the dual of $N(A^*)$; for if $(h,\phi) = 0$ for all ϕ as above, then $h = 0$ in $X - U$, and consequently $h = 0$ in X. Now, given g in $H^{-\infty}(F)$, choose a C^{∞} section ϕ vanishing in U such that $(h,\phi) = (h,g)$ for all h in $N(A^*)$. Then $g - \phi = g$ in U, and $g - \phi$ is orthogonal to $N(A^*)$, so there is an f with $Af = g - \phi$.

COROLLARY. *If A is in $\mathscr{E}_r(E,E)$ and x_0 in X is given, then there is a neighborhood U of x_0 such that A is onto with respect to U.*

Proof. Since $N(A^*)$ is finite dimensional, x_0 must have a neighborhood U satisfying the conditions of Theorem 9.3.

REMARK. If A is a differential operator, then the above corollary holds if we assume only that $\sigma(A)(\xi)$ is an isomorphism for all ξ over x_0. For then a neighborhood V of x_0 may be embedded in a torus of dimension n, and the coefficients of A extended from V so as to be elliptic over the whole torus. The same method allows one to deduce from the corollary of Theorem 8.1 a local regularity theorem for elliptic differential operators.

10. Some expansion theorems.

THEOREM 10.1. *Let A be in $\mathscr{E}_r(E,E)$, $r \neq 0$, $A = A^*$. Then there is a basis $\{\phi_m\}$ of $H^0(E)$ of orthonormal eigensections of A. The eigenvalues satisfy $|\lambda_m|^r \to \infty$ as $m \to \infty$. A section f in $H^{-\infty}(E)$ is in $H^{kr}(E)$ if and only if $\sum |(\phi_m,f)|^2 |\lambda_m|^{2k} < \infty$.*

Proof. If $r < 0$, the extension A_0 of A to $H_0(E)$ is a compact Hermitian operator, by Lemma 8.1. Thus the eigenfunctions exist, and $|\lambda_m|^r \to \infty$. By Theorems 8.3, and 8.1, corollary, the null space $N(A_0)$ of A_0 is finite dimensional and consists of H^∞ sections. Hence the orthogonal projection P_0 on $N(A_0)$ is of order $\leqq m$ for every m. Thus $A' = A + P_0$ is in $\mathscr{E}_r(E,E)$, $N(A') = 0$; and $\operatorname{ind}(A') = 0$ since A' is formally self-adjoint. As in Theorem 9.1, A' is an isomorphism of $H^k(E)$ with $H^{k-r}(E)$ for every k. Thus f is in $H^{kr}(E)$ if and only if $(A')^k f$ is in $H^0(E)$, which is equivalent to $\sum|\lambda_n|^{2k}|(\phi_n,f)|^2 = \sum|(\phi_n,(A')^k f)|^2 < \infty$.

In case $r > 0$, again let P_0 be the projection on the null space of A, and consider the inverse of $A + P_0$.

For a result on uniform convergence, the following lemma is helpful.

Lemma 10.1. *If S is any operator of order $< -n/2$, from $H^\infty(E)$ to $H^\infty(F)$, then the extension of S to $H^0(E)$ is a Hilbert-Schmidt operator.*

By saying S is Hilbert-Schmidt, we mean there is a section K in $H^0(\mathrm{HOM}(E,F))$ such that $Sf(x) = \int_X K(y,x)f(y)dv_y$. Recall that $\mathrm{HOM}(E,F)$ is a bundle over $X \times X$, whose fibre at (y,x) is the linear transformations of E_y into F_x.

To find the kernel K, use a partition of unity $\sum\phi_j = 1$ such that the support of each ϕ_j is contained in a neighborhood U_j with a coordinate map into $\mathbf{R}^n$, and such that E and F both have local bases over U_j. Then $S = \sum_{jk}\phi_j S\phi_k$, and each $\phi_j S\phi_k$ can be represented as a system of operators on $H^\infty(\mathbf{R}^n)$. By Lemma 5.3, each component of this system has a Hilbert-Schmidt kernel, which yields a kernel K_{jk} in $H^0(\mathrm{HOM}(E,F))$ representing $\phi_j S\phi_k$. Then $K = \sum K_{jk}$ is the desired kernel for S.

If S is Hilbert-Schmidt, then S is also compact, so S^*S, acting on $H^0(E)$, has an eigenfunction expansion $S^*Sf = \sum\lambda_m(f,\phi_m)\phi_m$, with $\lambda_m \geqq 0$. If S is Hilbert-Schmidt, it follows as usual that $\sum\lambda_m < \infty$.

An operator $D: H^\infty(E) \to H^\infty(F)$ is *of finite order* if it is of order $\leqq m$ for some $m < \infty$.

Theorem 10.2. *Let A be in $\mathscr{E}_r(E,E)$, $r \neq 0$, and $A = A^*$. Let $\{\phi_m\}$ be the basis of orthonormal eigensections of A, $A\phi_m = \lambda_m\phi_m$. Then f is in $H^\infty(E)$ if and only if $\sum|(f,\phi_m)|\,|D\phi_m|$ converges uniformly for each operator D of finite order*(3).

Proof. By $|D\phi_m|$ we mean the function on S such that

$$|D\phi_m|(x) = ((D\phi_m(x), D\phi_m(x)))^{1/2}.$$

The convergence condition is clearly sufficient for f to be a C^∞ section, and therefore in $H^\infty(E)$.

Let P_0 be the projection on the null space of A. Then $A + P_0$ has the same eigensections as A, so it suffices to consider an invertible A. But then we may assume $r < 0$. Further, A^n has the same eigensections as A, so we may consider $r < -n/2$. Then A is Hilbert-Schmidt and the eigenvalues λ_m satisfy $\sum|\lambda_m|^2 < \infty$.

(3) *Added in proof.* This criterion has been used in special cases by Kodaira and Spencer, and by Calderón and Zygmund in [3].

Since A is an isomorphism: $H^k(E) \to H^{k-r}(E)$, we may give the topology in $H^{kr}(E)$ by the norm $\|f\|_{kr} = \|A^k f\|_0$. From the Soboleff inequalities, $((f(x),f(x))) \leqq C^2(\|f\|_{-r})^2$. We may take D to be of order $dr > 0$ for some $d<0$, so that $\|Df\|_{jr} \leqq C_j \|f\|_{(j+d)r}$. Thus $|D\phi_m|(x) \leqq C\|D\phi_m\|_{-r} \leqq C'\|\phi_m\|_{r(d-1)} = C'\|A^{d-1}\phi_m\|_0 = C'|\lambda_m|^{d-1}$.

Now from Theorem 10.1, $|(f,\phi_m)|\,|\lambda_m|^{d-3}$ is a bounded function of m, so $\sum|(f,\phi_m)|\,|D\phi_m| \leqq M\sum|\lambda_m|^2 < \infty$.

REMARK. Theorem 10.1 shows that f is $H^\infty(E)$ if and only if $\sum|(f,\phi_m)|^2|\lambda_m|^k < \infty$ for every k. This has an interesting analog in the case of a compact real analytic manifold X and a self-adjoint analytic elliptic differential operator A of order r. Then f is real analytic if and only if $\sum|(\phi_m,f)|^2 t^{u_m} < \infty$ for some $t > 1$, where $(u_m)^r = |\lambda_m|$. The proof of this rests on solutions of the Cauchy problem and analyticity of solutions of analytic equations, and would be out of place here.

11. **Closure in L^2 operator norm.** Denote by $\mathscr{A}_0(E,F)$ the closure of the B^∞ operators of order zero from $H^\infty(E)$ to $H^\infty(F)$, in the norm $\|A\| = \sup\|Af\|_0$ for $\|f\|_0 = 1$. Thus $\mathscr{A}_0(E,F)$ is a norm-closed subspace of the bounded operators from $H^0(E)$ to $H^0(F)$. $\mathscr{A}_0$ includes the set $\mathscr{K}$ of all compact operators. For, operators of finite rank whose range lies in $H^\infty(F)$ are of order < -1, and so are B^∞ operators of order zero; and such operators are dense in $\mathscr{K}$.

In order to state the first theorem of this section, denote by $C_0(E,F)$ the complete normed vector space of all continuous, fibre-preserving maps of $T'(X)$ into $\mathrm{Hom}(E,F)$ which are homogeneous of degree zero on each fibre of $T'(X)$. If F is such a map and τ the projection of $T'(X)$ onto X, then $\|F(\xi)\|$ is the norm of $F(\xi)$ as a linear operator from $E_{\tau(\xi)}$ to $F_{\tau(\xi)}$, and the norm in $C_0(E,F)$ is $|F| = \sup\|F(\xi)\|$.

THEOREM 11.1. (i) *The symbol map σ extends to a map σ' of $\mathscr{A}_0(E,F)$ onto $C_0(E,F)$.*

(ii) $\sigma'(\mathscr{K}) = 0$, *and σ' induces a linear isometry of $\mathscr{A}_0/\mathscr{K}$ onto $C_0(E,F)$.*

(iii) *If $A \in \mathscr{A}_0(E,F)$, $B \in \mathscr{A}_0(F,G)$, then $BA \in \mathscr{A}_0(E,G)$ and $\sigma'(BA) = \sigma'(B)\sigma'(A)$.*

(iv) *If $A \in \mathscr{A}_0(E,F)$, then $A^* \in \mathscr{A}_0(F,E)$ and $\sigma(A^*)(\xi) = [\sigma(A)(\xi)]^*$.*

(v) *If $A \in \mathscr{A}_0(E,F)$, then $\sigma(A)(\xi)$ is an isomorphism for all ξ if and only if A is an F-operator from $H^0(E)$ to $H^0(F)$.*

The theorem is a consequence of

LEMMA 11.1. *If A is a B^∞ operator of order zero from $H^\infty(E)$ to $H^\infty(F)$, and $A + \mathscr{K}$ its equivalence class in $\mathscr{A}_0/\mathscr{K}$, then $|\sigma(A)| = \|A + \mathscr{K}\|$.*

Proof. That $|\sigma(A)| \leqq \|A + \mathscr{K}\|$ follows from Lemma 7.2. For the other inequality consider the operator $A^\sim$ on $H^0(E \oplus F) = H^0(E) \oplus H^0(F)$ defined by $A^\sim(e \oplus f) = A^*f \oplus Ae$. Here $E \oplus F$ denotes the orthogonal direct sum (Whitney sum) of the bundles E and F, and $H^0(E)\oplus H^0(F)$ is the orthogonal direct sum of the Hilbert spaces $H^0(E)$ and $H^0(F)$. We have $\| A^\sim \| = \| A\|$ and $|\sigma(A^\sim)| = |\sigma(A)|$; and A is a self-adjoint B^∞ singular integral operator. The lemma depends on showing that the spectrum $\mathrm{sp}(A^\sim)$ consists of the union over ξ of the eigenvalues of $\sigma(A^\sim)(\xi)$, plus isolated points of finite multiplicity.

Note that for any F-operator B, the range of B is the orthogonal complement of the null space of B^*. In particular, when B is a self-adjoint F-operator, B is invertible if and only if its null space is trivial. Now suppose λ_0 is an eigenvalue of $A^\sim$ and $|\lambda_0| > |\sigma(A^\sim)|$. Then $\lambda_0 I - A^\sim$ is an elliptic singular integral operator, and by Theorem 8.3 it has a finite dimensional null space. Then the orthogonal projection P_0 on this null space is compact, and since λ_0 is real $\lambda_0 I - A^\sim - P_0$ is a self-adjoint F-operator. Since it has trivial null space, it is invertible; and further there is a δ such that $0<\delta<|\lambda_0| - |\sigma(A^\sim)|$, and $\lambda I - A^\sim - P_0$ is invertible for $|\lambda-\lambda_0|<\delta$. It follows that the real numbers λ with $0<|\lambda-\lambda_0|<\delta$ are not in $\mathrm{sp}(A^\sim)$. For if λ is real, $0<|\lambda-\lambda_0|<\delta$, and $\lambda f - A^\sim f = 0$, then f is orthogonal to the range of P_0 and $\lambda f - A^\sim f - P_0 f = 0$; hence such an $f=0$, and $\lambda I - A^\sim$ is invertible.

Now let $\varepsilon>0$ be given, and let $\lambda_1,\cdots,\lambda_N$ be the eigenvalues of $A^\sim$ in $\{|\lambda| > |\sigma(A)| + \varepsilon\}$, and $P_1,\cdots,P_N$ the projections on the corresponding eigenspaces. Then $\sum \lambda_j P_j$ is compact, and $\mathrm{sp}(A^\sim - \sum\lambda_j P_j)$ lies in $\{|\lambda| \leqq |\sigma(A)| + \varepsilon\}$, so $\| A^\sim - \sum \lambda_j P_j \| \leqq |\sigma(A^\sim)| + \varepsilon = |\sigma(A)| + \varepsilon$.

Now A can be represented by a matrix

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix},$$

and we have found a matrix K of compact operators such that

$$\begin{pmatrix} K_{11} & A^* + K_{12} \\ A + K_{21} & K_{22} \end{pmatrix}$$

has norm $\leqq |\sigma(A)| + \varepsilon$. Since the norm of each entry in such a matrix is dominated by the norm of the operator represented by the matrix, we have $\| A + K_{21} \| \leqq |\sigma(A)| + \varepsilon$, and the lemma is proved.

Remark. The general idea of considering $A^\sim$ instead of A is due to Singer.

Proof of Theorem 11.1. By Lemma 11.1, $A_m \to A$ in L^2 norm only if $\sigma(A_m)$ converges in $C_0(E,F)$; we let $\sigma'(A) = \lim \sigma(A_m)$. Now the equality of Lemma 1.1 carries immediately from the B^∞ operators to all of $\mathscr{A}^0$, and consequently $\mathscr{A}_0/\mathscr{K}$ is isometric to a subspace of $C_0(E,F)$. Since the symbols of B^∞ operators are dense in $C_0(E,F)$, the isometry is onto $C_0(E,F)$. This proves parts (i) and (ii) of the theorem; parts (iii) and (iv) follow immediately by taking limits. For (v), suppose $\sigma'(A)(\xi)$ is 1-1 and onto for each ξ. Then $F: F(\xi) = [\sigma'(A)(\xi)]^{-1}$ is continuous, and is therefore the symbol of an operator A' in $\mathscr{A}_0(F,E)$. Since

$\sigma'(A'A) = I = \sigma'(AA')$, $AA' = I + K$ and $A'A = I + K'$ for some compact operators K and K', so A is an F-operator by Theorem 8.2. For the converse, observe that Theorem 7.3 carries over immediately to operators in $\mathscr{A}_0$, so that the necessary part of the proof of Theorem 9.2 can be repeated. This completes Theorem 11.1. We will write σ for σ', on the basis of part (i).

The set $\mathscr{A}_0$ is the appropriate setting for a result on the homotopy invariance of the index.

THEOREM 11.2. *Let A and B be elliptic operators in $\mathscr{A}_0(E,F)$, and suppose there is a homotopy F_t of $\sigma(A)$ to $\sigma(B)$, such that for each t and ξ, F_t is in $C_0(E,F)$ and $F_t(\xi)$ is nonsingular. Then* $\operatorname{index}(A) = \operatorname{index}(B)$.

Proof. According to parts (iv) and (v) of Theorem 8.2, the set $S_k = \{A + \mathscr{K} : A \in \mathscr{A}_0(E,F), \sigma(A)$ is nonsingular and $\operatorname{index}(A) = k\}$ is open in $\mathscr{A}_0/\mathscr{K}$. For if $\operatorname{index}(A) = k$ and $\| B + \mathscr{K} - (A + \mathscr{K}) \| < \varepsilon$, there is a compact K such that $\| B + K - A \| < \varepsilon$. Choosing ε as in 8.2(v), we have $\operatorname{index}(A) = \operatorname{index}(B + K) = \operatorname{index}(B)$. Thus index is continuous from the elliptic members of $\mathscr{A}_0/\mathscr{K}$ into the integers, and Theorem 11.2 follows by the usual connectedness argument and the isomorphism of $\mathscr{A}_0/\mathscr{K}$ with $C_0(E,F)$.

12. **Closure in the order zero topology.** Theorem 11.1 extends several properties of the B^∞ operators to a rather large class. However, the operators in this larger class $\mathscr{A}_0(E,F)$ do not preserve H^k for $k \neq 0$, and no regularity theorem can be expected for solutions of elliptic equations. We can retain these properties by considering the closure in a finer topology, the order zero topology on the space of operators of order zero from $H^\infty(E)$ to $H^\infty(F)$. This topology is easy to describe by giving $H^k(E)$ and $H^k(F)$ a norm, $\| \ \|_k$. Then set $||| A |||_m = \sup \| Af \|_k / \| f \|_k$, the sup being for $|k| \leqq m$ and $f \neq 0$ in $H^\infty(E)$. A base of neighborhoods of zero in the order zero topology is given by the sets $U_m = \{A : ||| A |||_m < 1/m\}$. Our results on the closure in the order zero topology rest on results for closure in m-norm.

For integer $m \geqq 0$, let $\mathscr{A}_m(E,F)$ be the closure of the B^∞ operators of order zero (as operators on $H^{-m}(E)$) in the m-norm $||| \ |||_m$. A member A of $\mathscr{A}_m(E,F)$ is essentially a $2m+1$-tuple $A = (A_{-m}, \cdots, A_m)$ of operators, where A_j is continuous from $H^j(E)$ to $H\ (F)$ and A_j is a restriction of A_{j-1}. Then since $H^{-r}(E)$ is the anti-dual of $H^r(E)$, we have $A^* = (A_m{}^*, \cdots, A_{-m}{}^*) = (A^*{}_{-m}, \cdots, A^*{}_m)$, and if B is in $\mathscr{A}_m(F,G)$ $BA = (B_{-m}A_{-m}, \cdots, B_m A_m)$. If B is a B^∞ operator from $H^\infty(E)$ to $H^\infty(F)$, then of course B_j is its extension to $H^j(E) \to H^j(F)$. $A^{(j)} \to A$ in $\mathscr{A}_m$ means $A_k^{(j)} \to A_k$ in operator norm, for each $|k| \leqq m$.

$\mathscr{A}_\infty(E,F)$ is the inverse limit of the $\mathscr{A}_m(E,F)$. An *$\mathscr{A}_\infty$ operator of order r* is any of the form AP^r, where $A \in \mathscr{A}_\infty(E,F)$ and P is an isomorphism in $\mathscr{E}_1(E,E)$, i.e., an elliptic B^∞ operator of order one, inducing an isomorphism $P_k : H^k(E) \to H^{k-1}(E)$ for each k. A trivial check (as in Lemma 12.1 below) shows that the class of $\mathscr{A}_\infty$ operators of order r does not depend on the choice of P, and could in fact be defined as the operators of the form $Q^r A'$, with Q an isomorphism in $\mathscr{E}_1(F,F)$ and A' in $\mathscr{A}_\infty(E,F)$. The symbol of such an operator is defined to be $\sigma(AP^r) = \sigma(A)\sigma(P^r)$, and is likewise independent of the choice of P. The $\mathscr{A}_\infty$ operators of order r are also the closure of the B^∞ operators of order r in a fairly obvious topology. The representation AP^r allows us to reduce the analysis of operators of order r to the case $r = 0$.

Since $\mathscr{A}_m \subset \mathscr{A}_0$, the symbol map σ extends naturally to $\mathscr{A}_m$, and parts (iii), (iv) and (v) of Theorem 11.1 apply to $\mathscr{A}_m$.

LEMMA 12.1. *Let $A \in \mathscr{A}_m(E,F)$, and ρ be a C^∞ function on $T'(X)$, homogeneous of degree one. Let $P \in \mathscr{E}_1(E,E)$ be such that P_k is an isomorphism of $H^k(E) \to H^{k-1}(E)$ for all k, and $\sigma(P)(\xi) = \rho(\xi)I$; let $Q \in \mathscr{E}_1(F,F)$ be such that Q_k is an isomorphism for all k and $\sigma(Q)(\xi) = \rho(\xi)I$. Then $Q^r A P^{-r}$ is in $A_{m-r}(E,F)$ for $0 \leqq r \leqq m$, and $\sigma(Q^r A P^{-r}) = \sigma(A)$.*

Proof. If $A^{(j)}$ is a sequence of B^∞ operators and $A^{(j)} \to A$, then for $|k| \leqq m, Q^r A_k^{(j)} P^{-r} \to Q^r A_k P^{-r}$ in the norm of operators from $H^{k-r}(E)$ to $H^{k-r}(F)$, and $\sigma(Q^r A^{(j)} P^{-r}) = \sigma(A^{(j)}) \to \sigma(A)$.

THEOREM 12.1. *If $A = (A_{-m}, \cdots, A_m) \in \mathscr{A}_m(E,F)$, then*

(i) *for each k, $\sigma(A) = 0$ if and only if A_k is compact;*

(ii) *for each k, A is elliptic if and only if A_k is an F-operator;*

(iii) *if A is elliptic, then* $\operatorname{index}(A_k) = \operatorname{index}(A_0)$.

Proof. (i) Let $B = Q^k A P^{-k}$ as in Lemma 1.1. Then $\sigma(B_0) = \sigma(A)$, and $A_k = (Q^{-k})_0 B_0 (P^k)_k$ is compact if and only if B_0 is compact. From Theorem 11.1, B_0 is compact if and only if $\sigma(B_0) = 0$.

The proof of (ii) imitates the proof of (i).

For (iii), let $A^{(j)}$ be a sequence of B^∞ operators, $A^{(j)} \to A$, $\sigma(A^{(j)}) \to \sigma(A)$. Since A is elliptic, so are the $A^{(j)}$ for j sufficiently large. By Theorem 8.3(iii), $\operatorname{index}(A_k^{(j)}) = \operatorname{index}(A_0^{(j)})$. By Theorem 8.2(v), $\operatorname{index}(A_k^{(j)}) \to \operatorname{index}(A_k)$ for $|k| \leqq m$, and the result follows.

The following restricted regularity result is a corollary of part (iii) of Theorem 12.1.

LEMMA 12.2. *If $A = (A_{-m}, \cdots, A_m) \in \mathscr{A}_m(E,F)$ is elliptic, then for each k the null space of A_k lies in $H^m(E)$.*

Proof. Let $\nu(A_k)$ denote the dimension of the null space of A_k. Then $\nu(A_k)$ is a nonincreasing function of k, and $-\nu(A_k{}^*) = -\nu(A^*{}_{-k})$ is likewise a nonincreasing function of k. Since $\operatorname{index}(A_k) = \nu(A_k) - \nu(A_k^*)$ is constant, $\nu(A_k)$ must be constant and the lemma is proved.

THEOREM 12.2. *Let $A \in \mathscr{A}_m(E,F)$ be elliptic. Suppose $f \in H^{-m}(E)$ and $Af = A_{-m}f \in H^k(F)$ for some $k \leqq m$. Then $f \in H^k(E)$.*

Proof. Let P_0 be orthogonal projection (in $H^0(E)$) on the null space of A^*A. By Lemma 12.2, $B = A^*A + P_0$ is in $\mathscr{A}_m(E,E)$. Moreover, $\operatorname{index}(B_k) = \operatorname{index}(B_0) = 0$, while $\nu(B_k) = 0$, so B_k is an isomorphism on $H^k(E)$, for each $|k| \leqq m$. Since $Af \in H^k(F)$, A^* maps $H^k(F)$ into $H^k(E)$, and P_0 projects onto a subspace of $H^m(E)$, we have $Bf \in H^k(E)$. Since B_k is an isomorphism on $H^k(E)$, it follows that $f \in H^k(E)$.

REMARK 12.1. Theorems 12.1 and 12.2 extend immediately, with appropriate changes of indices, to $\mathscr{A}_\infty$ operators of order r.

Since it is difficult to characterize the symbols of the operators in $\mathscr{A}_\infty(E,F)$ by some properties such as smoothness (as was the case for the B^∞ operators and the $\mathscr{A}_0$ operators), it is worth while to answer more modest questions such as whether the inverse of the symbol of an elliptic operator in $\mathscr{A}_m(E,F)$ is the symbol of some operator in $\mathscr{A}_m(F,E)$. The remaining results in this section are in this direction. They are not applied in the remaining sections.

LEMMA 12.3. *Let $A \in \mathscr{A}_m(E,F)$, and suppose A_k is invertible for some k, $|k| \leqq m$. Then A_j is invertible for all $|j| \leqq m$, $A^{-1} = ((A_{-m})^{-1}, \cdots, (A_m)^{-1}) \in \mathscr{A}_m(F,E)$, and $\sigma(A^{-1})(\xi) = [\sigma(A)(\xi)]^{-1}$.*

Proof. Since A_j is an F-operator and $\operatorname{index}(A_j) = \operatorname{index}(A_k) = 0$, A_j is invertible if it is either one-one or onto. If $j > k$, A_j has the left inverse $(A_k)^{-1}$, so A_j is one-one. If $j < k$, the range of A_j includes the range $(A_k) = H^k(F)$. Since $H^k(F)$ is dense in $H^j(F)$, and range (A_j) is closed, A_j is onto.

For the last two assertions, let $A^{(l)}$ be a sequence of B^∞ operators converging to A. Then by the continuity of inverses, $A_j^{(l)}$ is invertible for sufficiently large l, and $(A_j^{(l)})^{-1} \to A_j^{-1}$. This completes the proof, since $(A^{(l)})^{-1}$ is a B^∞ operator by Theorem 8.1.

THEOREM 12.4. *If $A \in \mathscr{A}_m(E,F)$ is elliptic, then there is an A' in $\mathscr{A}_m(F,E)$ with $\sigma(A')\sigma(A) = I$.*

Proof. Consider $A^\sim \in \mathscr{A}_m(E \oplus F, E \oplus F)$, as in Lemma 11.1, and let P_0 be projection on the null space of $A^\sim$. Then $B = A^\sim + P_0$ is elliptic, self-adjoint, and invertible; and by Lemma 12.2, B is in $\mathscr{A}_m(E \oplus F, E \oplus F)$. By Lemma 12.3, B^{-1} is in $\mathscr{A}_m(E \oplus F, E \oplus F)$, and one sees easily that we can let A' be the entry in row one and column two of the matrix representation of B^{-1}, this being the entry that maps $H^0(F)$ into $H^0(E)$.

Another immediate consequence of Lemma 12.3 is that, if $A \in \mathscr{A}_m(E,E)$, then $\operatorname{spectrum}(A_k) = \operatorname{spectrum}(A_j)$ for $|j| \leqq m$, $|k| \leqq m$. Thus it is reasonable to speak of $\operatorname{sp}(A) = \operatorname{spectrum}(A_0)$.

THEOREM 12.5. Let $A \in \mathscr{A}_m(E,E)$, *and suppose ϕ is analytic on* $\operatorname{sp}(A)$. *Then $\phi(A) = (\phi(A_{-m}), \cdots, \phi(A_m))$ is in $\mathscr{A}_m(E,E)$, and $\sigma(\phi(A)) = \phi(\sigma(A))$.*

Proof. $\mathscr{A}_m$ is a complete normed space; thus if Γ is a contour surrounding $\mathrm{sp}(A)$, $(1/2\pi i)\int_\Gamma \phi(\lambda)(\lambda - A)^{-1}d\lambda$ converges in $\mathscr{A}_m$. By the obvious extension of Theorem 7.3 to $\mathscr{A}_0$, we have $\mathrm{spectrum}(\sigma(A)(\xi)) \subset \mathrm{spectrum}(A_0)$; thus the spectrum of the symbol lies inside Γ, and $(1/2\pi i)\int_\Gamma \phi(\lambda)(\lambda - \sigma(A)(\xi))^{-1}d\lambda$ converges to $\phi(\sigma(A)(\xi))$, which concludes the theorem.

Our final result in this direction is a partial generalization of Theorem 12.4. If $A \in \mathscr{A}_m(E,E)$, then let $\mathrm{sp}(\sigma(A))$ denote the union over all ξ in $T'(X)$ of spectrum $(\sigma(A)(\xi))$, i.e., the union of all the eigenvalues of $\sigma(A)$. If ϕ is analytic on $\mathrm{sp}(\sigma(A))$, then $\phi(\sigma(A))$, defined by $\phi(\sigma(A))(\xi) = \phi(\sigma(A)(\xi))$, is a continuous function on $T'(X)$, homogeneous of degree zero.

THEOREM 12.6. *If $A \in \mathscr{A}_m(E,E)$ and ϕ is analytic on* $\mathrm{sp}(\sigma(A))$, *then there is a B in $\mathscr{A}_m(E,E)$ with $\sigma(B) = \phi(\sigma(A))$.*

Proof. Let Γ be a path in the complex plane lying in the domain of ϕ, and surrounding $\mathrm{sp}(\sigma(A))$. Then for $\lambda \in \Gamma$, $A - \lambda$ is elliptic; but $\mathrm{index}(A - \lambda)$ need not be zero, so there is not necessarily an invertible operator with the same symbol as $A - \lambda$. To avoid this difficulty consider $A(\lambda) = (A - \lambda)^\sim$, as in Lemma 11.1. For each λ_0 in Γ there is a compact operator $P(\lambda_0)$ (projection on the null space of $A(\lambda_0)$), such that $A(\lambda) - P(\lambda_0)$ is invertible for $|\lambda - \lambda_0| < \varepsilon(\lambda_0)$. Since Γ is compact, we may break it into finitely many disjoint curves $\Gamma_1, \cdots, \Gamma_k$ and choose $\lambda_j \in \Gamma_j$ so that $A(\lambda) - P(\lambda_j)$ is invertible for $\lambda \in \Gamma_j$. Thus the entry in row one and column two of $B^\sim = \sum_j \int_{\Gamma_j}(A(\lambda) - P(\lambda_j))^{-1}\phi(\lambda)d\lambda$ is in $\mathscr{A}_m(E,E)$. Since $\sigma((A(\lambda) - P(\lambda_j))^{-1}) = \sigma(A(\lambda))^{-1}$, the entry in row one and column two of $\sigma((A(\lambda) - P(\lambda_j))^{-1})$ is $(\sigma(A) - \lambda)^{-1}$, and the corresponding entry in $\sigma(B^\sim)$ is $\phi(\sigma(A))$.

13. **Tensor products.** Here we consider the tensor product $A \otimes B$ of an operator A acting on bundles over a manifold X, and an operator B acting on bundles over Y. The object is to obtain some further algebraic structure related to the index (Theorem 13.2) pointed out by Atiyah and Singer. The technical difficulty is that even if A and B are in the class $\mathscr{A}_0$, $A \otimes B$ is not in $\mathscr{A}_0$ except in trivial cases. However, we can fit the tensor product into the $\mathscr{A}_\infty$ framework by considering operators of order greater than zero.

Consider vector bundles E over X ($\dim X = n$), and G over Y ($\dim Y = m$). Then $E \otimes G$ is a bundle over $X \times Y$ isomorphic to $\mathrm{HOM}(E^*, G)$, where E^* is the dual bundle of E. As in §6, we assume a Hermitian inner product on E and G, which induces a similar structure on $E \otimes G$. If $f \in H^0(E)$ and $g \in H^0(G)$, then $f \otimes g \in H^0(E \otimes G)$. A multiple Fourier series argument shows that finite sums $\sum f_j \otimes g_j$, with f_j in $H^\infty(E)$ and g_j in $H^\infty(G)$, are dense in $H^k(E \otimes G)$ for each k. The inner product in $H^0(E \otimes G)$ is determined by $(f \otimes g,\, h \otimes k) = (f,h)(g,k)$. If A maps $H^\infty(E) \to H^\infty(F)$, and B maps $H^\infty(G) \to H^\infty(G)$, then $A \otimes B$ is defined on the (dense) subset of finite tensor sums by $(A \otimes B)(\sum f_j \otimes g_j) = \sum Af_j \otimes Bg_j$. The justification of this can be based on the following remarks.

REMARK 13.1. If r is an integer $\geqq 0$ and A is an operator of order $\leqq r$ on $H^{\infty}(\boldsymbol{R}^n)$, then for each $k \geqq 0$, $A \otimes I$ extends from the finite sums $\sum f_j \otimes g_j$ to a bounded operator from $H^{k+r}(\boldsymbol{R}^{n+m})$ to $H^k(\boldsymbol{R}^{n+m})$. To show this suppose, for ϕ in $H^{\infty}(\boldsymbol{R}^n)$, that $\int |(A\phi)\hat{}(\xi)|^2(1+|\xi|^2)^k d\xi \leqq C^2 \int |\phi\hat{}(\xi)|^2(1+|\xi|^2)^{k+r} d\xi$ for $0 \leqq k \leqq K$. Then from $(1+|\xi|^2+|\eta|^2)^k = \sum \gamma_j(1+|\xi|^2)^{k-j}|\eta|^{2j}$ we get

$$\int |((A\otimes I)f)\hat{}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^k d\xi$$
$$\leqq C^2 \sum \gamma_j |\eta|^{2j} \int |f\hat{}(\xi,\eta)|^2(1+|\xi|^2)^{k-j}(1+|\xi|^2)^r d\xi.$$

Replacing $(1+|\xi|^2)^r$ by $(1+|\xi|^2+|\eta|^2)^r$ and integrating with respect to η yields

$$\iint |((A\otimes I)f)\hat{}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^k d\xi d\eta$$
$$\leqq C^2 \iint |f\hat{}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^{k+r} d\xi d\eta.$$

REMARK 13.2. If A is a Euclidean operator of order $\leqq r$ $(r \geqq 0)$, so is $A \otimes I$, and $\|(A\otimes I)_k\| \leqq \sup_{|m|\leqq k}\|A_k\|$, where A_k denotes the extension of A to $H^k(\boldsymbol{R}^n)$. The inequality follows directly from the previous remark for $k \geqq r$, and indirectly from the same remark for $k \leqq 0$, by taking adjoints. For $0<k<r$ we have

$$\iint |((A\otimes I)f)\hat{}(\xi,\eta)|^2(1+|\xi|^2+|\eta|^2)^{k-r} d\xi d\eta$$
$$\leqq \iint |((A\otimes I)f)\hat{}(\xi,\eta)|^2(1+|\xi|^2)^{k-r} d\xi d\eta$$
$$\leqq \|A_k\|^2 \iint |f\hat{}(\xi,\eta)|^2(1+|\xi|^2)^k d\xi d\eta$$
$$\leqq \|A_k\|^2(\|f\|_{k+r})^2.$$

The following lemma follows from the previous two remarks by using a partition of unity and local bases of sections.

LEMMA 13.1. *Let r be an integer $\geqq 0$, and A be any operator of order $\leqq r$ from $H^{\infty}(E)$ to $H^{\infty}(F)$ (E and F bundles over X), and G be a bundle over Y. Then there is a unique continuous linear operator $A \otimes I$ of order $\leqq r$ from $H^{\infty}(E\otimes G)$ to $H^{\infty}(F\otimes G)$, such that $(A\otimes I)(f\otimes g) = (Af)\otimes g$. Let A_k be the extension of A to a map: $H^k(E)\to H^{k-r}(F)$. There is a constant c_k independent of A such that $\|(A\otimes I)_k\| \leqq c_k \sup_{|m|\leqq k}\|A_m\|$.*

Note this result applies to $A \otimes B$, since $A\otimes B = (A\otimes I)(I\otimes B)$.

THEOREM 13.1. *Let r be a positive integer, A be an $\mathscr{A}_\infty$ operator of order r from $H^\infty(E)$ to $H^\infty(F)$, and let I be the identity on $H^\infty(G)$. Then $A \otimes I$ is an $\mathscr{A}_\infty$ operator of order r, from $H^\infty(E \otimes G)$ to $H^\infty(F \otimes G)$; and $\sigma(A \otimes I)(\xi) = \sigma(A)(\xi) \otimes \sigma(I)(\xi)$.*

Proof. By Lemma 13.1, we may assume A is a B^∞ operator of order r. Let $\{\phi_j\}$ be a partition of unity on $X = \pi(E) = \pi(F)$, and $\{\psi_k\}$ a partition of unity on $Y = \pi(G)$. Suppose support$(\phi_i) \cup$ support(ϕ_j) lies in a coordinate neighborhood in X over which E and F are trivial, whenever support$(\phi_i) \cap$ support(ϕ_j) is not empty; and suppose the support of each ψ_k lies in a coordinate neighborhood of Y over which G is trivial. Then write $A \otimes I = \sum_{i,j,k} \phi_i A \phi_j \otimes \psi_k I$. If the supports of ϕ_i and ϕ_j are disjoint, then $\phi_i A \phi_j$ is of order $< r$, and hence by Lemma 13.1, $\phi_i A \phi_j \otimes I$ is of order $< r$, and $\phi_i A \phi_j \otimes I$ is a B^∞ operator of order r with symbol zero. If the supports of ϕ_i and ϕ_j are not disjoint, we can introduce local coordinates and local bases of E, F, and G, and thus reduce the question to the following lemma.

LEMMA 13.2. *Let A be a B^∞ singular integral operator on $H^\infty(\mathbf{R}^n)$, let Λ_0 on $H^\infty(\mathbf{R}^n)$ be defined by $(\Lambda_0 f)\hat{}(\xi) = |\xi| f\hat{}(\xi)$, let Λ on $H^\infty(\mathbf{R}^{n+m})$ be defined by $(\Lambda g)\hat{}(\xi,\eta) = (1 + |\xi|^2 + |\eta|^2)^{1/2} g\hat{}(\xi,\eta)$ $(\xi \in \mathbf{R}^n,\ \eta \in \mathbf{R}^m)$, and let r be a positive integer. Then there is a sequence A^j of B^∞ singular integral operators on $H^\infty(\mathbf{R}^{n+m})$, and an operator S_{-1} of order $\leqq -1$, such that $A^j \to (A(\Lambda_0)^r \otimes I)\Lambda^{-r} + S_{-1}$ in the order zero topology of operators on $H^\infty(\mathbf{R}^{n+m})$; and $\sigma(A_j)(x,y;\xi,\eta) \to |\xi|^r(|\xi|^2 + |\eta|^2)^{-r/2}\sigma(A)(x,\xi)$.*

Proof. First, write $((\Lambda_0)^r \otimes I)\Lambda^{-r} = H + S$, where

$$(Hg)\hat{}(\xi,\eta) = |\xi|^r(|\xi|^2 + |\eta|^2)^{-r/2} g\hat{}(\xi,\eta).$$

Then considering the Fourier transform of S, it is easy to show S is of order minus two on $H^\infty(\mathbf{R}^{n+m})$. If R^α is a Riesz operator on $H^\infty(\mathbf{R}^n)$,

$$(R^\alpha f)\hat{}(\xi) = (\xi/|\xi|)^\alpha f\hat{}(\xi),$$

then

$$((R^\alpha \otimes I)Hg)\hat{}(\xi,\eta) = (\xi/|\xi|)^\alpha |\xi|^r(|\xi|^2 + |\eta|^2)^{-r/2} g\hat{}(\xi,\eta) = h_\alpha(\xi,\eta) g\hat{}(\xi,\eta).$$

The factor $|\xi|^r$ makes h_α continuous in $\{|\xi|^2 + |\eta|^2 \geqq 1\}$, and h_α is homogeneous of degree 0. Hence h_α can be uniformly approximated by a sequence $h_{\alpha,j}(\xi,\eta)$ of functions homogeneous of degree zero and C^∞ in $\{|\xi|^2 + |\eta|^2 \geqq 1\}$. The $h_{\alpha,j}$ are symbols of B^∞ singular integral operators $H_{\alpha,j}$ on $H^\infty(\mathbf{R}^{n+m})$; and $H_{\alpha,j} \to (R^\alpha \otimes I)H$ in the order zero topology. Thus for any finite sum $A = \sum a_\alpha R^\alpha$ (a_α in $B^\infty(\mathbf{R}^n)$), there is a sequence A^j of B^∞ singular integral operators such that $A^j \to (A \otimes I)H$ in the order zero topology. Since sums of the form $\sum a_\alpha R^\alpha$ are dense in the B^∞ singular integral operators in the order zero topology (Theorem 2.1), there is such a sequence A^j for any B^∞ singular integral operator A. Finally, $A^j \to (A \otimes I)H = (A(\Lambda_0)^r \otimes I)\Lambda^{-r} - (A \otimes I)S$.

Turning now to the index question, let E_1 and F_1 be vector bundles over X, E_2 and F_2 vector bundles over Y. Then $G_1 = (E_1 \otimes E_2) \oplus (F_1 \otimes F_2)$ and $G_2 = (F_1 \otimes E_2) \oplus (E_1 \otimes F_2)$ are bundles over $X \times Y$. If A is an $\mathscr{A}_\infty(E_1, F_1)$ operator of order $r > 0$, and B an $\mathscr{A}_\infty(E_2, F_2)$ operator of order $r > 0$, then $A \# B$ is the $\mathscr{A}_\infty(G_1, G_2)$ operator of order r such that $(A \# B)(e_1 \otimes e_2 \oplus f_1 \otimes f_2) = (Ae_1 \otimes e_2 - f_1 \otimes B^* f_2) \oplus (e_1 \otimes Be_2 + A^* f_1 \otimes f_2)$. $A \# B$ is conveniently represented by the matrix

$$\begin{pmatrix} A \otimes I & -I \otimes B^* \\ I \otimes B & A^* \otimes I \end{pmatrix}.$$

Lemma 13.1 shows $A \# B$ is an $\mathscr{A}_\infty(G_1, G_2)$ operator of order r. Its adjoint $(A \# B)^*$ is represented by the matrix

$$\begin{pmatrix} A^* \otimes I & I \otimes B^* \\ -I \otimes B & A \otimes I \end{pmatrix},$$

and $(A \# B)^*(A \# B)$ by the diagonal matrix with diagonal entries $A^*A \otimes I + I \otimes B^*B$ and $I \otimes BB^* + AA^* \otimes I$. From this it is easy to see that $A \# B$ is elliptic if and only if A and B are elliptic.

Theorem 13.2. $\operatorname{index}(A \# B) = \operatorname{index}(A) \cdot \operatorname{index}(B)$.

Proof. Let $\nu(A), \nu(B)$, etc., denote the dimension of the null space of A, B, etc. Let $\{\phi_j^1\}$ be an orthonormal basis of eigensections of A^*A with eigenvalues λ_j. Let $\{\psi_j^1\}$, $\{\phi_j^2\}$, $\{\psi_j^2\}$; and μ_j, ν_j, τ_j, be the corresponding objects for AA^*, B^*B, and BB^* respectively. Let $\phi_{ij} = (\phi_i^1 \otimes \phi_j^2) \oplus 0$ and $\psi_{ij} = 0 \oplus (\psi_i^1 \otimes \psi_j^2)$. Then the set $\{\phi_{ij}, \psi_{kl}\}$ is an orthonormal basis of $H^0(G_1)$, and $(A \# B)^*(A \# B)\phi_{ij} = (\lambda_i + \nu_j)\phi_{ij}$, $(A \# B)^*(A \# B)\psi_{ij} = (\mu_i + \tau_j)\psi_{ij}$. Thus $\{\phi_{ij}, \psi_{kl}\}$ is a basis of eigensections of $(A \# B)^*(A \# B)$, and since $\lambda_i + \nu_j = 0$ if and only if $\lambda_i = \nu_j = 0$, with the same for $\mu_i + \tau_j$, it is easy to see that $\nu((A \# B)^*(A \# B)) = \nu(A^*A)\nu(B^*B) + \nu(AA^*)\nu(BB^*)$. Since for any elliptic operator C, $\nu(C^*C) = \nu(C)$, we find $\nu(A \# B) = \nu(A)\nu(B) + \nu(A^*)\nu(B^*)$. Replacing A by A^* and B by $-B$, we have $\nu((A \# B)^*) = \nu(A^*)\nu(B) + \nu(A)\nu(B^*)$, and Theorem 3.2 follows immediately.

14. **The maximal ideal space of $\mathscr{A}_0(C \times X, C \times X)$.** Here we consider again the closure, in L^2 operator norm, of the space of B^∞ operators of order zero, in the special case of a trivial one-dimensional bundle $C \times X$. These operators form an algebra, which turns out to have a simple set of generators; and its maximal ideal space is the unit cosphere bundle. This gives the symbol $\sigma(A)$ a natural invariant interpretation as the representing function of A on the maximal ideal space.

We suppose X has a Riemannian metric in $T^*(X)$, represented in local coordinates by $|\xi| = |\sum z_j d\chi_j(x)| = \sum z_j z_k g^{jk}(x)$. In these coordinates, the Laplacian is $\Delta = -(1/v)\sum(\partial/\partial\chi_j)vg^{jk}(\partial/\partial\chi_k)$. This is a B^∞ operator of order two, and $\sigma(\Delta)(\xi) = |\xi|^2$. When $E = C \times X$ we write $H^k(X)$ for $H^k(E)$.

LEMMA 14.1. *There is an operator J on $H^\infty(X)$ such that $\sigma(J)(\xi) = |\xi|^{-1}$, and*

(i) $\Delta J^2 = I + S_{-1}$, $J^2\Delta = I + S^1_{-1}$,

(ii) *J is an isomorphism*: $H^k(X) \to H^{k+1}(X)$ *for each* k,

(iii) $VJ = JV + S_{-1}$ *for each vector field V, and*

(iv) *for each ξ in $T'(X)$ there is a sequence $\{\phi_\xi^m\}$ such that* $\| VJ\phi_\xi^m + i|\xi|^{-1}\xi(V(x_0))\phi_\xi^m\| \to 0$ as $m \to \infty$. *Here ξ lies over x_0, ϕ_ξ^m converges weakly to zero, and* $\|\phi_\xi^m\| = 1$.

The existence of a J with the given symbol and property (ii) is given in Theorem 9.1. Since $\sigma(\Delta)(\xi) = \sigma(J^2)(\xi)^{-1}$, (i) follows from Theorem 7.2 on composition. The same theorem establishes (iii), since $\sigma(VJ) = \sigma(V)\sigma(J) = \sigma(J)\sigma(V) = \sigma(JV)$. Finally since $\sigma(VJ) = -i|\xi|^{-1}\xi(V(x_0))$, (iv) follows from Theorem 7.3.

Now let $\mathscr{A}$ be the closure (in the norm of operators on $H^0(X)$) of the algebra generated by the operators of the form VJ (with V ranging over the C^∞ vector fields), and the operators of order $\leqq -1$. We note first that $\mathscr{A}$ includes the ideal $\mathscr{K}$ of all compact operators on $H^0(X)$. For $\mathscr{A}$ includes all operators of the form $A: Af = (f,g)h$, where g and h are in $H^\infty(X)$. Since $H^\infty(X)$ is dense in $H^0(X)$, $\mathscr{A}$ includes all operators of the form $A: Af = (f,g)h$, with g and h in $H^0(X)$, hence includes all operators of finite dimensional range, and finally includes all compact operators.

With the metric on $T^*(X)$ we pick out $S^*(X)$, the subbundle of $T^*(X)$ consisting of unit vectors.

THEOREM 14.1. *The algebra $\mathscr{A}$ is the closure in operator norm of the algebra of B^∞ operators of order zero. The factor algebra $\mathscr{A}/\mathscr{K}$ has $S^*(X)$ as its space of maximal ideals, and is isometric to the algebra of continuous functions on $S^*(X)$. For each B^∞ operator A, $\sigma(A)$ is the representing function of the image of A in $\mathscr{A}/\mathscr{K}$.*

We begin the proof by observing that $\mathscr{A}$ contains all operators of the form $M_\phi: M_\phi f(x) = \phi(x)f(x)$, where ϕ is a continuous function on X. Suppose ψ is an H^∞ function with support in a coordinate neighborhood with coordinates χ, and let $V_j = \psi\partial/\partial\chi_j$. Then $\psi^2\Delta = -\sum V_j g^{jk} V_k + S_1$ (with S_1 of order 1), and $M_{\psi^2} = \psi^2\Delta J^2 + S_{-1} = -\sum V_j J g^{jk} V_k J + S'_{-1}$ is in $\mathscr{A}$, since $g^{jk}V_k$ is a C^∞ vector field. Since squares of such H^∞ functions generate $C_0(X)$ in sup norm, and hence in operator norm, we find all multipliers M_ϕ are in $\mathscr{A}$.

From property (iii) of Lemma 14.1, it follows that $V_1JV_2J - V_2JV_1J$ is an operator of order minus one, and hence a compact operator, by Lemma 5.1. Hence $\mathscr{A}/\mathscr{K}$ is commutative. Since $\mathscr{A}$ is a B^*-algebra (i.e., $\|A^*A\| = \|A\|^2$), $\mathscr{A}/\mathscr{K}$ is a commutative B^* algebra (see [13, p. 249]), and by the Gelfand representation theory $\mathscr{A}/\mathscr{K}$ is isometric to the algebra of continuous functions on its maximal ideal space.

To identify this space, note first that any homomorphism h of $\mathscr{A}/\mathscr{K}$ onto C induces a homomorphism of $C_0(X)$; for the multipliers M_ϕ are isometrically embedded in $\mathscr{A}/\mathscr{K}$. Thus to each h corresponds a point x_h in X such that $h(M_\phi) = \phi(x_h)$.

Further, if V is a real vector field then $(iVJ)^* = iVJ + S_{-1}$ (by direct calculation using (iii) of Lemma 14.1), so $h(iVJ)$ is a real linear functional on the vector fields V. Since $h(i\phi VJ) = \phi(x_h)h(iVJ)$, it follows that $h(iVJ)$ depends only on $V(x_h)$ and hence $h(iVJ) = \xi(V(x_h))$ for some ξ in the cotangent plane at x_h. To evaluate $|\xi|$, consider $A = -\sum V_j J g^{jk} V_k J$, where $-\sum V_j g^{jk} V_k = \psi^2\Delta + S_1$ as before. Then $A = M_{\psi^2} + S_{-1}$, so $h(A) = \psi^2(x_h) = \sum h(iV_jJ)h(g^{jk}iV_kJ) = \sum \xi(V_j(x_h))g^{jk}(x_h)\xi(V_k(x_h)) = \psi^2(x_h)|\xi|^2$; or $|\xi|^2 = 1$.

Conversely, given ξ in $S^*(X)$ we construct a homomorphism $h = h_\xi$ as follows. Set $h(\sum_j\prod_k(V_{jk}J)) = \sum_j\prod_k\xi(-iV_{jk}(x_0))$, where ξ lies over x_0. Since $\|V_{jk}J\phi_\xi^m - \xi(-iV_{jk}(x_0))\phi_\xi^m\| \to 0$, we have $\|\sum_j\prod_k(V_{jk}J\phi_\xi^m) - h(\sum\prod V_{jk}J)\phi_\xi^m\| \to 0$, and therefore $|h(A)| \leqq \|A\|$. Since ϕ_ξ^m converges weakly to zero, we get in the same way $|h(A+K)| \leqq \|A+K\|$ for every compact operator K, so that for the image $A+\mathscr{K}$ of A in $\mathscr{A}/\mathscr{K}$, we have $|h(A)| \leqq \|A+\mathscr{K}\|$. Thus $h(A)$ is independent of the representation $\sum_j\prod_k(V_{jk}J)$, and extends to a homomorphism of $\mathscr{A}/\mathscr{K}$ onto C.

Thus the maximal ideals of $\mathscr{A}/\mathscr{K}$ corresponds to the points of $S^*(X)$. Since the representing functions $A^\wedge(h)$ are continuous for the usual topology of $S^*(X)$, and separate points, it follows that the maximal ideal space of $\mathscr{A}/\mathscr{K}$ is topologically $S^*(X)$. Since the representing function of VJ coincides with its symbol, the same holds for the (unclosed) algebra generated by the $\{VJ\}$. It is then natural to call the representing function of an operator A in $\mathscr{A}$ its symbol, and to write it $\sigma(A)$.

It remains to identify $\mathscr{A}$ with the closure of the B^∞ operators of order zero, the $\mathscr{A}_0$ of §11. Since VJ is in $\mathscr{A}_0$, we have $\mathscr{A} \subset \mathscr{A}_0$ and $\mathscr{A}/\mathscr{K} \subset \mathscr{A}_0/\mathscr{K}$. Also the σ on $\mathscr{A}$ agrees with σ on $\mathscr{A}_0$. Since σ induces an isomorphism of $\mathscr{A}/\mathscr{K}$ onto $C_0(S^*)$, and of $\mathscr{A}_0/\mathscr{K}$ onto $C_0(S^*)$ (by Theorem 11.1), we have $\mathscr{A} = \mathscr{A}_0$, and Theorem 14.1 is established.

15. **Operators with closed range.** Theorem 11.1(v) characterizes the operators A in $\mathscr{A}_0(E,F)$ with closed range and finite null space, and such that A^* also has finite null space. These turn out to be the operators whose symbols are isomorphisms, the elliptic operators. Here we show to what extent $\sigma(A)$ determines whether or not the range of A is closed. Denote the range of A by $R(A)$, and

the null space by $N(A)$. In (iii) and (iv) we do not assume the sphere bundle $S(X)$ to be connected.

THEOREM 15.1. (i) *If $\sigma(A)(\xi)$ is one-one for each ξ, then $N(A)$ is finite dimensional and $R(A)$ is closed.*

(ii) *If $\sigma(A)(\xi)$ is onto for each ξ, then $N(A^*)$ is finite dimensional and $R(A)$ is closed.*

(iii) *If $\sigma(A)$ has continuous but not maximal rank, then there are compact operators K and K' such that $R(A+K)$ is closed and $R(A+K')$ is not closed.*

(iv) *If $\sigma(A)$ has discontinuous rank, then $R(A)$ is not closed.*

This result follows easily from Theorem 11.1, together with two elementary observations.

LEMMA 15.1. *If A is a normal operator on a Hilbert space, then $R(A)$ is closed if and only if A is invertible, or zero is isolated in spectrum(A).*

This follows rather directly from the spectral theorem.

LEMMA 15.2. *If any of $R(A)$, $R(A^*)$, $R(AA^*)$, $R(A^*A)$ are closed, so are the others.*

This depends on the fact that $R(A)$ is closed if and only if $R(A)$ is the orthogonal complement of $N(A^*)$, which holds if and only if $R(A^*)$ is the orthogonal complement of $N(A)$ [4, pp. 487–488].

We recall also part of the proof of Lemma 11.1.

LEMMA 15.3. *If A is a Hermitian singular integral operator, then spectrum(A) contains the union over ξ of the eigenvalues of $\sigma(A)(\xi)$, and spectrum(A) minus these values consists of isolated points of finite multiplicity.*

Parts (i) and (ii) of Theorem 15.1 are now clear. If $\sigma(A)(\xi)$ is onto for each ξ, then so is $\sigma(AA^*)(\xi)$, so that AA^* is elliptic and has closed range. In either case, Lemma 15.2 applies.

For part (iii), we show first that if $\sigma(A)$ is not one-one, but has continuous rank, then $A+K$ has closed range for some compact K. Let $P=(A^*A)^{1/2}$, and $A=UP$, where U maps the closure of $R(P)$ isometrically onto the closure of $R(A)$, and the orthogonal complement of $R(P)$ onto zero. Since $\mathscr{A}_0(E,E)$ is closed in uniform norm, P is in $\mathscr{A}_0(E,E)$ and $\sigma(P)(\xi)=(\sigma(A^*A)(\xi))^{1/2}$. Since rank $\sigma(P)=$ rank $\sigma(A)$ is continuous, $\sigma(P)$ must have an isolated zero eigenvalue. By Lemma 15.3, the spectrum of P near zero consists of zero plus, perhaps, a sequence λ_n of eigenvalues of finite multiplicity, $\lambda_n \to 0$. If E_n is projection on the null space of $P-\lambda_n I$, then $K_1=-\sum \lambda_n E_n$ is compact, and zero is isolated in the spectrum of $P+K_1$. Thus $R(P+K_1)$ is closed, and so is $R(A+UK_1)=R(U(P+K_1))$. Thus the K of Theorem 15.1(iii) can be taken as UK_1.

Now suppose $\sigma(A)$ has continuous but not maximal rank. By the previous paragraph we may assume $R(A)$ is closed. Then $N(A)$ is infinite dimensional; for otherwise A^*A would have closed range and finite dimensional null space, and would be elliptic, which implies $\sigma(A)$ is one-one. Likewise $N(A^*)$ is infinite dimensional. Now let $\{\phi_n\}$ be an orthonormal basis of $N(A)$, and $\{\psi_n\}$ a like basis of $N(A^*)$. Set $K'\phi_n = \psi_n/n$, $K' = 0$ on the orthogonal complement of $N(A)$. Then $R(A+K')$ includes all the ψ_n, but includes $\sum a_n\psi_n$ if and only if $\sum n^2|a_n|^2 < \infty$. Consequently, $R(A+K')$ is not closed.

For part (iv), suppose nullity of $\sigma(A)$ (which = nullity of $\sigma(A^*A)$) is discontinuous. Then a simple argument given below shows that $\sigma(A^*A)$ has arbitrarily small nonzero eigenvalues. Since these eigenvalues are in spectrum(A^*A) (Lemma 15.3), Lemma 15.1 shows $R(A^*A)$ is not closed.

To show $\sigma(A^*A)$ has arbitrarily small eigenvalues, suppose the opposite. Then there is a circle γ about the origin in the complex plane which does not pass through or surround any nonzero eigenvalue of $\sigma(A^*A)(\xi)$, for any ξ. Then the function given by $F(\xi) = (1/2\pi i)\int_\gamma[\sigma(A^*A)(\xi) - \lambda]^{-1}d\lambda$ is continuous and projection-valued. Therefore F has continuous rank, equal to the nullity of $\sigma(A)$.

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Appendix(1)

The object of this appendix is to extend, by more or less elementary methods, some results of Vol'pert [9](2), Dynin [3], Mihlin [4], [5] and the author [7] on the index of elliptic systems of singular integral operators on compact mani-

(1) *Added in proof*, May 18, 1964.

(2) Numbers in square brackets refer to the list of references at the end of this appendix.

folds; and to derive an elementary formula expressing the index of elliptic systems on Euclidean space $\boldsymbol{R}^n$ as $1/(n-1)!$ times the degree of a certain map. We use information on the homotopy groups of the unitary group [2], a little fibre bundle theory, the Hopf classification of maps into spheres and a tensor product technique used by Atiyah and Singer [1]. It has been stated in several papers [4], [5] that results for Euclidean space could be obtained from corresponding facts for the sphere by a stereographic projection. Here we use a technique of "transplanting" that seems more flexible; in particular, our results for compact manifolds and Euclidean space are obtained by transplanting certain system of operators on the torus. We note that, conversely, singular integral operators are defined in [6], [3], and the first part of this paper, as sums of "transplants" of singular integral operators on $\boldsymbol{R}^n$.

The main result we obtain can be deduced from the general formula in [1]. The point is that our methods are relatively elementary, and produce results in some important special cases. They may therefore shed some light on the general question of the index.

The author is indebted to his colleagues D. Arlt, E. Connell, and H. Levine for their patient topological assistance and to I. M. Singer for some helpful discussions. Singer has also observed that Theorem 1 below (transplanting) is true, and called it "excision." His proof appeared to involve the construction of a boundary value problem, however.

§A1 describes the transplanting techinque in the form of a theorem; §A2 gives our two basic lemmas; §A3 discusses their application to compact manifolds and §A4 to Euclidean space. §A5 sketches a generalization to arbitrary open manifolds of the theory of singular integral operators on compact manifolds, by requiring that the operators treated be close to multiplication operators in neighborhoods of infinity.

The notation of this appendix is the same as in the preceding article.

A1. **Transplanting.** In the following theorem E_1 is a bundle over a compact manifold X_1, U_1 is an open subset of X_2, $E_1|_{U_1}$ denotes the restriction of E_1 to U_1, and E_1' denotes the bundle E_2 pulled back over $S^*(X)$, the cosphere bundle of X.

THEOREM 1. *Let X_1 and X_2 be compact manifolds, E_j a bundle over X_j, U_1 an open subset of X_1, and A_1 an elliptic operator on sections of E_1 with $\sigma(A_1)(\xi_x) = I$ for $x \notin C_1$, where C_1 is some compact subset of U_1. Suppose there is an isomorphism ϕ of $E_1|_{U_1}$ onto $E_2|_{U_2}$ (U_2 an appropriate open set in X_2). Let ϕ' be the induced isomorphism of the part of E_1' over U_1 onto the part of E_2' over U_2. Then there is an elliptic operator A_2 on sections of E_2, such that* $\operatorname{index}(A_2) = \operatorname{index}(A_1)$, *$\phi'$ induces a transformation of $\sigma(A_1)$ into $\sigma(A_2)$ over U_2, and $\sigma(A_2) = I$ outside of U_2.*

Proof. Let ψ be a real C^∞ function on X_1 with support in U_1, $\psi = 1$ on C_1. Let I_{E_j} denote the identity operator on sections of E_j, and set $A_1' = \psi A_1 \psi + (1-\psi^2)I_{E_1}$. Then $\sigma(A_1) = \sigma(A_1')$, so also $\operatorname{index}(A_1) = \operatorname{index}(A_1')$. Now define a function ψ_2 on X_2 such that $\psi_2 = 0$ off U_2, and $\psi_2 f = \phi(\psi \cdot \phi^{-1}(f))$ for each section f of $E_2|_{U_2}$. Set $A_2 = \phi\psi A_1 \phi^{-1}\psi_2 + (1-\psi_2^2)I_{E_2}$. Then ϕ' induces a transformation of $\sigma(A_1)$ into $\sigma(A_2)$, and $\sigma(A_2) = I$ off U_2. Also, for sections g of E_2 with support in U_2, we have $A_2 g = \phi A_1' \phi^{-1} g$, and $A_2^* g = \phi A_1'^* \phi^{-1} g$. (We may suppose, by altering the volume element on X_2 and the inner product on E_2, that ϕ preserves these structures.) Since the null spaces of A_1', A_2, and their adjoints have support in U_1 or U_2 respectively, it follows that ϕ induces isomorphisms of these null spaces, and $\operatorname{index}(A_2) = \operatorname{index}(A_1') = \operatorname{index}(A_1)$.

A2. **Two lemmas.** Let M denote an oriented $2n-1$ manifold (in our application, $M = S^*(X)$). If f and g are maps of M into $U(n)$, $f \cdot g$ denotes the pointwise product, $f \cdot g(m) = f(m) \cdot g(m)$. Let p denote the projection of $U(n)$ onto S^{2n-1} obtained by taking the first row of a matrix. Here S^{2n-1} is the oriented unit sphere in C^n. Let $s_0 = (1, 0, \cdots, 0)$, and $U(n-1)$ denote $p^{-1}(s_0)$, a subgroup of $U(n)$. The following lemma has been given for $n = 2$ by Vol'pert [9] and Mihlin [4](3). The proof below, due to D. Arlt, is obtained by first "localizing" to an n-cell of M, then following the proof that the homotopy groups of a topological group are induced by pointwise multiplication of maps.

LEMMA 1. $\operatorname{degree}(p \circ (f \cdot g)) = \operatorname{degree}(p \circ f) + \operatorname{degree}(p \circ g)$.

Proof. Let I^{2n-1} denote an oriented $2n-1$ cube coherently embedded in M, and M' denote the complement of I^{2n-1}. Let Σ^{2n-1} denote M with M' collapsed to a point, or I^{2n-1} with its boundary collapsed to a point. Then any map ϕ of M into S^{2n-1} mapping M' into s_0 yields a $\phi^\sim : \Sigma^{2n-1} \to S^{2n-1}$, and $\operatorname{degree}(\phi^\sim) = \operatorname{degree}(\phi)$.

Given $f: M \to U(n)$, we can find f' homotopic to f such that $f'(M') \subset U(n-1)$. To see this, let ψ map Σ^{2n-1} into S^{2n-1} with degree $(\psi) = \operatorname{degree}(p \circ f)$, and thus obtain $\phi: M \to S^{2n-1}$ with $\phi(M') = s_0$, $\operatorname{degree}(\phi) = \operatorname{degree}(p \circ f)$. By the Hopf classification theorem, ϕ is homotopic to $p \circ f$; then the covering homotopy theorem shows there is an f' homotopic to f with $p \circ f' = \phi$.

Now given f and g, let f' and g' be obtained as above; it suffices to prove the lemma for f' and g', restricted to I^{2n-1}. Denote these restrictions by F and G, and realize I^{2n-1} as $0 \leqq t_j \leqq 1$ $(j = 1, \cdots, 2n-1)$. Set $F_s(t) = F(st_1, t_2, \cdots, t_{2n-1}) \cdot G(0, t_2, \cdots, t_{2n-1})$ for $0 \leqq t_1 \leqq 1 - 1/s$, $F_s = F(st_1, t_2, \cdots, t_{2n-1}) G(1 - s + st_1, t_2, \cdots, t_{2n-1})$ for $1 - 1/s \leqq t_1 \leqq 1/s$, and $F_s = F(1, t_2, \cdots, t_{2n-1}) G(1 - s + st_1, t_2, \cdots, t_{2n-1})$ for $1/s \leqq t_1 \leqq 1$. Then for each s, $1 \leqq s \leqq 2$, F_s maps I^{2n-1} into $U(n)$, and its boundary into $U(n-1)$.

(3) See also B. Bojarski, *On the index problem for systems of singular integral equations*, Bull. Acad. Polon. Sci. Ser. Sci. Math. **11** (1963), 653–655.

Also $p \circ F_1 = p \circ (F \cdot G)$ and $p \circ F_2 = p \circ F + p \circ G$, with addition in the sense of homotopy groups. Thus $p \circ (F \cdot G)$ is homotopic to $p \circ F + p \circ G$, and since degree yields an isomorphism of $\pi_{2n-1}(S^{2n-1})$ with the integers, the lemma follows.

The next lemma relies on the tensor product technique of §13, used by Atiyah and Singer in connection with the index problem.

LEMMA 2. *For each integer $n \geqq 1$, there exists on the n-torus T^n an $n \times n$ elliptic system A_n of singular integral operators such that* $\text{index}(A_n) = 1$, *$\sigma(A_n)(\xi_x)$ is a unitary matrix for each ξ_x in $S^*(T^n)$, and $\sigma(A_n) = I$ except over one n-simplex of a triangulation of T^n. The degree of the map of $S^*(T^n)$ into S^{2n-1}, obtained from $\sigma(A)$ as in Lemma* 1, *is* $(n-1)!$ *when the appropriate orientations are chosen.*

We let T^n be $\boldsymbol{R}^n$ reduced mod 2π in each variable. Then $T^*(X)$ is naturally $T^n \times \boldsymbol{R}^n$, and $S^*(X)$ is $T^n \times S^{n-1}$. The n-simplex of the lemma will be $\{x : |x_j| < \pi/2, j = 1,2,\cdots,n\}$.

The operator A_1 is nearly classical. Let H be the harmonic conjugate operator, $H(\sum_{-\infty}^{\infty} a_n e^{inx}) = \sum_{-\infty}^{\infty} \text{sgn}(n) a_n e^{inx}$, where $\text{sgn}(n) = n/|n|$ if $n \neq 0$, $\text{sgn}(0) = 0$. The bundle $S^*(T^1)$ is the disjoint union of two copies of T^1 on one of which $\sigma(H) = +1$, on the other of which $\sigma(H) = -1 : \sigma(H)(x,\xi) = \xi$ for $x \in T^1$, $|\xi| = 1$. A direct check by Fourier series shows that if $A = (1/2)(1 + e^{ix})H + (1/2)(1 - e^{ix})$, then the null space of A is spanned by $(1 + e^{-ix})$ and the null space of A^* is trivial, so $\text{index}(A) = 1$. Also $|\sigma(A)| = 1$, and $\sigma(A) = 1$ on the part of $S^*(T^1)$ over the point $x = \pi$, so $\sigma(A)$ is homotopic to a symbol $\sigma(A_1)$ which is the identity on all points of $S^*(T^1)$ over $\pi/2 \leqq |x| \leqq \pi$.

We proceed by induction, supposing A_n has been constructed. Let Λ_n be the operator on n-tuples of functions on T^n given by $\Lambda_n(\sum_\alpha a_\alpha e^{i\alpha \cdot x}) = a_0 + \sum_\alpha |\alpha| a_\alpha e^{i\alpha \cdot x}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of integers, and $|\alpha|^2 = \sum \alpha_j^2$; and Λ_1 the corresponding operator on functions on T^1. Denote points in $T'(T^n)$ by (x,ξ), $x \in T^n$, $|\xi| \neq 0$. Then $\sigma(\Lambda_n)(x,\xi) = |\xi|$, and Λ_n is an isomorphism of H^{k+1} onto H^k for $k = 0,1,\cdots$. Let I_n be the identity operator on n-tuples of functions on T^n. Set $B = C_1C_2C_3C_4$, with $C_j = B_j\Lambda_{n+1}^{-1}$, and

$$B_1 = \begin{pmatrix} A_n\Lambda_n \otimes I_1 & -I_n \otimes \Lambda_1 A_1^* \\ I_n \otimes A_1\Lambda_1 & \Lambda_n A_n^* \otimes I_1 \end{pmatrix};$$

B_2 is obtained from B_1 by replacing A_n with the identity, B_3 is obtained from B_1 by replacing A_1 with the identity, and B_4 is obtained from B_3 by replacing A_n with the identity; and Λ_{n+1} operates on $2n$-tuples of functions on T^{n+1}. B_1 is the product $A_n\Lambda_n \# A_1\Lambda_1$ discussed in Theorem 13.2 where it is shown that $\text{index}(A \# B) = \text{index}(A) \cdot \text{index}(B)$. Thus $\text{index}(B_1) = 1$. Further $B_2 = A_n\Lambda_n \# \Lambda_1$, so $\text{index}(B_2) = 0$; likewise $\text{index}(B_3) = \text{index}(B_4) = 0$. Since Λ_{n+1} is an isomorphism, and the index of a product is the sum of the indices of the factors, we find $\operatorname{index}(B) = 1$.

To consider the symbol, let a point in $T'(T^{n+1})$ be denoted by $(x_1,\cdots,x_n,y;\xi_1,\cdots,\xi_n,\eta)$, with all entries real and $\eta^2 + \sum\xi_j^2 \neq 0$. Since $\sigma(\Lambda_1) = |\eta|$, $\sigma(\Lambda_n) = |\xi|$, $\sigma(\Lambda_{n+1}) = \sqrt{(\xi^2 + \eta^2)}$, and $\sigma(A_1)$ and $\sigma(A_n)$ are unitary, one computes readily that $\sigma(B)$ is the identity if $|y| \geqq \pi/2$ or any $|x_j| \geqq \pi/2$; for Theorem 13.2 asserts that the symbol of $A \# A'$ is $\sigma(A) \# \sigma(A')$. It remains only to reduce B from a $2n$ by $2n$ system to an $n+1$ by $n+1$ system. But it follows readily from the covering homotopy theorem, together with the fact that $U(k)$ is a bundle over S^{2k-1} (as in Lemma 1), that any map of a manifold M of dimension $2n+3$ into $U(k)$ $(k > n+1)$ is homotopic to a map of M into unitary matrices of the form

$$\begin{pmatrix} 1 & \cdot & 0 & \cdots & \\ 0 & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & 1 & \\ \cdot & & & & U \end{pmatrix}$$

where U is in $U(n+1)$. Applying this with M taken as the part of $S^*(T^{n+1})$ lying over $\{|x_j| \leqq \pi/2,\ |y| \leqq \pi/2\}$, we find $\sigma(B)$ is homotopic to a map F into matrices of this special type, with $F = $ identity except over $\{|x_j| \leqq \pi/2, |y| \leqq \pi/2\}$. There is then an $(n+1)$ by $(n+1)$ system A_{n+1} with $\sigma(A_{n+1}) = $ the lower right-hand corner of F, i.e., if

$$F(x,y;\xi,\eta) = \begin{pmatrix} 1 & \cdot & 0 & \cdots & \\ 0 & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & 1 & \\ \cdot & & & & U \end{pmatrix},$$

then $\sigma(A_{n+1})(x,y;\xi,\eta) = U$. A_{n+1} meets the conditions of the lemma, except that we do not yet know the degree of $p \circ \sigma(A_{n+1})$.

Let $\xi_0 \in S^{n-1}$, and denote by G the group of maps ϕ of $S^*(T^{n+1})$ into $U(n)$, with $\phi(x,\xi) = 1$ when $\xi = \xi_0$, or when any $|x_j| \geqq \pi/2$. The group operation in G is pointwise multiplication. For each ϕ in G we have an $n \times n$ system A_ϕ with $\sigma(A_\phi) = \phi$; and $\operatorname{ind}(\phi) = \operatorname{index}(A_\phi)$ is a homomorphism of G which induces a homomorphism of the group H of homotopy classes of maps in G. By the first part of the lemma, ind maps H onto the integers; for the operator M which multiplies n-tuples of functions by $\sigma(A_n)(\cdot\,,\xi_0)^{-1}$, is an isomorphism, and $\sigma(MA_n)$ is in G, with $\operatorname{index}(MA_n) = \operatorname{index}(A_n) = 1$. On the other hand, as in Lemma 1, H is isomorphic to $\pi_{2n-1}(U(n))$, and thus isomorphic to the integers Z, so ind is essentially an isomorphism of $\pi_{2n-1}(U(n))$ onto Z.

Now let p denote the projection of $U(n)$ on S^{2n-1} as in Lemma 1. Since $\pi_{2n-1}(U(n)) = Z = \pi_{2n-1}(S^{2n-1})$, $\pi_{2n-2}(U(n-1)) = Z_{(n-1)!}$, and $\pi_{2n-2}(U(n)) = 0$ [2], the homotopy exact sequence of the bundle $U(n)$ over S^{2n-1} shows that $\phi \to \text{degree}(p \circ \phi)/(n-1)!$ induces an isomorphism of $\pi_{2n-1}(U(n))$ onto Z, which must then be $\pm$ the isomorphism given by ind.

A3. **An elementary index formula on compact manifolds.** We consider now elliptic $n \times n$ systems A of singular integral operators on a compact n-manifold X, and suppose $\sigma(A)(\xi_x)$ is unitary for each ξ_x in $S^*(X)$. Again let p denote the projection of $U(n)$ on S^{2n-1}.

THEOREM 2. *$S^*(X)$ and S^{2n-1} can be oriented so that for each n by n system A as above $(n-1)!\,\text{index}(A) = \text{degree}(p \circ \sigma(A))$, if and only if each elliptic $(n-1)$ by $(n-1)$ system on X has index zero.*

Thus the formula of the theorem holds for any manifold of dimension $\leqq 2$ (see [7]), for spheres S^n (see [3]), and generally when X is the boundary of a region in $\mathbf{R}^n$ [1, p. 429](4). I. M. Singer points out that the vanishing of the index of all $(n-1)$ by $(n-1)$ systems is equivalent to the condition that the Todd class of the manifold be 1. This requires the general formula of [1].

For the proof, suppose first the formula holds, and B is an elliptic $(n-1)$ by $(n-1)$ system with unitary symbol. Adjoining an identity operator to B, we have an n by n system A with $\text{index}(A) = \text{index}(B)$, and $p \circ \sigma(A)$ mapping onto one point s_0, so $\text{index}(A) = 0$.

For the converse, let the n by n system A be given. As in the "localization" part of the proof of Lemma 1, we may deform $\sigma(A)$ so that $p \circ \sigma(A)(\xi_x) = s_0$ except for ξ_x in some $2n-1$ cube. Then $p \circ \sigma(A)$ induces an element of $\pi_{2n-1}(U(n))$, so degree $(p \circ \sigma(A)) = k(n-1)!$ for some integer k. (See the end of the proof of Lemma 2.) Using Lemma 2 and the transplant method of Theorem 1, we find an n by n system A' with $\text{index}(A') = -k$ and $\text{degree}(p \circ \sigma(A')) = -k(n-1)!$. Then by Lemma 1, $\text{degree}(p \circ \sigma(A'A)) = 0$, so $\sigma(A'A)$ is homotopic to a system B with $p \circ \sigma(B)$ mapping into s_0. Thus B may be chosen so that it is obtained by adjoining an identity operator to an $(n-1)$ by $(n-1)$ system B', so

$$0 = \text{index}(B) = \text{index}(A'A) = \text{index}(A') + \text{index}(A) = -k + \text{index}(A).$$

A4. **A formula for the index on Euclidean space.** In [8] there was discussed a class of singular integral operators on Euclidean space $\mathbf{R}^n$; it was shown that the index exists and depends only on the homotopy class of the symbol.

THEOREM 3. *For an $n \times n$ elliptic system A of operators on Euclidean space, with unitary symbol, the formula of Theorem 2 holds*: $(n-1)!\,\text{index}(A) = \text{degree}(p \circ \sigma(A))$.

(4) See also I. M. Singer, *On the index of elliptic operators*, Outlines of the Joint Soviet American Symposium on Partial Differential Equations.

The proof uses methods of the previous results. First, we may replace A by an operator A' such that $\operatorname{index}(A) = \operatorname{index}(A')$, $\operatorname{degree}(p \circ \sigma(A)) = \operatorname{degree}(p \circ \sigma(A'))$, and $\lim_{|x| \to \infty} \sigma(A')(x,\xi) = I$, $\sigma(A')(x,\xi_0) = I$ for some fixed ξ_0 and all x. The proof now concludes as in the end of Lemma 2. In this case, we prove "index" maps onto the integers by transplanting the system A_n of Lemma 2 from T^n onto $\boldsymbol{R}^n$, as in the proof of Theorem 1.

A5. **Singular integral operators on open manifolds.** The previous methods suggest how to construct a theory of singular integral operators on open manifolds. Classically, differential operators on open manifolds require some boundary conditions to obtain well-posed problems, and in particular, in the elliptic case, the number of boundary conditions is half the order of the equation times the rank of the system. Since singular integral operators have order zero, no boundary conditions should be necessary. On the other hand, singular integrals behave rather badly at boundary points, so some assumption must be made about the nature of the operator near the boundary. Hence the definition given below. In this section, X denotes an open manifold (i.e. not necessarily compact, but no boundary points are in the manifold), carrying a C^∞ volume element; and E and F are bundles over X with complex Hermitian structure. In this case $H^0(E)$ depends on the choice of volume element and Hermitian structure.

DEFINITION. An operator A from sections of E to sections of F is a special singular integral operator (ssiop) if and only if:

(i) For each pair of C^∞ functions ϕ and ψ with disjoint compact supports, $\phi A \psi$ is a compact operator from $H^0(E)$ to $H^0(F)$ ($\phi A \psi$ may be subjected to more stringent restrictions for various classes of operators).

(ii) When ϕ and ψ have support in a coordinate neighborhood over which E and F are trivial, then $\phi A \psi$ can be represented by a matrix of singular integral operators in Euclidean space, using the coordinates on X and the local trivializations of E and F.

(iii) There is a function ϕ of compact support in X, and a section ψ of $\operatorname{Hom}(E,F)$, such that $A = \phi A \phi + (1 - \phi^2) M_\psi$, where M_ψ is the map of sections of E to sections of F induced by ψ.

Condition (iii) is automatically fulfilled if X is compact, and then the definition reduces to that of [6] and the first part of this paper. The whole theory of singular integral operators as developed there and in this paper holds without essential change for the operators just defined. Further, the index may be computed by reducing to the compact case as follows. Choose in X a manifold Y with smooth boundary bY, such that the closure of Y is compact in X, and Y contains the support of the function ϕ of part (iii) of the above definition. Then we may double Y with respect to bY, and extend A by reflection to the reflection of Y, since A is simply a multiplication operator in a neighborhood of bY. Arguing as in Theorem 1, the index of A is half the index of this doubled operator.

Finally, we may take limits and extend the theory to a larger class of operators, as in §§11 and 12, or [8, §IV].

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