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Intelligent inventory management with autonomation and service strategy

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Abstract

The manufacturer's service to the customer is one of the critical factors in maximizing profit. This study proposes the innovative (Q, r) inventory policy integrated with autonomated inspection and service strategy for service-dependent demand. First, an advanced autonomated inspection makes the product error-free. Therefore, this makes customers more satisfied and increases profit. The proposed model decides the optimal investment for such autonomated inspection. Second, three types of services are considered in the study: unpaid, partially paid, and fully paid services. Each type of service has a different service level and the amount of the customer's payment. Our model finds the optimal service strategy based on the variable conditions along with the optimal quantity and reorder level of inventory policy. Numerical analyses are made for different service strategies, along with a sensitivity analyses for various critical parameters. Results show that the full paid service is 84.88% beneficial compared to the unpaid service, and the autonomated inspection policy is 5.02% beneficial compared to the traditional ones. The increase in unit servicing costs always increases the profit of the company.

Keywords Inspection · Profit maximization · Service dependent demand · (Q, r) model

Mathematics Subject Classification 90B05 · 90B50 · 90B25 · 90B36

Introduction

The main objective of companies is to maximize their profit. One of the traditional strategies is enhancing customer satisfaction via error-free products and differentiated service. This study proposes an innovative (Q, r) inventory model implementing such a strategy. First, autonomated inspection is considered in manufacturing to make the product error-free. It could eventually make more profit. Second, various types of services are considered under service-dependent demand. Each service has a different quality based on the amount of customer's payment.

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In general, defective items may be delivered to the retailer from the manufacturing house. Managing inventory under such imperfect items is an essential job for industry managers. How much to order and when to order while considering the possibility of defective items are important decisions, and many relevant inventory models are studied (Barron & Baron, 2020; Cárdenas-Barrón et al., 2014; Chang et al., 2005). Especially, some (Q, r) models were developed under the consideration of shortages cost (Khan et al., 2011; Maddah et al., 2010; Manna & Chaudhuri, 2006). The shortage cost due to the defective items affects the optimal Q and r, and the company's profit. In such a situation, effective inspection to identify those defective products, which may be reworked or sold at a lower price, is helpful to reduce the shortage cost and eventually increase the company's profit. Safety stock allows the decision-maker to control the expected unplanned shortages, which directly helps to reduce the total cost. During the inspection period, all items were inspected. Due to the imperfect manufacturing system, some defective items were produced randomly. This random vari-

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able's distribution depends on the fraction of defective items in a batch (Karakatsoulis & Skouri, 2021).

The traditional inventory model was considered human-based inspection, which may be error-prone (Sarkar & Saren, 2016). However, there was a chance of errors in this type of inspection, which had a major impact in optimizing total system cost (Sarkar & Saren, 2016). Furthermore, this inspection error leads to a tremendous shortage cost (Tajbakhsh, 2010). For example, due to the low quality of masks, 1.3 million face masks were banned, and huge shortage costs occurred during the COVID-19 pandemic as per Euronews on March 21, 2020 (https://www.euronews.com). To overcome a massive shortage and calculate proper reorder level, it is necessary to perform an error-free inspection (Cheikhrouhou et al., 2018). Thus, checking the quality and defectiveness of the product is critical for any (Q, r) inventory system. A machine-based autonomated inspection policy was very much effective in detecting defective products (Lin et al., 2019; Sarkar et al., 2020). This autonomation helps prevent the product's defectiveness and decreases the shortage cost (Dey et al., 2021b). Hence, this present study attempts to formulate a (Q, r) inventory model integrated with an autonomated inspection policy, making the process more intelligent and profitable than the traditional one.

On the other hand, customers become more careful about the service provided by the company (Guajardo & Rönnqvist, 2015). Before buying the products, customers find which type of services the company will offer. They can get service instantly if problems arise in the product's life span (Rebaiaia & Ati-kadi, 2021; Rezg et al., 2008). Different types of servicing or maintaining a particular product are crucial in those days to optimize the profit of companies (Höller et al., 2020). Such service strategies of companies directly affect the demand and profit. Several existing studies were concentrated on maintaining the production process in terms of preventive or corrective maintenance (Haidar et al., 2022). However, as per the authors' knowledge, different services like home delivery, customer support, and repairing a sold product during the product's life cycle are in an inventory system still not considered by any existing literature.

Hence, an intelligent inventory model to address these issues is proposed in this current study. Machine-based autonomated inspection is implied to detect faulty items, control the shortages, and market demand along with enhancement in the profit of the inventory model. When customer demand is service-dependent, the best servicing strategy among the unpaid, partially paid, and full paid services are determined under various conditions. Some investments are incorporated to upgrade the servicing strategies. It helps to keep the company's brand image and enhance the industry's total profit.

In brief, the following significant issues for a (Q, r) inventory model are solved in this study:

(i) Most traditional (Q, r) models are considered a human-based inspection policy to detect imperfect or defective items (Sana et al., 2007; Karakatsoulis & Skouri, 2021). The autonomated inspection policy (Dey et al., 2021) for the (Q,r) inventory model was still not considered in the literature. Hence, a machine-based autonomated inspection policy is applied to detect the faulty product for the (Q, r) inventory model, which provides an error-free inspection process and makes the (Q, r) model more profitable.

(ii) The traditional (Q, r) inventory model deals with constant or deterministic demand (Widyadana & Wee, 2010). Some inventory and production models were recently developed under the consideration of selling price-dependent demand (Cárdenas-Barrón et al., 2021), and some production models were developed under quality-dependent demand (Dey et al., 2021b). However, a service-dependent (Q, r) inventory model is not sufficiently discussed. Thus, an effort is made in this current manuscript to fill this research gap.

(iii) Several inventory models set the service as a constraint (Albrecht, 2017; Sereshti et al., 2021). However, profit optimization by selective services to the customers, by the company, during the life cycle of the products is still not tried in any existing literature. Hence, in this study, an effort is made to optimize the company's profit based on the company's servicing strategy.

The detailed gaps in research and literature review are discussed in section "Previous studies related to this field". The (Q, r) model is illustratively described along with notations and assumption in section "Model description & formulation". Section "Solution methodology" contains the solution methodology, whereas Section "Numerical examples and analyses" deals with numerical examples and case studies. The sensitivity of the critical parameters is provided in the Sensitivity analysis section "Sensitivity analysis". The industrial benefits are discussed in section "Managerial insights" as managerial insights. Finally, some concluding remarks and feature extensions are described in the Conclusion, section "Conclusion".

Previous studies related to this field

This section discusses the in-depth analysis of the existing literature along with the research gaps, research questions, and the necessity of this study.

Necessity of autonomation

One traditional assumption for the EOQ model is that all produced products were perfect in condition. But in reality, not all produced items may be of excellent quality due to various issues. An EOQ model, along with the consideration of defective items, was proposed by Salameh and Jaber (2000). They considered that a random portion of defective items was delivered with a probability distribution function which is independent of lot size. Moreover, they considered a 100% inspection process to detect the defective quality item in a batch through the human inspection process. Those imperfect quality items were sold in a secondary market with less price. Manna and Chaudhuri (2006) proposed an inventory model, where shortages arose due to defective products, and they solved both the case with and without shortages. Moreover, their model was developed for a deteriorating product. An optimal buffer policy was constructively stated to overcome the shortage situation by Sana (2012). An inventory system with defective items was proposed by Sarkar (2012), where system reliability was increased through some investment. Strong bonding between producer and buyer was always beneficial for enhancing the profit of any imperfect system (Das Roy et al., 2012). Das Roy et al. (2012) discussed a just-in-time-based imperfect production system with shortages and backorder. An imperfect manufacturing-remanufacturing production model for the green product was developed by Sarkar et al. (2022b). In this model, they discussed environmental and economic sustainability, however, they neglected the concept of intelligent inspection.

The main theme of a (Q, r) inventory system is to manage the lot size Q and calculate the safety factor r properly to handle the situation of shortages. The effect of safety stock in an (Q, r) inventory model under stochastic demand in the fuzzy environment was studied by De and Sana (2018); Kumar et al. (2016). Recently, an (Q, r) model was developed under the consideration of available lead time and backorder by Barron and Baron (2020). The number of generations of faulty products increases in an imperfect production system's out-of-control state (Mahata, 2017). Mahata (2017) also discussed the effect of learning in his model. Recently, Karakatsoulis and Skouri (2021) considered an (Q, r) inventory system and calculated the optimum order level and safety factor in shortages. They constructed their study based on constant demand and without a proper inspection strategy.

An inspection is required to control the shortage situation due to imperfect production (Sana et al., 2007). One can calculate the exact amount of shortages of an (Q, r) inventory model if the product inspection is 100% error-free. However, 100% error-free inspection through a human-based inspection strategy for an inventory system is near impossible (Tiwari et al., 2020). Tiwari et al. (2020) proved that inspection error was vital in determining the ordered quantity, demand, system cost, and profit. Therefore, a machine-based inspection is required to perform an error-free inspection, which directly helps to calculate the safety factor exactly for an intelligent (Q, r) inventory model (Sett et al., 2020b). In a similar direction, an automated system was proposed by Lin et al. (2019) to detect the defectiveness of LED bulbs, which enhances the system profit. They prove that automated inspection increases the system's reliability up to 5.04%. An autonomated inspection is always beneficial for an imperfect production system to perform an error-free inspection and to identify defective items properly (Sarkar et al., 2020; Dey et al., 2022). Thus, for performing an error-free inspection and calculating the exact value of the safety factor, the concept of an intelligent machine-based autonomated inspection policy is very much beneficial for imperfect production systems (Dey et al., 2021b) (See Fig. 1). If one considered constant demand, neglected the autonomation-based inspection strategy and calculated the total system cost, this model shifted to Karakatsoulis and Skouri (2021) model. However, existing literature still does not consider a machine-based intelligent autonomated inspection strategy for an (Q, r) inventory model, which is necessary to control the shortages properly. Thus, performing an error-free inspection for an (Q, r) inventory model and controlling the shortages situation appropriately, where the defective rate is random and follows specific probability distribution was studied in this current research.

Service-dependent demand

Traditional inventory models deal with constant or deterministic demand patterns. However, it is almost impossible to determine the exact demand for a particular product. An intelligent service strategy for the customers is important to increase the demand for any product. Each company provides different services like home delivery, product installation, customer support, and many more. It isn't easy to provide 100% services to their customer all the time. There must be some limitations in these services. Thus, the determination of the exact service level is a very much crucial task.

The selling price of the product is essential to determine the demand for an EOQ model (Sana, 2010). Sana (2010) developed an ordering policy under selling price-dependent demand. Pal et al. (2015) discussed the effect of selling price and product quality on demand for an SCM model. In a similar direction, Taleizadeh et al. (2015) proposed a Vendor Managed Inventory model considering price-dependent demand. In this regard, an inventory model was proposed by Taleizadeh et al. (2018), where the selling price of the product was optimized. In a similar direction, a selling price-dependent integrated inventory model was proposed by Dey et al. (2019), where they used the concept of safety stock. An online-to-offline (O2O) retailing under the consideration of imperfect production was elaborated by Sett et al. (2020a), where they assumed that the demand for the product varies with the service level, selling price, and quality of the products. An inventory model for the perishable item

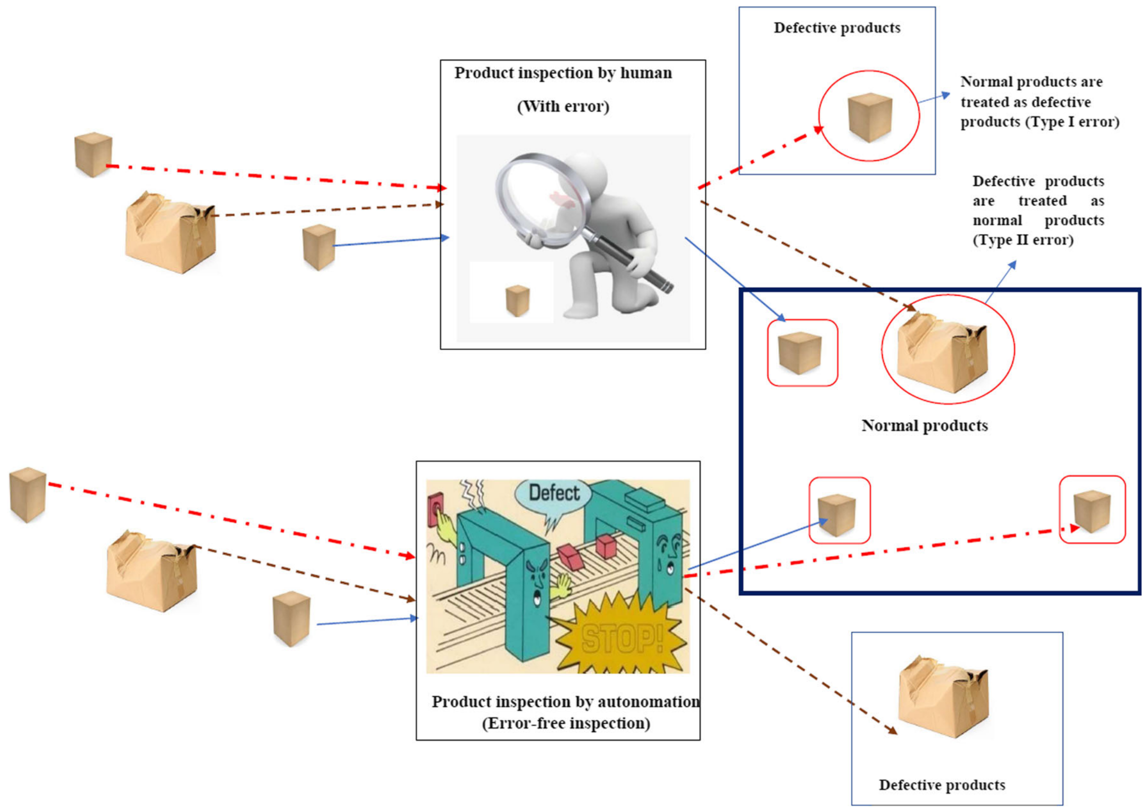


Fig. 1 Necessity of autonomation

Table 1 Research gaps and contributions of previous author(s)

Author(s)	Model type	Demand	Inspection policy	Safety factor	Maintenance strategy
Salameh and Jaber (2000)	IP	Constant	NA	NA	NA
Maddah et al. (2010)	Inv	Constant	HB	Shortage	NA
Khan et al. (2010)	EOQ	Constant	LE	NA	NA
Hsu and Hsu (2013)	EOQ	Constant	HE	Shortage	NA
Taleizadeh et al. (2015)	EOQ	Constant	NA	PB	NA
Sarkar and Saren (2016)	EPQ	Constant	HE	NA	NA
Dey et al. (2019)	II	SPD	NA	Variable	NA
Dey et al. (2021b)	IP	SPD	AUI	Variable	NA
Karakatsoulis and Skouri (2021)	(Q, r)	Constant	HB	Variable	NA
This model	(Q, r)	SLD	AUI	Variable	UP, PP, & FP

NA not applicable; *Inv.* Inventory; *EOQ* Economic Order Quantity; *EPQ* Economic Production Quantity; *HE* Human inspection with error; *IP* Imperfect production; *II* Integrated inventory; *SPD* Selling price dependent; *AUI* Autonomated inspection; *SLD* Service level dependent; *PB* Partial backorder; *HB* Human based inspection; *LE* Learning effect; (Q, r): (Q, r) inventory; *UP* Unpaid or free maintenance; *PP* Partially paid maintenance; *FP* full paid maintenance

was established by Khan et al. (2020), where the demand for the deteriorating items depends on selling price and advertisement. A replenishment policy for inventory system under selling price varying demand was developed by Duan and Ventura (2021).

Several studies were conducted based on demand variability, where the demand for the products depends on different realistic issues like selling price, advertisement, and quality of the products. However, in recent days, customers have been more careful about the company's service. Customers always prefer to buy those company's products, which will provide the best servicing. Nowadays, it is well known to every consumer that every product has an expiration date, and the product may transfer to an imperfect or faulty product during its life span. Thus, every customer was very much aware of the servicing or maintenance policy provided by the company.

Rezg et al. (2008) presented an optimum strategy to control inventory, where they adopted the preventive maintenance policy for the production system. Servicing strategy and product end-of-life (EOL) management in a sustainable way was illustrated by (Shokohyar et al., 2014). Based on the company's service level, an (Q, r, L) inventory model was proposed by Moon et al. (2014). This model minimized total system cost using the Min-Max distribution-free approach. Simultaneously, they provided the best policy for ordering quantity and reordered points. Moon et al. (2014) model was extended by Sarkar et al. (2015) by considering variable setup cost and investment to improve the process and service-level constraint. A service-levels-based inventory system with different spare parts was developed by Guajardo and Rönnqvist (2015). They proposed the Minimum Deviation from Service Level Referential Cost Method (MIND) to optimize the service level. Shin et al. (2016) proposed an inventory model where service level is a constraint. Moreover, they calculate the exact lead time for their model. Albrecht (2017) proposed an inventory model under the consideration of service level constraints. He also optimized the safety stock for his model. The effect of service level for an inventory model was studied by Gruson et al. (2018). Recently, a just-in-time inventory system under preventive maintenance and servicing was proposed by Haidar et al. (2022). Under an uncertain fuzzy environment, Bhuniya et al. (2021) proposed a supply chain model under the consideration of service level constraints. They discussed the transportation discount under distribution-free approaches.

Most of the inventory model deals with the service level as a constraint (Lin & Yang, 2011; Tsai & Zheng, 2013; Tavaghof-Gigloo & Minner, 2021; Kilic et al., 2018). A modified flower pollination algorithm was developed by Saha et al. (2021). They discussed the dynamic investment for promotion, which increased customer satisfaction, and an upgraded service was provided to the customers. The circular economy plays a major role in providing better service to the customers and keeping the environment clean from waste (Sarkar et al., 2022a). Since the provided service of the company can not be infinite, and this is the reason to assume the company's service level is up to a specific limit. Now, if it is possible to calculate the exact service level, the company undoubtedly benefited from the value of the optimal service level. In other words, the company's services enhance the demand for a particular product. Thus, this model was developed under the consideration of service-dependent demand, which is rare in literature. This study also incorporated some investments to develop the servicing strategies, which helps to optimize the profit. Moreover, the traditional model considered a maintenance policy for the production system, and sometimes a free minimal warranty policy was adopted by different existing literature (Rezg et al., 2008; Shokohyar et al., 2014; Rebaiaia & Ati-kadi, 2021). Whereas optimizing the inventory model by different types of servicing (unpaid, partially paid, fully paid) for the customer is unique. The concept of different servicing depends on the amount payable by customers for an (Q, r) inventory model with a faulty product, and intelligent autonomated inspection policy to calculate accurate safety factor is a new concept and an extensive research gap, which was fulfilled by the current study.

Model description & formulation

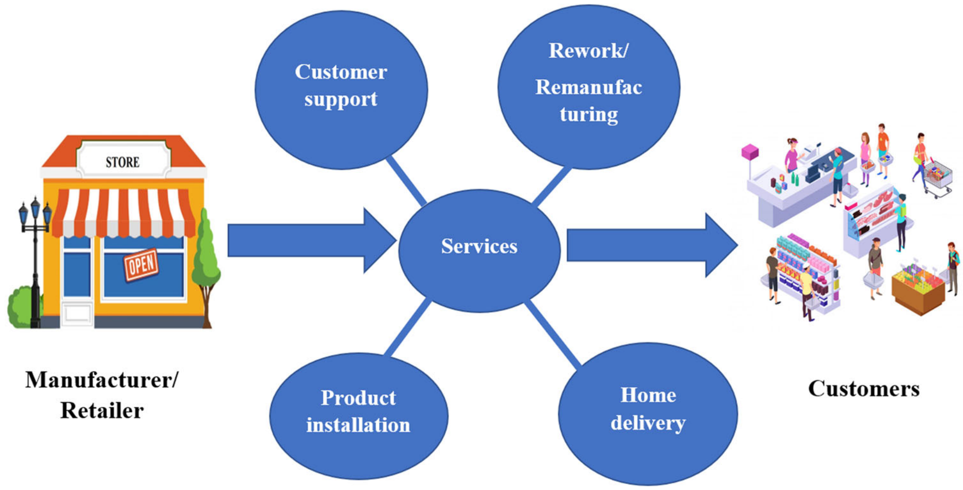
This section contains notations, assumptions, and a detailed description of the formulation of the model. A graphical representation (Fig. 2) is provided to show different types of services provided to the customers by the company.

Notation

To construct the model, the following notations are used

Decision	Variables
A	Investment for autonomated inspection ($/unit time)
s	Percentage of service provided by the company (percentage)
s^a	Percentage of service provided by the company during no shortage (percentage)
s^b	Percentage of service provided by the company during unplanned shortage (percentage)
r	Reorder point
Q	Order quantity (units)
Q^a	Order quantity for no shortage case (units)
Q^b	Order quantity for unplanned shortage case (units)
Parameters	
A_0	Initial fixed cost for inspection ($/unit time)
p	Fraction of faulty items (percentage)
μ	Percentage of servicing fees paid By the customers (percentage) $(0 \leq \mu \leq 1)$
B_Q	Number of non-faulty items in a batch of size Q (unit)
λ	Shape parameter related to service
$B(t)$	Number of non-faulty items detected at time t (unit)
γ	Shape parameter related to the investment for service
f_p	Probability density function of p
ξ	Shape parameter related to the investment for autonomation
P_Q	Probability density function of B_Q
P_t	Probability density function of $B(t)$
η	Scaling parameter related to service
D	Demand rate, which depends on service level (unit), $D = \lambda s^{\eta}$
h	Cost for holding per unit per unit time ($/unit/unit time)
T	Cycle length (time unit)
t_0	Time at which shortages occurs (time unit)

Fig. 2 Services provided to the customers



x	Rate of screening (percentage)
z	Rate of defectiveness per batch D/x
K	Fixed ordering cost ($/cycle)
C_{su}	Servicing/maintance cost per unit ($/unit)
b	backordering cost per unit per unit time ($/unit/unit time)
$ES(.)$	Expected unplanned shortages per cycle
TC_a	Expected total cost for no shortage case
TP_a	Expected total profit for no shortage case
TC_b	Expected total cost for unplanned shortage case
TP_b	Expected total profit for unplanned shortage case

Assumption

1. The demand rate, $D = \lambda s^{\eta}$, which depends on the percentage of service provided by the company. It is clear that if the company's services increase, the demand automatically increases, and simultaneously profit of the entire system is increased.
2. The company provides three types of servicing scheme to their final customers, namely free servicing, which mean no amount will be charged to the customer for this servicing, second is partial payment servicing, which means customers have to pay a certain percentage of the total servicing amount, and the company will pay rest amount. The third is the full payment servicing, e.g., the consumer will pay the total servicing fees.
3. An order of size, Q, is placed every time the inventory level drops to $r \geq 0$. The time between two consecutive orders is defined as a cycle, and it is of length T.
4. In each batch of size Q, a fraction, p, of items are defective. This implies that each batch contains a random number, B_Q, of non-defective items with a pdf P_Q. The number of defective items in a cycle is independent of the numbers in other cycles. Two different cases about B_Q are considered. In the first case, every item has a constant and known probability, m, to be defective. In the second case, $B_Q = (1 - p)Q$, where p is a random variable, with pdf f_p, $p \in [\alpha, \beta]$, $\alpha < \beta < 1$ (so independent of Q) with $E(p) = m$. These two cases affect the probability function of the number of non-defective items during the screening period and, at the same time, make $Var(p)$ dependent or independent of the order quantity Q. Hence, from now on, the first case will be referred to as the case with $Var(p)$ dependent of Q, while the second case will be referred to as the case with $Var(p)$ independent of Q (Karakatsoulis & Skouri, 2021).
5. Each batch is subject to 100% and an error-free screening process through a smart autonomation strategy at a finite rate $x > D$. During the autonomated inspection process, some unplanned shortages may appear, which are entirely backlogged, costing b per unit per unit time.
6. In each cycle, the number of non-faulty items, B_Q, is at least equal to the demand during the screening process with probability 1, i.e. $P(B_Q \geq DQ/x) = 1$.
7. Holding costs for perfect and faulty products are the same as they are kept in the same warehouse (Karakatsoulis & Skouri, 2021). The planning horizon is infinite, and lead time is negligible.

Formulation of model

The current study is based on an inventory policy to determine the optimal order quantity and safety stock to control shortages. Moreover, demand is considered service-dependent, as most existing literature discusses deterministic constant demand (Karakatsoulis & Skouri, 2021). However, stochastic service-dependent demand provides more realistic solutions. Simultaneously, instead of a traditional human inspection policy, a machine-based autonomated inspection policy (Dey et al., 2021b) is implied to perform an error-free inspection. A shortage occurs when all the warehouse products are sold out; in other words, one can say that shortages only occur after the depletion of the safety stock r. Due to huge demand and faulty production processes, stock out or shortages may occur after the safety stock's end. As per the assumption of this model, the chance of occurrence of unplanned shortages is zero for $t \geq t_0$ as $P(Y_Q \geq DQ/x) = 1$. Therefore, based on the assumption, different situations may occur, where one case is considered without shortages and occurs only when $r \geq Dt_0$. However, unplanned shortages may take account if $r < Dt_0$. Thus, two different cases: one with shortages and another without shortages, are considered as follows.

Case I: $r \geq Dt_0$

This case is when there are no shortages. Most of the cost functions are taken from Karakatsoulis and Skouri (2021). The cost related to this without shortages case are elaborated as follows:

Ordering cost (OC) To avoid the complexity of the model, the current study was created under the thought of conventional fixed ordering costs. The ordering cost for the entire cycle is given by

$$\text{OC} = \frac{K\lambda s^\eta}{Q(1-m)} \tag{1}$$

Holding cost (HC) Since this case is developed under the consideration of without shortages, thus initial inventory level is provided by $Q+r$. Due to the demand rate D, the inventory level diminishes with a linear rate of $-D$.

Therefore, the total expected holding cost for the entire cycle is given by

$$\text{HC} = \frac{\lambda s^\eta}{1-m}\left(\frac{hr(1-m)}{\lambda s^\eta} + \frac{hQ[E_p(1-p)^2 + 2mz]}{2\lambda s^\eta}\right) \tag{2}$$

Autonomation inspection cost (AIC) The traditional (Q, r) model is considered a human-based inspection, with a chance of error in the inspection process, which increases the system cost as well as decreers the brand image of the company. Thus, in this current study, an intelligent machine-based autonomated inspection policy (Sett et al., 2020b) is utilized to determine the faulty product.

Thus, the total cost for autonomation along with investment is given by

$$\text{AIC} = AQ + \xi \log\left(\frac{A_0}{A}\right) \tag{3}$$

Investment for service To satisfy and attract the customers, some service is provided by the company to their customers during the product's life cycle.

Now, to provide those services, some cost or investment is needed (Sarkar & Bhuniya, 2022). The investments for those services are given by

$$\text{Investment for service} = \frac{\gamma s^2}{2} \tag{4}$$

When $r \geq Dt_0$, the total cost of the system per unit time, $TC_a(r, Q, s, A)$ is given by:

$$TC_a(r, Q, s, A) = OC + HC + AIC + \frac{\gamma s^2}{2} \tag{5}$$

Now, if it is considered that P is the unit selling price, C_{us} is the unit service cost for the portion μ, and the unit purchasing cost is C_p, then the profit of the system is provided as

$$\begin{aligned} &TP_a(r, Q, s, A) \\ &\quad = (P - C_p + \mu C_{su})\lambda s^\eta - TC_a(r, Q, s, A) \end{aligned} \tag{6}$$

One can rewrite Eq. (6) as

$$\begin{aligned} &TP_a(r, Q, s, A) \\ &\quad = (P - C_p + \mu C_{su})\lambda s^\eta \\ &\quad\quad - \left[\frac{\lambda s^\eta}{1-m}\left\{\frac{K}{Q} + \frac{hr(1-m)}{\lambda s^\eta} + \frac{hm(1-m)}{2\lambda s^\eta}\right.\right. \\ &\quad\quad \left. + \frac{h[(1-m)^2 + 2mz]Q}{2\lambda s^\eta}\right\} \\ &\quad\quad \left. + AQ + \xi \log\left(\frac{A_0}{A}\right) + \frac{\gamma s^2}{2}\right] \end{aligned} \tag{7}$$

Case II: $r < Dt_0$

In this case, the situation of unplanned shortages is discussed. After performing the autonomated inspection, perfect or non-defective items are sold in the market, also treated as serviceable items. Hence, perfect but non-inspected products cannot be used to satisfy the demand. Therefore, the serviceable within the time interval $t \in (0, Q/x)$ is $r + B(t)$. $B(t)$ is a random variable with pdf $f_t(y)$, $y \in (0, \ldots, xt)$.

Now, when $y(t) + r \geq Dt$ that is enough inspected items in hand to satisfy the demand, then the expected holding cost is given by

$$h\int_0^{Q/x}\int_{Dt-r}^{xt} (Q + r - Dt) f_t(y)dydt \tag{8}$$

Again when $y(t) + r < Dt$ that is the number of serviceable or inspected perfect items are limited, then some unplanned backorders are taken into account, and in that case holding cost is given by

$$h\int_{0}^{Q/x}\int_{0}^{Dt-r}(Q-y)f_t(y)dydt \tag{9}$$

After some calculation, the expected holding cost in $(0, Q/x)$ is given by:

$$\frac{hzQ^2}{2\lambda s^\eta}(2-z)+\frac{hzrQ}{\lambda s^\eta}+hES(r,Q,S,A) \tag{10}$$

and the holding cost during $(Q/x, T)$ is given by

$$\frac{hQ^2E_p(1-p-z)^2}{2\lambda s^\eta}+\frac{hr(1-m-z)Q}{\lambda s^\eta} \tag{11}$$

Now, the expected shortages per cycle are given by

$$ES(r,Q,s,A)=\int_{r/D}^{t_0}\int_{0}^{Dt-r}(\lambda s^\eta t-r-y)f_t(y)dydt$$

Then, the expected cost for unplanned backlogged inventory is given by

$$bES(r,Q,S,A) \tag{12}$$

Therefore, using the Renewal Reward Theorem, and by using the concept from Karakatsoulis and Skouri (2021) the total cost of the system per unit time is calculated by summing ordering cost, holding cost, backlogging cost, autonomated investment cost, and investment for service, where ordering cost, autonomated investment cost, and investment for services are the same as case $r \geq Dt_0$.

Then, the total expected cost of the entire system is given by:

$$\begin{aligned}TC_b(r,Q,s,A) \; &= \frac{\lambda s^\eta}{1-m}\left\{\frac{K}{Q}+\frac{hr(1-m)}{\lambda s^\eta}\right.\\&+\frac{h[E_p(1-p)^2+2mz]Q}{2\lambda s^\eta}\\&\left.+\frac{(h+b)}{Q}ES(r,Q)\right\}+AQ\\&+\xi\log\left(\frac{A_0}{A}\right)+\frac{\gamma s^2}{2}\end{aligned} \tag{13}$$

Now, if it is considered that P is the unit selling price and the unit purchasing cost is C_p, then the profit of the system is provided as

$$TP_b(r,Q,s,A)=(P-C_p+\mu C_{su})\lambda s^\eta-TC_b(r,Q,s,A) \tag{14}$$

Therefore, the total profit of the system per unit time, $TP(r, Q, s, A)$, is:

$$TP(r,Q,s,A)=\begin{cases}TP_a(r,Q,s,A), & \text{for r} \geq \lambda s^\eta t_0\\ TP_b(r,Q,s,A), & \text{for r} \; < \lambda s^\eta t_0\end{cases} \tag{15}$$

The objective is to determine the values of r, Q, s, A that maximize the $TP(r, Q, s, A)$ i.e., to solve the problem:

$$\max_{r\geq 0,Q\geq 0,s\geq 0,A\geq 0} TP(r,Q,s,A)$$
$$\text{s.t. } P(Y_Q \geq \lambda s^\eta Q/x)=1$$

Solution methodology

Since the service level and investment values for autonomated inspection are independent of $E_p(1-p)^2$ and P_t. Therefore, the values of the service level and the investment for autonomated inspection to detect the faulty product for both cases are obtained by taking the first ordered partial derivative of the function $TP(r, Q, s, A)$ equating to zero, concerning the decision variable under the conditions $Q \geq 0$, and $r \geq 0$. Thus, the optimum values of the service level and investment for autonomated inspection are obtained as follows:

$$A^* = \frac{\xi}{Q} \tag{16}$$

To obtain the optimal value of the ordered quantity and safety stock, one must calculate the value of the $E_p(1-p)^2$ and p_t. The value of $Var(p)$ is required to find the value of $E_p(1-p)^2$. As per assumption 4, to find the optimal result, it is required to find whether the value of P_t depends on Q or not. Simultaneously, to find the value of Q, it is essential to consider whether the value of $Var(p)$ depends on Q or not.

When *Var*(*p*) depends on the value of *Q*

This section is developed to find the value of Q, when the value of $Var(p)$ depends on Q. To calculate the optimal values, following proposition is contracted under the assumption that the value of p_t is independent of Q.

Proposition 1 *Let non-defective items in a batch Q be represented by the random variable B_Q. Also, every item has a constant and independent of the other items' probability m to be defective. Mathematically, $B_Q \sim B(Q, 1-m)$. Moreover, it is assumed that the number of inspecting items is $xt \in N$, $t \in (0, Q/x)$. Then the number of non-defective items, i.e., perfect items, will be found as a random variable with*

$$P(B(t)=k)=\binom{xt}{k}(1-m)^k m^{xt-k},\quad k\in\{0,\ldots,xt\} \tag{17}$$

Since, the above Proposition guaranteed the independency of P_t on Q. Besides it $B_Q \sim B(Q,1-m)$ implies that $Var(B_Q)=Qm(1-m)$. Moreover, $Var(p=\frac{B_Q}{Q})=\frac{m(1-m)}{Q}$, that implies the dependency of B_Q on Q.

One can conclude the following corollary from proposition 1.

Corollary 1 *The likelihood of the event of an unplanned shortage at time t is independent of Q.*

From the corollary, it is clear that time is independent of Q. However, it depends on safety stock, i.e., depends on r, then one can write $ES(r,Q)=ES(r)$.

Now, the total system profit is given by

$$\begin{aligned}&TP_a(r,Q,s,A)\\&=(P-C_p+\mu C_{su})\lambda s^\eta\\&\quad-\left[\frac{\lambda s^\eta}{1-m}\left\{\frac{K}{Q}+\frac{hr(1-m)}{\lambda s^\eta}+\frac{hm(1-m)}{2\lambda s^\eta}\right.\right.\\&\quad\left.+\frac{h[(1-m)^2+2mz]Q}{2\lambda s^\eta}\right\}\\&\quad\left.+AQ+\xi\log\left(\frac{A_0}{A}\right)+\frac{\gamma s^2}{2}\right]\end{aligned}$$

and

$$\begin{aligned}&TP_b(r,Q,s,A)\\&=(P-C_p+\mu C_{su})\lambda s^\eta\\&\quad-\left[\frac{\lambda s^\eta}{1-m}\left\{\frac{K}{Q}+\frac{hr(1-m)}{\lambda s^\eta}\right.\right.\\&\quad+\frac{(h+b)}{Q}ES(r)+\frac{h[(1-m)^2+2mz]Q}{2\lambda s^\eta}\\&\quad\left.\left.+\frac{hm(1-m)}{2\lambda s^\eta}\right\}+AQ+\xi\log\left(\frac{A_0}{A}\right)+\frac{\gamma s^2}{2}\right]\end{aligned}$$

Proof See Appendix A.

Now, one has to find the optimum values of the decision variables and prove the concavity of the profit function $TP(r,Q,s,A)$ with the help of the optimum values of the decision variables.

Thus, the optimum values for the decision variables for constant r, and for $r\geq\lambda s^\eta$ is given by

$$Q^a=\sqrt{\frac{2\lambda s^\eta K}{2A(1-m)+h\left((1-m)^2+2mz\right)}} \tag{18}$$

$$s^a=\left[\frac{(1-m)Q\gamma}{\eta\lambda\left((1-m)(P-c_p+\mu C_{su})-K\right)}\right]^{\frac{1}{\eta-2}} \tag{19}$$

□

Theorem 1 *To find the concavity of the profit function, the following properties must be held:*

1. *The profit function $TP_a(r,Q,s,A)$ is decreasing when $r\geq\lambda s^\eta t_0$, and concave in $Q\geq 0$, $s\geq 0$, $A\geq 0$, when $\Omega_3<0$, and $\Gamma_3<0$.*
2. *The profit function $TP_b(r,Q,s,A)$ is concave when $r\leq\lambda s^\eta t_0$ for constant Q, s, A.*
3. *The profit function $TP_b(r,Q,s,A)$ is concave in $Q\geq 0$, $s\geq 0$, $A\geq 0$, when $\Omega_7<0$, and $\Gamma_5<0$ for constant r.*

Proof See Appendix B. □

Theorem 2 *It can be proven easily that the total system profit function is continuous for $r\geq 0$ owing to*

$$\lim_{r\to Dt_0} TP_b(r,Q,s,A)=TP_a(\lambda s^\eta t_0,Q,s,A)$$

Now, the aim is to maximize $TP(r,Q,s,A)$. Moreover, $TP_a(r,Q,s,A)$ takes it maximum value, when $r=\lambda s^\eta t_0$. Therefore, to prove that the profit function $TP(r,Q,s,A)$ is optimum, it is sufficient to optimize $TP_b(r,Q,s,A)$ over $(Q,r)\in[0,\lambda s^\eta t_0]\times(0,\infty)$.

Now, one must find the value of Q^* and r^* by using the TP_b.

Thus, the maximum value of $TP(r,Q,s,A)$ is obtained with (Q^*,r^*,s^*,A^*), the value of Q^* in terms of Q^b is obtained as

$$Q^b=\sqrt{\frac{2\lambda s^\eta\left(K+(h+b)ES(r^b)\right)}{2A(1-m)+h\left((1-m)^2+2mz\right)}} \tag{20}$$

$$s^b=\left[\frac{(1-m)Q\gamma}{\eta\lambda\left((1-m)(P-c_p+\mu C_{su})-K-(h+b)ES(r^b)\right)}\right]^{\frac{1}{\eta-2}} \tag{21}$$

$$\frac{\lambda s^\eta}{Q^b(1-m)}\int_{\frac{r^b}{\lambda s^\eta}}^{t_0}F_t\left(\lambda s^\eta t-r^b\right)dt=\frac{h}{h+b};\ if\ r^b\geq 0 \tag{22}$$

or,

$$\begin{aligned}&maxTP(r,Q,s,A)\\&\quad=TP\left(0,\sqrt{\frac{2\lambda s^\eta\left(K+(h+b)ES(0)\right)}{2A(1-m)+h\left((1-m)^2+2mz\right)}},\right.\\&\qquad\left.s^b,A^*\right),\ if\ r^b<0,\end{aligned}$$

which gives that

$$\frac{\lambda s^{\eta}[(1-m)^2+2mz]}{(1-m)^2}\int_{\frac{r^b}{\lambda s^{\eta}}}^{t_0} P_t\left(\lambda s^{\eta}t-r^b\right)dt>\frac{h}{h+b} \tag{23}$$

Thus, one can conclude that

$$\begin{aligned}&maxTP(r^*,Q^*,s^*,A^*)\\&=max\left\{TP\left(\lambda s^{\eta}t_0,\sqrt{\frac{2\lambda s^{\eta}\,(K)}{2A(1-m)+h\left((1-m)^2+2mz\right)}},s^*,A^*\right),\right.\\&\left.TP\left(0,\sqrt{\frac{2\lambda s^{\eta}\,(K+(h+b)ES(0))}{2A(1-m)+h\left((1-m)^2+2mz\right)}},s^*,A^*\right)\right\}\end{aligned} \tag{24}$$

It proves that an almost closed-form solution is obtained for the profit expression with optimal order quantity, optimum percentage of service, and optimum investment for autonomation, and the solutions are provided in Eqs. (20), and (21).

Proof See Appendix C. □

When *Var*(*p*) is independent of the value of *Q*

When $Var(p)$ is independent of Q, P_t depends on Q . Now, by using Bayes' Theorem and utilizing the concept of (Karakatsoulis & Skouri, 2021) model. The profit functions are obtained as follows. Therefore, the total profit of the system per unit time, $TP(r,Q,s,A)$, is given by:

$$TP(r,Q,s,A)=\begin{cases}TP_a(r,Q,s,A), & \text{for r} \geq \beta Qz\\ TP_b(r,Q,s,A), & \text{for r } <\beta Qz\end{cases} \tag{25}$$

where

$$\begin{aligned}TP_a(r,Q,s,A)&=(P-C_p+\mu C_{su})\lambda s^{\eta}\\&-\left[\frac{\lambda s^{\eta}}{1-m}\left\{\frac{K}{Q}+\frac{hr(1-m)}{\lambda s^{\eta}}\right.\right.\\&\left.+\frac{h[E_p(1-p)^2+2mz]Q}{2\lambda s^{\eta}}\right\}\\&\left.+AQ+\xi\log\left(\frac{A_0}{A}\right)+\frac{\gamma s^2}{2}\right]\end{aligned}$$

and

$$\begin{aligned}TP_b(r,Q,s,A)&=(P-C_p+\mu C_{su})\lambda s^{\eta}\\&-\left[\frac{\lambda s^{\eta}}{1-m}\left\{\frac{K}{Q}+\frac{hr(1-m)}{\lambda s^{\eta}}\right.\right.\\&+\frac{(h+b)}{Q}ES(r,Q)\\&\left.+\frac{h[E_p(1-p)^2+2mz]Q}{2\lambda s^{\eta}}\right\}\\&\left.+AQ+\xi\log\left(\frac{A_0}{A}\right)+\frac{\gamma s^2}{2}\right]\end{aligned}$$

where

$$ES(r,Q,s,A)=\int_{r/D}^{t_0}\int_{0}^{Dt-r}(Dt-r-y)f_t(y)dydt$$

and $E_p(1-p)^2$ is independent of Q (as f_p is independent of Q).

The objective is the determination of $r(\geq 0)$, $Q(\geq 0)$, $s(\geq 0)$, and $A(\geq 0)$ that maximize $TP(r,Q,s,A)$. The next theorem provides properties in this direction.

Thus, the optimum values for the decision variables for constant r, and for $r\geq\beta Qz$ is given by

$$Q^a=\sqrt{\frac{2\lambda s^{\eta}K}{2A(1-m)+h\left(E_p(1-p)^2+2mz\right)}} \tag{26}$$

$$s^a=\left[\frac{(1-m)Q\gamma}{\eta\lambda\left((1-m)(P-c_p+\mu C_{su})-K\right)}\right]^{\frac{1}{\eta-2}} \tag{27}$$

Theorem 3 *To find the concavity of the profit function, the following properties must be hold:*

1. *The profit function* $TP_a(r,Q,s,A)$ *is decreasing when* $r\geq\beta zQ$, *and concave in* $Q\geq 0$, $s\geq 0$, $A\geq 0$, *when* $\Upsilon_1<0$, *and* $\Delta_1<0$.
2. *The profit function* $TP_b(r,Q,s,A)$ *is concave when* $r<\beta zQ$ *for constant* Q, s, A.
3. *The profit function* $TP_b(r,Q,s,A)$ *is concave in* $Q\geq 0$, $s\geq 0$, $A\geq 0$, *when* $\Upsilon_3<0$, *and* $\Delta_2<0$ *for constant* r.

Proof See Appendix D. □

Theorem 4 *It can be proven easily that the total system profit function is continuous for* $r\geq 0$ *owing to*

$$\lim_{r\to\beta zQ}TP_b(r,Q,s,A)=TP_a(\lambda s^{\eta}t_0,Q,s,A)$$

Now, the aim is to maximize $TP(r,Q,s,A)$. *Moreover,* $TP_a(r,Q,s,A)$ *takes it maximum value, when* $r=\beta zQ$. *Therefore, to prove that the profit function* $TP(r,Q,s,A)$ *is optimum, it is sufficient to optimize* $TP_b(r,Q,s,A)$ *over* $(Q,r)\in[0,\beta zQ\times(0,\infty)$.

Now, one have to find the value of Q^* *and* r^*, *and* s^* *by using the* TP_b.

Thus, the maximum value of $TP(r,Q,s,A)$ *is obtain with* (Q^*,r^*,s^*,A^*), *the value of* Q^* *in terms of* Q^b *is obtained as*

$$Q^b = \sqrt{\frac{2\lambda s^\eta \left(K + (h+b)ES(r^b)\right)}{2A(1-m) + h\left(E_p(1-p)^2 + 2mz\right)}} \quad (28)$$

$$s^b = \left[\frac{(1-m)Q\gamma}{\eta\lambda\left((1-m)(P - c_p + \mu C_{su}) - K - (h+b)ES(r^b)\right)}\right]^{\frac{1}{\eta-2}} \quad (29)$$

$$\frac{\lambda s^\eta}{Q^b(1-m)} \int_{\frac{r^b}{\lambda s^\eta}}^{t_0} F_t\left(\lambda s^\eta t - r^b\right) dt = \frac{h}{h+b};\ if\ r^b \geq 0 \quad (30)$$

or,

$$maxTP(r, Q, s, A) = TP\left(0, \sqrt{\frac{2\lambda s^\eta \left(K + (h+b)ES(0)\right)}{2A(1-m) + h\left(E_p(1-p)^2 + 2mz\right)}}, s^*, A^*\right),\ if\ r^b < 0,$$

which gives that

$$\frac{\lambda s^\eta[E_p(1-p)^2 + 2mz]}{(1-m)^2} \int_{\frac{r^b}{\lambda s^\eta}}^{t_0} P_t\left(\lambda s^\eta t - r^b\right) dt > \frac{h}{h+b} \quad (31)$$

Thus, one can conclude that

$$maxTP(r^*, Q^*, s^*, A^*) = max\left\{TP\left(\beta Qz, Q = \sqrt{\frac{2\lambda s^\eta K}{2A(1-m) + h\left(E_p(1-p)^2 + 2mz\right)}}, s^*, A^*\right), TP\left(0, \sqrt{\frac{2\lambda s^\eta \left(K + (h+b)ES(0)\right)}{2A(1-m) + h\left(E_p(1-p)^2 + 2mz\right)}}, s^*, A^*\right)\right\} \quad (32)$$

It proves that an almost closed-form solution is obtained for the profit expression with optimal order quantity, optimum percentage of service, and optimum investment for autonomation. The solutions are provided in Eqs. (28), and (29).

Proof See Appendix E. □

Numerical examples and analyses

The following data are utilized to perform the numerical experiment, the values of the parameters are taken from Karakatsoulis and Skouri (2021) and Dey et al. (2021b). The solution is obtained by Mathematica 11.0 at Windows 10 64-bit operating system with an Intel i7 processor and 16 GB of RAM. The computation time is within 0.3 s. Since this is a general nonlinear function with four variables, the time complexity of the solutions does not matter.

Numerical examples when *Var*(*p*) depends on the value of *Q*

The value of holding cost is 5$ (per unit), setup cost is 85$ (per setup), initial fixed inspection cost is $250, unit selling price is $30 (per unit) and the parches price is $26 per unit, the value of the scaling and shape parameters are $\lambda = 16,500$, $\eta = 2$, $\xi = 10,000$, unit service charge is $C_{su} = \$7$ per unit, investment for service is $500 per cycle, ordering cost is $85.

Optimum result under different defective rates and different percentage amount of customer's service fee when $r \geq \lambda s^\eta t_0$

If one varies the defective rate, the system profit under full paid service, partially paid service, and unpaid services are provided in Table 2. From Table 2, it is clear that system profit is optimum if the system produces all perfect items and the customer pays the full amount for service. In contrast, the percentage of service is 79%, safety stock is 30, optimum order quantity is almost 85 unit, and optimum total expected profit is $84, 920.60. The investment for the intelligent autonomated inspection strategy is $116.69. Similarly, suppose customers paid 70% of the total amount for services and the company paid 30%. In that case, the system profit is $63, 290.50, whereas if customers paid 50% of the total amount for services and the company paid 50% for the service, the system profit is $48, 873.80. If customers paid 30% for service, and the company pays the rest, the system profit is $34, 457.10. Finally, if the company provides full service free of cost, customers do not need to pay any amount for services, the system profit is $12, 832.10.

When the defective rate is 3%, the total system profit under full paid service is $84, 600.40, and the optimal order quantity is 88.25 units with the percentage of service 79%. Simultaneously, safety stock is 31 units, and investment for autonomation is $113.31. Again, suppose customers paid 70% of the total amount for services and the company paid 30% for the service. In that case, the system profit is $62, 975.30, whereas, if customers paid 50% of the total amount for services and the company paid 50% for the service, then system profit is $48, 558.60. If customers paid 30% for service, and the company pays the rest, then system profit is $34, 141.90. Finally, if the company provides full service free of cost, that is, customers no need to pay any amount for services, then system profit is $12, 516.80 (see Table 2).

Table 2 Optimum values for different defective rates under different percentage amounts of service fee

DR	SL	OQ	SS	IA	TP(FP)	TP(70%)	TP(50%)	TP(30%)	TP(UP)
0	0.79	85.69	30	116.69	84920.60	63290.50	48873.80	34457.10	12832.10
0.03	0.79	88.25	31	113.31	84600.40	62975.30	48558.60	34141.90	12516.80
0.05	0.76	83.48	31	119.79	76747.90	56734.00	43391.50	30048.90	10035.10
0.07	0.76	85.21	28	117.35	76542.60	56528.80	43186.20	29843.70	9829.82

DR Defective rate (%); *SL* Service level (%); *OQ* Order quantity (units); *SS* Safety stock (units); *IA* Investment for autonomated inspection ($); TP(FP): Total profit under full paid service ($/cycle); TP(70%): Total profit under 70% paid service ($/cycle); TP(50%): Total profit under 50% paid service ($/cycle); TP(30%): Total profit under 30% paid service ($/cycle); TP(UP): Total profit under unpaid or free service ($/cycle)

A similar discussion can draw for defective rate 5% and 7%.

From Table 2, it is clear that total profit is reduced with an increasing rate of defectiveness. From Table 2, it is also clear that if the defective rate increase, the percentage of service of the company is reduced, and investment for autonomation increases up to a specific limit and is then reduced.

Table 2 shows that the value of all decision variables is the same; that is, the optimum value of service, order quantity and cost for autonomated inspection are the same. However, system profit was different due to the variation in the amount paid for the service. Thus, it is clear that the service provided by the company to their customers during the product's life cycle takes a significant role in profit optimization.

The concavity of the profit function with respect to the optimal value of order quantity and investment for autonomation is graphically presented in Fig. 3.

Numerical examples when $Var(p)$ depends on the value of Q and $r < \lambda s^{\eta} t_0$

Here the explanation of the situation when the value of $Var(p)$ depends on the value of Q and unplanned shortages may occur is discussed. The parameter values are taken as the previous case with the value of b =20. From the previous discussion, it is clear that payment for servicing the product during its life cycle takes a vital role in profit optimization. Thus, in this section, full-paid service and free service cases are discussed. The optimum values for paid service and free service are presented in Table 3. Moreover, the concavity of the function is graphically illustrated in Fig. 4.

Similar to the previous discussion, profit is optimum when both cases' defective rate is zero. The optimum profit for full paid service and free service is $75, 690.60 and $3607.02, and the investment for autonomation is $48.62. The optimum ordered quantity is 205 units, and the percentage of service is 79%. From Table 3, it is clear that profit is quite less compared to the case when $r \geq \lambda s^{\eta} t_0$.

Table 3 Optimum values for different defective rates under full paid and unpaid services

DR	SL	OQ	SS	IA	TP(FPS)	TP(UPS)
0	0.79	205.67	22	48.62	75690.60	3607.02
0.03	0.77	178.69	20	55.96	71592.40	3112.48
0.05	0.77	182.36	20	54.84	71378.10	2898.18
0.07	0.74	149.97	18	66.68	65291.50	2043.66

DR Defective rate (%); *SL* Service level (%); *OQ* order quantity (units); *SS* safety stock (units); *IA* investment for autonomated inspection ($); *TP* total profit ($/cycle); *FPS* full paid service; *UPS* unpaid service

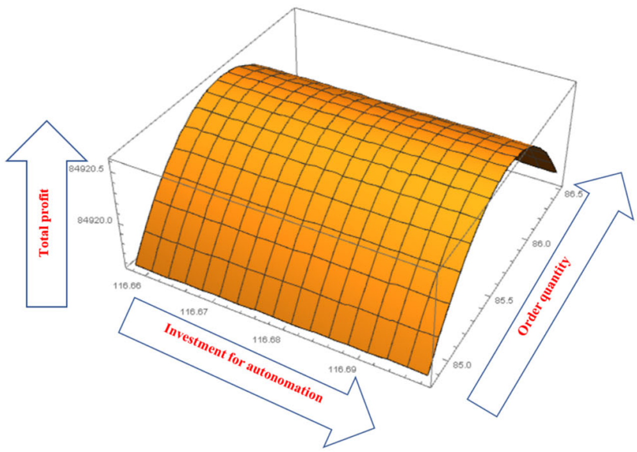
Special case I: without autonomated inspection

All parameters' values are taken similarly to previous examples. Only the value of the scaling parameter related to autonomous inspection is set at zero and eliminates the function related to autonomated inspection. This model transfers to the general EOQ model under traditional inspection. Then the profit for full paid and free service are provided in Tables 4 and 5, respectively. The concavity of the function concerning optimum order quantity and service level is graphically presented in Fig. 5.

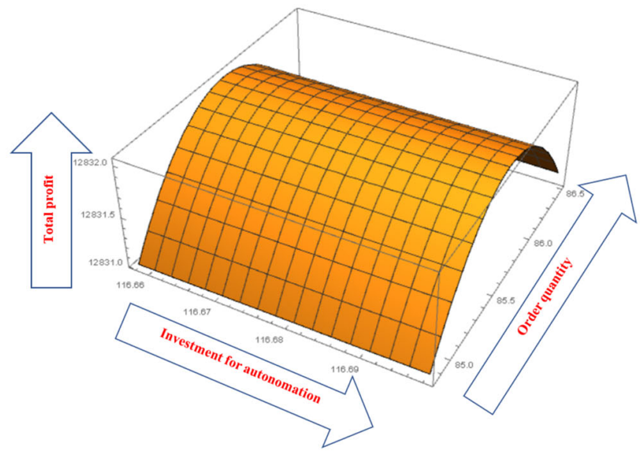
Similar to the previous discussion, profit is optimum when the defective rate is zero for both the cases and optimum profit for full paid service, and free services are $80660.00 and $9768.55. Owing to the intelligent autonomated inspection strategy, the company will benefit by $4260 for full paid service, whereas for free service, the company will benefit by $3064. Thus, intelligent autonomated inspection is very much beneficial for any inventory model.

Special case II: without autonomated inspection and investment in service

The parametric values all are the same, only the investment function related to service and revenue due to the service is neglected. Then the current model shifted to a traditional inventory model, and the optimum results are provided in Table 6.



(a) Concavity under full paid service.



(b) Concavity under free service.

Fig. 3 Concavity of total profit with respect to investment for autonomation and ordered quantity under full paid and free service

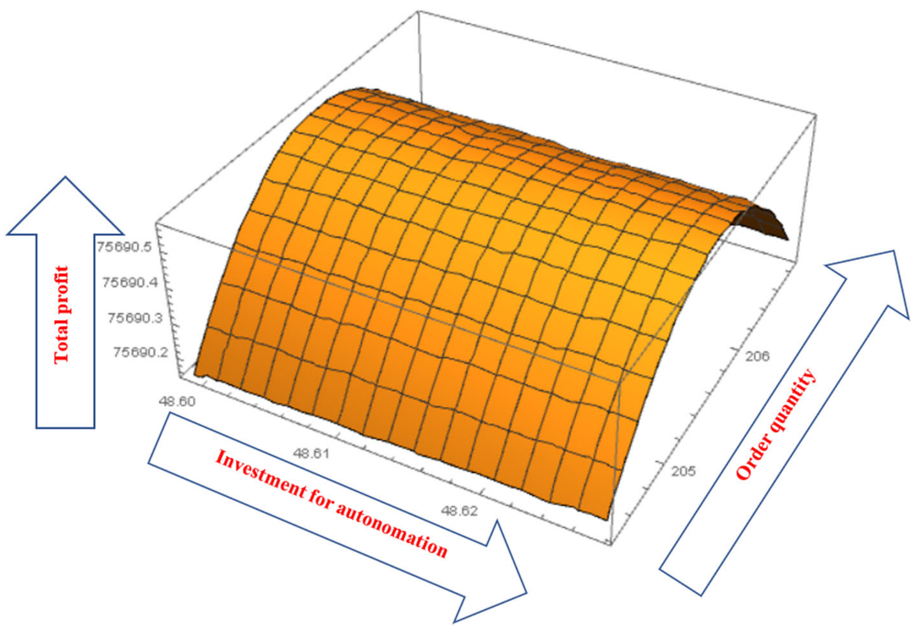


Fig. 4 Concavity of total profit with respect to investment for autonomation and ordered quantity

Table 4 Optimum values for different defective rates without autonomation

Defective rate (%)	Service level (%)	Order quantity	Safety stock	Total Profit ($/cycle)(Full paid service)
0	0.72	129.29	30	80,660.00
0.03	0.72	132.07	30	80,609.30
0.05	0.69	133.95	35	73,569.00
0.07	0.69	135.86	35	73,532.10

Table 5 Optimum values for different defective rates without autonomation (unpaid/free service)

Defective rate (%)	Service level (%)	Order quantity	Safety stock	Total Profit ($/cycle)
0	0.82	318.41	30	9768.55
0.03	0.82	325.26	30	9753.82
0.05	0.82	329.91	35	9718.05
0.07	0.82	334.61	35	9706.44

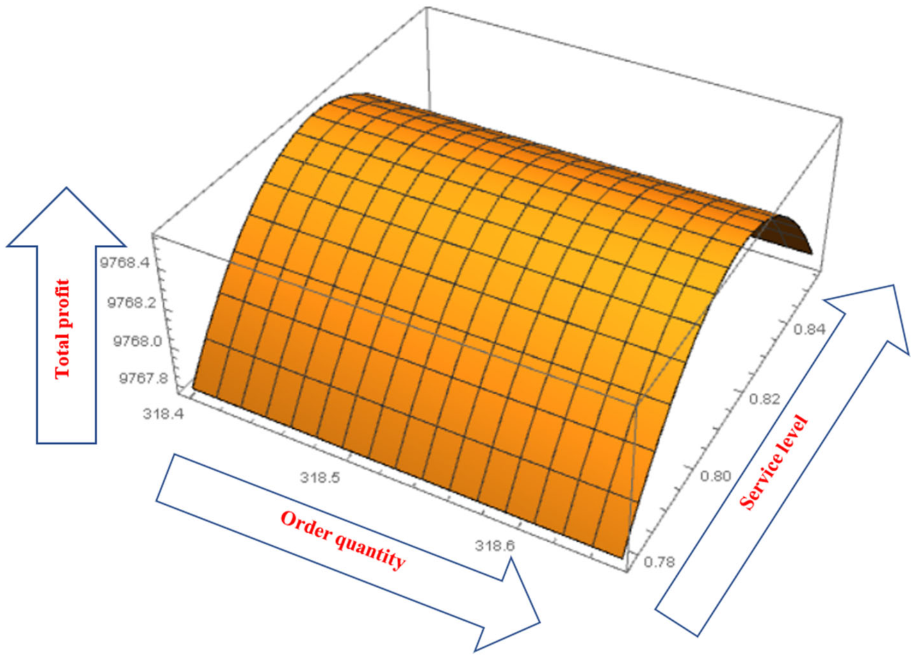


Fig. 5 Concavity of total profit with respect to ordered quantity and service level without autonomation

Table 6 Optimum values for different defective rates without autonomation and investment for service

Defective rate (%)	Service level (%)	Order quantity	Safety stock	Total Profit ($/cycle)
0	0.75	561.75	68	33,726.30
0.03	0.75	573.84	65	33,715.30
0.05	0.74	574.28	65	32,751.00
0.07	0.73	574.59	69	31,778.90

From Table 6, it is clear that the optimum profit is \$33, 726.30, where the optimum order quantity is 561, and the percentage of service is 75%. From Table 6, it is also clear that profit, in this case, is much less compared to previous cases. Using the concept of autonomous inspection and revenue from service, the system profit is \$84, 920, whereas the profit, in this case, is \$33, 726.30. Thus, using an autonomation strategy and different service policies optimizes total system profit by more than two and half times the traditional inventory model. The optimality of the profit graphically presented in Fig. 6.

Numerical examples when $Var(p)$ is independent of the value of Q

Now, some numerical examples are provided in this section when $Var(p)$ is independent of the value of Q. From the previous discussion, it is clear that paid service is always beneficial. Thus, it is enough to show the profit for full paid service and profit for unpaid service.

Numerical examples under different defective rates when $r \geq \lambda s^{\eta} t_0$

Tables 2 and 7 clearly show that there are no changes in total system profits and the optimum value of the decision variables. Thus, it is clear that the dependency of $Var(p)$ on Q does not numerically affect the total system profit or cost. It is only an effect in analytic solutions. The solution methodology section states that the dependency of $Var(p)$ on Q provides a more closed-form solution.

Similar discussion one can draw for the case when $r < \lambda s^{\eta} t_0$.

Comparisons with existing literature

In this current section, a comparison based on numerical results is performed. The optimum result for different cases compared to the present study are provided in Table 8. The settings of the present study is not directly matched with those provided in existing studies. Hence, it is impossible to use all parameters precisely the same. We tried to use the values of the parameters from existing studies at their best fit in this model for comparison. An (Q, r, l) inventory model was developed by Sarkar (2012) where the production rate is vari-

Fig. 6 Concavity of total profit with respect to ordered quantity without autonomation and investment for service

Table 7 Optimum values for different defective rates under full paid service

DR	SL	OQ	SS	IA	TP(FP)	TP(UP)
0	0.79	85.69	30	116.69	84,920.60	12,832.10
0.03	0.79	88.25	31	113.31	84,600.40	12,516.80
0.05	0.76	83.48	31	119.79	76,747.90	10,035.10
0.07	0.76	85.21	28	117.35	76,542.60	9829.82

DR Defective rate (%); *SL* Service level (%); *OQ* Order quantity (units); *SS* Safety stock (units); *IA* Investment for autonomated inspection ($); *TP* Total profit ($/cycle); *FP* Full paid service; *UP* Unpaid service

able, and some investments were incorporated to upgrade the quality of the production process. Moreover, he formulated the model for the imperfect production process. However, in his model, he ignored the company's service strategy. He also did not consider the machine-based autonomation strategies to identify defective products. The total system profit for Sarkar (2012) model was $66775.90. In a similar direction, Cárdenas-Barrón et al. (2021) calculated the profit of an (Q, r) inventory under time-dependent holding cost. The profit for Cárdenas-Barrón et al. (2021) model was $6350.92. In contrast, due to the use of different service and autonomated inspection strategies, the current study provides better results compared to Sarkar (2012), and Cárdenas-Barrón et al. (2021) models. Therefore, it is clear that different service strategies and autonomated inspection policies are more beneficial for any inventory system.

Sensitivity analysis

The effect of critical parameters is discussed in this section. The present study is for a general (Q, r) inventory problem for a smart product industry (like a smartphone, laptops, etc.). It was evident that every parameter value can change for different industries. To illustrate this situation, we perform the sensitivity analysis for the key parameters. From the sensitivity analysis section, one can find the effect of parametric values on total profit, and we change the values of the parameters within the range $\pm 50\%$. Thus, even though we use some particular parametric value from the existing literature to perform the numerical results, our model can use in the (Q, r) inventory sector. Sensitivity examination decides how diverse values of an autonomous variable influence a specific dependent variable under a given set of assumptions is sensitivity examinations, how different sources of vulnerability in numerical results contribute to the models by and significant instability. If the value of a particular parameter change up to -50%, -25%, $+25\%$, $+50\%$. At the same time, the value of the reaming parameters is fixed, then the effect of those particular parameters on total system profit is presented in Table 9.

From Table 9, one can conclude the effect of the parameters as follows:

(i) The value of the scaling parameter related to demand is affected only when it decreases to $+50\%$, within a decreasing rate in scaling parameter, total system profit is also reduced up to -53%, whereas an increasing rate does not affect in total system profit.

(ii) The shape parameter related to demand is also very much sensitive. The shape parameter related to demand is inversely proportional to the total profit that is the increasing rate in the shape parameter decreases the total system profit. If one increases the shape parameter's value up to $+50\%$, system cost reduces to -21.44%, and if one increases up to $+25\%$, then

Table 8 Comparison with existing literature numerically

Major findings	Sarkar (2012)	Cárdenas-Barrón et al. (2021)	This study
Model	(Q, r, l)	(Q, r)	(Q, r)
Defective rate	0.03	NC	0.03
Service	NC	NC	cons.
Autonomation	NC	NC	cons.
Total profit ($)	66775.90	6350.92	84,600.40

NC not considered; *cons.* considered

Table 9 Percentage change in total profit

Parameters	Changes (in percentage)	CPI^* (in percentage)	Parameters	Changes (in percentage)	CPI^* (in percentage)
Scaling	+50	NF	Shape	+50	−21.44
parameter	+25	NF	parameter	+25	−11.34
related to	−25	−26.77	related to	−25	+12.72
demand	−50	−53.15	demand	−50	NF
	+50	−0.32		+50	−1.77
Ordering	+25	−0.16	Holding	+25	−0.95
cost	−25	+0.17	cost	−25	+1.16
	−50	+0.34		−50	+2.71
Scaling	+50	−1.09	Unit	+50	+33.41
parameter	+25	−0.69	servicing	+25	+16.71
related to	−25	NF	cost	−25	−16.71
investment	−50	NF		−50	−33.41
Initial	+50	−1.62	Investment	+50	−0.004
inspection	+25	−0.71	related to	+25	−0.002
cost	−25	+0.91	service	−25	+0.002
	−50	+2.19		−50	+0.004

*CPI (Combined profit increase) = $\frac{Change\ of\ profit}{orginal\ profit} \times 100\%$, NF stands for Not effected

system profit reduces up to −11.34%. Again if one reduces up to −25%, then total system profit increases to +12.72. Since the demand depends on the service level, scaling and shape parameter related to this demand is quite sensitive.

(iii) The ordering cost is a little bit sensitive. If one changes ordering cost up to +50%, the system profit reduces up to −0.32%, and if reduced up to −50%, then total system profit increases up to +0.34%.

(iv) The holding cost always has an impact on total system profit. It is quite natural that an increasing rate in holding costs is always harmful to any inventory system. If one increases the holding cost to +50%, system profit reduces up to −1.77%. Similarly, if one minimizes the holding cost to −50%, the total system cost increases up to +2.71%.

(v) Service always plays a vital role in determining the total system profit. If any industry charges for every service, the company will obviously benefit. However, if the company provides free service, it can attract more customers and increase its system profit. Thus, unit servicing cost significantly impacts optimizing total system profit. If one increases the unit servicing price to +50%, total system profit will increase to +33.41%, whereas if one reduces the unit servicing cost to −50%, then total system cost reduces up to −33.41%.

(vi) Rest parameters are a little bit sensitive to total system profit.

Managerial insights

Taking the right decision on inventory is one of the most critical issues for any industry to maximize its profit. The managers of any industry can make several significant decisions based on the current study.

1. From this study, the industry managers can decide how much service is beneficial to increasing the demand for any product and system profit.

2. Moreover, managers can decide on investment for an autonomated inspection policy, which provides an error-free inspection.
3. Different service strategies are also beneficial in maximizing of the industry's profit. One can decide on the percentage of the amount that will be paid for the services.
4. Managers can also decide on the safety stock and reorder point, which are crucial for any industry.
5. This current study also determines the effect of planned or unplanned backorder. From this proposed research work, the managers of industry can make several important decisions, which directly increase the industry's profit.

Conclusion

This model applies autonomated inspection and service strategies in inventory management to maximize the company's profit. The proposed model shows the enhancement of profit increment of the company in terms of earned revenue through the fees paid for services. Implementation of autonomated inspection provides the exact amount of safety stock. It increases the company's profit to 5.02%, when customers pay the full-service fee. A significant decision can be taken by managers on safety stock and optimum order quantity, which are the most common pillar for inventory systems. From the numerical section, it is clear that the dependency of $Var(p)$ on Q does not numerically affect total system profit but provides a more closed-form solution analytically. In contrast, the full paid service becomes the best strategy to maximize profit. On the other hand, free service attracts more customers. Thus, industry managers can decide on the percentage of paid service to obtain maximum profit. Therefore, the current study is beneficial for industry to maximize system profit.

Lead time and infinite planning horizon are the limitations of this study. One can develop this model by considering lead time and extending this model as the (Q, r, l) model (Moon et al., 2014). One can extend this model by considering some warranty periods given by the company (Rebaiaia & Ati-kadi, 2021). All servicing or maintenance will be free during that period, attracting more customers while system profit was increased simultaneously. In the future, one can consider the concept of trade credit to extend this model (Mandal et al., 2017). Industry can use the idea of third-party logistics (3PL) to provide the service.

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Code availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A Proof of proposition 1

If xt items are screened, then the probability, $P(B(t) = k)$, k of them to be non-defective

$$\begin{aligned}
P(B(t)=k) &= \sum_{y=0}^{Q} P(B(t)=k : Y_Q = y) P(Y_Q = y) \\
&= \sum_{y=0}^{Q} \frac{\binom{y}{k}\binom{Q-y}{xt-k}}{\binom{Q}{xt}} \binom{Q}{y} (1-m)^y m^{Q-y} \\
&= \sum_{y=k}^{Q-xt+k} \frac{\binom{y}{k}\binom{Q-y}{xt-k}}{\binom{Q}{xt}} \binom{Q}{y} (1-m)^y m^{Q-y} \\
&= \binom{xt}{k} (1-m)^k m^{xt-k} \\
&\quad \sum_{y=k}^{Q-xt+k} \binom{Q-xt}{y-k} (1-m)^{y-k} m^{Q-xt-y+k} \\
&= \binom{xt}{k} (1-m)^k m^{xt-k} \\
&\quad \sum_{\lambda=0}^{Q-xt} \binom{Q-xt}{\lambda} (1-m)^{\lambda} m^{Q-xt-\lambda} \\
&= \binom{xt}{k} (1-m)^k m^{xt-k}, k \in \{0, ..., xt\}
\end{aligned}$$

Appendix B: Proof of Theorem 1

1.
$\frac{\partial TC_a(r,Q,s,A)}{\partial r} = -h < 0$, hence $TC_a(r, Q, s, A)$ is decreasing in $r \geq Dt_0$. Thus,$TC_a(r, Q, s, A)$ attains its maximum at $r = Dt_0$ which is independent of Q.
Also, since r is independent of Q

$$\frac{\partial TP_a(r, Q, s, A)}{\partial Q} = -A - \frac{\lambda s^\eta}{1-m} \left\{ \frac{-K}{Q^2} + \frac{h((1-m)^2 + 2mz)}{2\lambda s^\eta} \right\}$$

and

$$\frac{\partial^2 TP_a(r, Q, s, A)}{\partial Q^2} = -\frac{2K\lambda s^\eta}{(1-m)Q^3} = \Omega_1(say) < 0$$

therefore $TP_a(r, Q, s, A)$ is concave in Q.

Due to the independency of Q on r, one can calculate the Hessian matrix for three variables, i.e., for Q, s, and A, and calculate the value of the principal minors as follows: First of all, one can calculate all second-order partial derivatives with respect to the decision variables as follows:

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial A^2} = -\frac{\xi}{A^2} = \Omega_2(say) < 0$$

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial s^2} = \eta\lambda s^{\eta-2}(\eta-1)(P-C_p+\mu C_{su})-\gamma - \frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q}+\frac{h(1-m)ms^{-\eta}}{2\lambda} + \frac{h(1-m)rs^{-\eta}}{\lambda}+\frac{hQs^{-\eta}\left((1-m)^2+2mz\right)}{2\lambda}\right) - \left(\frac{2h\eta}{s^2(1-m)}+\frac{h\eta(-\eta-1)}{2(1-m)s^2}\right)\left(-m(1-m) + 2r(1-m)+Q\left((1-m)^2+2mz\right)\right) = \Omega_3(say)$$

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial Q\partial A} = \frac{\partial^2 TP_a(r,Q,s,A)}{\partial A\partial Q} = -1$$

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial Q\partial s} = \frac{\partial^2 TP_a(r,Q,s,A)}{\partial s\partial Q} = \frac{h\eta\left((1-m)^2+2mz\right)}{2(1-m)s} - \frac{\eta\lambda s^{\eta-1}}{1-m}\left(\frac{-K}{Q^2}+\frac{h\left((1-m)^2+2mz\right)}{2\lambda s^{\eta}}\right) = \Omega_5(say)$$

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial A\partial s} = \frac{\partial^2 TP_a(r,Q,s,A)}{\partial s\partial A} = 0$$

Now, one have to calculate the value of all principal diagonal minors

$$|H_{11}| = \left|\frac{\partial^2 TP_a(.)}{\partial Q^2}\right| = -\frac{2K\lambda s^{\eta}}{(1-m)Q^3} = \Omega_1 < 0$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 TP_a(.)}{\partial Q^2} & \frac{\partial^2 TP_a(.)}{\partial Q\partial A} \\ \frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A^2} \end{vmatrix} = \frac{2K\xi\lambda s^{\eta}}{A^2(1-m)Q^3} - 1 > 0$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 TP_a(.)}{\partial Q^2} & \frac{\partial^2 TP_a(.)}{\partial Q\partial A} & \frac{\partial^2 TP_a(.)}{\partial Q\partial s} \\ \frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A^2} & \frac{\partial^2 TP_a(.)}{\partial A\partial s} \\ \frac{\partial^2 TP_a(.)}{\partial s\partial Q} & \frac{\partial^2 TP_a(.)}{\partial s\partial A} & \frac{\partial^2 TP_a(.)}{\partial s^2} \end{vmatrix}$$

$$= \frac{\partial^2 TP_a(.)}{\partial Q^2} \times \begin{vmatrix} \frac{\partial^2 TP_a(.)}{\partial A^2} & \frac{\partial^2 TP_a(.)}{\partial A\partial s} \\ \frac{\partial^2 TP_a(.)}{\partial A\partial s} & \frac{\partial^2 TP_a(.)}{\partial s^2} \end{vmatrix} - \frac{\partial^2 TP_a(.)}{\partial Q\partial A} \times \begin{vmatrix} \frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial s\partial A} \\ \frac{\partial^2 TP_a(.)}{\partial s\partial Q} & \frac{\partial^2 TP_a(.)}{\partial s^2} \end{vmatrix} + \frac{\partial^2 TP_a(.)}{\partial Q\partial s} \times \begin{vmatrix} \frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A^2} \\ \frac{\partial^2 TP_a(.)}{\partial s\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A\partial s} \end{vmatrix}$$

$$= \Omega_1(\Omega_2\Omega_3) - \Omega_3 - \Omega_5^2\Omega_2 = \Gamma_3(say)$$

2. Notice that $\int_0^{\lambda s^{\eta}t_0-r} f_{t_0}(y)dy = 0$ (using Eq. (1)). Hence,

$$\frac{\partial ES(r)}{\partial r} = -\int_{r/\lambda s^{\eta}}^{t_0}\int_0^{\lambda s^{\eta}t-r} f_t(y)dydt$$

$$\frac{\partial^2 ES(r)}{\partial r^2} = \int_{r/\lambda s^{\eta}}^{t_0} f_t(\lambda s^{\eta}t-r)dt \geq 0$$

So,

$$\frac{\partial TP_b(r,Q,s,A)}{\partial r} = -\frac{\lambda s^{\eta}}{1-m} \times \left\{-\frac{(h+b)}{Q}\int_{r/\lambda s^{\eta}}^{t_0}\int_0^{\lambda s^{\eta}t-r} f_t(y)dydt + \frac{h(1-m)}{2\lambda s^{\eta}}\right\}$$

and

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial r^2} = -\frac{\lambda s^{\eta}(h+b)}{(1-m)Q}\int_{r/\lambda s^{\eta}}^{t_0} f_t(\lambda s^{\eta}t-r)dt \leq 0$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial A^2} = -\frac{\xi}{A^2} = \Omega_2(say) < 0$$

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial s^2} = \eta\lambda s^{\eta-2}(\eta-1)(P-C_p+\mu C_{su})-\gamma - \frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q}+\frac{h(1-m)ms^{-\eta}}{2\lambda} + \frac{(b+h)ES(r)}{Q}+\frac{h(1-m)rs^{-\eta}}{\lambda} + \frac{hQs^{-\eta}\left((1-m)^2+2mz\right)}{2\lambda}\right) - \left(\frac{2h\eta}{s^2(1-m)}+\frac{h\eta(-\eta-1)}{2(1-m)s^2}\right)\left(-m(1-m)+2r(1-m)+Q\left((1-m)^2+2mz\right)\right) = \Omega_7(say)$$

which means that $TP_b(r, Q, s, A)$ is concave in $r < Dt_0$, for constant Q, s, A.

3.
Since $ES(r)$ is independent of Q, s, A then general Hessian matrix is calculated as follows to prove the concavity of the function $TP_b(r, Q, s, A)$

$$\frac{\partial TP_b(r,Q,s,A)}{\partial Q} = -A - \frac{\lambda s^\eta}{1-m} \times \left\{ \frac{-K}{Q^2} + \frac{h(1-m)^2 + 2mz}{2\lambda s^\eta} - \frac{(h+b)ES(r)}{Q^2} \right\}$$

and

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial Q^2} = -\frac{2\lambda s^\eta}{(1-m)Q^3}[K + (h+b)ES(r)] = \Omega_8 < 0$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial s^2} = \eta\lambda s^{\eta-2}(\eta-1)(P - C_p + \mu C_{su}) - \gamma - \frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q} + \frac{h(1-m)ms^{-\eta}}{2\lambda} + \frac{(b+h)ES(r)}{Q} + \frac{h(1-m)rs^{-\eta}}{\lambda} + \frac{hQs^{-\eta}\left((1-m)^2 + 2mz\right)}{2\lambda}\right) - \left(\frac{2h\eta}{s^2(1-m)} + \frac{h\eta(-\eta-1)}{2(1-m)s^2}\right)\left(-m(1-m) + 2r(1-m) + Q\left((1-m)^2 + 2mz\right)\right) = \Omega_7(say)$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial Q \partial A} = \frac{\partial^2 TP_b(r,Q,s,A)}{\partial A \partial Q} = -1$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial Q \partial s} = \frac{\partial^2 TP_b(r,Q,s,A)}{\partial s \partial Q} = \frac{h\eta\left((1-m)^2 + 2mz\right)}{2(1-m)s} - \frac{\eta\lambda s^{\eta-1}}{1-m}\left(\frac{-K}{Q^2} - \frac{(b+h)ES(r)}{Q^2} + \frac{h\left((1-m)^2 + 2mz\right)}{2\lambda s^\eta}\right) = \Omega_9(say)$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial A \partial s} = \frac{\partial^2 TP_b(r,Q,s,A)}{\partial s \partial A} = 0$$

Now, one have to calculate the value of all principal diagonal minors

$$|H_{11}| = \left|\frac{\partial^2 TP_b(.)}{\partial Q^2}\right| = -\frac{2\lambda s^\eta}{(1-m)Q^3}[K + (h+b)ES(r)] = \Omega_8 < 0$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial Q^2} & \frac{\partial^2 TP_b(.)}{\partial Q \partial A} \\ \frac{\partial^2 TP_b(.)}{\partial A \partial Q} & \frac{\partial^2 TP_b(.)}{\partial A^2} \end{vmatrix} = \frac{2(K + (h+b)ES(r))\xi\lambda s^\eta}{A^2(1-m)Q^3} - 1 > 0$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial Q^2} & \frac{\partial^2 TP_b(.)}{\partial Q \partial A} & \frac{\partial^2 TP_b(.)}{\partial Q \partial s} \\ \frac{\partial^2 TP_b(.)}{\partial A \partial Q} & \frac{\partial^2 TP_b(.)}{\partial A^2} & \frac{\partial^2 TP_b(.)}{\partial A \partial s} \\ \frac{\partial^2 TP_b(.)}{\partial s \partial Q} & \frac{\partial^2 TP_b(.)}{\partial s \partial A} & \frac{\partial^2 TP_b(.)}{\partial s^2} \end{vmatrix} = \frac{\partial^2 TP_b(.)}{\partial Q^2} \times \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial A^2} & \frac{\partial^2 TP_b(.)}{\partial A \partial s} \\ \frac{\partial^2 TP_b(.)}{\partial A \partial s} & \frac{\partial^2 TP_b(.)}{\partial s^2} \end{vmatrix} - \frac{\partial^2 TP_b(.)}{\partial Q \partial A} \times \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial A \partial Q} & \frac{\partial^2 TP_b(.)}{\partial s \partial A} \\ \frac{\partial^2 TP_b(.)}{\partial s \partial Q} & \frac{\partial^2 TP_b(.)}{\partial s^2} \end{vmatrix} + \frac{\partial^2 TP_b(.)}{\partial Q \partial s} \times \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial A \partial Q} & \frac{\partial^2 TP_b(.)}{\partial A^2} \\ \frac{\partial^2 TP_b(.)}{\partial s \partial Q} & \frac{\partial^2 TP_b(.)}{\partial A \partial s} \end{vmatrix} = \Omega_8\,(\Omega_2\Omega_7) - \Omega_7 - \Omega_9^2\Omega_2 = \Gamma_5(say)$$

As a result, $TP_b(r, Q, s, A)$ is concave in Q, s, and A, for constant r.

Appendix C: Proof of Theorem 2

The first order derivatives of$TP_b(r, Q, s, A)$, w.r.t. Q, r, s, and A are:

$$\frac{\partial TP_b(r,Q,s,A)}{\partial Q} = -A - \frac{\lambda s^\eta}{1-m}\left\{\frac{-K}{Q^2} + \frac{h(1-m)^2 + 2mz}{2\lambda s^\eta} - \frac{(h+b)ES(r)}{Q^2}\right\} \tag{C1}$$

and

$$\frac{\partial TP_b(r,Q,s,A)}{\partial r} = \frac{\lambda s^\eta}{1-m}\left\{-\frac{(h+b)}{Q}\int_{r/\lambda s^\eta}^{t_0}\int_0^{\lambda s^\eta t - r} f_t(y)dydt + \frac{h(1-m)}{2\lambda s^\eta}\right\} \tag{C2}$$

So, using relations (C1) and (C2) the first order conditions for maximum give:

$$Q^b = \sqrt{\frac{2\lambda s^\eta (K + (h+b) ES(r^b))}{2A(1-m) + h[(1-m)^2 + 2mz]}} \tag{C3}$$

$$\frac{\lambda s^\eta}{(1-m)Q^b} \int_{r^b/\lambda s^\eta}^{t_0} F_t(\lambda s^\eta t - r^b) dt = \frac{h}{h+b} \tag{C4}$$

From the second order condition for maximum, the solution (r^b, Q^b), obtained from (C3), (C4), is optimal if

$$\frac{(h+b)[(1-m)^2 + 2mz]\lambda s^\eta}{(1-m)^2} \int_{r^b/\lambda s^\eta}^{t_0} f_t(\lambda s^\eta t - r^b) dt < -h \tag{C5}$$

The relation (C4) does not ensure non-negative solution for r^b, but notice that,

$$lim_{r \to \lambda s^\eta T_0} \frac{\partial TP_b(r, Q, s, A)}{\partial r} = -\frac{h(1-m)}{\lambda s^\eta} < 0 \tag{C6}$$

Hence, if in the solution of (C3) and (C4) $r^b \geq 0$ then (r^b, Q^b) is the optimal policy, while if $r^b < 0$ then

$$\begin{aligned} &max TP(r^*, Q^*, s^*, A^*) \\ &= max \left\{ TP\left(Dt_0, \sqrt{\frac{2\lambda s^\eta K}{2A(1-m) + h\left((1-m)^2 + 2mz\right)}}, s^*, A^* \right), \right. \\ &\left. TP\left(0, \sqrt{\frac{2\lambda s^\eta \left(K + (h+b)ES(0)\right)}{2A(1-m) + h\left((1-m)^2 + 2mz\right)}}, s^*, A^*\right) \right\} \end{aligned}$$

Appendix D: Proof of Theorem 3

1. $\frac{\partial TP_a(r,Q,s,A)}{\partial r} = -h < 0$, hence $TP_a(r, Q, s, A)$ is decreasing in r. Thus,$TP_a(r, Q, s, A)$ attains its maximum at $r = \beta Qz$ which is independent of Q.
Also, since r is independent of Q

$$\begin{aligned} &\frac{\partial TP_a(r, Q, s, A)}{\partial Q} \\ &= -A - \frac{\lambda s^\eta}{1-m} \left\{ \frac{-K}{Q^2} + \frac{h(E_p(1-p)^2 + 2mz)}{2\lambda s^\eta} \right\} \end{aligned}$$

and

$$\frac{\partial^2 TP_a(r, Q, s, A)}{\partial Q^2} = -\frac{2K\lambda s^\eta}{(1-m)Q^3} = \Omega_1(say) < 0$$

$$\begin{aligned} &\frac{\partial^2 TP_a(r, Q, s, A)}{\partial s^2} \\ &= \eta\lambda s^{\eta-2}(\eta-1)(P - C_p + \mu C_{su}) - \gamma \\ &\quad -\frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q} + \frac{h(1-m)\beta Qzs^{-\eta}}{\lambda} \right. \\ &\quad \left. + \frac{hQs^{-\eta}\left(E_p(1-p)^2 + 2mz\right)}{2\lambda}\right) \\ &\quad + \frac{h\eta(\eta-1)}{s^2}\left(\beta Qz + \frac{Q[E_p(1-p)^2 + 2mz]}{2(1-m)}\right) \\ &= \Upsilon_1(say) \end{aligned}$$

therefore $TP_a(r, Q, s, A)$ is concave in Q.

Due to the independency of Q on r, one can calculate the Hessian matrix for three variables, i.e., for Q, s, and A, and calculate the value of the principal minors as follows: First of all, one can calculate all second-order partial derivatives with respect to the decision variables as follows:

$$\begin{aligned} &\frac{\partial^2 TP_a(r, Q, s, A)}{\partial A^2} \\ &= -\frac{\xi}{A^2} = \Omega_2(say) < 0 \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 TP_a(r, Q, s, A)}{\partial s^2} \\ &= \eta\lambda s^{\eta-2}(\eta-1)(P - C_p + \mu C_{su}) - \gamma \\ &\quad -\frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q} + \frac{h(1-m)\beta Qzs^{-\eta}}{\lambda} \right. \\ &\quad \left. + \frac{hQs^{-\eta}\left(E_p(1-p)^2 + 2mz\right)}{2\lambda}\right) \\ &\quad + \frac{h\eta(\eta-1)}{s^2}\left(\beta Qz + \frac{Q[E_p(1-p)^2 + 2mz]}{2(1-m)}\right) \\ &= \Upsilon_1(say) \end{aligned}$$

$$\frac{\partial^2 TP_a(r, Q, s, A)}{\partial Q \partial A} = \frac{\partial^2 TP_a(r, Q, s, A)}{\partial A \partial Q} = -1$$

$$\begin{aligned} &\frac{\partial^2 TP_a(r, Q, s, A)}{\partial Q \partial s} \\ &= \frac{\partial^2 TP_a(r, Q, s, A)}{\partial s \partial Q} = \frac{h\eta\left(E_p(1-p)^2 + 2mz\right)}{2(1-m)s} \\ &\quad -\frac{\eta\lambda s^{\eta-1}}{1-m}\left(-\frac{K}{Q^2} + \frac{h\left(E_p(1-p)^2 + 2mz\right)}{2\lambda s^\eta}\right) \\ &= \Upsilon_2(say) \end{aligned}$$

$$\begin{aligned} &\frac{\partial^2 TP_a(r, Q, s, A)}{\partial A \partial s} \\ &= \frac{\partial^2 TP_a(r, Q, s, A)}{\partial s \partial A} = 0 \end{aligned}$$

Now, one have to calculate the value of all principal diagonal minors

$$\left|H_{11}\right|=\left|\frac{\partial^2 TP_a(.)}{\partial Q^2}\right|=-\frac{2K\lambda s^\eta}{(1-m)Q^3}=\Omega_1<0$$

$$\left|H_{22}\right|=\begin{vmatrix}\frac{\partial^2 TP_a(.)}{\partial Q^2} & \frac{\partial^2 TP_a(.)}{\partial Q\partial A}\\ \frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A^2}\end{vmatrix}=\frac{2K\xi\lambda s^\eta}{A^2(1-m)Q^3}-1>0$$

$$\left|H_{33}\right|=\begin{vmatrix}\frac{\partial^2 TP_a(.)}{\partial Q^2} & \frac{\partial^2 TP_a(.)}{\partial Q\partial A} & \frac{\partial^2 TP_a(.)}{\partial Q\partial s}\\ \frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A^2} & \frac{\partial^2 TP_a(.)}{\partial A\partial s}\\ \frac{\partial^2 TP_a(.)}{\partial s\partial Q} & \frac{\partial^2 TP_a(.)}{\partial s\partial A} & \frac{\partial^2 TP_a(.)}{\partial s^2}\end{vmatrix}$$

$$=\frac{\partial^2 TP_a(.)}{\partial Q^2}\times\begin{vmatrix}\frac{\partial^2 TP_a(.)}{\partial A^2} & \frac{\partial^2 TP_a(.)}{\partial A\partial s}\\ \frac{\partial^2 TP_a(.)}{\partial A\partial s} & \frac{\partial^2 TP_a(.)}{\partial s^2}\end{vmatrix}-\frac{\partial^2 TP_a(.)}{\partial Q\partial A}\times\begin{vmatrix}\frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial s\partial A}\\ \frac{\partial^2 TP_a(.)}{\partial s\partial Q} & \frac{\partial^2 TP_a(.)}{\partial s^2}\end{vmatrix}$$

$$+\frac{\partial^2 TP_a(.)}{\partial Q\partial s}\times\begin{vmatrix}\frac{\partial^2 TP_a(.)}{\partial A\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A^2}\\ \frac{\partial^2 TP_a(.)}{\partial s\partial Q} & \frac{\partial^2 TP_a(.)}{\partial A\partial s}\end{vmatrix}$$

$$=\Omega_1\left(\Omega_2\Upsilon_1\right)-\Upsilon_1-\Upsilon_2^2\Omega_2=\Delta_1(say)$$

2. Notice that $\int_0^{\lambda s^\eta t_0-r} f_{t_0}(y)dy=0$. Hence,

$$\frac{\partial ES(r)}{\partial r}=-\int_{r/\lambda s^\eta}^{t_0}\int_0^{\lambda s^\eta t-r} f_t(y)dydt$$

$$\frac{\partial^2 ES(r)}{\partial r^2}=\int_{r/\lambda s^\eta}^{t_0} f_t(\lambda s^\eta t-r)dt\geq 0$$

So,

$$\frac{\partial TP_b(r,Q,s,A)}{\partial r}=-\frac{\lambda s^\eta}{1-m}\left\{-\frac{(h+b)}{Q}\int_{r/\lambda s^\eta}^{t_0}\int_0^{\lambda s^\eta t-r} f_t(y)dydt+\frac{h(1-m)}{2\lambda s^\eta}\right\}$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial r^2}=-\frac{\lambda s^\eta(h+b)}{(1-m)Q}\int_{r/\lambda s^\eta}^{t_0} f_t(\lambda s^\eta t-r)dt\leq 0$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial A^2}=-\frac{\xi}{A^2}=\Omega_2(say)<0$$

$$\frac{\partial^2 TP_a(r,Q,s,A)}{\partial s^2}=\eta\lambda s^{\eta-2}(\eta-1)(P-C_p+\mu C_{su})-\gamma-\frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q}+\frac{(b+h)ES(r)}{Q}+\frac{h(1-m)\beta z s^{-\eta}}{\lambda}+\frac{hQs^{-\eta}\left(E_p(1-p)^2+2mz\right)}{2\lambda}\right)+\frac{h\eta(\eta-1)}{s^2}\left(\beta z+\frac{Q[E_p(1-p)^2+2mz]}{2(1-m)}\right)=\Upsilon_3(say)$$

which means that $TP_b(r,Q,s,A)$ is concave in $r<\beta zQ$, for constant Q,s,A.

3. Since, $ES(r)$ is independent of Q,s,A then general Hessian matrix is calculated as follows to prove the concavity of the function $TP_b(r,Q,s,A)$

$$\frac{\partial TP_b(r,Q,s,A)}{\partial Q}=-A-\frac{\lambda s^\eta}{1-m}\left\{\frac{-K}{Q^2}+\frac{h(E_p(1-p)^2+2mz)}{2\lambda s^\eta}-\frac{(h+b)ES(r)}{Q^2}\right\}$$

and

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial Q^2}=-\frac{2\lambda s^\eta}{(1-m)Q^3}[K+(h+b)ES(r)]=\Omega_8<0$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial s^2}=\eta\lambda s^{\eta-2}(\eta-1)(P-C_p+\mu C_{su})-\gamma-\frac{\eta(\eta-1)\lambda s^{\eta-2}}{1-m}\left(\frac{K}{Q}+\frac{(b+h)ES(r)}{Q}+\frac{h(1-m)\beta z s^{-\eta}}{\lambda}+\frac{hQs^{-\eta}\left(E_p(1-p)^2+2mz\right)}{2\lambda}\right)+\frac{h\eta(\eta-1)}{s^2}\left(\beta z+\frac{Q[E_p(1-p)^2+2mz]}{2(1-m)}\right)=\Upsilon_3(say)$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial Q\partial A}=\frac{\partial^2 TP_b(r,Q,s,A)}{\partial A\partial Q}=-1$$

$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial Q\partial s}=\frac{\partial^2 TP_b(r,Q,s,A)}{\partial s\partial Q}$$

$$= \frac{h\eta\left(E_p(1-p)^2+2mz\right)}{2(1-m)s} - \frac{\eta\lambda s^{\eta-1}}{1-m}\left(\frac{-K}{Q^2} - \frac{(h+b)ES(r)}{Q^2} + \frac{h\left(E_p(1-p)^2+2mz\right)}{2\lambda s^{\eta}}\right)$$
$$= \Upsilon_4(say)$$
$$\frac{\partial^2 TP_b(r,Q,s,A)}{\partial A\partial s} = \frac{\partial^2 TP_b(r,Q,s,A)}{\partial s\partial A} = 0$$

Now, one have to calculate the value of all principal diagonal minors

$$|H_{11}| = \left|\frac{\partial^2 TP_b(.)}{\partial Q^2}\right| = -\frac{2\lambda s^{\eta}}{(1-m)Q^3}[K+(h+b)ES(r)] = \Omega_8 < 0$$

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial Q^2} & \frac{\partial^2 TP_b(.)}{\partial Q\partial A} \\ \frac{\partial^2 TP_b(.)}{\partial A\partial Q} & \frac{\partial^2 TP_b(.)}{\partial A^2} \end{vmatrix} = \frac{2(K+(h+b)ES(r))\xi\lambda s^{\eta}}{A^2(1-m)Q^3} - 1 > 0$$

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial Q^2} & \frac{\partial^2 TP_b(.)}{\partial Q\partial A} & \frac{\partial^2 TP_b(.)}{\partial Q\partial s} \\ \frac{\partial^2 TP_b(.)}{\partial A\partial Q} & \frac{\partial^2 TP_b(.)}{\partial A^2} & \frac{\partial^2 TP_b(.)}{\partial A\partial s} \\ \frac{\partial^2 TP_b(.)}{\partial s\partial Q} & \frac{\partial^2 TP_b(.)}{\partial s\partial A} & \frac{\partial^2 TP_b(.)}{\partial s^2} \end{vmatrix} = \frac{\partial^2 TP_b(.)}{\partial Q^2} \times \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial A^2} & \frac{\partial^2 TP_b(.)}{\partial A\partial s} \\ \frac{\partial^2 TP_b(.)}{\partial A\partial s} & \frac{\partial^2 TP_b(.)}{\partial s^2} \end{vmatrix} - \frac{\partial^2 TP_b(.)}{\partial Q\partial A} \times \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial A\partial Q} & \frac{\partial^2 TP_b(.)}{\partial s\partial A} \\ \frac{\partial^2 TP_b(.)}{\partial s\partial Q} & \frac{\partial^2 TP_b(.)}{\partial s^2} \end{vmatrix} + \frac{\partial^2 TP_b(.)}{\partial Q\partial s} \times \begin{vmatrix} \frac{\partial^2 TP_b(.)}{\partial A\partial Q} & \frac{\partial^2 TP_b(.)}{\partial A^2} \\ \frac{\partial^2 TP_b(.)}{\partial s\partial Q} & \frac{\partial^2 TP_b(.)}{\partial A\partial s} \end{vmatrix}$$
$$= \Omega_8(\Omega_2\Upsilon_3) - \Upsilon_3 - \Upsilon_4^2\Omega_2 = \Delta_2(say)$$

As a result, $TP_b(r, Q, s, A)$ is concave in Q, s, and A, for constant r.

Appendix E: Proof of Theorem 4

The proof is similar to the proof of Theorem 2. First of all, the first order derivatives of $TP_b(r, Q, s, A)$ w.r.t. Q r, s, and A correspondingly, are

$$\frac{\partial TP_b(s,r,Q,A)}{\partial Q} = -A - \frac{\lambda s^{\eta}}{1-m}\left\{\frac{-K}{Q^2} + \frac{h[E_p(1-p)^2+2mz]}{2\lambda s^{\eta}} - \frac{(h+b)ES(r)}{Q^2}\right\} \quad \text{(E7)}$$

$$\frac{\partial TP_b(s,r,Q,A)}{\partial r} = -\frac{\lambda s^{\eta}}{1-m}\left\{-\frac{(h+b)}{Q}\int_{r/\lambda s^{\eta}}^{t_0}\int_0^{\lambda s^{\eta}t-r} f_t(y)dydt + \frac{h(1-m)}{\lambda s^{\eta}}\right\} \quad \text{(E8)}$$

So using relations (E7), (E8) the first order conditions for maximum give:

$$Q^b = \sqrt{\frac{2K\lambda s^{\eta} + 2\lambda s^{\eta}(h+b)ES(r^b)}{2A(1-m) + h[E(1-p)^2+2mz]}} \quad \text{(E9)}$$

$$\frac{\lambda s^{\eta}}{(1-m)Q^b}\int_{r^b/\lambda s^{\eta}}^{t_0} F_t(\lambda s^{\eta}t - r^b)dt = \frac{h}{h+b} \quad \text{(E10)}$$

From the second order condition for minimum, the solution (r^b, Q^b), obtained from (E9), (E10) is optimal if

$$\frac{(h+b)[E_p(1-p)^2+2mz]\lambda s^{\eta}}{(1-m)^2}\int_{r^b/\lambda s^{\eta}}^{t_0} f_t(\lambda s^{\eta}t - r^b)dt < h \quad \text{(E11)}$$

The relation (E10) does not ensure non-negative solution for r^b, but

$$lim_{r\to\beta Qz}\frac{\partial TP_b(r,Q^b,s^b,A^*)}{\partial r} = -\frac{h(1-m)}{\lambda s^{\eta}} < 0 \quad \text{(E12)}$$

Hence, if in the solution of (E9) and (E10) $r^b \geq 0$ then (r^b, Q^b) is the optimal policy, while if $r^b < 0$ then

$$maxTP(r,Q,s,A) = maxTP \times \left(0, \sqrt{\frac{2K\lambda s^{\eta} + 2\lambda s^{\eta}(h+b)ES(0)}{2A(1-m) + h[E(1-p)^2+2mz]}}, s, A\right)$$

If inequality (E12) does not hold, then: max $TP(r, Q, s, A)$= max$\left\{TP\left(\beta Qz, Q = \sqrt{\frac{2K\lambda s^{\eta}}{2A(1-m)+h[E(1-p)^2+2z(m+(1-m)\beta)]}}, s, A\right) TP\left(0, \sqrt{\frac{2K\lambda s^{\eta}+2\lambda s^{\eta}(h+b)ES(0)}{2A(1-m)+h[E(1-p)^2+2mz]}}, s, A\right)\right\}$

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