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Intensity–duration models based on bivariate gamma distributions

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Abstract. Bivariate and univariate gamma distributions are some of the most popular models for hydrological processes (Yue *et al.*, 2001). In fact, the *intensity* and the *duration* of most hydrological variables are frequently modeled by gamma distributions. This raises the important question: what is the distribution of the *total amount* = *intensity* × *duration*? In this paper, the exact distribution of $P = XY$ and the corresponding moment properties are derived when the random vector (X, Y) has two of the most flexible bivariate gamma distributions. The expressions turn out to involve several special functions.

1. Introduction

Bivariate and univariate gamma distributions are some of the most popular models for hydrological processes. The reader is referred to the review paper by Yue *et al.* (2001) for detailed references. It is known that gamma distributions are popular models for the *intensity* and the *duration* of most hydrological variables and that often *intensity* and *duration* are correlated (Yue, 2001). This raises the important question: what is the distribution of the *total amount* = *intensity* × *duration*? In particular,

1. if X denotes the rainfall intensity and Y denotes the rainfall duration then what is the distribution of the amount of rainfall $P = XY$?
2. if X denotes the drought severity and Y denotes the drought duration then what is the distribution of the magnitude of drought $P = XY$?
3. if X denotes the rate of stream-flow into a reservoir's catchment and Y denotes the storm duration then what is the distribution of the amount $P = XY$ of water received during the storm?
4. if X denotes the snowfall intensity and Y denotes the snowfall duration then what is the distribution of the amount of snow $P = XY$?

Thus, it is important that the distribution of the product of components of bivariate gamma distributions is studied. In this paper, we consider two of

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the most flexible bivariate gamma distributions that have received hydrological applications.

The distribution of XY has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, Abu-Salih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). However, there is relatively little work of this kind when X and Y are correlated random variables. The only work known to the authors is the one by Garg *et al.* (2002) for Dirichlet family.

In this paper, we derive the exact distribution of $P = XY$ when X and Y are correlated gamma random variables arising from the following distributions:

1. McKay's bivariate gamma distribution (McKay, 1934) given by the joint pdf
$$f(x, y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1}(y-x)^{q-1} \exp(-ay) \tag{1}$$
for $y > x > 0$, $a > 0$, $p > 0$ and $q > 0$. For hydrological applications of this model, see Clarke (1979, 1980).
2. Kibble's bivariate gamma distribution (Kibble, 1941) given by the joint pdf
$$f(x, y) = \frac{(xy)^{(\alpha-1)/2}}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \exp\left(-\frac{x+y}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right) \tag{2}$$
for $x > 0$, $y > 0$, $\alpha > 0$ and $0 \leq \rho < 1$, where $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order ν defined by
$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4}\right)^k.$$
For hydrological applications of this model, see Izawa (1965) and Phatarford (1976).

The explicit expressions for the pdfs and the moments of $P = XY$ for these two distributions are derived in Sections 2–4. The calculations involve several special functions, including the ${}_1F_2$ hypergeometric function defined by
$${}_1F_2(a; b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!},$$

the Jacobi polynomial defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\},$$

the modified Bessel function of the first kind defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{1}{(\nu+1)_k k!}\left(\frac{x^2}{4}\right)^k,$$

the modified Bessel function of the third kind of order ν defined by

$$K_\nu(x) = \frac{\pi\{I_{-\nu}(x) - I_\nu(x)\}}{2\sin(\nu\pi)}$$

with $K_0(\cdot)$ interpreted as the limit

$$K_0(x) = \lim_{\nu\to 0} K_\nu(x),$$

and the modified Laguerre polynomial defined by

$$L_n^\nu(x) = \frac{x^{-\nu}\exp(x)}{n!}\frac{d^n}{dx^n}\{x^{n+\nu}\exp(-x)\},$$

where $(e)_k = e(e+1)\ldots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

LEMMA 1 (Equation (2.3.7.3), Prudnikov *et al.*, 1986, volume 1). *For* $\beta > 0$ *and* $p > 0$,

$$\begin{aligned}
&\int_a^\infty x^{\alpha-1}(x^2-a^2)^{\beta-1}\exp(-px)dx \\
&\quad = \frac{1}{2}a^{\alpha+2\beta-2}B\left(\beta, 1-\beta-\frac{\alpha}{2}\right){}_1F_2\left(\frac{\alpha}{2};\frac{1}{2},\frac{\alpha}{2}+\beta;\frac{a^2p^2}{4}\right) \\
&\qquad - \frac{1}{2}pa^{\alpha+2\beta-1}B\left(\beta,\frac{1-\alpha}{2}-\beta\right){}_1F_2\left(\frac{1+\alpha}{2};\frac{3}{2},\frac{1+\alpha}{2}+\beta;\frac{a^2p^2}{4}\right) \\
&\qquad + (2p)^{2-\alpha-2\beta}\Gamma(\alpha+2\beta-2){}_1F_2\left(1-\beta;2-\beta-\frac{\alpha}{2},\frac{3-\alpha}{2}-\beta;\frac{a^2p^2}{4}\right).
\end{aligned}$$

LEMMA 2 (Equation (2.3.16.1), Prudnikov *et al.*, 1986, volume 1). *For* $p > 0$ *and* $q > 0$,

$$\int_0^\infty x^{\alpha-1}\exp(-px-q/x)dx = 2\left(\frac{q}{p}\right)^{\alpha/2}K_\alpha(2\sqrt{pq}).$$

LEMMA 3 (Equation (2.15.5.4), Prudnikov *et al.*, 1986, volume 2). *For* $p > 0$ *and* $\nu > -n-1$,

$$\int_0^\infty x^{\nu+2n+1} \exp(-px^2) I_\nu(cx) dx = \frac{n! c^\nu}{2^{\nu+1} p^{n+\nu+1}} \exp\left(\frac{c^2}{4p}\right) L_n^\nu\left(-\frac{c^2}{4p}\right).$$

LEMMA 4 (Equation (2.19.3.2), Prudnikov *et al.*, 1986, volume 2). *For* $p > 0$ *and* $\alpha > 0$,

$$\int_0^\infty x^{\alpha-1} \exp(-px) L_n^\lambda(cx) dx = \frac{\Gamma(\alpha)}{p^\alpha} P_n^{(\lambda, \alpha-\lambda-n-1)}\left(1 - \frac{2c}{p}\right).$$

The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. McKay's bivariate gamma distribution

Theorem 1 derives the pdf of $U = XY$ when X and Y are distributed according to (1).

THEOREM 1. *If* X *and* Y *are jointly distributed according to* (1) *then*

$$\begin{aligned} f_U(u) = \frac{a^{p+q} u^{q-1}}{2\Gamma(p)\Gamma(q)} \Bigg\{ & u^{(p-q)/2} B\left(q, \frac{p-q}{2}\right) {}_1F_2\left(1 - \frac{p+q}{2}; \frac{1}{2}, 1 + \frac{q-p}{2}; \frac{a^2 u}{4}\right) \\ & - a u^{(p-q+1)/2} B\left(q, \frac{p-q-1}{2}\right) {}_1F_2\left(\frac{3-p-q}{2}; \frac{3}{2}, \frac{3+q-p}{2}; \frac{a^2 u}{4}\right) \\ & + 2^{1+p-q} (au)^{p-q} \Gamma(q-p) {}_1F_2\left(1-q; \frac{2+p-q}{2}, \frac{1+p-q}{2}; \frac{a^2 u}{4}\right) \Bigg\} \end{aligned} \tag{3}$$

for $0 < u < \infty$.

PROOF. From (1), the joint pdf of $(X, U) = (X, XY)$ becomes

$$f(x, u) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-q-1} (u - x^2)^{q-1} \exp(-au/x). \tag{4}$$

Thus, the pdf of U can be written as

$$f_U(u) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} J(u), \tag{5}$$

where

$$J(u) = \int_0^{\sqrt{u}} x^{p-q-1} (u - x^2)^{q-1} \exp(-au/x) dx.$$

Substituting $w = 1/x$, the integral $J(u)$ can be rewritten as

$$J(u) = u^{q-1} \int_{1/\sqrt{u}}^{\infty} w^{1-p-q} (w^2 - 1/u)^{q-1} \exp(-auw) dw. \tag{6}$$

Direct application of Lemma 1 shows that (6) can be calculated as

$$\begin{aligned} J(u) = 2^{-1} u^{q-1} \Bigg\{ & u^{(p-q)/2} B\left(q, \frac{p-q}{2}\right) {}_1F_2\left(1 - \frac{p+q}{2}; \frac{1}{2}, 1 + \frac{q-p}{2}; \frac{a^2 u}{4}\right) \\ & - a u^{(p-q+1)/2} B\left(q, \frac{p-q-1}{2}\right) {}_1F_2\left(\frac{3-p-q}{2}; \frac{3}{2}, \frac{3+q-p}{2}; \frac{a^2 u}{4}\right) \\ & + 2^{1+p-q} (au)^{p-q} \Gamma(q-p) {}_1F_2\left(1-q; \frac{2+p-q}{2}, \frac{1+p-q}{2}; \frac{a^2 u}{4}\right) \Bigg\}. \end{aligned} \tag{7}$$

The result of the theorem follows by combining (5) and (7). ▲

Using special properties of the hypergeometric functions, one can derive elementary forms for the pdf in (3). This is illustrated in the remark below.

REMARK 1. *If X and Y are jointly distributed according to* (1) *and if $q = 1$ then*

$$f_U(u) = \frac{a^{2p} u^{p-1}}{\Gamma(p)} \Gamma(1-p, a\sqrt{u})$$

for $0 < u < \infty$.

Now, we derive the moments of $U = XY$ when X and Y are distributed according to (1).

REMARK 2. *If X and Y are jointly distributed according to* (1) *then*

$$E(U^n) = \frac{\Gamma(p+n)\Gamma(p+q+2n)}{a^{2n}\Gamma(p)\Gamma(p+q+n)}$$

for $n \geq 1$.

PROOF. Follows by writing $E(U^n) = E(X^n Y^n) = E((X/Y)^n Y^{2n})$ and using the fact X/Y and Y are independent. Note X/Y has the beta distribution with shape parameters p and q while Y has the gamma distribution with scale parameter a and shape parameter $p+q$. ▲

3. Kibble's bivariate gamma distribution

Theorem 2 derives the pdf of $P = XY$ when X and Y are distributed according to (2).

THEOREM 2. *If X and Y are jointly distributed according to* (2) *then*

$$f_P(p) = \frac{2p^{(\alpha-1)/2}}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} I_{\alpha-1}\left(\frac{2\sqrt{p\rho}}{1-\rho}\right) K_0\left(\frac{2\sqrt{p}}{1-\rho}\right) \tag{8}$$

for $0 < p < \infty$.

PROOF. From (2), the joint pdf of $(X, P) = (X, XY)$ becomes

$$f(x,p) = \frac{p^{(\alpha-1)/2}}{x\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \exp\left\{-\frac{p+x^2}{x(1-\rho)}\right\} I_{\alpha-1}\left(\frac{2\sqrt{p\rho}}{1-\rho}\right). \tag{9}$$

Thus, the pdf of P can be written as

$$f_P(p) = \frac{p^{(\alpha-1)/2}}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} I_{\alpha-1}\left(\frac{2\sqrt{p\rho}}{1-\rho}\right) J(p), \tag{10}$$

where

$$J(p) = \int_0^\infty \frac{1}{x} \exp\left\{-\frac{p+x^2}{x(1-\rho)}\right\} dx.$$

Direct application of Lemma 2 shows that $J(p)$ can be calculated as

$$J(p) = 2K_0\left(\frac{2\sqrt{p}}{1-\rho}\right). \tag{11}$$

The result of the theorem follows by combining (10) and (11). ▲

Using special properties of the Bessel function of the first kind, one can derive elementary forms for the pdf in (8). This is illustrated in the remark below.

REMARK 3. *If X and Y are jointly distributed according to* (2) *and if* $\alpha \geq 3/2$ *is a half integer then*

$$\begin{aligned} f_P(p) = {} & \frac{2\sqrt{2}p^{(\alpha-1)/2}}{\sqrt{\pi}\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}\sqrt{z}} K_0\left(\frac{2\sqrt{p}}{1-\rho}\right) \\ & \times \left\{ a \sum_{j=0}^{[(2|\alpha-1|-3)/4]} \frac{(-1)^j (2j+|\alpha-1|+1/2)!(2z)^{-2j-1}}{(2j+1)!(-2j+|\alpha-1|-3/2)!} \right. \\ & \left. - q \sum_{j=0}^{[(2|\alpha-1|-1)/4]} \frac{(-1)^j (2j+|\alpha-1|-1/2)!}{(2j)!(-2j+|\alpha-1|-1/2)!(2z)^{2j}} \right\}, \end{aligned}$$

where $z = 2\sqrt{p\rho}/(1-\rho)$, $a = \cos(\pi(2\alpha-3)/4 - z)$, *and* $b = \sin(\pi(2\alpha-3)/4 - z)$.

Now, we derive the moments of $P = XY$ when X and Y are distributed according to (2). We need the following lemma.

LEMMA 5. *If X and Y are jointly distributed according to* (2) *then*

$$E(X^m Y^n) = \frac{n!\Gamma(m+\alpha)(1-\rho)^n}{\Gamma(\alpha)} P_n^{(\alpha-1, m-n)}\left(\frac{1+\rho}{1-\rho}\right)$$

for $m \geq 1$ and $n \geq 1$.

PROOF. One can express

$$\begin{aligned}
E(X^m Y^n) &= \frac{1}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \int_0^\infty x^{m+(\alpha-1)/2} \exp\left(-\frac{x}{1-\rho}\right) \\
&\quad \times \int_0^\infty y^{n+(\alpha-1)/2} \exp\left(-\frac{y}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right) dy dx \\
&= \frac{2}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \int_0^\infty x^{m+(\alpha-1)/2} \exp\left(-\frac{x}{1-\rho}\right) \\
&\quad \times \int_0^\infty w^{2n+\alpha} \exp\left(-\frac{w^2}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{x\rho}w}{1-\rho}\right) dw dx \\
&= \frac{n!(1-\rho)^n}{\Gamma(\alpha)} \int_0^\infty x^{m+\alpha-1} \exp(-x) L_n^{\alpha-1}\left(-\frac{x\rho}{1-\rho}\right) dx, \qquad (12)
\end{aligned}$$

which follows after setting $w = \sqrt{y}$ and applying Lemma 3. The integral in (12) can be calculated by direct application of Lemma 4 to yield

$$\int_0^\infty x^{m+\alpha-1} \exp(-x) L_n^{\alpha-1}\left(-\frac{x\rho}{1-\rho}\right) dx = \Gamma(m+\alpha) P_n^{(\alpha-1, m-n)}\left(\frac{1+\rho}{1-\rho}\right). \quad (13)$$

The result of the lemma follows by combining (12) and (13). ▲

The moments of $P = XY$ are now simple consequences of this lemma as illustrated in the following remark.

REMARK 4. *If X and Y are jointly distributed according to* (2) *then*

$$E(P^n) = \frac{n!\Gamma(n+\alpha)(1-\rho)^n}{\Gamma(\alpha)} P_n^{(\alpha-1, 0)}\left(\frac{1+\rho}{1-\rho}\right) \qquad (14)$$

for $n \geq 1$.

PROOF. Follows by writing $E(P^n) = E(X^n Y^n)$ and applying Lemma 5 with $m = n$. ▲

Using special properties of the Jacobi polynomials, one can derive elementary forms for (14) for particular values of n. This is shown in the remark below.

REMARK 5. *If X and Y are jointly distributed according to* (2) *then*

$$E(P) = (\alpha + \rho)\alpha,$$

$$E(P^2) = \alpha(1+\alpha)(\alpha + 4\alpha\rho + \alpha^2 + 4\rho + 2\rho^2),$$

$$E(P^3) = \alpha(1+\alpha)(2+\alpha)(18\rho + 2\alpha + 27\alpha\rho + 36\rho^2 + 18\alpha\rho^2 + 3\alpha^2 + 9\alpha^2\rho + 6\rho^3 + \alpha^3),$$

$$E(P^4) = \alpha(1+\alpha)(2+\alpha)(3+\alpha)(96\rho + 6\alpha + 176\alpha\rho + 432\rho^2 + 360\alpha\rho^2 + 11\alpha^2 + 96\alpha^2\rho + 72\alpha^2\rho^2 + 96\alpha\rho^3 + 16\alpha^3\rho + 288\rho^3 + 6\alpha^3 + 24\rho^4 + \alpha^4).$$

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