Interacting Individuals Leading to Zipf's Law

Matteo Marsili[1,2] and Yi-Cheng Zhang[1]
[1]*Institut de Physique Théorique, Université de Fribourg, CH-1700 Fribourg, Switzerland*
[2]*International School for Advanced Studies (SISSA) and INFM Unit, V. Beirut 2-4, I-34014 Trieste, Italy*
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We present a general approach to explain the Zipf's law of city distribution. If the simplest interaction (pairwise) is assumed, individuals tend to form cities in agreement with the well-known statistics. [S0031-9007(98)05632-4]

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Zipf [1], half a century ago, found that city sizes obey an astonishingly simple distribution law, which is attributed to the more generic *least effort principle of human behavior*. Let us denote by q_m the number of cities having the population size m, then $R(m) = \int_m^\infty q_{m'}\, dm'$ defines the rank; i.e., the largest city has $R = 1$, the second largest $R = 2$, etc. Zipf found that empirically, $R(m) \sim 1/m^\gamma$, with $\gamma \approx 1$ (see Fig. 1). Remarkably Zipf showed that the scaling exponent $\gamma = 1$ is very close to reality for many different societies and during various time periods. More recent data [2,3] shows some variations from the pure $\gamma = 1$ result as also shown in Fig. 1. Countries which have a unique social structure, such as the former USSR or China, do not follow Zipf's law. For other developed countries Zipf's law remains a rather good approximation. Such a generic law calls for a generic explanation, since different countries (e.g., Germany and the U.S.) have different cultural and economic structures, and their people have innumerable reasons to choose whether to live in a big or small city. Yet collectively the society self-organizes, without express wishes of authorities, to obey Zipf's law.

Human settlements on the Earth's surface appear to be clustered, hence cities. This is because individuals *interact* with each other through social, economic, and cultural ties. In very primitive times an individual (or a family) performed all basic activities to survive; there was no need to form a large cluster beyond the size of a tribe. In modern times, ever refined mutual cooperation/competition brings people to live together. Yet this tendency does not seem to lead to a single "megacity" in the world: Many of us may prefer to live in a big city, but equally as many escape it for all its negative impacts. Somehow the ensuing compromise results in a robust statistical distribution, Zipf's law.

All these very general features call for an equally general approach to model city distribution. Below we propose a general framework using master equations. Let there be Q cities and m_i citizens in the ith city. The model is defined in terms of a master equation, assigning transition rates for the growth $w_a(m_i)$ or decrease $w_d(m_i)$ of the population of a city of size m_i. In other words we assume that with a probability $w_a(m_i)dt$ a new citizen arrives in city i in the time interval $(t, t + dt)$, so that $m_i \to m_i + 1$. With a probability $w_d(m_i)dt$ one of the m_i citizens departs so that $m_i \to m_i - 1$. We also assume that there is a small probability $p\,dt$ that a new city is created, with a single citizen. Assigning the transition rates $w_a(m)$ and $w_d(m)$ specifies the model. Note that birth and death are not explicitly considered; these can be included in the transition probabilities. Note also that once an individual leaves a city, he/she will not necessarily settle in another city right away; thus the total number of city dwellers is not conserved. Thus the whole system is composed of the city dwellers and a reservoir of unsettled travelers whose number is unregistered.

One can study this problem introducing the average number $q_{m,t}$ of cities of size m at time t, which satisfies the master equation:

$$\partial_t q_{m,t} = w_d(m + 1)q_{m+1,t} - w_d(m)q_{m,t} + p\delta_{m,1} + w_a(m - 1)q_{m-1,t} - w_a(m)q_{m,t}\,. \quad (1)$$

The parameters of the equation are the transition rates $w_d(m)$, p, $w_a(m)$.

The total number $n(t) = \sum_m mq_{m,t}$ of persons and the total number Q of cities are generally not constant:

$$\partial_t Q = p - w_d(1)q_{1,t}\,, \quad (2)$$

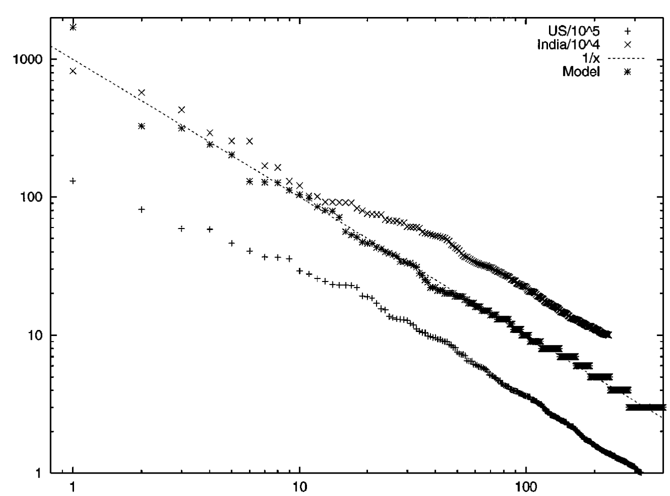

FIG. 1. Zipf plot for cities of population larger than 10^5 in the U.S. and in India (1994 estimate [2]), and for a simulation of the master equation with $p = 10^{-3}$, $W = 10^6$, and $\alpha = 2$.

$$\partial_t n = p - \sum_{m=1}^{\infty} [w_d(m) - w_a(m)] q_{m,t} . \quad (3)$$

Let us focus on the stationary state solutions $\partial_t q_{m,t} = 0$, for which $q_{m,t}$ is simply denoted q_m, independent of time, and n and Q are constant on average. This leads us to the equation

$$w_d(m+1) q_{m+1} - w_d(m) q_m + p\delta_{m,1} + w_a(m-1) q_{m-1} - w_a(m) q_m = 0, \quad (4)$$

where the rates w are also independent of time. The problem can be readily handled with the aid of the generating function $g(s) = \sum_m s^m q_m$.

Let us first consider the linear case $w_d(m) = Dm$ and $w_a(m) = Am$. This corresponds to independent decisions by the individuals. It is convenient to choose the constants to be $A = (1-p)/n, D = 1/n$, where n counts the average total number of citizens (constant). In general there is a deficit for each existing city $w_d > w_a$, since a departing individual has a finite chance to create a new city. The above equation is readily solved with the result

$$q_m = \frac{np}{1-p} \frac{(1-p)^m}{m} \sim \frac{1}{m} e^{-pm}. \quad (5)$$

One can verify that the average number of citizens is indeed $\sum m q_m = n$ while the average number of cities is

$$Q = \frac{np|\ln p|}{1-p}. \quad (6)$$

Several remarks are in order: due to the above deficit, the odds are always against the existing cities. If the deficit is small (or $p \ll 1$), fluctuations can still make a large city arise. The parameter p sets the cutoff size $m^* \simeq 1/p$ of the power law behavior $q_m \sim 1/m$. The condition $p \ll 1$ is equivalent to the statement that the above deficit is small, or that the creation of a new city is a rare event, which seems realistic. The distribution $q_m \sim 1/m$ is very broad. Inverting the rank function $R(m)$, we find $m(R) = m^* \exp(-R + 1)$, very different from the observed Zipf's law.

The linear model implies no interaction among citizens. From an individual's viewpoint, the chance to leave (or arrive at) a city is independent of the city's size m; everybody is free to move around. The simplest interaction we may consider is pairwise type; this leads us to $w \sim m^2$. In this case we choose $w_a(m) = (1-p)m^2/W$ and $w_d(m) = m^2/W$. The calculation is, again, standard and the result is that

$$n = \frac{pW|\ln p|}{1-p},$$

$$Q = \frac{n}{|\ln p|} \sum_{m=1}^{\infty} \frac{(1-p)^m}{m^2} \simeq \frac{\pi^2 n}{6|\ln p|},$$

and

$$q_m = \frac{Wp}{1-p} \frac{(1-p)^m}{m^2}. \quad (7)$$

Note that $q_m \sim 1/m^2$, for $m \ll m^* \simeq 1/p$. Again using the rank relation, we find $R(m) \sim 1/m$, or in the more familar form of Zipf $m(R) \sim 1/R$. Therefore we may draw the conclusion that it is the pairwise interaction that is behind Zipf's law for city distribution, a general explanation indeed. Recently Zanette and Manrubia have proposed a city formation model [3]. They use a multiplication and diffusion process and find that their results also reproduce Zipf's law. A possible motivation for such a multiplicative process is that citizens of the same city are subject to the same aggregate shocks, which tend to increase or decrease the population size. The implicit assumption is that the strength of such random shocks does not depend on the size m_i of city i. Our present approach is complementary to theirs: since there are more or less m^2 departures and arrivals, the net increase (or decrease) δm of the population of a city in a unit time interval, is *linearly* proportional to m. This leads to a system with multiplicative noise. This argument shows, on one side that multiplicative noise results from a pairwise interaction, on the other that aggregate shocks, i.e., random events which affect equally each citizen of a given city, also would lead to the m^2 transition rates in our model.

On the other hand, the linear case $w_i \sim m$ is characterized by fluctuations δm which are proportional to $\sqrt{m}$. As has been discussed in another reproduction-diffusion system [4], this is typical of a system of noninteracting individuals.

These results are confirmed by numerical simulations. We show in Fig. 1 the Zipf plot of a population of cities obtained for $w_a(m)$, $w_d(m) \sim m^2$. This compares well with actual data [1].

One may argue that, in reality, *both* linear and square terms should be present, since an individual's decision must have both his independent as well as interactive parts. Therefore it is natural to consider the mixed case. For the ease of presentation we assume the transition probabilities to be

$$w_a(m) = \frac{m^2}{m_0} + m, \quad w_d(m) = e^{1/m^*}\left(\frac{m^2}{m_0} + am\right). \quad (8)$$

The simplest way to derive the distribution of city sizes, in this case, is to use detailed balance: The number of cities of size m becoming of size $m-1$ is $q_{m,t} w_d(m)$. In the steady state, this has to balance the number of cities of size $m-1$ becoming of size m, i.e., $q_{m-1,t} w_a(m-1)$. This readily gives

$$q_m = C \frac{e^{-m/m^*}}{m} \frac{\Gamma(m + m_0)}{\Gamma(m + 1 + am_0)},$$

where the constant C depends on the rate p at which new cities are created.

For sizes $m \ll m_0$ the distribution is practically the same as that for the linear case, $q_m \sim 1/m$. If the cutoff size $m^* \ll m_0$, we conclude that the quadratic term is not relevant. If, on the other hand $m^* \gg m_0$, the linear behavior $q_m \sim 1/m$ holds for $m \ll m_0$ and then it crosses over to a power law behavior

$$q_m \sim m^{-2-(a-1)m_0}, \quad \text{for } m_0 \ll m \ll m^*.$$

The Zipf exponent is therefore $\gamma = 1/[1 + (a - 1)m_0]$. Zipf's law $\gamma \simeq 1$ obtains only if the pairwise interaction is dominant, i.e., more precisely if $|m_0(a - 1)| \ll 1$. In the original Zipf's work [1], the scaling law is valid only for large cities. The above results with $|a - 1| \ll 1$ allow for a scenario where the distribution of city sizes crosses over from Zipf's law with $\gamma \cong 1$ for large cities $m \gg m_0$ to a noninteracting situation $q_m \sim 1/m$ for small towns $m \ll m_0$. The population dynamics in small towns is dominated by the linear term in the transition rates, which describes noninteracting individuals. In large cities, on the other hand, interaction dominates and it leads to Zipf's law. Note that Zipf's law, with a general value of γ, can also occur if the condition $|m_0(a - 1)| \ll 1$ is not met.

It is also interesting to consider a general model with $w_a(m) = (1 - p)m^\alpha/W$ and $w_d(m) = m^\alpha/W$. This indeed allows us to investigate the effects of multiperson interactions. For example $\alpha = 3$ would represent a three person interaction. Having found that for $\alpha = 1, 2$ the solution is

$$q_m = \frac{pW}{1 - p} \frac{(1 - p)^m}{m^\alpha}, \tag{9}$$

we can try this solution in the equation. Treating separately the cases $m = 1$ and $m > 1$ we see indeed that this is the solution. Such a solution is also readily found using detailed balance. The numbers n and Q are given by

$$n = \frac{pW}{\Gamma(\alpha - 1)} \int_0^\infty \frac{t^{\alpha-2} dt}{e^t - 1 + p}. \tag{10}$$

If $\alpha > 2$ the integral is finite as $p \to 0$ and one has simply $n \sim pW$. For $1 < \alpha < 2$ the integral diverges as $p \to 0$ and the leading term is $n \sim p^{\alpha-1}W$. In the same way one can calculate the number of cities

$$Q = \sum_{m=1}^{\infty} q_m = \frac{pW}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1} dt}{e^t - 1 + p}. \tag{11}$$

For $\alpha > 1$ the integral is finite for $p = 0$, so that $Q \sim pW$.

The interesting point here is that the average size of cities is finite for $\alpha > 2$, $n/Q \simeq \zeta(\alpha - 1)/\zeta(\alpha)$, where $\zeta(x)$ is Riemann's zeta function. This observation raises the question of what happens when the population density $\rho = n/Q$ grows beyond the value $\rho_c = \zeta(\alpha - 1)/\zeta(\alpha)$.

The answer is that the excess population concentrates in just one city, which therefore has a finite, large fraction of the total population. This effect has been studied recently [5] in an equilibrium model where a Hamiltonian $H = \sum_i \ln m_i$ was considered. The equilibrium distribution at inverse temperature $\beta = \alpha$ is clearly given by a power law distribution. However, for $\alpha > 2$, there are two phases: a fluid phase for $\rho < \rho_c(\alpha)$ which is well described by a power law distribution with an exponential cutoff, and a *condensed* phase where a megacity nucleates, containing a finite fraction of the total population. It is easy to understand this transition from equilibrium considerations: Let us consider the state in which we assign to each city i a random number of citizens m_i drawn from a power law distribution with exponent $\alpha > 2$. This state minimizes the entropy and it has a density which is given by $\rho = n/Q = \langle m_i \rangle = \rho_c(\alpha)$. If we want to find a state with a lower density, one can introduce a chemical potential, which is equivalent to an exponential cutoff on the distribution of m_i. On the other hand if one wants to build a state of higher density $\rho > \rho_c$ one gets into trouble. There are two ways out: The first is to put some $\delta m_i = \rho - \rho_c$ extra citizens in each city. This results in a state which has a free energy cost $\delta F \simeq Q \ln(\rho/\rho_c)$. The second way out is to put all the $Q(\rho - \rho_c)$ excess citizens in only one city. This leads to a free energy cost $\delta F \simeq \ln[Q(\rho/\rho_c - 1)]$. This only grows logarithmically with the system size whereas the first variant leads to an extensive increase of the free energy. It is then clear that in the equilibrium state the system prefers to create a megacity to accommodate the excess population.

This discussion clearly refers to an equilibrium model. Metropolis or Monte Carlo dynamics of this equilibrium system [6] is different from the dynamics of our master equation. Furthermore, and more importantly, we deal with a system in which neither the number of cities Q nor the population size n is fixed. In other words the density ρ is not fixed. For $p \ll 1$ the average density is very close to $\rho_c(\alpha)$. The density, no matter how it fluctuates, will sweep across ρ_c in time. This suggests that, in our model, dynamic nucleation of a megacity should occur for $\alpha > 2$. This agrees indeed with the results of numerical simulations. We show in Fig. 2 the distribution of the fraction $m_{\max}/n$ of citizens who live in the biggest city. For $\alpha = 3$, the distribution is peaked at values very close to 1 (for $\alpha = 2$ the critical density diverges $\rho_c = \infty$. Some precursor effects of the transition are however visible).

This leads to our second observation, that processes with $\alpha > 2$ must result in the nucleation of a megacity, which is not very realistic. We can conclude that interactions of higher orders (than pairwise) are not relevant in the dynamics of city formation.

We see that the interaction leading to Zipf's law is, on one hand, the simplest possible (pairwise interaction). On the other it is a rather special one, since it is the "lowest order" of interaction which does not lead to the formation

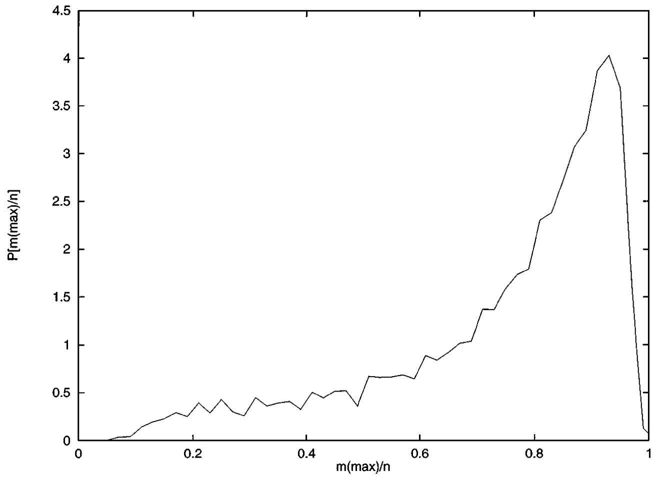

FIG. 2. Distribution of the fraction of citizens in the largest city.

of a megacity, which draws a good portion of the whole population. The absence of a megacity suggests that in an expansion of the transition rates in powers of m, we should neglect terms of order higher than the second. This leads us to the mixed interaction case, which gives very realistic results.

In many disparate societies, it is not unnatural to assume that individuals make their city-dwelling decision based on their own opinions as well as on their interaction with other citizens. If this indeed is the case, we show that the larger cities obey approximately Zipf's law.

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[1] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley, Cambridge, 1949).
[2] Data are drawn from the same sources as Ref. [3].
[3] D. Zanette and S. Manrubia, Phys. Rev. Lett. **79**, 523 (1997).
[4] Y.-C. Zhang, M. Serva, and M. Polikarpov, J. Stat. Phys. **58**, 849 (1990); Y.-C. Zhang, Phys. Rev. E **55**, R3817 (1997).
[5] P. Bialas, Z. Burda, and D. Johnston, e-print cond-mat/9609264.
[6] C. Godreche and J. M. Luck, e-print cond-mat/9707052.