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**INTERACTION OF ELECTROMAGNETIC AND
ELASTIC FIELDS IN SOLIDS**

A.A.F. VAN DE VEN

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PROEFSCHRIFT

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DOOR DE PROMOTOREN**

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PREFACE

The aim of the present study is to construct a rigorous phenomenological theory of the interactions of electromagnetic and elastic fields in solids, with particular attention to ferromagnetic materials. This theory will be based on a set of postulates, as, for example, the invariance under rigid-body motions of the energy balance, a principle first stated by GREEN and RIVLIN, and the principle of COLEMAN and NOLL. We shall use a finite-strain concept, as this to our opinion greatly clarifies the derivation of the basic relations. In cases that a small-strain approximation is justified, this approximation will be made after the deduction of the general equations. Moreover, we shall assume that we may apply the methods of thermodynamics. The first law of thermodynamics serves as the basis for our local equations of balance and jump conditions, while from the second law some of the needed constitutive equations may be obtained, while for the other restrictions are found.

In this way we shall set up a complete, general nonlinear system of equations of balance, constitutive equations and jump conditions. The present theory deals especially with ferromagnetic media, without mechanical or electromagnetic dissipation, and effects as gyromagnetic action, magnetostriction, exchange interaction, thermomagnetic and thermoelectric effects, etc. will be discussed. The theory may be extended to include dissipation effects as for instance was done by ALBLAS. Moreover, Cosserat-media, i.e. media with internal mechanical moments or higher-order electromagnetic moments, can be studied in an analogous way.

A system of linear equations and boundary conditions was extracted from the general nonlinear equations derived in the first part of this thesis. Although this system is a linear one, it is still very complex. However, if numerical values based on existing experimental data, are used, it turns out that many of the terms in these equations are negligible com-

pared to a few which are dominant. To find the connection between our work and the physical and experimental literature, we have interpreted the coefficients occurring in our linearized equations in terms of known technical effects, as for instance magnetic anisotropy, magnetostriction, thermoelectric effects etc. This was done for the practical important case of a ferromagnetic material with cubic symmetry. The general theory is illustrated by two examples: the first concerning the vibrations of a cylinder in a magnetic field and the second one dealing with the buckling of magnetoelastic plates. For the latter problem, the equations are simplified for the case of a soft-ferromagnetic material.

Throughout this thesis Gaussian units are used and the Maxwell-equations are written in the Minkowski-formulation. For the conversions from Gaussian-units to Giorgi-units and from the Minkowski-formulation to the Chu-formulation we refer to the Appendices I and III, respectively. On the whole we have employed a Cartesian tensor notation whereby the summation convention is applied. This means that summation over any repeated subscript must be executed, where the summation runs from 1 to 3. References to literature are denoted by a number in square brackets (e.g. [1]), sometimes preceded by the name of the author and/or followed by a further indication. We shall number the equations in each chapter independently. When referring for example to equation (1) of Chapter I, we write (1) in Chapter I and I.(1) in the other chapters.

LIST OF SYMBOLS

X_α	Lagrange coordinates
x_i	Euler coordinates
J	Jacobian
$\underline{V}, \underline{v}$	Velocity
V	Material volume
S	Complete boundary of V
ρ	Density
S, s	Entropy
θ, θ	Temperature
r	Heat source
$\Sigma(t)$	Surface of discontinuity
$\underline{N}, \underline{n}$	Unit normal vector at $\Sigma(t)$
\underline{W}_n	Normal velocity of a surface of discontinuity
$\underline{D}, \underline{d}$	Electric displacement
$\underline{E}, \underline{e}$	Electric field intensity
$\underline{P}, \underline{p}$	Polarization per unit of mass
$\underline{B}, \underline{b}$	Magnetic induction
$\underline{H}, \underline{h}$	Magnetic field intensity
$\underline{M}, \underline{m}$	Magnetization per unit of mass
$\underline{J}, \underline{j}$	Electric current density
Q, q	Free charge
c	Speed of light in vacuum
T_{ij}, t_{ij}	Stress tensor
M_{ijk}	Third order couple-stress tensor
Q_i, q_i	Heat flux
ρU	Internal energy
ρK	Kinetic energy due to the spin of the magnetization vector
M_s	Saturation magnetization per unit of mass
Π_{ij}, π_{ij}	Second order couple-stress tensor
Γ	Gyromagnetic ratio
\underline{G}^*	Effective magnetic field
$\underline{\Phi}$	Entropy flux
Σ	A thermodynamic energy functional
$F_{i\alpha}$	Deformation gradient

$E_{\alpha\beta}$	Deformation tensor
$\Lambda_{\alpha}, \Delta_{\alpha}, G_{\alpha\beta}$	Invariant constitutive variables
ξ	Intermediate state
$\underline{U}, \underline{u}$	Displacement
c_{11}, c_{12}, c_{44}	Elastic constants
G	Shear modulus
E	Young's modulus
ν	Poisson's ratio
K_1	Anisotropy coefficient
B_0, B_1, B_2	Magnetostriction coefficients
κ_1	Thermal conductivity
c_w	Specific heat
α	Linear thermal expansion coefficient
α_1	Exchange coefficient
$\chi^{(e)}$	Electric susceptibility
r	Resistivity
$R(t)$	Thermal coefficient of resistivity
k_1, k_2, k_3	Magneto-resistivity coefficients
β	Thermoelectric coefficient
μ	Magnetic permeability

Mathematical symbols

$\frac{\partial}{\partial t}$	Partial time derivative (\underline{x} fixed)
$\frac{d}{dt}$ or \cdot	Material time derivative (\underline{X} fixed)
$\frac{D}{Dt}$ or ∇	Jaumann time derivative
$,_i = \frac{\partial}{\partial x_i}$	
$,^{\alpha} = \frac{\partial}{\partial X_{\alpha}}$	
δ_{ij}	Kronecker delta
e_{ijk}	Permutation tensor
$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$	
$A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$	
$[[A]] = A^+ - A^-$	Jump of A over a surface of discontinuity.

I. BASIC CONCEPTS

I.1. Motion and thermodynamics

Let us consider a finite body \mathcal{B} with material points X and identify the material point X with its position \underline{X} in a fixed reference configuration (viz. Fig. I.1).

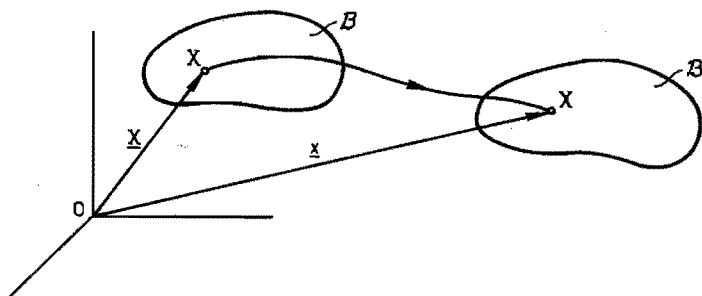


Fig. I.1.

The region of space occupied by the body in its undeformed state is chosen as reference configuration. A motion of the body is defined by a sufficiently smooth vector function $\underline{\chi}$ which assigns position

$$(1) \quad \underline{x} = \underline{\chi}(\underline{X}, t),$$

to each material point X at each instant of time t (cf. [1], pp. 325-328). We restrict our attention to motions in which mass elements are conserved for each material volume of \mathcal{B} . The components of \underline{X} and \underline{x} with respect to a fixed cartesian coordinate system are designated with X_α ($\alpha = 1, 2, 3$) and x_i ($i = 1, 2, 3$), respectively. The coordinates X_α are called material or Lagrange coordinates and x_i space or Euler coordinates.

The mapping describing the motion is assumed to be one-to-one, so that

(1) has a unique inverse

$$(2) \quad \underline{X} = \underline{\chi}^{-1}(\underline{x}, t) =: \underline{\kappa}(\underline{x}, t) .$$

Furthermore, we take the Jacobian J to be positive:

$$(3) \quad J = \det \left(\frac{\partial x_i}{\partial X_\alpha} \right) > 0 .$$

The velocity \underline{V} is defined by

$$(4) \quad v_i := \frac{\partial X_i(\underline{X}, t)}{\partial t} .$$

Throughout this thesis we shall use the symbol $\frac{d}{dt}$ or a superposed dot to denote differentiation with respect to t holding the material coordinates X_α fixed. Thus

$$(5) \quad \frac{d\phi}{dt} \equiv \dot{\phi} := \frac{\partial \bar{\phi}(\underline{X}, t)}{\partial t} .$$

$\dot{\phi}$ is called the material derivative of ϕ .

By the symbol $\frac{\partial}{\partial t}$ we denote differentiation with respect to t, whereby the space coordinates x_i are assumed to be fixed. Hence

$$(6) \quad \frac{\partial \phi}{\partial t} := \frac{\partial \tilde{\phi}(\underline{x}, t)}{\partial t} .$$

It is easy to establish that the following relation holds between these two derivatives:

$$(7) \quad \frac{d\phi}{dt} = \frac{\partial \tilde{\phi}}{\partial t} + v_i \tilde{\phi}_{,i} ,$$

where

$$(8) \quad \phi_{,i} := \frac{\partial \tilde{\phi}(\underline{x}, t)}{\partial x_i} .$$

We shall also use the so called Jaumann-derivative, defined by

$$(9) \quad \frac{D}{Dt} \phi_{i_1 \dots i_n} \equiv \overset{V}{\phi}_{i_1 \dots i_n} :=$$

$$= \dot{\phi}_{i_1 \dots i_n} + \sum_{\nu=1}^n \phi_{i_1 \dots i_{\nu-1} j i_{\nu+1} \dots i_n} V_{[j, i_\nu]}$$

where ϕ is an arbitrary tensor of order n and the symbol $[]$ stands for the asymmetric part of a two-tensor

$$(10) \quad A_{[ij]} := \frac{1}{2}(A_{ij} - A_{ji}) .$$

It follows immediately from the definition that there is no difference between the material derivative and the Jaumann-derivative of a scalar function. Hence, for $n = 0$,

$$(11) \quad \overset{\nabla}{\phi} = \dot{\phi} .$$

Following Green and Rivlin [2], we consider motions of the continuum which differ from those given by (1) only by superposed rigid-body motions. Thus

$$(12) \quad x_i^* = \chi_i^*(X, t) = b_i(t) + Q_{ij}(t)\chi_j(X, t) ,$$

where $b_i(t)$ is a uniform vector and $Q_{ij}(t)$ a uniform, orthogonal tensor. We say that a quantity is invariant under superposed rigid-body motions if the transformation $x_i \rightarrow x_i^*$ does not change this quantity. The Jaumann-derivative of an invariant quantity remains invariant.

In this thesis, we shall base the derivations of the equations of balance and the constitutive equations on two postulates, i.e. the first and second law of thermodynamics ([3], pp. 9-11).

We write the first law in the form

$$(13) \quad \dot{\bar{E}} = \bar{W} + \bar{D} ,$$

where \bar{E} is the internal energy, \bar{W} the net working per unit of time and \bar{D} the supply of heat per unit of time, not from mechanical origin. We suppose (13) not only to hold for the total volume of \bar{B} , but also for any partial material volume.

As a second axiom, we postulate the Clausius-Duhem inequality in the form ([3], eqs (2.25) and (2.27))

$$(14) \quad \rho \dot{S} - \phi_{i,i} - \frac{\rho \sigma}{\theta} \geq 0 ,$$

where ρ is the density, S the entropy per unit of mass, ϕ_i the entropy influx per unit of surface and unit of time, θ the temperature and σ the heat supply per unit of mass and unit of time, not from mechanical origin.

The quantities \mathcal{D} and σ are related by

$$(15) \quad \mathcal{D} = \int_V \rho \sigma \, dV - \oint_S h_i n_i \, dS,$$

where V is a material volume with complete boundary S , h_i the heat efflux and n_i the unit normal on S .

Usually, the entropy flux is taken to be equal to

$$(16) \quad \phi_i = - \frac{h_i}{\theta}.$$

However, we shall not assume (16) a priori, but we shall derive this relation in Chapter III (viz. p. 40) as a constitutive relation for the material considered there.

1.2. General balance equations

Underlying all purely mechanical theories of elastic bodies are four fundamental principles of conservation. These are:

- i) conservation of mass,
- ii) conservation of linear momentum,
- iii) conservation of angular momentum,
- iv) conservation of energy.

The following integral equations of balance express these basic principles of mechanics in a mathematical form sufficiently general for our purposes.

i) *Mass*

$$(17) \quad \frac{d}{dt} \int_V \rho \, dV = 0.$$

ii) *Linear momentum*

$$(18) \quad \frac{d}{dt} \int_V \rho p_i dV = \oint_S t_{ij} n_j dS + \int_V \rho F_i dV .$$

iii) *Angular momentum or moment of momentum*

$$(19) \quad \frac{d}{dt} \int_V \rho \{s_{ij} + x_{[i} p_{j]}\} dV = \oint_S \{m_{ijk} + x_{[i} t_{j]k}\} n_k dS + \\ + \int_V \rho \{L_{ij} + x_{[i} F_{j]}\} dV .$$

iv) *Energy*

$$(20) \quad \frac{d}{dt} \int_V \rho E dV = \oint_S \{t_{ij} v_i + m_{ikj} \Omega_{ik} - h_j\} n_j dS + \\ + \int_V \rho \{F_i v_i + L_{ij} \Omega_{ij} + \sigma\} dV .$$

In these formulae the region of integration V is, in general, a moving region that contains the same set of material points at each instant t (material volume). Further, S is the complete boundary of V and \underline{n} is the outward unit normal on S . The quantities that occur in (17) to (20) are named as follows:

ρ	= mass density	,	m_{ijk}	= couple-stress	,
p_i	= momentum density	,	L_{ij}	= extrinsic body couple,	
t_{ij}	= stress tensor	,	E	= energy density	,
F_i	= extrinsic body force,		h_i	= heat efflux	,
s_{ij}	= spin density	,	σ	= heat supply	,

while the contributions $m_{ikj} \Omega_{ik}$ and $L_{ij} \Omega_{ij}$ represent the energy influx caused by the couple-stresses and the energy supply owing to the body couple, respectively.

All components are taken with respect to a cartesian frame of reference. We remark that these equations of balance hold for arbitrary regions V . The fluxes in the equations (17) - (20) have a certain indefiniteness, i.e. it is always possible to replace a part of the flux by a volume

source or vice versa (equivalence of surface and volume sources, cf. [1], p. 469). Therefore, we first take a certain form for our surface sources, consistent with (17) - (20), from which constitutive equations and boundary conditions for t_{ij} , m_{ijk} and h_i can be derived.

Each of the equations of balance has the typical structure

$$(21) \quad \frac{d}{dt} \int_V \rho \Psi \, dV = \oint_S \theta_i n_i \, dS + \int_V \rho \phi \, dV,$$

where Ψ is the density of the quantity in balance, θ_i is its flux and ϕ is its supply.

Let us consider a material volume V within which there occurs a surface $\Sigma(t)$ that is a singular surface with respect to Ψ and possibly also with respect to \underline{v} . The singular surface, assumed smooth, may be in motion with velocity \underline{w} . Examples of singular surfaces are shock waves, slipstreams, as well as the boundary of a solid body.

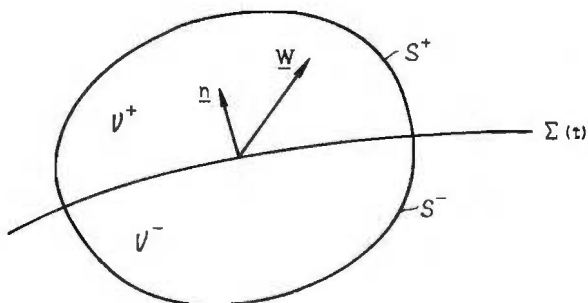


Fig. I.2.

We assign to the surface $\Sigma(t)$ a unit normal \underline{n} (cf. Fig. I.2). Further, we assume that $\Sigma(t)$ divides V into two regions V^+ and V^- . The same holds for the boundary S , the two parts being S^+ and S^- . In general, the regions and surfaces V^+ , V^- , S^+ and S^- fail to be material. We use the notation $[A]$ for the difference $A^+ - A^-$ of the limiting values A^+ and A^- of the quantity A as the surface of discontinuity is

approached from either side.

We say that $\Sigma(t)$ is a material singular surface if

$$(22) \quad W_n := W_i n_i = V_i^+ n_i = V_i^- n_i .$$

Let the following conditions be met: the quantities $\frac{\partial(\rho\Psi)}{\partial t}$ and ϕ are bounded in the neighbourhood of $\Sigma(t)$, while on each side of $\Sigma(t)$ the quantities $\rho\Psi$, $V_i n_i$ and $\theta_i n_i$ approach limits that are continuous functions of position. Then it can be shown ([1], section 157, 193) that the global equation of balance (21) is equivalent to the following local equation

$$(23) \quad \rho \dot{\Psi} + (\dot{\rho} + \rho V_{i,i}) \Psi = \theta_{i,i} + \phi ,$$

together with the jump condition on $\Sigma(t)$

$$(24) \quad \llbracket \Psi(V_i n_i - W_n) - \theta_i n_i \rrbracket = 0 , \quad \text{on } \Sigma(t) .$$

If $\Sigma(t)$ is a material singular surface, the condition (24) reduces to

$$(25) \quad \llbracket \theta_i n_i \rrbracket = 0 , \quad \text{on } \Sigma(t) .$$

In the next section we shall not only deal with balance equations of the form (21), but also with

$$(26) \quad \frac{d}{dt} \int_S \psi_i n_i \, dS = \oint_C \theta_i \, d\delta_i + \int_S \varphi_i n_i \, dS ,$$

where S is a material surface with complete boundary C .

According to [1] (section 80, 277 and 278), equation (26) is equivalent to the following system of local balance equations and jump conditions

$$(27) \quad \frac{\partial \psi_i}{\partial t} + e_{ijk} (-\theta_k + e_{klm} \psi_l V_{m,j} + \psi_{j,j} V_i - \varphi_i) = 0 ,$$

$$(28) \quad \llbracket \psi_i (W_n - V_j n_j) + \psi_j n_j (W_n n_i - V_i) + e_{ijk} \theta_k n_j \rrbracket = 0 , \quad \text{on } \sigma(t) .$$

In these equations, e_{ijk} is the permutation tensor, $\sigma(t)$ is a line of discontinuity formed by the intersection of $\Sigma(t)$ with S . Furthermore, W_n is the component of the velocity of $\sigma(t)$ along \underline{n} , \underline{n} being the unit

normal on $\Sigma(t)$.

I.3. Electromagnetic equations

The following five global equations of balance can serve as a basic system for the electromagnetic theory of moving media. In Gaussian units, we have

i) *Faraday's law*

$$(29) \quad \frac{1}{c} \frac{d}{dt} \int_S B_i n_i dS = - \oint_C (E_i + \frac{1}{c} e_{ijk} v_j B_k) ds_i .$$

ii) *Gauss' first law*

$$(30) \quad 0 = \oint_S B_i n_i dS .$$

iii) *Ampère's law*

$$(31) \quad \frac{1}{c} \frac{d}{dt} \int_S D_i n_i dS = \oint_C (H_i - \frac{1}{c} e_{ijk} v_j D_k) ds_i + \\ - \frac{4\pi}{c} \int_S (J_i - Qv_i) dS .$$

iv) *Gauss' second law*

$$(32) \quad 0 = \oint_S D_i n_i dS - 4\pi \int_V Q dV .$$

v) *Conservation of charge*

$$(33) \quad \frac{d}{dt} \int_V Q dV = - \oint_S (J_i - Qv_i) n_i dS .$$

We note that the latter equation is not independent of the preceding ones, as it is a direct consequence of the laws iii) and iv).

The quantities which occur in the above equations are named as follows:

\underline{E} = electric field intensity, \underline{H} = magnetic field intensity,
 \underline{D} = electric displacement , \underline{B} = magnetic induction ,
 \underline{J} = electric current density, Q = free charge ,

and

$$c = 2,998 \times 10^{10} \text{ cm/sec} = \text{speed of light in vacuum.}$$

We note that surface charges and surface currents are excluded in the above formulae.

By using (23), (24), (27) and (28) we can derive from (29) to (33) the following system of local balance equations with jump conditions

$$(34) \quad \begin{aligned} \frac{1}{c} \frac{\partial B_i}{\partial t} &= - e_{ijk} E_{k,j}, & B_{i,i} &= 0, \\ \frac{1}{c} \frac{\partial D_i}{\partial t} + \frac{4\pi}{c} J_i &= e_{ijk} H_{k,j}, & D_{i,i} &= 4\pi Q, \\ \frac{\partial Q}{\partial t} + J_{i,i} &= 0, \end{aligned}$$

and

$$(35) \quad \begin{aligned} [e_{ijk} E_j n_k + \frac{1}{c} B_i W_n] &= 0, & [D_i n_i] &= 0, \\ [e_{ijk} H_j n_k - \frac{1}{c} D_i W_n] &= 0, & [B_i n_i] &= 0, \\ [J_i n_i - Q W_n] &= 0, & & \text{on } \Sigma(t). \end{aligned}$$

The equations (34) are the well-known Maxwell equations. They constitute a system of seven independent equations for the sixteen unknowns E_i , D_i , H_i , B_i , J_i and Q . In order to obtain a complete set of equations, we shall derive constitutive equations for E_i , H_i and J_i in Chapter III.

We introduce the magnetization per unit of mass \underline{M} and the polarization per unit of mass \underline{P} by means of the equations ([4], p. 11)

$$(36) \quad B_i = H_i + 4\pi\rho M_i,$$

$$(37) \quad D_i = E_i + 4\pi\rho P_i.$$

We note that the electromagnetic fields \underline{E} , \underline{D} , etc., are not invariant under superposed rigid-body motions. Therefore, it is desirable to introduce the convective quantities \underline{E}^* , \underline{D}^* , etc., that are the values measured by an observer translating with velocity \underline{V} with respect to the

inertial space.

In the sequel, we shall neglect all terms proportional to V^2/c^2 . In this nonrelativistic approximation, the following relations for the convective fields hold:

$$\begin{aligned}
 D_i^* &= D_i + \frac{1}{c} e_{ijk} V_j H_k, & B_i^* &= B_i - \frac{1}{c} e_{ijk} V_j E_k, \\
 E_i^* &= E_i + \frac{1}{c} e_{ijk} V_j B_k, & H_i^* &= H_i - \frac{1}{c} e_{ijk} V_j D_k, \\
 P_i^* &= P_i - \frac{1}{c} e_{ijk} V_j M_k, & M_i^* &= M_i + \frac{1}{c} e_{ijk} V_j P_k, \\
 J_i^* &= J_i - QV_i, & Q^* &= Q.
 \end{aligned}
 \tag{38}$$

It has to be noted that these convective fields are invariant under superposed rigid-body translations.

Let us consider a region of space containing moving ponderable charges in vacuum. The variables of this problem, i.e. E_i , D_i , H_i , B_i , J_i , Q and V_i , must satisfy the following systems of equations:

- i) the Maxwell equations (34),
- ii) the constitutive relations

$$(39) \quad D_i = E_i, \quad H_i = B_i, \quad J_i = QV_i,$$

- iii) the momentum balance (cf. [4], p. 104, eq. (43))

$$(40) \quad \int_V (QE_i + e_{ijk} J_j B_k) dV = \frac{d}{dt} \int_V \rho V_i dV.$$

Using these relations, it can be proved that the following balance of energy holds

$$\begin{aligned}
 (41) \quad & \frac{d}{dt} \int_V \left\{ \frac{1}{8\pi} (E_i E_i + H_i H_i) + \frac{1}{2} \rho V_i V_i \right\} dV = \\
 & = \oint_S \left\{ -\frac{c}{4\pi} e_{ijk} E_j H_k + \frac{1}{8\pi} (E_j E_j + H_j H_j) V_i \right\} n_i dS.
 \end{aligned}$$

We remark that it is also possible to derive from (41) the jump condi-

tions (35).

For reasons, that will become clear in the next chapter, it will be useful to express the left-hand side of equation (41) in convective quantities. Taking into account (38) the balance of energy (41) can be rewritten into the form

$$\begin{aligned}
 (42) \quad & \frac{d}{dt} \int_V \left\{ \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) + \frac{1}{2} \rho v_i v_i + \frac{2}{4\pi c} e_{ijk} E_j^* H_k^* v_i \right\} dV = \\
 & = \oint_S \left\{ -\frac{c}{4\pi} e_{ijk} E_j H_k + \frac{1}{8\pi} (E_j^* E_j^* + H_j^* H_j^*) v_i + \right. \\
 & \quad \left. + \frac{2}{4\pi c} e_{jkl} E_k^* H_l^* v_j v_i \right\} n_i dS .
 \end{aligned}$$

1.4. Constitutive principles

As the equations of balance, discussed in the sections I.2, I.3 constitute an incomplete set, they must be supplemented by a system of constitutive relations expressing the various fluxes, densities and supplies which appear in these balance laws in terms of an independent set of variables. The equations of balance are common to all mechanical theories. On the other hand, the constitutive relations distinguish one continuum theory from another; in fact they serve to define the material under consideration.

In order to set up a system of constitutive equations, we divide all variables into a set of independent variables

$$(43) \quad \Xi := \{q^{(1)}, q^{(2)}, \dots, q^{(n)}\} ,$$

and a set of dependent variables, the latter being functions of the dependent ones.

The constitutive theory, presented in this thesis, will be based on the following series of postulates.

- i) *The principle of equipresence* ([1], p. 703-704), according to which the same independent variables should appear in all constitutive relations unless their presence contradicts the equations of balance, the entropy inequality, the principle of objectivity

stated below, or some material symmetry.

Hence, for every dependent variable ϕ the following relation holds

$$(44) \quad \phi = \phi(q^{(1)}, q^{(2)}, \dots, q^{(n)}; \underline{X}, t) .$$

ii) *The principle of objectivity*

We state this principle in the following way: the properties of a material are not influenced by rigid-body motions. As a consequence of this principle, the dependence of the functions in (44) can occur only through dependence on objective combinations of the independent variables.

As a consequence of this principle one has the following theorem of Cauchy (cf. [5], pp. 887-888, 901-904):

A function of a system of n vectors

$$\phi = \phi(v_i^{(1)}, v_i^{(2)}, \dots, v_i^{(n)}) ,$$

that is invariant under rigid-body rotations Q_{ij} , by which the vectors transform according to

$$v_i^{(a)} \rightarrow Q_{ij} v_j^{(a)} ,$$

can only depend on the scalar products

$$v_i^{(a)} v_i^{(b)} , \quad a, b = 1, 2, \dots, n ,$$

and on the determinants

$$e_{ijk} v_i^{(a)} v_j^{(b)} v_k^{(c)} , \quad a, b, c = 1, 2, \dots, n .$$

iii) *The principle of Coleman and Noll ([6]).*

Before stating this principle we define a thermodynamic process as the set of all variables (dependent as well as independent), satisfying the equations of balance. Moreover, such a process is called admissible if it is compatible with the constitutive assumptions (44).

The said principle of Coleman and Noll can now be formulated in the following way:

For every thermodynamic process admissible in a body of a given material and for every part of the body and at every time t the entropy inequality (14) is valid.

Throughout this thesis we restrict ourselves to elastic media. The theory of elasticity is concerned with the mechanics of deformable bodies which recover their original shape upon the removal of all forces causing the deformation. An elastic body possesses a natural state, being the unloaded state of the body.

Moreover, as an elastic material has no memory, the constitutive equations are not influenced by the history of the motion. Therefore, we posit:

iv) *The principle of momentary action*

The value of a dependent variable at time t is determined by the values of the independent variables at the same time t .

II. EQUATIONS OF BALANCE

II.1. Introduction

In this chapter, we shall derive a system of local equations of balance of mass, momentum and moment of momentum, for a conducting, polarizable and magnetizable medium. The equation for the moment of momentum is derived under the restriction that the magnetization is saturated. Moreover, we shall not consider magnetic dissipation effects.

We first postulate a global equation of balance of energy, from which we derive, in the way described in Section I.2, a local equation. We proceed by stating the following postulate:

The energy balance equation is invariant under superposed rigid-body translations and rotations.

By making some a priori assumptions concerning the invariance of the quantities involved, we then arrive at the equations of balance we are looking for. This method is first formulated by Green and Rivlin [2]. In an analogous way we shall construct a system of jump conditions for the density, the stresses, the couple-stresses and the heat fluxes. In the last section we shall set up a system of global equations of balance, equivalent to the local equations and the jump conditions obtained in the foregoing sections.

In the next chapter, these equations of balance will be supplemented by a set of constitutive equations.

II.2. Balance of energy

We postulate the following integral balance of energy, as a generalization of equation I(42) for conducting, polarizable, magnetizable and deformable media.

$$\begin{aligned}
 (1) \quad \frac{d}{dt} \int_V \rho E \, dV &= \int_V \{ \rho r + \rho F_i^{(m)} V_i \} dV + \\
 &+ \oint_S \{ T_{ij} V_i + M_{ikj} \Omega_{ik} - Q_j - \frac{c}{4\pi} e_{jkl} E_k H_l + \\
 &+ \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) V_j + \frac{2}{4\pi c} e_{ikl} E_k H_l V_j V_i + R_j \} n_j \, dS .
 \end{aligned}$$

In this equation, ρE is the total energy density that can be divided into the following terms

$$(2) \quad \rho E = \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) + \rho U + \frac{1}{2} \rho V_i V_i + \rho T ,$$

where

$\frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*)$ = the electromagnetic energy of the long-range interaction and of the external field;

ρU = the short-range energy or internal energy, i.e. the deformation energy, the anisotropy energy, the polarization energy, etc.;

$\frac{1}{2} \rho V_i V_i$ = the classical kinetic energy;

ρT = the remaining part of the kinetic energy, descended for instance from the electromagnetic momentum and the spin of the magnetization vector. This term will be specified by means of invariance requirements. According to equation I(42), ρT must satisfy

$$(3) \quad \rho T = \frac{2}{4\pi c} e_{ijk} E_j H_k V_i , \quad \text{if } P_i = M_i = 0 .$$

Equation (1) says that the time rate of change of the sum of electromagnetic field energy, internal energy and kinetic energy is equal to the rate at which work is done by the mechanical body forces ($\rho F_i^{(m)} V_i$), the surface tractions ($T_{ij} V_i n_j$) and the couple-stresses ($M_{ikj} \Omega_{ik} n_j$), plus the heat source (ρr) and the heat flux ($-Q_j n_j$), supplemented by the flux of electromagnetic energy across the surface S . This latter contribution consists of the Poynting-vector ($-\frac{c}{4\pi} e_{ijk} E_k H_l$), the

convective flux of electromagnetic energy ($\frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) V_j$), and of electromagnetic momentum ($\frac{2}{4\pi c} e_{ijk} E_k H_\ell V_j V_i$), plus a term, representing the flux of electromagnetic dipole energy (R_j), the form of which is still undetermined (cf. also [8]). By means of requirements of invariance, we shall obtain an explicit expression for \underline{R} . According to equation I(42) this vector \underline{R} must satisfy the relation

$$(4) \quad \underline{R} = \underline{0}, \quad \text{if } \underline{P} = \underline{M} = \underline{0}.$$

In order to facilitate the forthcoming calculations, we split up \underline{R} into

$$(5) \quad R_i = \rho (E_j P_j + H_j M_j) V_i + \frac{\rho^2}{c} e_{jkl} (P_k H_\ell + E_k M_\ell) V_j V_i + \tilde{R}_i.$$

The choice of the expression $\frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*)$ for the electromagnetic energy is motivated by the form of this energy in a vacuum. We notice, that it is always possible to take a distinct expression for this energy, that also coincides with the vacuum-energy. Such an alternative expression could be, for instance: $\frac{1}{8\pi} (E_i^* E_i^* + B_i^* B_i^*)$. However, an analogous theory, as the one that will be described in the forthcoming sections, could be set up on the basis of this alternative energy. In this case, only the constitutive equations would alter. For instance, we should obtain a different stress tensor. By a transformation of the energy functional, however, it is always possible to get back the constitutive equations that we will derive in the next chapter.

The global balance of energy (1) yields, in the way described in Section I.2, a local equation of balance. By using (2) and (5) and the electromagnetic equations of Section I.3, this local equation can be worked out into the form

$$(6) \quad \begin{aligned} & - (Q E_i + \frac{1}{c} e_{ijk} J_j B_k) V_i - \rho (P_j E_{j,i} + M_j H_{j,i}) V_i + \\ & - \frac{1}{4\pi c} \frac{d}{dt} [e_{ijk} (D_j B_k + E_j H_k) V_i] + \frac{4\pi\rho}{c} \frac{d}{dt} (\rho e_{ijk} P_j M_k V_i) + \\ & - \frac{1}{4\pi c} e_{ijk} (D_j B_k + E_j H_k) V_{\ell,\ell} V_i - \tilde{R}_{i,i} - \rho E_i \dot{P}_i - \rho H_i \dot{M}_i + \\ & + \rho \dot{U} + \rho (\dot{V}_i - F_i^{(m)}) V_i + \rho \dot{T} + (U + \frac{1}{2} V_i V_i + T) (\dot{\rho} + \rho V_{j,j}) - \rho r + \end{aligned}$$

$$-T_{ij,j}V_i - T_{ij}V_{i,j} - (M_{ijk}\Omega_{ij})_{,k} + Q_{i,i} - J_i^*E_i^* = 0.$$

Starting from the principle, that states that this relation is invariant under superposed rigid-body translations and rotations, we shall derive in the next two sections equations of balance of mass, of momentum and of moment of momentum. A similar method is used by Alblas [7]-[9] and Parkus [10].

II.3: Balance of mass and of momentum

We assume that the quantities: ρ , $(\dot{V}_i - F_i^{(m)})$, U , r , T_{ij} , $(M_{ijk}\Omega_{ij})$ and Q_i are invariant under superposed rigid-body translations with velocity $\underline{b}(t)$, while the remaining quantities occurring in (6) transform according to

$$\begin{aligned} V_i &\Rightarrow V_i - b_i, & T &\Rightarrow T + \tau_i b_i, \\ D_i &\Rightarrow D_i + \frac{1}{c} e_{ijk} b_j H_k, & B_i &\Rightarrow B_i - \frac{1}{c} e_{ijk} b_j E_k, \\ (7) \quad E_i &\Rightarrow E_i + \frac{1}{c} e_{ijk} b_j B_k, & H_i &\Rightarrow H_i - \frac{1}{c} e_{ijk} b_j D_k, \\ P_i &\Rightarrow P_i - \frac{1}{c} e_{ijk} b_j M_k, & M_i &\Rightarrow M_i + \frac{1}{c} e_{ijk} b_j P_k, \\ J_i &\Rightarrow J_i - Q b_i, & Q &\Rightarrow Q, & \tilde{R}_i &\Rightarrow \tilde{R}_i + \tilde{R}_{ij}^{(1)} b_j + \tilde{R}_{ijk}^{(2)} b_j b_k. \end{aligned}$$

We note that τ_i , $\tilde{R}_{ij}^{(1)}$ and $\tilde{R}_{ijk}^{(2)}$ are unknowns, that will be determined in the following.

We transform (6) by superposing a rigid-body translation with velocity $\underline{b}(t)$, and we subtract the original equation from the transformed one. After the neglect of terms proportional to c^{-2} , and after some rearranging, we obtain

$$\begin{aligned} (8) \quad &[-\tilde{R}_{kij,k}^{(2)} - \frac{4\pi\rho^2}{c} e_{jkl} P_k M_l]_{,i} b_i b_j + \frac{1}{2} (\dot{\rho} + \rho V_{j,j}) b_i b_i + \\ &+ \frac{2}{4\pi c} e_{ijk} E_j H_k + \rho \tau_i \dot{b}_i + [(\dot{\rho} + \rho V_{j,j})(\tau_i - V_i) - \rho \dot{V}_i + \rho F_i^{(m)}] + \end{aligned}$$

$$\begin{aligned}
& + T_{ij,j} + Q E_i + \frac{1}{c} e_{ijk} J_j B_k + \rho P_j E_{j,i} + \rho M_j H_{j,i} + \\
& + \frac{1}{4\pi c} \frac{d}{dt} \{e_{ijk} (D_j B_k + E_j H_k)\} + \frac{1}{4\pi c} e_{ikl} (D_k B_l + E_k H_l) V_{j,j} + \\
& - \tilde{R}_{ji}^{(1)} + \rho \dot{\tau}_i + \left(\frac{4\pi\rho^2}{c} e_{ikl} P_{kM} V_{j,j} \right) b_i = 0 .
\end{aligned}$$

The first term of (8) can be eliminated by choosing

$$(9) \quad \tilde{R}_i = - \frac{4\pi\rho^2}{c} e_{jkl} P_{kM} V_{j,i} ,$$

because in that case

$$(10) \quad \tilde{R}_{kij}^{(2)} = - \frac{4\pi\rho^2}{c} \delta_{ki} e_{jlm} P_{lM} .$$

Besides this, (9) also gives

$$(11) \quad \tilde{R}_{ji}^{(1)} = \frac{4\pi\rho^2}{c} e_{ikl} P_{kM} V_j + \delta_{ij} \frac{4\pi\rho^2}{c} e_{klm} P_{kM} V_m .$$

We note that the relation (8) is valid for arbitrary $\underline{b}(t)$. Hence, the coefficients of $b_i b_i$, \dot{b}_i and b_i have to be zero, which leads us to the following relations

$$(12) \quad \dot{\rho} + \rho V_{i,i} = 0 ,$$

$$(13) \quad \rho \tau_i = - \frac{2}{4\pi c} e_{ijk} E_j H_k ,$$

$$\begin{aligned}
(14) \quad \rho \dot{v}_i = & T_{ij,j} + \rho F_i^{(m)} + Q E_i + \frac{1}{c} e_{ijk} J_j B_k + \rho P_j E_{j,i} + \\
& + \rho M_j H_{j,i} + \frac{1}{4\pi c} \frac{d}{dt} [e_{ijk} (D_j B_k - E_j H_k)] + \\
& + \frac{1}{4\pi c} e_{ijk} (D_j B_k - E_j H_k) V_{l,l} - \left(\frac{4\pi\rho^2}{c} e_{jkl} P_{kM} V_j \right)_{,i} .
\end{aligned}$$

On deriving (14), the equations (11) - (13) are used.

We note that (12) expresses the conservation of mass and (14) represents the balance of momentum.

The relation (13) can be satisfied by taking

$$(15) \quad \rho T = \frac{2}{4\pi c} e_{ijk} E_j H_k V_i + \rho K,$$

where ρK is invariant under superposed rigid-body translations. Consequently, ρK may not depend on the velocity. It follows from (3) that ρK must be zero if polarization and magnetization are absent.

By the results obtained above, we have specified \underline{R} and ρT but for some invariant parts. However, these parts do not need to be modified any further, because it is always possible to replace them by a surface or volume source.

Defining the electromagnetic volume force $\underline{F}^{(e)}$ by

$$(16) \quad \begin{aligned} F_i^{(e)} &:= Q E_i + \frac{1}{c} e_{ijk} J_j B_k + \rho P_j E_{j,i} + \rho M_j H_{j,i} + \\ &+ \frac{1}{4\pi c} \frac{d}{dt} [e_{ijk} (D_j B_k - E_j H_k)] + \frac{1}{4\pi c} e_{ijk} (D_j B_k - E_j H_k) V_{\ell,\ell} + \\ &- \left(\frac{4\pi \rho^2}{c} e_{jkl} P_k M_\ell V_j \right)_{,i} = \\ &= Q E_i + \frac{1}{c} e_{ijk} J_j B_k + \rho P_j E_{j,i} + \rho M_j H_{j,i} + \\ &+ \frac{\rho}{c} \frac{d}{dt} [e_{ijk} (P_j B_k + E_j M_k)] - \left(\frac{4\pi \rho^2}{c} e_{jkl} P_k M_\ell V_j \right)_{,i}, \end{aligned}$$

the equation of balance of momentum reduces to

$$(17) \quad \rho \dot{V}_i = T_{ij,j} + \rho F_i^{(m)} + F_i^{(e)}.$$

With the aid of the electromagnetic equations of Section I.3, the electromagnetic volume force can be rewritten as

$$(18) \quad F_i^{(e)} = \rho \frac{d}{dt} \left[-\frac{1}{4\pi \rho c} e_{ijk} E_j H_k \right] + T_{ij,j}^M + \left[\frac{1}{4\pi c} e_{jkl} E_k H_\ell V_i \right]_{,j},$$

where

$$\frac{1}{4\pi c} e_{ijk} E_j H_k$$

is the electromagnetic momentum, and T_{ij}^M a Maxwell stress tensor defined by

$$(19) \quad T_{ij}^M := \frac{1}{4\pi} [E_i^* D_j^* + H_i^* B_j^* - \frac{1}{2} \delta_{ij} (E_k^* E_k^* + H_k^* H_k^*)].$$

After substitution of (9), (12), (14) and (15) into (6), the local balance of energy reduces to

$$(20) \quad \rho \dot{U} + \rho \dot{K} - \rho r - T_{ij} V_{i,j} + Q_{i,i} - (M_{ijk} \Omega_{ij})_{,k} + \\ - \rho E_i^* \dot{P}_i^* - \rho H_i^* \dot{M}_i^* - J_i^* \dot{E}_i^* = 0 .$$

11.4. Balance of moment of momentum

The moment of momentum equation that we shall derive in this section, is valid for a more limited class of materials than the equations obtained in the last section. In this section, and in what follows, the following restrictions are imposed:

i) The magnetization is saturated, thus

$$(21) \quad M_i^* M_i^* = M_s^2 ,$$

where the constant M_s is the saturation magnetization.

ii) Magnetic dissipation is not taken into account.

iii) Only the spin of the magnetization vector contributes to the kinetic energy term ρK .

If these restrictions are satisfied, the kinetic energy K must be constant (cf. equation IV.(53)). Hence

$$(22) \quad \rho \dot{K} = 0 .$$

We have not yet specified the tensor Ω_{ij} occurring in (1) in combination with the couple-stress M_{ijk} . By choosing an explicit expression for Ω_{ij} , we define at the same time the tensor M_{ijk} . We shall identify the tensor Ω_{ij} with the angular velocity of the magnetization vector, so

$$(23) \quad \Omega_{ij} = \frac{2}{M_s^2} M_{[i}^* \dot{M}_{j]}^* .$$

By using (23) we obtain

$$(24) \quad M_{ijk} \Omega_{ij} = \frac{2}{M_s^2} M_{ijk} M_{[i}^* \dot{M}_{j]}^* = \Pi_{jk} \dot{M}_j^* ,$$

where the tensor Π_{ij} is defined by

$$(25) \quad \Pi_{ij} := \frac{2}{M_s^2} M_{kij} M_k^* .$$

In (24) we have used the fact that we may take

$$M_{ijk} = M_{[ij]k} ,$$

without losing any information.

After substitution of (22) and (24) into (20), the energy balance becomes

$$(26) \quad \rho \dot{U} - \rho r + Q_{i,i} - T_{ij} V_{i,j} - \Pi_{ij,j} \dot{M}_i^* - \Pi_{ij} (\dot{M}_i^*)_{,j} + \\ - \rho E_i^* P_i^* - \rho H_i^* M_i^* - J_i^* E_i^* = 0 .$$

According to the principle stated in the Introduction, this relation has to be invariant under superposed rigid-body rotations. We note that the material derivative of an invariant quantity is not necessarily invariant too. A derivative that preserves invariance is the Jaumann-derivative, defined in Section I.1. Therefore, we replace in (26) the material derivatives by Jaumann-derivatives, obtaining

$$(27) \quad \rho \overset{\nabla}{U} - \rho r + Q_{i,i} - T_{ij} V_{i,j} - \Pi_{ij,j} \overset{\nabla}{M}_i^* - \Pi_{ij} (\overset{\nabla}{M}_i^*)_{,j} + \\ + (\Pi_{ik} M_j^* V_{[j,i]})_{,k} - \rho E_i^* \overset{\nabla}{P}_i^* + \rho E_i^* P_j^* V_{[j,i]} + \\ - \rho H_i^* \overset{\nabla}{M}_i^* + \rho H_i^* M_j^* V_{[j,i]} - J_i^* E_i^* = 0 .$$

Let us consider a superposed rigid-body rotation, described by

$$(28) \quad V_i \Rightarrow V_i + e_{ijk} \omega_j x_k ,$$

where $\underline{\omega}$ is an arbitrary but uniform vector.

We state that all quantities occurring in (27) are invariant under these rotations, except the velocity \underline{V} and the magnetic field \underline{H}^* , that transforms according to

$$(29) \quad H_i^* \Rightarrow H_i^* - \frac{1}{\Gamma} \omega_i ,$$

where Γ is the gyromagnetic ratio which is of the order c^{-1} . We note that this transformation is a consequence of the Barnett-effect (cf. [7]).

Transforming (27) by superposition of the rigid-body rotation (28), and subtracting the original equation yields

$$(30) \quad \omega_i \left\{ \frac{\rho}{\Gamma} (\dot{M}_i^* + M_j^* v_{[j,i]}) + e_{ijk} [T_{jk} + (\Pi_{j\ell} M_k^*)_{,\ell} + \rho E_j^* P_k^* + \rho H_j^* M_k^*] \right\} = 0 .$$

Since (30) is valid for arbitrary $\underline{\omega}$, the following relation must hold

$$(31) \quad \frac{1}{\Gamma} \dot{M}_i^* = e_{ijk} \{ P_j^* E_k^* + M_j^* H_k^* - \frac{1}{\rho} T_{[jk]} + \frac{1}{\rho} (M_j^* \Pi_{k\ell})_{,\ell} \} .$$

This equation represents the local balance of moment of momentum for a polarizable, magnetically saturated, nondissipative medium.

Another quantity, frequently used in the theory of magnetodynamics, is the so called effective magnetic field \underline{G}^* . This field can be defined by the equations

$$(32) \quad \frac{1}{\Gamma} \dot{M}_i^* = e_{ijk} M_j^* G_k^* , \quad G_i^* M_i^* = 0 .$$

By means of (32) the relation (31) can also be written as

$$(33) \quad M_{[i}^* G_{j]}^* = P_{[i}^* E_{j]}^* + M_{[i}^* H_{j]}^* - \frac{1}{\rho} T_{[ij]} + \frac{1}{\rho} (M_{[i}^* \Pi_{j]k})_{,k} .$$

Multiplying (33) by M_j^* , we obtain the following expression for G_i^*

$$(34) \quad G_i^* = \frac{2}{M_s^2} M_j^* \{ P_{[i}^* E_{j]}^* + M_{[i}^* H_{j]}^* - \frac{1}{\rho} T_{[ij]} + \frac{1}{\rho} (M_{[i}^* \Pi_{j]k})_{,k} \} ,$$

where we have used (21) and (32)².

II.5. Jump conditions

We substitute (2), (5), (9), (15) and (24) into the global balance of energy (1), in order to obtain

$$\begin{aligned}
(35) \quad & \frac{d}{dt} \int_V \left\{ \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) + \rho U + \frac{1}{2} \rho V_i V_i + \right. \\
& \left. + \frac{2}{4\pi c} e_{ijk} E_j H_k V_i + \rho K \right\} dV = \\
& = \oint_S \left\{ T_{ij} V_j + \Pi_{ij} \dot{M}_i^* - Q_j - \frac{c}{4\pi} e_{jkl} E_k H_l + \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) V_j + \right. \\
& \left. + \frac{2}{4\pi c} e_{ikl} E_k H_l V_j V_i + \rho (E_i^* P_i^* + H_i^* M_i^*) V_j + \right. \\
& \left. + \frac{\rho^2}{2} e_{ikl} (P_k^* B_l^* + E_k^* M_l^*) V_j V_i \right\} n_j dS + \\
& + \int_V \left\{ \rho r - J_i^* E_i^* + \rho F_i^{(m)} V_i \right\} dV .
\end{aligned}$$

Equation (35) has the typical structure of the general balance equation I.(21). Hence, we can derive from (35) a discontinuity condition similar to equation I.(24). By using I.(36)-(37), this condition may be written as

$$\begin{aligned}
(36) \quad & \mathbb{I} \left\{ \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) + \rho U + \frac{1}{2} \rho V_i V_i + \frac{2}{4\pi c} e_{ijk} E_j H_k V_i + \right. \\
& \left. + \rho K \right\} (V_l n_l - W_n) - \left\{ T_{ij} V_j + \Pi_{ij} \dot{M}_i^* - Q_j - \frac{c}{4\pi} e_{jkl} E_k H_l + \right. \\
& \left. - \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) V_j + \frac{2}{4\pi c} e_{ikl} E_k H_l V_j V_i + \right. \\
& \left. + \frac{1}{4\pi} (E_i^* D_i^* + H_i^* B_i^*) V_j + \frac{\rho^2}{c} e_{ikl} (P_k^* B_l^* + E_k^* M_l^*) V_j V_i \right\} n_j \mathbb{I} = 0 ,
\end{aligned}$$

on $\Sigma(t)$.

With the relations

$$\begin{aligned}
(37) \quad & \mathbb{I} \left[- \frac{c}{4\pi} e_{jkl} E_k H_l - \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) V_j + \frac{1}{4\pi} (E_i^* D_i^* + H_i^* B_i^*) V_j + \right. \\
& \left. + \frac{2}{4\pi c} e_{ikl} E_k H_l V_j V_i + \frac{\rho^2}{c} e_{ikl} (P_k^* B_l^* + E_k^* M_l^*) V_j V_i \right] n_j = \\
& = - 2\pi \mathbb{I} \left\{ (\rho M_i^* n_i)^2 + (\rho P_i^* n_i)^2 \right\} W_n - \mathbb{I} \left\{ \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) + \right. \\
& \left. - \frac{1}{4\pi} (E_i^* D_i^* + H_i^* B_i^*) - \frac{4\pi \rho^2}{c} e_{ijk} P_j M_k (V_i + W_n n_i) \right\} (V_j n_j - W_n) \mathbb{I} ,
\end{aligned}$$

which will be proved in Appendix II. , and

$$(38) \quad \begin{aligned} \frac{1}{8\pi} (E_i E_i + H_i H_i) &= \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) + \frac{1}{4\pi c} e_{ijk} V_i (E_j B_k + D_j H_k) , \\ \frac{1}{4\pi} (E_i D_i + H_i B_i) &= \frac{1}{4\pi} (E_i^* D_i^* + H_i^* B_i^*) + \frac{2}{4\pi c} e_{ijk} V_i (D_j B_k + E_j H_k) , \end{aligned}$$

which can be inferred directly from equation I. (38), the jump condition can be rewritten as

$$(39) \quad \begin{aligned} & \{ \frac{1}{2} \rho V_i V_i + \rho U + \rho K + \rho (E_i^* P_i^* + H_i^* M_i^*) + \\ & - \frac{\rho}{c} e_{ijk} V_i (P_j^* B_k^* + D_j^* M_k^*) - \frac{4\pi\rho^2}{c} e_{ijk} P_j^* M_k^* (V_i + W_n n_i) \} (V_\ell n_\ell - W_n) + \\ & + \{ - T_{ij} V_i - \Pi_{ij} \dot{M}_i^* + Q_j \} n_j + \\ & + 2\pi \{ (\rho M_i^* n_i)^2 + (\rho P_i^* n_i)^2 \} W_n \} = 0 , \quad \text{on } \Sigma(t) . \end{aligned}$$

Again, we postulate that the condition (39) has to be invariant under rigid-body motions.

Under a superposed rigid-body translation the velocities V_i and W_n transform according to

$$(40) \quad V_i \Rightarrow V_i - b_i \quad \text{and} \quad W_n \Rightarrow W_n - b_i n_i ,$$

while all other quantities occurring in (39) are invariant.

Transforming (39) by superposition of a rigid-body translation and subtracting the original equation results in the following jump conditions for the density and the stresses

$$(41) \quad \{ \rho (V_i n_i - W_n) \} = 0 , \quad \text{on } \Sigma(t) ,$$

$$(42) \quad \begin{aligned} \{ T_{ij} \} n_j &= 2\pi \{ (\rho M_j^* n_j)^2 + (\rho P_j^* n_j)^2 \} n_i + \\ &+ \{ \{ \rho V_i - \frac{\rho}{c} e_{ijk} (P_j B_k + D_j M_k) + \\ &- \frac{4\pi\rho^2}{c} e_{jkl} P_k M_l (\delta_{ij} + n_i n_j) \} (V_m n_m - W_n) \} , \quad \text{on } \Sigma(t) . \end{aligned}$$

The requirement of invariance under superposed rigid-body rotations yields in the usual way

$$(43) \quad \left[-\frac{\rho}{2\Gamma} e_{ijk} M_k^* (V_\ell n_\ell - W_n) + M_{[i}^* \Pi_{j]k} n_k \right] = 0, \quad \text{on } \Sigma(t),$$

where we have used (28), (29), (41) and (42).

These jump conditions are simplified considerably, if the discontinuity surface is a material one.

By substituting I.(22) into (41) - (43) we arrive at

$$(44) \quad [T_{ij}]n_j = 2\pi[(\rho M_j^* n_j)^2 + (\rho P_j^* n_j)^2]n_i, \quad \text{on } \Sigma(t),$$

$$(45) \quad [M_{[i}^* \Pi_{j]k}]n_k = 0, \quad \text{on } \Sigma(t),$$

$$(46) \quad [Q_j]n_j = [T_{ij}V_i - 2\pi\{(\rho M_i^* n_i)^2 + (\rho P_i^* n_i)^2\}V_j + \Pi_{ij}^* M_i^*]n_j,$$

on $\Sigma(t)$.

In some of the following chapters, we shall consider problems concerning solid bodies placed in a vacuum. Let S be the boundary of the body, let T_i^* and Q^* be prescribed surface forces and surface heat supply. Further, we take S to be free of surface moments. In this case, the system (41) to (43) reduces to the following set of boundary conditions

$$(47) \quad T_{ij}n_j = 2\pi\{(\rho M_j^* n_j)^2 + (\rho P_j^* n_j)^2\}n_i + T_i^*, \quad \text{on } S,$$

$$(48) \quad M_{[i}^* \Pi_{j]k} n_k = 0, \quad \text{on } S,$$

$$(49) \quad Q_j n_j = Q^*, \quad \text{on } S.$$

On deriving the boundary condition (49), the equations (29), (43) and (44) have been used.

11.6. Global equations of balance

In the preceding sections we have set up a system of local equations of balance, together with jump conditions, for the mass, the momentum, the moment of momentum and the energy. This system must be equivalent to a set of global equations of balance similar to equations I.(17)-(20). In this section we shall derive such a set of global equations.

The global balance of mass belonging to (12) and (41) is found most easily. Since there is no supply of mass, this one reads

$$(50) \quad \frac{d}{dt} \int_V \rho \, dV = 0 .$$

We arrive at the global balance of momentum by means of integration over the volume V of (17), into which the expression (18) for the electromagnetic volume force is substituted:

$$(51) \quad \begin{aligned} \frac{d}{dt} \int_V \{ \rho V_i + \frac{1}{4\pi c} e_{ijk} E_j H_k \} dV = \\ = \oint_S \{ T_{ij} + T_{ij}^M - \frac{1}{4\pi c} e_{jkl} E_k H_l V_i \} n_j \, dS + \int_V \rho F_i^{(m)} \, dV . \end{aligned}$$

Besides the local equation (17), we can derive from (51) also the jump conditions for the stresses (42), by means of equation I.(28) and the electromagnetic equations of Section I.3.

It is easy to see that (51) can be made to correspond with I.(48) by taking in the latter:

the moment density equal to

$$(52) \quad \rho p_i = \rho V_i + \frac{1}{4\pi c} e_{ijk} E_j H_k ,$$

the stress tensor to

$$(53) \quad t_{ij} = T_{ij} + T_{ij}^M - \frac{1}{4\pi c} e_{jkl} E_k H_l V_i ,$$

and the body force to

$$(54) \quad \rho F_i = \rho F_i^{(m)} .$$

Substituting these expressions into the balance of angular momentum I.(19) yields

$$(55) \quad \begin{aligned} \frac{d}{dt} \int_V \rho \{ s_{ij} + x_{[i} \langle V_{j]} + \frac{1}{4\pi c} e_{j]kl} E_k H_l \} dV = \\ = \oint_S \{ m_{ijk} + x_{[i} t_{j]k} \} n_k \, dS + \int_V \rho \{ L_{ij} + x_{[i} F_{j]}^{(m)} \} dV . \end{aligned}$$

As can be shown by some simple calculations, the local balance of moment of momentum (31) and the jump condition (43) can be inferred from (55) if we take:

the spin density

$$(56) \quad \rho s_{ij} = \frac{\rho}{2\Gamma} e_{ijk} M_k^*,$$

the couple stress

$$(57) \quad m_{ijk} = M_{[i}^* \Pi_{j]k},$$

and the body couple

$$(58) \quad \rho L_{ij} = 0.$$

The right-hand side of (56) represents the intrinsic angular momentum of the spin of the magnetization vector.

Substitution of (53), (56) to (58) into (55) results in the following global balance of moment of momentum

$$(59) \quad \begin{aligned} \frac{d}{dt} \int_V \{ \frac{\rho}{2\Gamma} e_{ijk} M_k^* + x_{[i} (V_{j]} + \frac{1}{4\pi\rho c} e_{j]kl} E_k H_l) \} dV = \\ = \oint_S \{ M_{[i}^* \Pi_{j]k} + x_{[i} (T_{j]k} + T_{j]k}^M - \frac{1}{4\pi c} e_{k\ell m} E_\ell H_m V_{j]} \} n_k dS + \\ + \int_V x_{[i} F_{j]}^{(m)} dV. \end{aligned}$$

With the relations (23), (54), (57) and (58) the global balance of energy I.(20) becomes

$$(60) \quad \begin{aligned} \frac{d}{dt} \int_V \rho E dV = \oint_S \{ t_{ij} v_i + \Pi_{ij} \dot{M}_i^* - h_j \} n_j dS + \\ + \int_V \{ \rho F_i^{(m)} v_i + \rho \sigma \} dV. \end{aligned}$$

Obviously, this relation corresponds with (1) if ρE is given by (2) and if we take

the heat flux

$$(61) \quad h_j = Q_j + \frac{c}{4\pi} e_{jkl} E_k H_l + \left\{ T_{ij}^M - \frac{1}{4\pi c} e_{jkl} E_k H_l V_i \right\} V_j,$$

and the heat supply

$$(62) \quad \rho\sigma = \rho r.$$

We conclude by noting that the four equations (50), (51), (58) and (59) constitute a system of global equations of balance for the mass, the moment, the moment of momentum and the energy, that is equivalent with the local equations and jump conditions found in the sections I.2-I.5.

III. CONSTITUTIVE EQUATIONS

III.1. Introduction

The balance equations, derived in Chapter II, are not sufficiently in number for the determination of the unknown variables. Therefore the system must be supplemented by a set of constitutive equations in order to obtain a complete system. In this chapter, we shall set up a system of constitutive equations for the entropy, the polarization, the stresses, the couple-stresses, the heat flux, the electric current density and the entropy flux. This theory will be based on an entropy inequality similar to equation I.(14).

Since the constitutive equations characterize the medium under consideration, we must first specify the class of materials we wish to regard in the present work. Throughout this thesis, we shall investigate a medium that is polarizable, magnetizable and thermoelastic, without mechanical or electromagnetic dissipation. Moreover, we take into account exchange interaction and heat conduction. These features of the material underly the choice of a set of independent variables. After such a selection has been made, we shall derive constitutive equations by means of the principle of Coleman and Noll, discussed in Section I.4. Following the second postulate of Section I.4, we proceed by rewriting the constitutive equations in a form that is invariant under superposed rigid-body rotations. This will be done by transforming our primary set of independent variables into a set of invariant variables. To this end, we shall use the theorem of Cauchy, formulated in Section I.4.

As we have already mentioned before, the stress tensor T_{ij} has a certain arbitrariness. On the analogy of the work of Brown ([11], Section 5.6), we shall introduce some alternative definitions of the stress tensor in Section 5 of this chapter.

In the last section, we shall compare the results of the present theory with some recent articles treating similar subjects.

III.2. Entropy inequality

One of the basic postulates, underlying the derivation of the constitutive equations, is formed by the entropy inequality or Clausius-Duhem inequality, formulated in Section I.1. In that paragraph, we have postulated the inequality mentioned above as follows (cf. equation I.(14))

$$(1) \quad \rho \dot{S} - \frac{\rho r}{\theta} - \phi_{i,i} \geq 0 .$$

We remark that the entropy flux ϕ , occurring in this inequality, is still undetermined. However, we shall derive a constitutive equation for this quantity in the present chapter. In order to facilitate the forthcoming calculations, we introduce the vector $\underline{\sigma}$ by

$$(2) \quad \sigma_i := \phi_i + \frac{Q_i}{\theta} .$$

In the following section we shall prove that, for the class of materials under consideration, the vector $\underline{\sigma}$ is equal to zero. Then, we have shown that the familiar expression for the entropy flux I.(16) is valid in our case.

Motivated by the special form of the energy balance II.(20) and by the selection of the independent variables in the next section, we replace the internal energy density U by the thermodynamic function Σ defined by

$$(3) \quad \Sigma := U - \theta S - P_i^* E_i^* .$$

Eliminating the heat supply r from (1) by means of the energy balance II.(20) and substituting (2) and (3) into the thus obtained relation, yields the following inequality

$$(4) \quad -\rho \dot{\Sigma} - \rho S \dot{\theta} - \rho P_i^* \dot{E}_i^* + \rho H_i^{**} \dot{M}_i^{**} + J_i^* \dot{E}_i^* + \Pi_{i,j} \dot{M}_i^{**} + \\ + \Pi_{ij} (\dot{M}_i^*)_{,j} + T_{ij} v_{i,j} - \theta \sigma_{i,i} - \frac{1}{\theta} Q_{i,\theta} \geq 0 .$$

Starting from the above inequality, we shall derive in the next section a system of constitutive equations.

III.3. Derivation of the constitutive equations

Before we are able to set up a system of constitutive equations, we first must specify the class of materials these relations refer to. This will be done by selecting a set of independent variables S . Since we are interested in magnetizable, polarizable and thermoelastic materials, this set must contain the variables $F_{i\alpha}$, M_i^* , E_i^* and θ , where

$$(5) \quad F_{i\alpha} := \frac{\partial x_i}{\partial X_\alpha} = \frac{\partial x_i(X, t)}{\partial X_\alpha}$$

the deformation gradient.

Of course, it is always possible to enter H_i^* or B_i^* instead of M_i^* and P_i^* or D_i^* instead of E_i^* in the set S . We do not choose H_i^* (or B_i^*) because this quantity is not invariant under superposed rigid-body rotations (cf. II.(29)). The reason that we have selected E_i^* and not P_i^* lies in the fact that we prefer a constitutive equation expressing P_i^* as a function of E_i^* to one that gives E_i^* as a function of P_i^* .

Further, we wish to take into account exchange interaction and heat conduction, what can be accomplished by including in S

$$(6) \quad M_{i\alpha} := \frac{\partial M_i^*}{\partial X_\alpha} \quad \text{and} \quad \theta_\alpha := \frac{\partial \theta}{\partial X_\alpha},$$

respectively.

We do not consider mechanical or electromagnetic dissipation. Hence, S does not contain time rates like \dot{x}_i or \dot{M}_i^* .

In this way, we arrive at the following set of independent variables

$$(7) \quad S := \{F_{i\alpha}, M_i^*, M_{i\alpha}, E_i^*, \theta, \theta_\alpha\}.$$

According to the principle of equipresence, discussed in Section I.4, each dependent variable must be a function of all independent variables, unless the contrary is proved. In particular, we have

$$(8) \quad \Sigma = \Sigma(F_{i\alpha}, M_{i\alpha}^*, E_i^*, \theta, \theta_\alpha),$$

and

$$(9) \quad \sigma = \sigma(F_{i\alpha}, M_{i\alpha}^*, E_i^*, \theta, \theta_\alpha).$$

We note that, just as the internal energy U , also the functional Σ should be invariant under superposed rigid-body rotations. This condition is fulfilled if Σ satisfies the following relation (cf. [11], p. 84)

$$(10) \quad \frac{\partial \Sigma}{\partial F_{[i\alpha}} F_{j]\alpha} + \frac{\partial \Sigma}{\partial M_{[i}^*} M_{j]}^* + \frac{\partial \Sigma}{\partial M_{[i\alpha}} M_{j]\alpha} + \frac{\partial \Sigma}{\partial E_{[i}^*} E_{j]}^* = 0.$$

The derivatives $\dot{\Sigma}$ and $\sigma_{i,i}$, occurring in (4), can be worked out by means of (8) and (9) into the form

$$(11) \quad \dot{\Sigma} = \frac{\partial \Sigma}{\partial F_{i\alpha}} \dot{F}_{i\alpha} + \frac{\partial \Sigma}{\partial M_i^*} \dot{M}_i^* + \frac{\partial \Sigma}{\partial M_{i\alpha}} \dot{M}_{i\alpha} + \frac{\partial \Sigma}{\partial E_i^*} \dot{E}_i^* + \frac{\partial \Sigma}{\partial \theta} \dot{\theta} + \frac{\partial \Sigma}{\partial \theta_\alpha} \dot{\theta}_\alpha,$$

$$(12) \quad \sigma_{i,i} = \frac{\partial \sigma_i}{\partial F_{j\alpha}} F_{j\alpha,i} + \frac{\partial \sigma_i}{\partial M_j^*} M_{j,i}^* + \frac{\partial \sigma_i}{\partial M_{j\alpha}} M_{j\alpha,i} + \frac{\partial \sigma_i}{\partial E_j^*} E_{j,i}^* + \frac{\partial \sigma_i}{\partial \theta} \dot{\theta}_{,i} + \frac{\partial \sigma_i}{\partial \theta_\alpha} \dot{\theta}_{\alpha,i}.$$

By substitution of (11) and (12) into (4), after elimination of \dot{M}_i^* from (4) by means of the angular momentum equation taken in the form II.(32)¹, and with the aid of the relations

$$(13) \quad \frac{\partial}{\partial X_\alpha} = F_{i\alpha} \frac{\partial}{\partial x_i}, \quad \dot{F}_{i\alpha} = v_{i,j} F_{j\alpha}, \quad (\dot{M}_i^*)_{,j} = \dot{M}_{i\alpha} \frac{\partial X_\alpha}{\partial x_j},$$

the following inequality is obtained

$$(14) \quad \Gamma e_{ijk} M_{jk}^* G_k^* (\rho H_i^* - \rho \frac{\partial \Sigma}{\partial M_i^*} + \Pi_{i\ell, \ell}) - \rho (\frac{\partial \Sigma}{\partial \theta} + S) \dot{\theta} + \\ - \rho \left(P_i^* + \frac{\partial \Sigma}{\partial E_i^*} \right) \dot{E}_i^* + \left(\Pi_{ij} \frac{\partial X_\alpha}{\partial x_j} - \rho \frac{\partial \Sigma}{\partial M_{i\alpha}} \right) \dot{M}_{i\alpha} +$$

$$\begin{aligned}
& + \rho \frac{\partial \Sigma}{\partial \theta_\alpha} \dot{\theta}_\alpha + (T_{ij} - \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha}) V_{i,j} - \frac{1}{\theta} Q_i \theta_{,i} + \\
& - \theta \frac{\partial \sigma_i}{\partial F_{j\alpha}} F_{j\alpha,i} - \theta \frac{\partial \sigma_i}{\partial M_j^*} M_{j,i}^* - \theta \frac{\partial \sigma_i}{\partial M_{j\alpha}} M_{j\alpha,i} - \theta \frac{\partial \sigma_i}{\partial E_i^*} E_{j,i}^* + \\
& - \theta \frac{\partial \sigma_i}{\partial \theta} \theta_{,i} - \theta \frac{\partial \sigma_i}{\partial \theta_\alpha} \theta_{\alpha,i} + J_i^* E_i^* \geq 0 .
\end{aligned}$$

According to the principle of Coleman and Noll, discussed in Section I.4, and to the constitutive assumptions (7), the quantities

$$\dot{\theta}, \dot{E}_i^*, V_{i,j}, \dot{M}_{i\alpha}, \dot{\theta}_\alpha, F_{j\alpha,i}, M_{j\alpha,i}, E_{j,i}^*, \theta_{\alpha,i}$$

can be chosen arbitrarily and independent of any other term in the above inequality. Therefore, in order that the inequality (14) is satisfied for every admissible thermodynamic process, the coefficients of the quantities listed above must be zero.

To illustrate this procedure, let us consider as an example the coefficient of $\dot{\theta}$. According to the constitutive assumption (7), this coefficient is independent of $\dot{\theta}$. Since all terms occurring in (14) are independent of $\dot{\theta}$, but for the second one, that is linear in $\dot{\theta}$, it is evident that the inequality (14) is only to satisfy for every value of $\dot{\theta}$ by taking the coefficient of $\dot{\theta}$ equal to zero.

Analogous reasoning for the other coefficients, leads to the following results:

$$(15) \quad S = - \frac{\partial \Sigma}{\partial \theta} ,$$

$$(16) \quad P_i^* = - \frac{\partial \Sigma}{\partial E_i^*} ,$$

$$(17) \quad T_{ij} = \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} ,$$

$$(18) \quad \Pi_{ij} \frac{\partial X_\alpha}{\partial x_j} - \rho \frac{\partial \Sigma}{\partial M_{i\alpha}} = 0 ,$$

$$(19) \quad \frac{\partial \Sigma}{\partial \theta_\alpha} = 0 ,$$

$$(20) \quad \frac{\partial \sigma_i}{\partial F_{j\alpha}} = \frac{\partial \sigma_i}{\partial M_{j\alpha}} = \frac{\partial \sigma_i}{\partial E_j^*} = \frac{\partial \sigma_i}{\partial \theta_\alpha} = 0 .$$

Multiplication of (18) by $F_{j\alpha}$ gives

$$(21) \quad \Pi_{ij} = \rho \frac{\partial \Sigma}{\partial M_{i\alpha}} F_{j\alpha} .$$

By using (16), (17) and (21) and the invariance condition (10), the first term of (14) can be shown to be equal to zero. To this end, we first prove

$$\begin{aligned} (22) \quad & e_{ijk} M_j^* (\rho H_i^* - \rho \frac{\partial \Sigma}{\partial M_i^*} + \Pi_{i\ell, \ell}) = \\ & = e_{ijk} \left\{ \rho H_i^* M_j^* + \Pi_{i\ell, \ell} M_j^* + \rho \frac{\partial \Sigma}{\partial F_{[i\alpha}} F_{j]\alpha} + \right. \\ & \left. + \rho \frac{\partial \Sigma}{\partial M_{[i\alpha}} M_{j]\alpha} + \rho \frac{\partial \Sigma}{\partial E_{[i}^*} E_{j]}^* \right\} = \\ & = \rho e_{ijk} \left\{ -P_{[i}^* E_{j]}^* + H_{[i}^* M_{j]}^* + \frac{1}{\rho} T_{[ij]} + \frac{1}{\rho} (\Pi_{[i\ell} M_{j]}^*),_{\ell} \right\} = \\ & = -\frac{\rho}{\Gamma} M_k^* , \end{aligned}$$

according to II.(31).

The first term of (14) becomes then

$$(23) \quad -\rho G_k^* M_k^* = 0 ,$$

as a consequence of equation II.(32)¹.

Substitution of these results reduces the inequality (14) to

$$(24) \quad -\left(\frac{1}{\theta} Q_i + \theta \frac{\partial \sigma_i}{\partial \theta}\right)_{,i} - \theta \frac{\partial \sigma_i}{\partial M_j^*} M_{j,i}^* + J_i^* E_i^* \geq 0 .$$

Furthermore, it follows from (20) that \underline{g} can be at most a function of θ

and \underline{M}^*

$$(25) \quad \sigma_i = \sigma_i(\theta, \underline{M}^*) .$$

In order to prove that $\underline{\sigma}$ is equal to zero, let us consider a process in which $M_{j,i}^*$ is arbitrary, but

$$\theta_{,i} = E_i^* = 0 .$$

In that case, it turns out that the inequality (24) is only to satisfy by taking

$$(26) \quad \frac{\partial \sigma_i}{\partial M_{j,i}^*} = 0 .$$

Hence,

$$(27) \quad \sigma_i = \sigma_i(\theta) ,$$

and (24) further reduces to

$$(28) \quad - \left(\frac{1}{\theta} Q_i + \theta \frac{d\sigma_i}{d\theta} \right) \theta_{,i} + J_i^* E_i^* \geq 0 .$$

Let us denote the left-hand side of (28), for the case that

$$F_{i\alpha} = M_i^* = M_{i\alpha} = E_i^* = 0 ,$$

by \mathcal{D} , thus

$$(29) \quad \mathcal{D} = \mathcal{D}(\theta, \theta_\alpha) := - \left[\frac{Q_i(0,0,0,0,\theta,\theta_\alpha)}{\theta} + \theta \frac{d\sigma_i(\theta)}{d\theta} \right] \theta_{,i} \geq 0 .$$

As follows from (29), \mathcal{D} attains its minimum value $\mathcal{D} = 0$ for $\theta_{,i} = 0$ (then also $\theta_\alpha = 0$).

Hence, we must have

$$(30) \quad \left. \frac{\partial \mathcal{D}}{\partial \theta_{,i}} \right|_{\theta_{,i}=0} = 0 ,$$

which implies that

$$(31) \quad \theta \frac{d\sigma_i(\theta)}{d\theta} + \frac{Q_i(0,0,0,0,\theta,0)}{\theta} = 0 .$$

We note that (31) is valid for every admissible temperature distribution, hence also for a uniform one. On physical reasons, it is unlikely that in an undeformed body, without electromagnetic interactions and with a uniform temperature field, there is a flux of energy. Hence, the right-hand side of (31) must be equal to zero, by which we have proved that

$$(32) \quad \frac{d\sigma_i(\theta)}{d\theta} = 0 .$$

Consequently, $\underline{\sigma}$ is a constant. This constant may be taken equal to zero, because only the derivative of $\underline{\sigma}$ enters the entropy inequality. At this point, we have shown that, for the class of materials under consideration, the well-known relation for the entropy flux

$$(33) \quad \phi_i = -\frac{Q_i}{\theta} ,$$

is valid.

There now only two terms remain in the entropy inequality

$$(34) \quad -\frac{1}{\theta} Q_{i,\alpha} + J_i^* E_i^* \geq 0 .$$

It is not possible to satisfy this relation in a general nonlinear way. However, if we assume that the dependence of Q and J^* on the independent variables θ_α (or $\theta_{,i}$) and E_i^* is a linear one, the inequality (34) yields conditions for the coefficients in these linear expressions.

Therefore, we take

$$(35) \quad Q_i = -\kappa_{ij} \theta_{,j} + \beta_{ij}^{(Q)} E_j^* ,$$

and

$$(36) \quad J_i^* = \sigma_{ij} E_j^* + \beta_{ij}^{(J)} \frac{\theta_{,j}}{\theta} .$$

The inequality (34) is now satisfied if κ_{ij} and σ_{ij} are positive definite tensors, and if

$$(37) \quad \beta_{ij}^{(Q)} = \beta_{ji}^{(J)} =: \beta_{ij} .$$

We note that the latter relation is not necessary but only sufficient. The relation (37) can also be derived by use of the Onsager-relations (cf. [12], p. 216).

The coefficients κ_{ij} , σ_{ij} and β_{ij} are, in general, functions of the elements of S . The constitutive equations (35) and (36) are known as Fourier's law and Ohm's law, respectively. The coefficients κ_{ij} and σ_{ij} are named thermal and electrical conductivity, respectively. The coefficients β_{ij} are responsible for thermoelectric effects, as there are the Seebeck-effect, the Peltier-heat and the Thomson-heat (cf. [12], Ch. 12).

By use of the equations II.(22), II.(24), (3), (8), (16), (17) and (21), the energy balance II.(20) reduces to

$$(38) \quad \rho \dot{\theta} S + Q_{i,i} = \rho r + J_i^* E_i^* .$$

In principle, at this point the general nonlinear theory is completed. For convenience, let us survey the equations that we have found. We have

- i) *The electromagnetic equations, i.e.*
 seven, independent, Maxwell-equations (I.(34)),
 six relations, defining \underline{M} and \underline{P} (I.(47)-(48)).
- ii) *The balance equations, i.e.*
 one balance of mass (I.(12)),
 three balances of momentum (I.(14)),
 three balances of moment of momentum (I.(31)),
 one balance of energy (38).
- iii) *The constitutive equations, i.e.*
 one for the entropy (15),
 three for the polarization (16),
 nine for the stresses (17),
 nine for the couple-stresses (21),
 three for the heat flux (35),
 three for the electric current density (36).

Together these 49 equations constitute a complete system for the 49 unknown variables

$$x_i, \rho, E_i, D_i, P_i, H_i, B_i, M_i, J_i, Q, \theta, S, T_{ij}, \Pi_{ij}, Q_i.$$

The jump conditions belonging to the above equations are given by I.(35), II.(41), II.(42), II.(43) and II.(39).

In concluding, we make the following remarks:

- i) The constitutive equations obtained in this chapter, are not written in a form that is invariant under superposed rigid-body rotations. This will be done in the next section.
- ii) Before we can work out the constitutive equations any further, we need an explicit expression for the functional Σ . Such an expression will be given in Chapter VI.
- iii) The system of equations, summarized above, is highly nonlinear and hard to solve in their present form. Therefore, we shall linearize these equations in Chapter V.

III.4. Invariant form of the constitutive equations

From the relation (19), it follows that the functional Σ is independent of the gradient of the temperature $\theta_{,\alpha}$, so

$$(39) \quad \Sigma = \Sigma(F_{i\alpha}, M_i^*, M_{i\alpha}^*, E_i^*, \theta).$$

We note that Σ cannot be any function of the above variables, because Σ must be invariant under a rigid rotation of the body. The theorem of Cauchy (cf. Section I.4) states that if Σ is a functional which is invariant under rigid-body rotations, Σ must reduce at most to a function of the scalar products and the determinants of the independent variables.

Under a rigid-body rotation, the coordinates transform according to

$$(40) \quad x_i \Rightarrow Q_{ij} x_j \quad \text{and} \quad X_\alpha \Rightarrow X_\alpha,$$

where Q_{ij} is a rotation tensor, satisfying

$$(41) \quad Q_{ki} Q_{kj} = Q_{ik} Q_{jk} = \delta_{jk} \quad \text{and} \quad \det\{Q_{ij}\} = +1.$$

The above rotation transforms the independent variables according to

$$(42) \quad \begin{aligned} F_{i\alpha} &\Rightarrow Q_{ij} F_{j\alpha} , & M_i^* &\Rightarrow Q_{ij} M_j^* , \\ M_{i\alpha} &\Rightarrow Q_{ij} M_{j\alpha} , & E_i^* &\Rightarrow Q_{ij} E_j^* , \\ \theta &\Rightarrow \theta . \end{aligned}$$

Under the conditions mentioned above, the said theorem of Cauchy states that Σ may depend only on

$$\theta ,$$

the scalar products, such as

$$F_{i\alpha} F_{i\beta} , \quad F_{i\alpha} M_i^* , \quad M_i^* M_i^* , \quad \text{etc.},$$

and the determinants, such as

$$e_{ijk} F_{i\alpha} F_{j\beta} F_{k\gamma} , \quad e_{ijk} F_{i\alpha} F_{j\beta} M_k^* , \quad e_{ijk} M_i^* M_j^* E_k^* , \quad \text{etc.}$$

We note that it can be proved that only the quantities

$$(43) \quad \begin{aligned} C_{\alpha\beta} &:= F_{i\alpha} F_{i\beta} , & \Lambda_\alpha &:= F_{i\alpha} M_i^* , \\ A_{\alpha\beta} &:= F_{i\alpha} M_{i\beta} , & \Delta_\alpha &:= F_{i\alpha} E_i^* , \end{aligned}$$

and

$$\theta ,$$

need to be considered without any loss of generality, since all of the remaining quantities are expressible in terms of the set (43). For instance, we have

$$(44) \quad M_i^* E_i^* = \Lambda_\alpha (C^{-1})_{\alpha\beta} \Delta_\beta .$$

For the proof of the above assertion, confer [13], p. 1309.

For reasons, that we shall give right away, we prefer to use instead of the right Cauchy-Green tensor $C_{\alpha\beta}$ the deformation tensor $E_{\alpha\beta}$, defined by

$$(45) \quad E_{\alpha\beta} := \frac{1}{2}(C_{\alpha\beta} - \delta_{\alpha\beta}) = \frac{1}{2}(F_{i\alpha} F_{i\beta} - \delta_{\alpha\beta}) .$$

The advantage of the tensor $E_{\alpha\beta}$ lies in the fact that it goes to zero in case of absence of deformation.

If the exchange energy is invariant, i.e. if the relation

$$(46) \quad \frac{\partial \Sigma}{\partial M_{[i\alpha}} M_{j]\alpha} = 0 ,$$

holds, the tensor $A_{\alpha\beta}$ may be replaced by the symmetric tensor $G_{\alpha\beta}$, defined by

$$(47) \quad G_{\alpha\beta} := M_{i\alpha} M_{i\beta} = A_{\beta\gamma} (C^{-1})_{\gamma\delta} A_{\alpha\delta} .$$

The proof can be found in [13], pp. 1310-1311. In what follows, we suppose that (46) is valid.

The definitions (45) and (47) enable us to replace (39) by

$$(48) \quad \Sigma = \Sigma(E_{\alpha\beta}, \Lambda_{\alpha}, G_{\alpha\beta}, \Delta_{\alpha}, \theta) .$$

We note that by Σ according to (48), the invariance condition (10) is satisfied identically.

The constitutive equations, derived in Section III.3, can be expressed in invariant variables. This results in the following system

$$(49) \quad S = - \frac{\partial \Sigma}{\partial \theta} ,$$

$$(50) \quad P_i^* = - \frac{\partial \Sigma}{\partial \Delta_{\alpha}} F_{i\alpha} ,$$

$$(51) \quad T_{ij} = \rho \frac{\partial \Sigma}{\partial E_{\alpha\beta}} F_{i\alpha} F_{j\beta} + \rho \frac{\partial \Sigma}{\partial \Lambda_{\alpha}} F_{j\alpha} M_i^* + \rho \frac{\partial \Sigma}{\partial \Delta_{\alpha}} F_{j\alpha} E_i^* ,$$

$$(52) \quad \Pi_{ij} = 2\rho \frac{\partial \Sigma}{\partial G_{\alpha\beta}} M_{i,k}^* F_{j\alpha} F_{k\beta} .$$

In this way we have obtained a system of constitutive equations, that is invariant under superposed rigid-body motions.

At this point, we would like to recapitulate the conditions under which the constitutive equations are valid. These are

- i) The material is thermoelastic, magnetizable, polarizable and electrically and thermally conductive.

- ii) Exchange interaction is taken into account.
- iii) Elastic and magnetic dissipation are excluded.
- iv) The magnetization is saturated.
- v) The exchange energy is invariant under superposed rigid-body rotations.
- vi) The heat flux and the convective electric current are linear functions of the temperature gradient and the convective electric field.

III.5. Alternative definitions of the stresses

So far, we have used a stress tensor T_{ij} , that is given by the constitutive equation (17) and that has to satisfy the momentum equation II.(17) together with the jump conditions II.(42). In principle, this stress tensor is characterized by the energy balance II.(1), and, more specified, by the vector \underline{R} , occurring in this equation. By choosing an other expression for \underline{R} , we arrive at a stress tensor differing from T_{ij} . Hence, we notice that, although they have to meet some common restrictions (e.g. invariance), the stresses are not unique.

In the monograph by Brown ([11], Section 5.6) a number of frequently occurring stress formulations is given. These stress tensors are only then completely defined, if the system consisting of the constitutive equations, the balance of momentum and the jump conditions is known. In this section we shall present these systems for some alternative stress definitions.

To render the equations amenable, we shall restrict ourselves to the static theory of a nonconducting body in a vacuum. Moreover, we shall neglect exchange interaction. We assign the stresses used in the preceding part by $T_{ij}^{(1)}$. Under the restrictions mentioned above, the following equations for $T_{ij}^{(1)}$ hold (cf. (17), II.(17), II.(31) and II.(47))

$$T_{ij}^{(1)} = \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} ,$$

$$T_{ij,j}^{(1)} + \rho P_{,j} E_{j,i} + \rho M_{,j} H_{j,i} + \rho F_i^{(m)} = 0 ,$$

(53)

$$T_{[ij]}^{(1)} = \rho P_{[i} E_{j]} + \rho M_{[i} H_{j]} ,$$

$$T_{ij}^{(1)} n_j = 2\pi \{ (\rho M_j n_j)^2 + (\rho P_j n_j)^2 \} n_i , \quad \text{on } S .$$

The stresses defined by (53) are called the stresses according to the Maxwell-model I. In this formulation, the electromagnetic body force, occurring in the momentum equation, can be written as the gradient of the Maxwell stress defined by II.(19).

An alternative stress tensor, named Maxwell-model II, is defined by

$$(54) \quad T_{ij}^{(2)} := T_{ij}^{(1)} - 2\pi\rho^2 (M_k M_k + P_k P_k) \delta_{ij} .$$

It is easy to show that for these stresses the following equations are valid

$$(55) \quad T_{ij}^{(2)} = \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} - 2\pi\rho^2 \delta_{ij} (M_k M_k + P_k P_k) ,$$

$$T_{ij,j}^{(2)} + \rho P_j D_{j,i} + \rho M_j B_{j,i} + \rho F_i^{(m)} = 0 ,$$

$$T_{[ij]}^{(2)} = \rho P_{[i} D_{j]} + \rho M_{[i} B_{j]} ,$$

$$T_{ij}^{(2)} n_j = 2\pi\rho^2 (M_t^2 + P_t^2) n_i , \quad \text{on } S ,$$

where M_t and P_t are the tangential components of \underline{M} and \underline{P} on S . We note that the role played by \underline{E} and \underline{H} in the first definition is taken over by \underline{D} and \underline{B} in the formulae (55). In case of a dynamic problem, we have to substitute the convective fields \underline{M}^* , \underline{P}^* , etc., for \underline{M} , \underline{P} , etc., respectively, in order to preserve invariance.

A stress tensor, to which will be referred as the Amperian-current model, is defined by

$$(56) \quad T_{ij}^{(3)} := T_{ij}^{(1)} - \rho P_i D_j - \rho M_i B_j + \frac{1}{2} \delta_{ij} \rho M_k (B_k + \mathbb{E}_k) + \\ + \frac{1}{2} \delta_{ij} \rho P_k (D_k + E_k) .$$

For these stresses the following relations can be deduced

$$\begin{aligned}
T_{ij}^{(3)} &= \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} - \rho P_{i,j} D_j - \rho M_{i,j} B_j + \frac{1}{2} \delta_{ij} \rho M_k (B_k + H_k) + \\
&\quad + \frac{1}{2} \delta_{ij} \rho P_k (D_k + E_k) , \\
T_{ij,j}^{(3)} &+ (\rho P_i)_{,j} D_j - (\rho P_j)_{,i} D_j + (\rho M_i)_{,j} B_j - (\rho M_j)_{,i} B_j + \\
&\quad + \rho F_i^{(m)} = 0 ,
\end{aligned}$$

(57)

$$T_{[ij]}^{(3)} = 0 ,$$

$$\begin{aligned}
T_{ij}^{(3)} n_j &= -\frac{1}{4\pi} [B_i B_j - \frac{1}{2} \delta_{ij} B_k B_k] n_j - \frac{1}{4\pi} [D_i D_j - \frac{1}{2} \delta_{ij} D_k D_k] n_j = \\
&= \rho \bar{B}_j (M_j n_i - M_i n_j) + \rho \bar{D}_j (P_j n_i - P_i n_j) - 2\pi \rho^2 (M_i^2 + P_i^2) n_i ,
\end{aligned}$$

on S .

In conclusion, we mention a stress tensor that, at least for the magnetic part, can be based on a magnetic pole model. This tensor is defined by

$$(58) \quad T_{ij}^{(4)} := T_{ij}^{(1)} + \rho E_i P_j + \rho H_i M_j ,$$

from which it can be inferred that

$$\begin{aligned}
T_{ij}^{(4)} &= \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} + \rho E_i P_j + \rho H_i M_j , \\
T_{ij,j}^{(4)} &- (\rho P_j)_{,i} E_i - (\rho M_j)_{,i} H_i + \rho F_i^{(m)} = 0 , \\
(59) \quad T_{[ij]}^{(4)} &= 0 , \\
T_{ij}^{(4)} n_j &= \rho \bar{E}_i P_j n_j + \rho \bar{H}_i M_j n_j + 2\pi \{ (\rho M_j n_j)^2 + (\rho P_j n_j)^2 \} n_i , \text{ on } S .
\end{aligned}$$

Brown, [11], introduces also a stress related to a magnetic dipole moment. However, because this stress is not a tensor, we leave it out of consideration.

For physical backgrounds of the several stresses defined in this section, we refer to [11], Sections 5.1 and 5.6.

III.6. Literature survey

The first consistent nonlinear treatment of an electrically polarized elastic continuum in interaction with an electrostatic field appeared in a paper by Toupin [5] in 1956. Toupin has extended the aforesaid theory by including dynamical problems in a paper [14] of 1963. Toupin started with the postulation of a set of global balance equations from which local balance equations, constitutive equations and jump conditions are derived, holding for perfectly elastic dielectrics. The results of these papers can be made to fit those of our theory by taking in the latter the convective magnetization \underline{M}^* equal to zero. However, we must note that in [14] Toupin has used a stress tensor that differs from the tensor T_{ij} , used throughout this chapter, by the amount

$$\rho(E_i + \frac{1}{c} e_{ikl} V_k B_l) P_j .$$

One among several other workers, who investigated the static behaviour of elastic media in interaction with electromagnetic fields, is Eringen ([15], [16]). At certain points, Eringen is slightly at variance with the present theory. For instance, the expression for the electromagnetic body force derived in [16] differs a term

$$\frac{4\pi\rho}{c} e_{ijk} J_j M_k$$

from the formula according to II.(16) in case of a static situation. In the paper [17], Tiersten treated the quasi-static behaviour of electrically polarized thermoelastic bodies. The resulting equations are derived by means of an application of the laws of continuum physics to a macroscopic model consisting of an electronic charge continuum coupled to a lattice continuum. The aforementioned author has also written two articles on quasi-static magnetoelastic interactions, [13] and [18]. In [13] a similar method as in [17] is used, but now the lattice continuum is coupled to an electronic spin continuum, while in [18] a variational method is used. In [13], also the linear equations for a small dynamic field superimposed on a large static field are obtained from the general system of nonlinear equations. Quite recently,

Tiersten and his co-writer Tsai, [19], derived the differential equations and boundary conditions describing the behaviour of a finitely deformable, polarizable, magnetizable and heat conducting insulator in interaction with an electromagnetic field. In this paper, a method is used that is an extended combination of those of [13] and [18]. Both ionic and electronic polarization are included in this treatment and the gradient of the polarization is included in the set of constitutive variables. The results of all these papers, as far as they refer to the problems considered here, are in agreement with those of this thesis. However, in [19] a stress tensor is used that differs from the one used in [13] and [18] (this can be seen from equations (6.17) with (5.14) of [19]). The paper [19] contains a detailed literature survey in its Introduction. We note that, to our opinion, the approach in the aforementioned papers is rather artificial and that the fundamental derivations are hard to understand.

Quasi-static magnetoelastic effects are also investigated by Alblas [7], Brown [11], Kaliski [20] and by Akhiezer, Baryakhtar and Peletninskii [21]. In the articles [7] and [20] magnetic dissipation is taken into account. Brown's monograph [11] is one of the fundamental works on magnetoelastic interactions, although it deals mainly with static magnetic fields. Two different methods are described in [11]. In the first method field and stress concepts play a dominant role, while in the second energy considerations were the basis of the treatment, where the key principle was the minimization of a thermodynamic potential. Underlying the derivation in [20], are a spin equation and an energy balance. The book [21] deals mainly with spin waves, but one chapter is devoted to magnetoelastic interactions, whereby an infinitesimal magnetostrictive approximation is used.

The ultimate equations of these papers all correspond with each other and also with the results of the present theory, if in the latter all terms proportional to c^{-1} are neglected.

In [9], Alblas has extended the theory of [7], by including polarization and dynamical effects. The same class of problems is reported in the monograph by Parkus [10]. The works of these two authors are based on an energy balance, that is of a similar kind as the one used by us.

This energy law was elaborated to yield balance equations with the aid of a method of Green and Rivlin ([2]), that is also described in Section II.1 of this thesis. Further, they also employed a Clausius-Duhem inequality for the derivation of the constitutive equations. These papers, just as the present thesis, do refer to "slowly" moving media, i.e. relativistic effects are neglected. This is achieved by omitting all terms that are proportional to c^{-2} . As a result, none of the aforementioned works is Lorentz invariant. In [9], the Chu-formulation of electrodynamics is used and applications to magnetoelastic wave propagation in the infinite space and in the half-space are given. When comparing the results of [9] and [10] with each other and with those of our work, complete agreement is found.

Parkus has also derived the magnetoelastic equations for a much more restrictive class, by means of a variational method, in [22], Ch. IV, by following a method developed by Brown. A variational principle was also used by Vlasov and Ishmukhametov [23]. They employed the principle of Hamilton, in order to obtain balance laws of momentum and of moment of momentum and constitutive equations for the electric field intensity and for the effective magnetic field, holding for an elastic medium that is both polarizable and magnetizable. In [23], an approximation, based on infinitesimal deformations is applied. We note that the balance of momentum found in [23] and the one according to equation II.(17) are slightly distinct. We will return to this subject at the end of the next chapter (viz. Section IV.8).

In the book of De Groot and Mazur [24], dealing with nonequilibrium thermodynamics, one chapter is devoted to irreversible processes in polarized media. A dynamic, but nonrelativistic, theory is presented, based on a set of electrodynamic balance laws, in which in a to some extent arbitrary way, but consistent with pertinent invariance conditions, mechanical and thermal contributions are introduced. The results seem to be equivalent to ours, but a complete comparison is difficult, as in [24] only constitutive equations for fluids are deduced.

To conclude this literature survey, we mention the book of Penfield and Haus [25]. By means of both the principle of virtual work and the

principle of Hamilton, these authors derive an expression for the electromagnetic body force in an elastic continuum that is polarizable and magnetizable. The equations of [25] are written in the Chu-formulation, whereas in the present work the Minkowski-formulation is used. After reformulating the results of [25] in the Minkowski-notation, it turns out that the expressions for the electromagnetic body force according to [25] and to equation II.(16) of our work, completely correspond with each other. For the details of the calculations underlying this statement, confer Appendix III.

IV. HAMILTON'S PRINCIPLE

IV.1. Introduction

In the chapters II and III, equations of balance, constitutive equations and jump conditions for the electromagnetic interactions with thermo-elastic media were derived by a method that was based on a balance of energy. In the present chapter the same system of equations will be derived in an other way, namely by means of Hamilton's principle. In this chapter, we shall restrict ourselves to the case of a medium that is neither electrically nor thermally conductive. This means that we do not consider thermal effects and we take equal to zero the charge Q and the current \underline{J} . Furthermore, we shall consider only material discontinuity surfaces.

Hamilton's principle is a variational principle. The so called action integral is required to be unchanged when certain variables are altered. The action integral is an integral over a time interval $[t_1, t_2]$ and over a material volume V of the Lagrangian \mathcal{L} . Denoting a variation by δ , Hamilton's principle can be formulated as

$$(1) \quad \delta \int_{t_1}^{t_2} \int_V \mathcal{L} \, dV dt = 0 .$$

The variations in (1) are zero at the times t_1 and t_2 and on the boundary of V .

The advantage of Hamilton's principle lies in the fact that the technique is straight away and very unsusceptible to errors once the Lagrangian is written down. However, the choice of an explicit expression for the Lagrangian will always remain open for discussion. In the next section we shall postulate a form for the Lagrangian.

The theory given in this chapter can serve two different purposes. On the one hand, we can use Hamilton's principle in order to corroborate the results obtained in the chapters II and III. On the other hand, we may state that, if Hamilton's principle yields equations that are identical to those of the chapters II and III, the used expression for the Lagrangian is the correct one for the problem under consideration.

Hence, in the latter case Hamilton's principle serves as a confirmation of the correctness of the form of the chosen Lagrangian. From the thus found expression, one can then easily derive the Lagrangians for every more restrictive class of problems. Furthermore, this expression can also serve as a basis for an extension to more general media, e.g. dissipative or conductive media.

A complete description of the features of Hamilton's principle is beyond the scope of this thesis. For an excellent treatment of this principle, we refer to [25], Chapter 6, and to the references mentioned there. In the references [18], [22] and [23], also variational principles are employed for the deduction of the equations governing the behaviour of elastic media in interaction with electromagnetic fields.

IV.2. Lagrangian

The basic step in the application of Hamilton's principle is formed by the choice of the Lagrangian. Here, we postulate the following expression for the Lagrangian \mathcal{L} , for a moving, polarizable, magnetizable, elastic medium

$$(2) \quad \mathcal{L} = \frac{1}{8\pi} (E_i^* E_i^* - B_i^* B_i^*) + \rho E_i^* P_i^* + \rho B_i^* M_i^* + \\ + \rho K - \rho U - 2\pi\rho^2 M_i^* M_i^* + \frac{1}{2}\rho V_i V_i .$$

In the following sections we shall show that, by using this expression for \mathcal{L} , Hamilton's principle will yield equations identical to those of the preceding two chapters.

We note that the form of the Lagrangian according to (2) corresponds with the expression used by Vlasov and Ishmukhametov in [23].

By use of the relations I.(36) and I.(37), the Lagrangian (2) can also be written as

$$(3) \quad \mathcal{L} = \frac{1}{8\pi} (\mathbf{E}_i^* \mathbf{E}_i^* - \mathbf{H}_i^* \mathbf{H}_i^*) + \rho \mathbf{E}_i^* \mathbf{P}_i^* + \rho \mathbf{K} - \rho U + \frac{1}{2} \rho \mathbf{V}_i \mathbf{V}_i .$$

In this form, the Lagrangian is identical to the one used by Penfield and Haus ([25], p. 174, eq. (6.51)) if the latter is rewritten into Minkowski-notation.

In this chapter, we shall employ the expression (2). We note that the kinetic energy of the magnetization vector \mathbf{K} is a function of $\underline{\mathbf{M}}^*$ and $\dot{\underline{\mathbf{M}}}^*$, i.e.

$$(4) \quad \mathbf{K} = \mathbf{K}(\underline{\mathbf{M}}^*, \dot{\underline{\mathbf{M}}}^*)$$

while, since thermal effects are left out of consideration, for the internal energy the relationship holds

$$(5) \quad U = U(F_{i\alpha}, M_i^*, M_{i\alpha}, P_i^*) .$$

Moreover, we note that all variations occurring in the above formulae will be taken with respect to a system that moves together with the medium.

Note. We remark that it is possible to include also the charge Q and the electric current $\underline{\mathbf{J}}$, by adding to $\delta \mathcal{L}$ the amounts (cf. [26], eq. (IX.147))

$$(6) \quad \frac{1}{c} \mathbf{J}_i \delta A_i - Q \delta \varphi ,$$

where $\underline{\mathbf{A}}$ and φ are the electromagnetic potentials that can be introduced by

$$(7) \quad \mathbf{E}_i = -\varphi_{,i} - \frac{1}{c} \frac{\partial A_i}{\partial t} , \quad \mathbf{B}_i = e_{ijk} A_{k,j} ,$$

(cf. [4], Section 1.9).

IV.3. Restrictions on the variations

As the region V on which the action integral is defined constitutes a material volume and since we take δ to be a variation with respect to a system that moves together with the medium, we have (compare the material derivative of an integral over a material volume)

$$(8) \quad \delta \int_{t_1}^{t_2} \int_V \mathcal{L} \, dV dt = \int_{t_1}^{t_2} \int_V \{ \delta \mathcal{L} + \mathcal{L} (\delta U_i)_{,i} \} dV dt = 0 ,$$

where \underline{U} is the displacement vector.

Furthermore, the variations must be restricted by the following requirements

i) The total mass of V is conserved.

ii) The two Maxwell-equations

$$(9) \quad \frac{1}{c} \frac{\partial B_i}{\partial t} = -e_{ijk} E_{k,j}, \quad B_{i,i} = 0 ,$$

and the jump conditions

$$(10) \quad \llbracket e_{ijk} E_j n_k + \frac{1}{c} B_i V_j n_j \rrbracket = 0, \quad \llbracket B_i \rrbracket n_i = 0 \text{ on } \Sigma(t)$$

where $\Sigma(t)$ is a material discontinuity surface, must be satisfied a priori. The other two Maxwell-equations plus jump conditions will be obtained as a result of Hamilton's principle.

iii) The magnetization is saturated.

The conservation of mass gives

$$(11) \quad \delta \int_{t_1}^{t_2} \int_V \rho \, dV dt = \int_{t_1}^{t_2} \int_V \{ \delta \rho + \rho (\delta U_i)_{,i} \} dV dt = 0 ,$$

from which it follows that

$$(12) \quad \delta \rho = -\rho (\delta U_i)_{,i} .$$

By putting

$$(13) \quad \mathcal{L} = \rho L$$

and by using (12), the relation (8) can be written as

$$(14) \quad \int_{t_1}^{t_2} \int_V \rho \delta L \, dV dt = 0 .$$

The interchange of the order of variation and derivations is governed by the following relations

$$\delta \left(\frac{\partial}{\partial \underline{x}_\alpha} \right) = \frac{\partial}{\partial \underline{x}_\alpha} \delta, \quad \delta \left(\frac{d}{dt} \right) = \frac{d}{dt} \delta,$$

$$(15) \quad \delta \left(\frac{\partial}{\partial \underline{x}_i} \right) = \frac{\partial}{\partial \underline{x}_i} \delta - (\delta U_j)_{,i} \frac{\partial}{\partial \underline{x}_j},$$

$$\delta \left(\frac{\partial}{\partial t} \right) = \frac{\partial}{\partial t} \delta - \frac{\partial}{\partial t} (\delta U_j) \frac{\partial}{\partial \underline{x}_j}.$$

The Maxwell-equations (9) can be satisfied by introducing the vector potential \underline{A} and the scalar potential φ by (cf. [4], Section 1.9)

$$(16) \quad E_i = -\varphi_{,i} - \frac{1}{c} \frac{\partial A_i}{\partial t}, \quad B_i = e_{ijk} A_{k,j}.$$

The jump conditions (10) are then satisfied if we assume \underline{A} and φ to be everywhere continuous, i.e.

$$(17) \quad [\varphi] = [A_i] = 0 \text{ on } \Sigma(t).$$

This can be shown by substituting (16) into (10), giving

$$(18) \quad -e_{ijk} [\varphi_{,j} n_k + \frac{1}{c} \frac{\partial A_j}{\partial t} n_k + \frac{1}{c} A_{j,k} v_l n_l] = 0,$$

$$e_{ijk} [A_{k,j}] n_i = 0, \quad \text{on } \Sigma(t).$$

When φ and \underline{A} are continuous on $\Sigma(t)$, then the tangential derivatives are too. Hence, (18)² is satisfied identically, while (18)¹ becomes

$$(19) \quad -\frac{1}{c} e_{ijk} \left[\frac{\partial A_j}{\partial t} n_k + A_{j,k} v_l n_l \right] = 0 \text{ on } \Sigma(t).$$

That this relation holds for continuous \underline{A} can be proved as follows: Since the tangential derivative of \underline{A} is continuous we have

$$(20) \quad [A_{i,j}] = \left[\frac{\partial A_i}{\partial n} \right] n_j \text{ on } \Sigma(t)$$

where $(\partial A_i / \partial n)$ is the normal derivative of A_i on $\Sigma(t)$.

With this (19) becomes

$$\begin{aligned}
(21) \quad & -\frac{1}{c} e_{ijk} \left[\frac{dA_j}{dt} n_k - A_{j,l} V_l n_k + A_{j,k} V_l n_l \right] = \\
& = -\frac{1}{c} e_{ijk} \left[\frac{dA_j}{dt} n_k - \frac{\partial A_j}{\partial n^l} n_l V_l n_k + \frac{\partial A_j}{\partial n^k} n_k V_l n_l \right] = \\
& = -\frac{1}{c} e_{ijk} \left[\frac{dA_j}{dt} \right] n_k = -\frac{1}{c} e_{ijk} \frac{d}{dt} [A_j] n_k = 0,
\end{aligned}$$

according to (17)².

We note the \underline{A} is not uniquely defined by (16)², for we may always add to \underline{A} the gradient of an arbitrary scalar function. Therefore, we impose on \underline{A} and φ the supplementary condition (cf. [4], p. 24, eq. (12))

$$(22) \quad A_{i,i} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0.$$

From (16), we obtain for the variations of \underline{E} and \underline{B} , with the aid of (15),

$$\begin{aligned}
(23) \quad \delta E_i &= -(\delta \varphi)_{,i} + \varphi_{,j} (\delta U_j)_{,i} - \frac{1}{c} \frac{\partial}{\partial t} (\delta A_i) + \frac{1}{c} A_{i,j} \frac{\partial}{\partial t} (\delta U_j), \\
\delta B_i &= e_{ijk} (\delta A_k)_{,j} - e_{ijk} A_{k,l} (\delta U_l)_{,j}.
\end{aligned}$$

The variation of \underline{M}^* is restricted by the condition that the magnetization is saturated. For

$$(24) \quad M_i^* M_i^* = M_s^2,$$

gives

$$(25) \quad M_i^* \delta M_i^* = 0,$$

which relation is satisfied, if

$$(26) \quad \delta M_i^* = e_{ijk} M_j^* \delta \mu_k,$$

for arbitrary $\delta \mu_k$.

The relations (12), (23) and (25) assure us that the constraints stated in the beginning of this section are not violated.

We now shall elaborate the relation (14), by taking into account (12), (23) and (25), for arbitrary variations of φ , \underline{A} , \underline{P}^* , \underline{u} (or \underline{M}^*) and \underline{U} . In the next sections, these variations will be studied successively.

IV.4. Variation of the scalar potential φ and the vector potential \underline{A}

In this section, we shall take only the variations $\delta\varphi$ and δA_i unequal to zero.

For convenience, we write the electromagnetic fields occurring in the first four terms of (2) in their nonconvective form, i.e.

$$(27) \quad \mathcal{L} = \frac{1}{8\pi} (E_i E_i - B_i B_i) + \rho E_i P_i + \rho B_i M_i + \\ + \rho K - \rho U - 2\pi\rho^2 M_i^* M_i^* + \frac{1}{2}\rho V_i V_i .$$

We now substitute (16) into (27) and we wish to apply Hamilton's principle for the thus obtained Lagrangian and for variations with respect to φ and A_i . However, as we have imposed on φ and \underline{A} the restriction (22), the Lagrangian (27) must be supplemented by a term

$$\lambda (A_{i,i} + \frac{1}{c} \frac{\partial \varphi}{\partial t}) ,$$

where λ is a Lagrange multiplier.

By use of (23), we then obtain from (1) and (27)

$$(28) \quad \delta \int_{t_1}^{t_2} \int_V \mathcal{L} dV dt = \frac{1}{4\pi} \int_{t_1}^{t_2} \int_V [(-\varphi_{,i} - \frac{1}{c} \frac{\partial A_i}{\partial t} + 4\pi\rho P_i) (-\delta\varphi_{,i} - \frac{1}{c} \frac{\partial}{\partial t} \delta A_i) + \\ - (e_{i\ell m} A_{m,\ell} - 4\pi\rho M_i) e_{ijk} \delta A_{k,j} + \lambda (\delta A_{i,i} + \frac{1}{c} \frac{\partial}{\partial t} \delta\varphi)] dV dt = \\ = \frac{1}{4\pi} \int_{t_1}^{t_2} \int_V [\{-\varphi_{,ii} - \frac{1}{c} \frac{\partial A_{i,i}}{\partial t} + (4\pi\rho P_i)_{,i} - \frac{1}{c} \frac{\partial \lambda}{\partial t}\} \delta\varphi + \\ + \{-\frac{1}{c} \frac{\partial \varphi_{,i}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2} + \frac{1}{c} \frac{\partial}{\partial t} (4\pi\rho P_i) - A_{j,ji} + A_{i,jj} +$$

$$\begin{aligned}
& + e_{ijk} (4\pi\rho M_k)_{,j} - \lambda_{,i} \delta A_i \rfloor dV dt + \\
& + \frac{1}{4\pi} \int_{t_1}^{t_2} \int_{\Sigma(t)} \left[\left[\varphi_{,i} + \frac{1}{c} \frac{\partial A_i}{\partial t} - 4\pi\rho P_i \right] n_i - \frac{1}{c} \llbracket \lambda \rrbracket V_{,j} n_j \right] \delta\varphi + \\
& + \left\{ -\frac{1}{c} \llbracket \varphi_{,i} + \frac{1}{c} \frac{\partial A_i}{\partial t} - 4\pi\rho P_i \rrbracket V_{,j} n_j + \right. \\
& \left. + \llbracket A_{,j,i} - A_{i,j} - 4\pi\rho e_{ijk} M_k \rrbracket n_j + \llbracket \lambda \rrbracket n_i \right\} \delta A_i \rfloor dS dt = 0 ,
\end{aligned}$$

where $\Sigma(t)$ is a material discontinuity surface that intersects V . Since (28) must hold for arbitrary $\delta\varphi$ and δA_i , we find from it, by putting equal to zero the coefficients of $\delta\varphi$ and δA_i , the equations

$$\begin{aligned}
& \varphi_{,ii} + \frac{1}{c} \frac{\partial A_{i,i}}{\partial t} - (4\pi\rho P_i)_{,i} = -\frac{1}{c} \frac{\partial \lambda}{\partial t} , \\
(29) \quad & \frac{1}{c} \frac{\partial \varphi_{,i}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 A_i}{\partial t^2} + A_{,j,ji} - A_{i,jj} - \frac{1}{c} \frac{\partial}{\partial t} (4\pi\rho P_i) + \\
& + e_{ijk} (4\pi\rho M_k)_{,j} = -\lambda_{,i} ,
\end{aligned}$$

and the jump conditions

$$\begin{aligned}
& \llbracket \varphi_{,i} + \frac{1}{c} \frac{\partial A_i}{\partial t} - 4\pi\rho P_i \rrbracket n_i = \frac{1}{c} \llbracket \lambda \rrbracket V_{,j} n_j , \\
(30) \quad & \frac{1}{c} \llbracket \varphi_{,i} + \frac{1}{c} \frac{\partial A_i}{\partial t} - 4\pi\rho P_i \rrbracket V_{,j} n_j + \\
& - \llbracket A_{i,j} - A_{j,i} + 4\pi\rho e_{ijk} M_k \rrbracket n_j = \llbracket \lambda \rrbracket n_i \text{ on } \Sigma(t) .
\end{aligned}$$

From the two relations (29) we can deduce that

$$(31) \quad \lambda_{,ii} - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} = 0 ,$$

while from (30) we find, by multiplication of the second condition by n_i ,

$$(32) \quad \llbracket \lambda \rrbracket = 0 \text{ on } \Sigma(t) .$$

We note that there are no begin conditions given for λ . If we assume that at a time t_0

$$(33) \quad \lambda = \frac{\partial \lambda}{\partial t} = 0 \text{ for } t = t_0 ,$$

we find as the solution of (31)

$$(34) \quad \lambda = 0 .$$

The precise meaning of the conditions (33) will be specified lateron.

We introduce the fields \underline{D} and \underline{H} by

$$(35) \quad \begin{aligned} D_{i,i} &= -\varphi_{,i} - \frac{1}{c} \frac{\partial A_{,i}}{\partial t} + 4\pi\rho P_i , \\ H_i &= e_{ijk} A_{k,j} - 4\pi\rho M_i . \end{aligned}$$

Assuming (32) to hold, we then find from (29) and (30) with (35)

$$(36) \quad \begin{aligned} D_{i,i} &= 0 , \\ \frac{1}{c} \frac{\partial D_{,i}}{\partial t} - e_{ijk} H_{k,j} &= 0 , \end{aligned}$$

and the discontinuity conditions

$$(37) \quad \begin{aligned} \llbracket D_{,i} \rrbracket n_i &= 0 , \\ \llbracket e_{ijk} H_{j,k} - \frac{1}{c} D_{,i} V_{,j} n_j \rrbracket &= 0 \text{ on } \Sigma(t) . \end{aligned}$$

Hence, by variation of the action integral with respect to φ and \underline{A} , with constant \underline{P}^* , \underline{M}^* and \underline{V} , we have derived the lacking two Maxwell-equations together with their discontinuity conditions.

Note. We recall that if λ was not taken equal to zero, the right-hand sides of the two relations (36) would be $\frac{1}{c} \frac{\partial \lambda}{\partial t}$ and $\lambda_{,i}$, respectively. This means that the conditions (33) express the fact that there exists a time t_0 at which the Maxwell-equations (36) are satisfied. Hence, we may conclude from the foregoing that, if the Maxwell-equations (36) are satisfied at one moment, they hold at every time t .

IV.5. Variation of the polarization P^*

Let the only variation unequal to zero be δP_i^* , i.e. let us hold φ , \underline{A} , \underline{M}^* and \underline{U} fixed. According to (4) and (5), we then have

$$(38) \quad \delta U = \frac{\partial U}{\partial P_i^*} \delta P_i^* \text{ and } \delta K = 0 ,$$

respectively.

In this case, (14) with (2) and (13) becomes

$$(39) \quad \int_{t_1}^{t_2} \int_V \{ \rho E_i^* \delta P_i^* - \rho \frac{\partial U}{\partial P_i^*} \delta P_i^* \} dV dt = 0 ,$$

which gives the constitutive equation

$$(40) \quad E_i^* = \frac{\partial U}{\partial P_i^*} .$$

By means of the Legendre transformation

$$(41) \quad \Sigma = U - P_i^* E_i^* ,$$

where

$$(42) \quad \Sigma = \Sigma(P_{i\alpha}, M_i^*, E_i^*, M_{i\alpha}) ,$$

equation (40) passes into

$$(43) \quad P_i^* = - \frac{\partial \Sigma}{\partial E_i^*} .$$

We note that this relation is identical to the constitutive equation III.(16).

IV.6. Variation of the magnetization M^*

When varying \underline{M}^* , with constant φ , \underline{A} , \underline{P}^* and \underline{U} , we find from (4)

$$(44) \quad \delta K = \frac{\partial K}{\partial M_i^*} \delta M_i^* + \frac{\partial K}{\partial \dot{M}_i^*} \delta \dot{M}_i^* = \left[\frac{\partial K}{\partial M_i^*} - \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{M}_i^*} \right) \right] \delta M_i^* + \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{M}_i^*} \delta \dot{M}_i^* \right) .$$

We define the vector \underline{G}^* by

$$(45) \quad G_i^* := \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{M}_i^*} \right) - \frac{\partial K}{\partial M_i^*} + \lambda M_i^* ,$$

where the coefficient λ will be chosen in such a way that

$$(46) \quad G_i^* M_i^* = 0 .$$

Relation (46) is satisfied if

$$(47) \quad \lambda = - \frac{1}{M_g^*} M_i^* \left[\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{M}_i^*} \right) - \frac{\partial K}{\partial M_i^*} \right] .$$

With the definition (45) and with allowance for the restriction (25), the relation (44) can be rewritten as

$$(48) \quad \delta K = -G_i^* \delta M_i^* + \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{M}_i^*} \delta M_i^* \right) .$$

For the variation of the internal energy U with respect to \underline{M}^* , we have according to (5)

$$(49) \quad \delta U = \frac{\partial U}{\partial M_i^*} \delta M_i^* + \frac{\partial U}{\partial M_{i\alpha}^*} \delta M_{i\alpha}^* = \frac{\partial \Sigma}{\partial M_i^*} \delta M_i^* + \frac{\partial \Sigma}{\partial M_{i\alpha}^*} (\delta M_{i\alpha}^*)_{,\alpha} ,$$

with Σ defined by (42).

With the foregoing results, relation (14) can be worked out for variations with respect to \underline{M}^* . Replacing $\delta \underline{M}^*$ by the expression (26), we then arrive, after some operations analogous to those of the preceding sections, at

$$(50) \quad e_{ijk} G_j^* M_k^* = e_{ijk} M_k^* [H_j^* - \frac{\partial \Sigma}{\partial M_j^*} + \frac{1}{\rho} (\rho \frac{\partial \Sigma}{\partial M_{j\alpha}^*} F_{\ell\alpha})_{,\ell}] ,$$

and

$$(51) \quad \llbracket \rho e_{ijk} M_j^* \frac{\partial \Sigma}{\partial M_{k\alpha}^*} F_{\ell\alpha} \rrbracket n_\ell = 0 \text{ on } \Sigma(t) .$$

It is easy to show that the discontinuity condition (51) is equivalent to II.(45), by substituting into the latter the constitutive equation III.(21).

We eliminate $(\partial\Sigma/\partial M_j^*)$ from (50) by means of the invariance condition III.(10). By this procedure the relation (50) transforms into

$$(52) \quad M_{[i}^* G_{j]}^* = - \frac{\partial\Sigma}{\partial E_{[i}^*} E_{j]}^* + M_{[i}^* H_{j]}^* - \frac{\partial\Sigma}{\partial F_{[i\alpha}} F_{j]\alpha} + \\ + \frac{1}{\rho} (\rho M_{[i}^* \frac{\partial\Sigma}{\partial M_{j]\alpha}} F_{k\alpha})_{,k} .$$

By use of the constitutive equations III.(16), III.(17) and III.(21), it can be proved that (52) corresponds to the angular momentum equation II.(33), provided that the vector \underline{G}^* represents the effective magnetic field, as introduced by II.(32). In order to be able to verify this last assertion, we need an explicit expression for the kinetic energy K. According to Brown [27], p. 42, eq. (3-52), the energy K is, in case of a magnetically saturated medium, equal to

$$(53) \quad K = \frac{M}{\Gamma} \frac{\gamma}{(1-\gamma)} (\alpha\dot{\beta} - \beta\dot{\alpha}) ,$$

where α , β and γ are the components with respect to some Cartesian coordinate system, of the unit vector $\underline{\theta}$ defined by

$$(54) \quad \underline{\theta} = \frac{1}{M_s} \underline{M}^* .$$

The expression (53) is equivalent to the one used by Vlasov and Ishmukhametov [23], eq. (23).

With (45), (47) and (53) we obtain

$$(55) \quad \underline{G}^* = \frac{1}{\Gamma} \dot{\underline{\theta}} \times \underline{\theta} ,$$

which after multiplication by M^* , leads to

$$(56) \quad e_{ijk} M_j^* G_k^* = \frac{1}{\Gamma} \dot{M}_i^* .$$

It is evident from the equations (46) and (56) that the vector \underline{G}^* defined by (45) is indeed identical to the effective magnetic field according to II.(32).

IV.7. Variation of the displacement U

Finally, let us consider the variations δU_i . In this case, we obtain from (4) and (5)

$$(57) \quad \delta K = 0 ,$$

$$\delta U = \frac{\partial U}{\partial F_{i\alpha}} \delta F_{i\alpha} = \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} (\delta U_i)_{,j} ,$$

where (41) is used.

With (2), (12), (13) and (57), the relation (14) becomes, when φ , \underline{A} , \underline{P}^* and $\underline{\mu}$ are held fixed,

$$(58) \quad \int_{t_1}^{t_2} \int_V \left\{ \frac{1}{8\pi} (E_j^* E_j^* - B_j^* B_j^*) (\delta U_i)_{,i} + \frac{1}{4\pi} (E_j^* + 4\pi \rho P_j^*) \delta E_j^* + \right.$$

$$\left. - \frac{1}{4\pi} (B_j^* - 4\pi \rho M_j^*) \delta B_j^* - \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} (\delta U_i)_{,j} + \rho V_i \frac{d}{dt} \delta U_i + \right.$$

$$\left. + 2\pi \rho^2 M_j^* M_j^* (\delta U_i)_{,i} \right\} dV dt = 0 .$$

By means of the transformation rules I.(38) and the relations (35), this equation can be rewritten as

$$(59) \quad \int_{t_1}^{t_2} \int_V \left\{ \frac{1}{8\pi} (E_j E_j - B_j B_j) (\delta U_i)_{,i} + \frac{1}{4\pi} D_j \delta E_j - \frac{1}{4\pi} H_j \delta B_j + \right.$$

$$\left. - \frac{\rho}{c} e_{ijk} (P_j B_k + E_j M_k) \frac{d}{dt} \delta U_i - \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} (\delta U_i)_{,j} + \right.$$

$$\left. + \rho V_i \frac{d}{dt} \delta U_i + 2\pi \rho^2 M_j M_j (\delta U_i)_{,i} + \frac{4\pi \rho^2}{c} e_{jkl} V_j P_k M_l (\delta U_i)_{,i} \right\} dV dt = 0 .$$

After substitution of (23), with $\delta \varphi = \delta A_i = 0$, into (59) this relation can be elaborated, by partial integration and with the aid of the well-known electromagnetic equations into the form

$$\begin{aligned}
(60) \quad & \int_{t_1}^{t_2} \int_V \{ [-\rho \dot{V}_i + (\rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha})_{,j} + \rho P_j E_{j,i} + \rho M_j H_{j,i} + \\
& + \frac{\rho}{c} \frac{d}{dt} [e_{ijk} (P_j B_k + E_j M_k)] - (\frac{4\pi\rho^2}{c} e_{jkl} P_k M_l V_j)_{,i}] \delta U_i dV dt = \\
& + \int_{t_1}^{t_2} \int_{\Sigma(t)} \{ [-\rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} - \frac{1}{4\pi} (E_i D_j + H_i B_j - \frac{1}{2} \delta_{ij} (E_k E_k + H_k H_k)) \} + \\
& - \frac{1}{4\pi c} e_{ikl} D_k B_l V_j + \delta_{ij} \frac{4\pi\rho^2}{c} e_{klm} V_k P_l M_m - \frac{1}{4\pi c} D_j \frac{\partial A_i}{\partial t} + \\
& + \frac{1}{4\pi} e_{jkl} H_l A_{i,k} - \frac{1}{4\pi c} V_j D_k A_{i,k}] \delta U_i n_j dS dt = 0 .
\end{aligned}$$

From this relation, the following momentum equation and boundary condition can be found

$$(61) \quad \rho \dot{V}_i = (\rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha})_{,j} + F_i^{(e)} ,$$

with $F_i^{(e)}$ according to II.(16), and

$$\begin{aligned}
(62) \quad & \{ [-\rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} - T_{ij}^M + \frac{1}{4\pi c} e_{jkl} E_k H_l V_i] n_j = \\
& = \frac{1}{4\pi} \{ \frac{1}{c} D_j (\frac{dA_i}{dt} - A_{i,k} V_k) - (e_{jkl} H_l - \frac{1}{c} D_k V_j) A_{i,k} \} n_j
\end{aligned}$$

on $\Sigma(t)$, where T_{ij}^M is given by II.(19).

With the conditions (17)², (20) and (37), it can be proved that the right-hand side of (62) is equal to zero.

We note that the relation (51) corresponds with the momentum equation II.(17), if into the latter the constitutive equation III.(17) is substituted.

As concerns the discontinuity condition (62), we state that this condition, in a way analogous to the one used by the derivation of II.(37), as described in Appendix II, can be transformed into

$$(63) \quad \llbracket \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} \rrbracket n_j = 2\pi [(\rho M_j^* n_j)^2 + (\rho P_j^* n_j)^2] n_i \text{ on } \Sigma(t) .$$

The latter equation can, with III.(17), be showed to be identical to the jump condition for the stresses II.(44). Confer also the jump condition that could be obtained from the global balance equation of momentum II.(51).

IV.8. Conclusions

As we have said in the introduction, the aim of the present chapter was to find an expression for the Lagrangian that corroborates the results of the preceding chapters by application of Hamilton's principle. We have succeeded in this goal, as far as the expression (2) for \mathcal{L} did yield the Maxwell-equations, the balance equations of momentum and of moment of momentum, together with their discontinuity conditions, and the constitutive equation for the polarization. The stress tensor and the couple-stress tensor, as introduced in Chapter II, are in the theory described in the present chapter, automatically eliminated.

In conclusion, we shall briefly review four papers, that also employed Hamilton's principle for the investigation of electromagnetoelastic interactions. First we name the works of Tiersten [18], Parkus [22] and Penfield & Haus [25]. We have discussed these references already in Section III.6, where we have seen that they are in correspondence with this thesis.

There remains the article of Vlasov and Ishmukhametov [23]. In this paper, an elastic medium that is polarizable as well as magnetizable is considered. The authors restricted themselves to the theory of small deformations, i.e. to the infinitesimal elasticity theory. We remark that, to our opinion, the structure of the paper is not very lucid. So they introduced, for instance, three kinds of variations:

- δ , a variation with respect to a laboratory system of coordinates,
- δ_1 , a variation in a coordinate system that translates together with the medium, and
- δ_2 , a variation in a system that moves in translation and rotation with the medium.

However, it is not always clear which variation is used for the derivation of the equations of [23]. E.g., notice that the sentence (cf. p. 144 of [23]):

Varying with respect to \underline{u} , \underline{P} and \underline{M} with constant \underline{A} , φ , \underline{E} and \underline{B} ,
is only compatible with the variation δ . Hence, we must go out from

$$\delta \int_{t_1}^{t_2} \int_V \mathcal{L} \, dV dt = 0 ,$$

with

$$\delta \varphi = \delta A_i = \delta E_i = \delta B_i = 0 ,$$

and then equate to zero the factors preceding δu_i ($\approx \delta_1 u_i$ in the linear approximation of [23]), $\delta_2 M_i^*$ and $\delta_2 P_i^*$ (instead of $\delta_1 M_i^*$ and $\delta_1 P_i^*$), in order to arrive at the equations (25)-(28) of [23]. The equation of motion (30), however, is not completely correct. It must be supplemented by the amount

$$-\left(\frac{4\pi\rho^2}{c} e_{jkl} V_j P_k M_l\right)_{,i}$$

while further the electromagnetic quantities in the last two terms should be replaced by its convective values. If we require the internal energy to be invariant under superposed rigid-body rotations, it can be shown, with the aid of III.(10), III.(16), III.(17), III.(20) and II.(33) that the last two terms of eq. (30) of [23] are equal to zero. Hence, the, corrected, momentum equation (30) of [23] is equal to II.(17).

V. LINEARIZATION WITH RESPECT TO AN INTERMEDIATE STATE

V.1. Introduction

The dynamic equations of a polarizable, magnetizable thermoelastic medium, as outlined in the previous chapters, are highly nonlinear and complex. Apart from the usual nonlinearities, as contained in the finite strain theory of elasticity, there emerge difficulties from the balance equations and the boundary conditions, since they are formulated in Eulerian coordinates.

To render the equations amenable, we shall linearize this system of nonlinear equations derived in the preceding chapters. To this end, we first introduce an intermediate state. In this state, the medium is in interaction with a finite electromagnetic field. On this field, a small dynamic field is superposed. The nonlinear equations will now be linearized in the disturbances caused by this small field.

In the final section, the thus obtained equations will be further simplified by taking account of the fact that, although the fields are large, the deformations in the intermediate state are small.

The linear equations that will be derived in the present chapter, can be used, for instance, for the investigation of the stability of the intermediate state. If the disturbances tend to grow without limit, we state that the intermediate state is unstable. This principle will be applied in the study of the stability of magnetoelastic plates in the final chapter of this thesis.

V.2. Statement of the problem

Let us consider an elastic medium, initially free from stress (state X). On the application of large electromagnetic fields, the state of the medium alters (intermediate state ξ). An extra field, that is infinitesimally small, is superposed on the basic fields. This extra field

brings the medium into the present configuration \underline{x} .

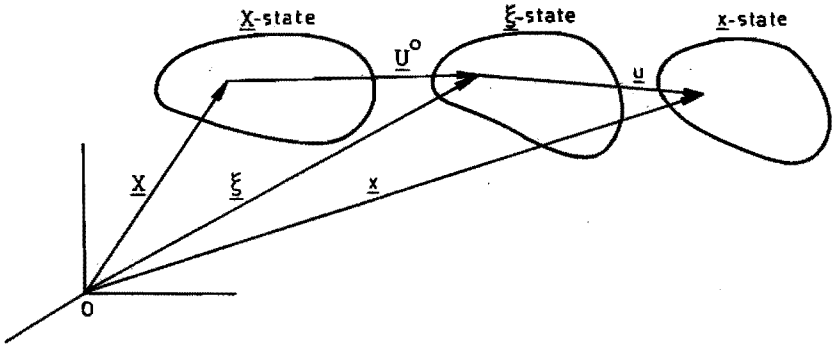


Fig. V.1.

Hence, we distinguish the following three configurations (cf. Fig. V.1).

- i) the undeformed or natural state \underline{X} (X_0);
- ii) the intermediate state $\underline{\xi}$ (ξ_1), in which only the large electromagnetic fields are applied;
- iii) the present or spatial state \underline{x} (x_1), that differs only slightly from the intermediate configuration.

We take the magnitude of the magnetic field in the intermediate state large enough to make the magnetization in the medium saturated. The quantities in the intermediate state will be labeled with an upper index $^{\circ}$, while the disturbances will be denoted by lower case letters. For instance, we have for the displacements

$$(1) \quad \underline{U} = \underline{U}^{\circ} + \underline{u},$$

where

$$(2) \quad \underline{U}^{\circ} = \underline{\xi} - \underline{X} \quad \text{and} \quad \underline{u} = \underline{x} - \underline{\xi}.$$

Moreover, we have

$$(3) \quad \underline{M}^* = \underline{M}^{\circ*} + \underline{m}^*, \quad \underline{E}^* = \underline{E}^{\circ*} + \underline{e}^*, \quad \theta = \theta^{\circ} + \theta, \quad \text{etc.,}$$

with the restrictions

$$(4) \quad \frac{|\underline{m}^*|}{|\underline{M}^{\circ*}|} \ll 1, \quad \frac{|e^*|}{|\underline{E}^{\circ*}|} \ll 1, \quad \frac{|\theta|}{|\theta^{\circ}|} \ll 1, \quad \text{etc.}$$

In the present chapter we wish to derive a system of equations, that is linear in the small quantities \underline{u} , \underline{m}^* , etc., under the assumption that the $\underline{\xi}$ -state is known.

As the magnetization is saturated, the following relations must hold

$$(5) \quad M_i^{\circ*} M_i^{\circ*} = M_s^2,$$

and

$$(6) \quad M_i^* M_i^* = (M_i^{\circ*} + m_i^*)(M_i^{\circ*} + m_i^*) = M_s^2,$$

from which, after neglectation of terms that are of second order in $|\underline{m}^*|$, it follows that

$$(7) \quad m_i^* M_i^{\circ*} = 0.$$

The balance equation of mass II.(12) yields the following linearized relation between the density in the present state (ρ) and the one in the intermediate state (ρ°)

$$(8) \quad \rho = \rho^{\circ} (1 - u_{i,i}).$$

In (8), i denotes differentiation with respect to x_i , but in the linear approximation this may be replaced by differentiation with respect to ξ_i . Hence, in this approximation we have

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial \xi_i},$$

as can easily be proved with the chain rule of differentiation. We note, that in the sequel it is always allowed to read $\partial/\partial \xi_i$ for the symbol i .

V.3. Linearization of the constitutive equations

We wish to express the constitutive equations, derived in Chapter III, in terms of the small quantities \underline{u} , \underline{m}^* , \underline{e}^* and θ . However, before we can do this, we must first linearize the independent variables, occurring in III.(48), with respect to these quantities.

From the definition III.(5), it follows, with the aid of (2)², that the following, linear, expression for the deformation gradient holds

$$(9) \quad F_{i\alpha} = \xi_{i,\alpha} + u_{i,j} \xi_{j,\alpha} .$$

With this relation, and with the definition III.(44), we can deduce the linearized expression for the deformation tensor

$$(10) \quad E_{\alpha\beta} = E_{\alpha\beta}^{\circ} + e_{ij} \xi_{i,\alpha} \xi_{j,\beta} ,$$

where

$$(11) \quad E_{\alpha\beta}^{\circ} = \frac{1}{2}(\xi_{i,\alpha} \xi_{i,\beta} - \delta_{\alpha\beta}) \quad \text{and} \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) .$$

In an analogous way, we can linearize the remaining objective constitutive variables, defined in Section III.4, obtaining

$$(12) \quad \Lambda_{\alpha} = \Lambda_{\alpha}^{\circ} + \lambda_j \xi_{j,\alpha} ,$$

where

$$(13) \quad \Lambda_{\alpha}^{\circ} = M_i^{\circ*} \xi_{i,\alpha} \quad \text{and} \quad \lambda_j = M_i^{\circ*} u_{i,j} + m_j^* ,$$

and

$$(14) \quad \Delta_{\alpha} = \Delta_{\alpha}^{\circ} + \delta_j \xi_{j,\alpha} ,$$

where

$$(15) \quad \Delta_{\alpha}^{\circ} = E_i^{\circ*} \xi_{i,\alpha} \quad \text{and} \quad \delta_j = E_i^{\circ*} u_{i,j} + e_j^* ,$$

and, in conclusion,

$$(16) \quad G_{\alpha\beta} = G_{\alpha\beta}^{\circ} + g_{ij} \xi_{i,\alpha} \xi_{j,\beta} ,$$

where

$$(17) \quad G_{\alpha\beta}^{\circ} = M_{i,\alpha}^{\circ*} M_{i,\beta}^{\circ*} \quad \text{and} \quad g_{ij} = M_{k,j}^{\circ*} m_{k,i}^* + M_{k,i}^{\circ*} m_{k,j}^* .$$

We now are able to linearize the constitutive equations. To this end, we write the partial derivatives of the functional Σ , with respect to $E_{\alpha\beta}$, Λ_{α} , $G_{\alpha\beta}$, Δ_{α} and θ , as a power series in the small variables defined above, retaining only linear terms.

Starting with the constitutive equation for the entropy III.(49), we get

$$(18) \quad \begin{aligned} s &= - \frac{\partial \Sigma}{\partial \theta} = - \left(\frac{\partial \Sigma}{\partial \theta} \right)^{\circ} - \left(\frac{\partial^2 \Sigma}{\partial \theta^2} \right)^{\circ} \theta - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial E_{\alpha\beta}} \right)^{\circ} e_{ij} \xi_{i,\alpha} \xi_{j,\beta} + \\ &\quad - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Lambda_{\alpha}} \right)^{\circ} \lambda_i \xi_{i,\alpha} - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Delta_{\alpha}} \right)^{\circ} \delta_i \xi_{i,\alpha} - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial G_{\alpha\beta}} \right)^{\circ} g_{ij} \xi_{i,\alpha} \xi_{j,\beta} = \\ &= s^{\circ} + {}^1\sigma_{\theta} + {}^2\sigma_{ij} u_{i,j} + {}^3\sigma_i m_i^* + {}^4\sigma_i e_i^* + {}^5\sigma_{ij} m_{ij}^* = s^{\circ} + s , \end{aligned}$$

where

$$(19) \quad \begin{aligned} s^{\circ} &= - \left(\frac{\partial \Sigma}{\partial \theta} \right)^{\circ} , \\ {}^1\sigma &= - \left(\frac{\partial^2 \Sigma}{\partial \theta^2} \right)^{\circ} , \\ {}^2\sigma_{ij} &= - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial E_{\alpha\beta}} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta} - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Lambda_{\alpha}} \right)^{\circ} M_i^{\circ*} \xi_{j,\alpha} + \\ &\quad - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Delta_{\alpha}} \right)^{\circ} E_i^{\circ*} \xi_{j,\alpha} , \\ {}^3\sigma_i &= - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Lambda_{\alpha}} \right)^{\circ} \xi_{i,\alpha} , \\ {}^4\sigma_i &= - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Delta_{\alpha}} \right)^{\circ} \xi_{i,\alpha} , \\ {}^5\sigma_{ij} &= - 2 \left(\frac{\partial^2 \Sigma}{\partial \theta \partial G_{\alpha\beta}} \right)^{\circ} M_{i,k}^{\circ*} \xi_{j,\alpha} \xi_{k,\beta} . \end{aligned}$$

Analogously, we obtain from III.(50), for the polarization

$$(20) \quad P_i^* = P_i^{\circ*} + {}^1n_{i,\theta} + {}^2n_{ijk}u_{j,k} + {}^3n_{ij}m_j^* + {}^4n_{ij}e_j^* + {}^5n_{ijk}m_{j,k}^* = \\ = P_i^{\circ*} + p_i^* ,$$

where

$$(21) \quad P_i^{\circ*} = - \left(\frac{\partial \Sigma}{\partial \Delta_\alpha} \right)^\circ \xi_{i,\alpha} , \\ {}^1n_i = - \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \theta} \right)^\circ \xi_{i,\alpha} , \\ {}^2n_{ijk} = - \left(\frac{\partial \Sigma}{\partial \Delta_\alpha} \right)^\circ \delta_{ij} \xi_{k,\alpha} - \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial E_{\beta\gamma}} \right)^\circ \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} + \\ - \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Lambda_\beta} \right)^\circ M_j^{\circ*} \xi_{i,\alpha} \xi_{k,\beta} - \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Delta_\beta} \right)^\circ E_j^{\circ*} \xi_{i,\alpha} \xi_{k,\beta} , \\ {}^3n_{ij} = - \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Lambda_\beta} \right)^\circ \xi_{i,\alpha} \xi_{j,\beta} , \\ {}^4n_{ij} = - \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Delta_\beta} \right)^\circ \xi_{i,\alpha} \xi_{j,\beta} , \\ {}^5n_{ijk} = - 2 \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial G_{\beta\gamma}} \right)^\circ M_{j,\ell}^{\circ*} \xi_{k,\beta} \xi_{\ell,\gamma} \xi_{i,\alpha} .$$

The relation for the couple-stress III.(52) leads to

$$(22) \quad \Pi_{ij} = \Pi_{ij}^\circ + {}^1\pi_{ij,\theta} + {}^2\pi_{ijkl}u_{k,\ell} + {}^3\pi_{ijk}m_k^* + {}^4\pi_{ijk}e_k^* + \\ + {}^5\pi_{ijk\ell}m_{k,\ell}^* = \Pi_{ij}^\circ + \pi_{ij} ,$$

where

$$\Pi_{ij}^\circ = 2\rho^\circ \left(\frac{\partial \Sigma}{\partial G_{\alpha\beta}} \right)^\circ M_{i,k}^{\circ*} \xi_{j,\alpha} \xi_{k,\beta} ,$$

$$\begin{aligned}
{}^1\pi_{ij} &= 2\rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial \theta} \right)^{\circ} M_{i,k}^{\circ*} \xi_{j,\alpha} \xi_{k,\beta}, \\
{}^2\pi_{ijkl} &= 2\rho^{\circ} \left(\frac{\partial \Sigma}{\partial G_{\alpha\beta}} \right)^{\circ} M_{i,p}^{\circ*} [-\delta_{kl} \xi_{j,\alpha} \xi_{p,\beta} + \delta_{jk} \xi_{l,\alpha} \xi_{p,\beta}] + \\
&\quad + 2\rho^{\circ} M_{i,p}^{\circ*} \xi_{j,\alpha} \xi_{p,\beta} \left[\left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial E_{\gamma\delta}} \right)^{\circ} \xi_{k,\gamma} \xi_{l,\beta} + \right. \\
(23) \quad &\quad \left. + \left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial \Lambda_{\gamma}} \right)^{\circ} M_k^{\circ*} \xi_{l,\gamma} + \left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial \Delta_{\gamma}} \right)^{\circ} E_k^{\circ*} \xi_{l,\gamma} \right], \\
{}^3\pi_{ijk} &= 2\rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial \Lambda_{\gamma}} \right)^{\circ} M_{i,p}^{\circ*} \xi_{j,\alpha} \xi_{p,\beta} \xi_{k,\gamma}, \\
{}^4\pi_{ijk} &= 2\rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial \Delta_{\gamma}} \right)^{\circ} M_{i,p}^{\circ*} \xi_{j,\alpha} \xi_{p,\beta} \xi_{k,\gamma}, \\
{}^5\pi_{ijkl} &= 2\rho^{\circ} \left(\frac{\partial \Sigma}{\partial G_{\alpha\beta}} \right)^{\circ} \delta_{ik} \xi_{j,\alpha} \xi_{l,\beta} + \\
&\quad + 4\rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial G_{\alpha\beta} \partial G_{\gamma\delta}} \right)^{\circ} M_{i,p}^{\circ*} M_{k,q}^{\circ*} \xi_{j,\alpha} \xi_{p,\beta} \xi_{l,\gamma} \xi_{q,\delta}.
\end{aligned}$$

The equation for the stress III.(51) can be linearized to

$$\begin{aligned}
(24) \quad T_{ij} &= T_{ij}^{\circ} + {}^1\tau_{ij\theta} + {}^2\tau_{ijkl} u_{k,l} + {}^3\tau_{ijk} m_k^* + {}^4\tau_{ijk} e_k^* + \\
&\quad + {}^5\tau_{ijkl} m_{k,l}^* = T_{ij}^{\circ} + \tau_{ij},
\end{aligned}$$

where

$$\begin{aligned}
T_{ij}^{\circ} &= \rho^{\circ} \left(\frac{\partial \Sigma}{\partial E_{\alpha\beta}} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta} + \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right)^{\circ} M_i^{\circ*} \xi_{j,\alpha} + \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Delta_{\alpha}} \right)^{\circ} E_i^{\circ*} \xi_{j,\alpha}, \\
{}^1\tau_{ij} &= \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \theta} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta} + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \theta} \right)^{\circ} M_i^{\circ*} \xi_{j,\alpha} + \\
&\quad + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial \theta} \right)^{\circ} E_i^{\circ*} \xi_{j,\alpha},
\end{aligned}$$

$$\begin{aligned}
{}^2\tau_{ijkl} &= \rho^\circ \left(\frac{\partial \Sigma}{\partial E_{\alpha\beta}} \right)^\circ [-\delta_{kl} \xi_{i,\alpha} \xi_{j,\beta} + \delta_{ik} \xi_{l,\alpha} \xi_{j,\beta} + \delta_{jk} \xi_{i,\alpha} \xi_{l,\beta}] + \\
&+ \rho^\circ \left(\frac{\partial \Sigma}{\partial \Lambda_\alpha} \right)^\circ [-\delta_{kl} M_i^{\circ*} \xi_{j,\alpha} + \delta_{jk} M_i^{\circ*} \xi_{l,\alpha}] + \\
&+ \rho^\circ \left(\frac{\partial \Sigma}{\partial \Delta_\alpha} \right)^\circ [-\delta_{kl} E_i^{\circ*} \xi_{j,\alpha} + \delta_{jk} E_i^{\circ*} \xi_{l,\alpha}] + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial E_{\gamma\delta}} \right)^\circ \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} \xi_{l,\delta} + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \Lambda_\gamma} \right)^\circ [M_k^{\circ*} \xi_{i,\alpha} \xi_{j,\beta} \xi_{l,\gamma} + M_i^{\circ*} \xi_{j,\gamma} \xi_{k,\alpha} \xi_{l,\beta}] + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \Delta_\gamma} \right)^\circ [E_k^{\circ*} \xi_{i,\alpha} \xi_{j,\beta} \xi_{l,\gamma} + E_i^{\circ*} \xi_{j,\gamma} \xi_{k,\alpha} \xi_{l,\beta}] + \\
(25) \quad &+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_\alpha \partial \Lambda_\beta} \right)^\circ M_i^{\circ*} M_k^{\circ*} \xi_{j,\alpha} \xi_{l,\beta} + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Delta_\beta} \right)^\circ E_i^{\circ*} E_k^{\circ*} \xi_{j,\alpha} \xi_{l,\beta} + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_\alpha \partial \Delta_\beta} \right)^\circ [M_i^{\circ*} E_k^{\circ*} \xi_{j,\alpha} \xi_{l,\beta} + E_i^{\circ*} M_k^{\circ*} \xi_{j,\beta} \xi_{l,\alpha}] , \\
{}^3\tau_{ijk} &= \rho^\circ \left(\frac{\partial \Sigma}{\partial \Lambda_\alpha} \right)^\circ \delta_{ik} \xi_{j,\alpha} + \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \Lambda_\gamma} \right)^\circ \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_\alpha \partial \Lambda_\beta} \right)^\circ M_i^{\circ*} \xi_{j,\alpha} \xi_{k,\beta} + \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Delta_\beta} \right)^\circ E_i^{\circ*} \xi_{j,\alpha} \xi_{k,\beta} , \\
{}^4\tau_{ijk} &= \rho^\circ \left(\frac{\partial \Sigma}{\partial \Delta_\alpha} \right)^\circ \delta_{ik} \xi_{j,\alpha} + \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \Delta_\gamma} \right)^\circ \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} + \\
&+ \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_\alpha \partial \Delta_\beta} \right)^\circ M_i^{\circ*} \xi_{j,\alpha} \xi_{k,\beta} + \rho^\circ \left(\frac{\partial^2 \Sigma}{\partial \Delta_\alpha \partial \Delta_\beta} \right)^\circ E_i^{\circ*} \xi_{j,\alpha} \xi_{k,\beta} ,
\end{aligned}$$

$$\begin{aligned}
{}^5\tau_{ijkl} &= 2\rho \circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial G_{\gamma\delta}} \right) \circ M_{k,p}^{\circ*} \xi_{i,\alpha} \xi_{j,\beta} \xi_{\ell,\gamma} \xi_{p,\delta} + \\
&+ 2\rho \circ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial G_{\beta\gamma}} \right) \circ M_i^{\circ*} M_{k,p}^{\circ*} \xi_{j,\alpha} \xi_{\ell,\beta} \xi_{p,\gamma} + \\
&+ 2\rho \circ \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial G_{\beta\gamma}} \right) \circ E_i^{\circ*} M_{k,p}^{\circ*} \xi_{j,\alpha} \xi_{\ell,\beta} \xi_{p,\gamma} .
\end{aligned}$$

For later reference, we rewrite the coefficient of $u_{k,\ell}$ in (24) in the form

$$(26) \quad {}^2\tau_{ijkl} = -\delta_{kl} T_{ij}^{\circ} + \delta_{jk} T_{il}^{\circ} + \delta_{ik} T_{jl}^{\circ} + \bar{c}_{ijkl} ,$$

where \bar{c}_{ijkl} follows from (25)³ and (26). In the next chapter, it will turn out that in a very good approximation, \bar{c}_{ijkl} may be taken equal to (viz. p. 116)

$$(27) \quad \bar{c}_{ijkl} = \rho \circ \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial E_{\gamma\delta}} \right) \circ \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} \xi_{\ell,\delta} .$$

In the angular momentum equation II.(31), there appears the antisymmetric part of the stress tensor. For this tensor, the following constitutive equation holds

$$(28) \quad T_{[ij]} = \rho \frac{\partial \Sigma}{\partial \Lambda_{\alpha}} M_{[i}^* F_{j]\alpha} + \rho \frac{\partial \Sigma}{\partial \Delta_{\alpha}} E_{[i}^* F_{j]\alpha} .$$

Linearization of this relation gives

$$\begin{aligned}
(29) \quad T_{[ij]} &= T_{[ij]}^{\circ} + {}^1\tau_{ij}^{(a)} \theta + {}^2\tau_{ijkl}^{(a)} u_{k,\ell} + {}^3\tau_{ijk}^{(a)*} m_k^* + {}^4\tau_{ijk}^{(a)*} e_k^* + \\
&+ {}^5\tau_{ijkl}^{(a)*} m_{k,\ell}^* = T_{[ij]}^{\circ} + t_{[ij]} ,
\end{aligned}$$

where

$$T_{[ij]}^{\circ} = \rho \circ \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right) \circ M_{[i}^{\circ*} \xi_{j],\alpha} + \rho \circ \left(\frac{\partial \Sigma}{\partial \Delta_{\alpha}} \right) \circ E_{[i}^{\circ*} \xi_{j],\alpha} ,$$

$$\begin{aligned}
1_{\tau_{ij}^{(a)}} &= \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \theta} \right)^{\circ} M_{[i \delta_{j}], \alpha}^{\circ * \ast} \xi_{j, \alpha} + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial \theta} \right)^{\circ} E_{[i \delta_{j}], \alpha}^{\circ * \ast} \xi_{j, \alpha}, \\
2_{\tau_{ijk\ell}^{(a)}} &= \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right)^{\circ} \{ M_{[i \delta_{j}] k}^{\circ * \ast} \xi_{\ell, \alpha} - \delta_{k\ell} M_{[i \delta_{j}], \alpha}^{\circ * \ast} \} + \\
&+ \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Delta_{\alpha}} \right)^{\circ} \{ E_{[i \delta_{j}] k}^{\circ * \ast} \xi_{\ell, \alpha} - \delta_{k\ell} E_{[i \delta_{j}], \alpha}^{\circ * \ast} \} + \\
&+ \rho^{\circ} M_{[i \delta_{j}], \alpha}^{\circ * \ast} \left\{ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial E_{\beta \gamma}} \right)^{\circ} \xi_{k, \beta} \xi_{\ell, \gamma} + \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} M_k^{\circ * \ast} \xi_{\ell, \beta} + \right. \\
&+ \left. \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Delta_{\beta}} \right)^{\circ} E_k^{\circ * \ast} \xi_{\ell, \beta} \right\} + \rho^{\circ} E_{[i \delta_{j}], \alpha}^{\circ * \ast} \left\{ \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial E_{\beta \gamma}} \right)^{\circ} \xi_{k, \beta} \xi_{\ell, \gamma} + \right. \\
(30) \quad &+ \left. \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} M_k^{\circ * \ast} \xi_{\ell, \beta} + \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial \Delta_{\beta}} \right)^{\circ} E_k^{\circ * \ast} \xi_{\ell, \beta} \right\}, \\
3_{\tau_{ijk}^{(a)}} &= \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right)^{\circ} \delta_{[ik} \xi_{j], \alpha} + \rho^{\circ} M_{[i \delta_{j}], \alpha}^{\circ * \ast} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} \xi_{k, \beta} + \\
&+ \rho^{\circ} E_{[i \delta_{j}], \alpha}^{\circ * \ast} \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} \xi_{k, \beta}, \\
4_{\tau_{ijk}^{(a)}} &= \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Delta_{\alpha}} \right)^{\circ} \delta_{[ik} \xi_{j], \alpha} + \rho^{\circ} M_{[i \delta_{j}], \alpha}^{\circ * \ast} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Delta_{\beta}} \right)^{\circ} \xi_{k, \beta} + \\
&+ \rho^{\circ} E_{[i \delta_{j}], \alpha}^{\circ * \ast} \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial \Delta_{\beta}} \right)^{\circ} \xi_{k, \beta}, \\
5_{\tau_{ijk\ell}^{(a)}} &= 2\rho^{\circ} M_{[i \delta_{j}], \alpha}^{\circ * \ast} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial G_{\beta \gamma}} \right)^{\circ} M_{k, p}^{\circ * \ast} \xi_{\ell, \beta} \xi_{p, \gamma} + \\
&+ 2\rho^{\circ} E_{[i \delta_{j}], \alpha}^{\circ * \ast} \left(\frac{\partial^2 \Sigma}{\partial \Delta_{\alpha} \partial G_{\beta \gamma}} \right)^{\circ} M_{k, p}^{\circ * \ast} \xi_{\ell, \beta} \xi_{p, \gamma}.
\end{aligned}$$

There are still two constitutive equations left to discuss, i.e. Fourier's law III.(35) and Ohm's law III.(36). Using the assumption, that the coefficients of conductivity are independent of the temperature

gradient, we can infer

$$(31) \quad Q_i = Q_i^{\circ} - \kappa_{ij}^{\circ} \theta_{,j} + {}^1 k_{i\theta} + {}^2 k_{ijk} u_{k,j} + {}^3 k_{ijm}^* + {}^4 k_{ij} e_j^* + \\ + {}^5 k_{ijk} m_{j,k}^* = Q_i^{\circ} + q_i ,$$

where

$$(32) \quad Q_i^{\circ} = - \kappa_{ij}^{\circ} \theta_{,j} + \beta_{ij}^{\circ} E_j^{\circ*} , \\ {}^1 k_{i\theta} = - \left(\frac{\partial \kappa_{ij}}{\partial \theta} \right)^{\circ} \theta_{,j} + \left(\frac{\partial \beta_{ij}}{\partial \theta} \right)^{\circ} E_j^{\circ*} , \\ {}^2 k_{ikl} = - \left[\left(\frac{\partial \kappa_{ij}}{\partial E_{\alpha\beta}} \right)^{\circ} \varepsilon_{k,\alpha} \varepsilon_{l,\beta} + \left(\frac{\partial \kappa_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} M_k^{\circ*} \varepsilon_{l,\alpha} + \right. \\ \left. + \left(\frac{\partial \kappa_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} E_k^{\circ*} \varepsilon_{l,\alpha} \right] \theta_{,j} + \left[\left(\frac{\partial \beta_{ij}}{\partial E_{\alpha\beta}} \right)^{\circ} \varepsilon_{k,\alpha} \varepsilon_{l,\beta} + \right. \\ \left. + \left(\frac{\partial \beta_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} M_k^{\circ*} \varepsilon_{l,\alpha} + \left(\frac{\partial \beta_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} E_k^{\circ*} \varepsilon_{l,\alpha} \right] E_j^{\circ*} , \\ {}^3 k_{ik} = - \left(\frac{\partial \kappa_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} \varepsilon_{k,\alpha} \theta_{,j} + \left(\frac{\partial \beta_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} \varepsilon_{k,\alpha} E_j^{\circ*} , \\ {}^4 k_{ik} = - \left(\frac{\partial \kappa_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} \varepsilon_{k,\alpha} \theta_{,j} + \beta_{ik}^{\circ} + \left(\frac{\partial \beta_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} \varepsilon_{k,\alpha} E_j^{\circ*} , \\ {}^5 k_{ikl} = - 2 \left(\frac{\partial \kappa_{ij}}{\partial G_{\alpha\beta}} \right)^{\circ} \varepsilon_{l,\alpha} \varepsilon_{p,\beta} M_k^{\circ*} \theta_{,j} - 2 \left(\frac{\partial \beta_{ij}}{\partial G_{\alpha\beta}} \right)^{\circ} \varepsilon_{l,\alpha} \varepsilon_{p,\beta} M_k^{\circ*} E_j^{\circ*} ,$$

while Ohm's law becomes

$$(33) \quad J_i^* = J_i^{\circ*} + {}^1 s_i \theta + {}^2 s_{ijk} u_{j,k} + {}^3 s_{ijm}^* + {}^4 s_{ij} e_j^* + {}^5 s_{ijk} m_{j,k}^* + \\ + \beta_{ji}^{\circ} \frac{\theta_{,j}}{\theta^{\circ}} = J_i^{\circ*} + j_i^* ,$$

where

$$\begin{aligned}
J_i^{\circ*} &= \sigma_{ij}^{\circ} E_j^{\circ*} + \beta_{ij}^{\circ} \frac{\theta^{\circ}}{\theta^{\circ}} \frac{j}{\theta^{\circ}}, \\
1s_i &= \left(\frac{\partial \sigma_{ij}}{\partial \theta} \right)^{\circ} E_j^{\circ*} - \beta_{ji}^{\circ} \frac{\theta^{\circ}}{(\theta^{\circ})^2} + \left(\frac{\partial \beta_{ij}}{\partial \theta} \right)^{\circ} \frac{\theta^{\circ}}{\theta^{\circ}} \frac{j}{\theta^{\circ}}, \\
2s_{ikl} &= \left[\left(\frac{\partial \sigma_{ij}}{\partial E_{\alpha\beta}} \right)^{\circ} \epsilon_{k,\alpha} \epsilon_{l,\beta} + \left(\frac{\partial \sigma_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} M_{k,\alpha}^{\circ*} \epsilon_{l,\alpha} + \right. \\
&\quad \left. + \left(\frac{\partial \sigma_{ij}}{\partial \Delta_{\alpha}} \right)^{\circ} E_{k,\alpha}^{\circ*} \epsilon_{l,\alpha} \right] E_j^{\circ*} + \left[\left(\frac{\partial \beta_{ij}}{\partial E_{\alpha\beta}} \right)^{\circ} \epsilon_{k,\alpha} \epsilon_{l,\beta} + \right. \\
&\quad \left. + \left(\frac{\partial \beta_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} M_{k,\alpha}^{\circ*} \epsilon_{l,\alpha} + \left(\frac{\partial \beta_{ij}}{\partial \Delta_{\alpha}} \right)^{\circ} E_{k,\alpha}^{\circ*} \epsilon_{l,\alpha} \right] \frac{\theta^{\circ}}{\theta^{\circ}} \frac{j}{\theta^{\circ}}, \\
3s_{ik} &= \left(\frac{\partial \sigma_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} \epsilon_{k,\alpha} E_j^{\circ*} + \left(\frac{\partial \beta_{ij}}{\partial \Lambda_{\alpha}} \right)^{\circ} \epsilon_{k,\alpha} \frac{\theta^{\circ}}{\theta^{\circ}} \frac{j}{\theta^{\circ}}, \\
4s_{ik} &= \sigma_{ik}^{\circ} + \left(\frac{\partial \sigma_{ij}}{\partial \Delta_{\alpha}} \right)^{\circ} \epsilon_{k,\alpha} E_j^{\circ*} + \left(\frac{\partial \beta_{ij}}{\partial \Delta_{\alpha}} \right)^{\circ} \epsilon_{k,\alpha} \frac{\theta^{\circ}}{\theta^{\circ}} \frac{j}{\theta^{\circ}}, \\
5s_{ikl} &= 2 \left(\frac{\partial \sigma_{ij}}{\partial G_{\alpha\beta}} \right)^{\circ} \epsilon_{l,\alpha} \epsilon_{m,\beta} M_{k,m}^{\circ*} E_j^{\circ*} + 2 \left(\frac{\partial \beta_{ij}}{\partial G_{\alpha\beta}} \right)^{\circ} \epsilon_{l,\alpha} \epsilon_{m,\beta} M_{k,m}^{\circ*} \frac{\theta^{\circ}}{\theta^{\circ}} \frac{j}{\theta^{\circ}}.
\end{aligned}
\tag{34}$$

At this point, we have decomposed each constitutive equation into a part related to the intermediate state, assumed to be known, plus a part that is linear in the disturbances: \underline{u} , \underline{m}^* , e^* and θ .

V.4. Linearization of the balance equations

In this section, we shall in a similar way as in the preceding section, split up the electromagnetic and the mechanical balance equations.

Let us start with the Maxwell-equation I.(34)¹

$$\frac{1}{c} \frac{\partial B_i}{\partial t} = - e_{ijk} \frac{\partial E_k}{\partial x_j}.$$

With the definition

$$(35) \quad \frac{\partial^\circ}{\partial t} = \frac{\partial}{\partial t} \Big|_{\underline{\xi}=\text{constant}},$$

and the decompositions

$$(36) \quad B_i = B_i^\circ + b_i, \quad E_i = E_i^\circ + e_i, \quad \text{etc.}, \quad \text{and} \quad V_i = V_i^\circ + v_i,$$

where

$$(37) \quad v_j = \dot{u}_j,$$

we find, after neglect of terms that are small of second order in the disturbances, that

$$(38) \quad \begin{aligned} \frac{\partial B_i}{\partial t} &= \frac{dB_i}{dt} - \frac{\partial B_i}{\partial x_j} V_j = \\ &= \dot{B}_i^\circ + \dot{b}_i - \frac{\partial B_i^\circ}{\partial \xi_j} V_j^\circ - b_{i,j} V_j^\circ - B_{i,j}^\circ v_j + B_{i,j}^\circ V_k^\circ u_{j,k} = \\ &= \frac{\partial^\circ B_i^\circ}{\partial t} + \frac{\partial^\circ b_i}{\partial t} - B_{i,j}^\circ v_j + B_{i,j}^\circ V_k^\circ u_{j,k}. \end{aligned}$$

Moreover, we obtain by utilizing the chain rule of differentiation,

$$(39) \quad \frac{\partial E_k}{\partial x_j} = \frac{\partial E_k^\circ}{\partial \xi_j} + e_{k,j} - E_{k,l}^\circ u_{l,j}.$$

We note that, if the electromagnetic fields in the $\underline{\xi}$ -state are uniform, the last two terms of (38) and the last one of (39) may be omitted.

In the sequel, we shall drop the index $^\circ$ in $\partial^\circ/\partial t$. Hence, in the remaining part of this chapter, we must read for $\partial/\partial t$ the partial time derivative referred to the $\underline{\xi}$ -state. We notice that, in the equations for the disturbances, it makes no difference whether $\partial/\partial t$ is taken with respect to the $\underline{\xi}$ - or the \underline{x} -state.

By using the foregoing results, we can decompose the electromagnetic equations of Section I.(3). We obtain the system

$$\frac{1}{c} \frac{\partial B_i^\circ}{\partial t} = -e_{ijk} E_{k,j}^\circ, \quad B_{i,i}^\circ = 0,$$

$$\begin{aligned}
 & \frac{1}{c} \frac{\partial D_i^\circ}{\partial t} + \frac{4\pi}{c} J_i^\circ = e_{ijk} H_{k,j}^\circ, \quad D_{i,i}^\circ = 4\pi Q, \\
 (40) \quad & \frac{\partial Q^\circ}{\partial t} + J_{i,i}^\circ = 0, \\
 & B_i^\circ = H_i^\circ + 4\pi\rho M_i^\circ, \\
 & D_i^\circ = E_i^\circ + 4\pi\rho P_i^\circ,
 \end{aligned}$$

for the $\underline{\xi}$ -state, while for the disturbances we get

$$\begin{aligned}
 & \frac{1}{c} \frac{\partial b_i}{\partial t} - \frac{1}{c} B_{i,j}^\circ v_j + \frac{1}{c} B_{i,j}^\circ V_k^\circ u_{j,k} = -e_{ijk} (e_{k,j} - E_{k,\ell}^\circ u_{\ell,j}), \\
 & b_{i,i} - B_{i,j}^\circ u_{j,i} = 0, \\
 (41) \quad & \frac{1}{c} \frac{\partial d_i}{\partial t} - \frac{1}{c} D_{i,j}^\circ v_j + \frac{1}{c} D_{i,j}^\circ V_k^\circ u_{j,k} + \frac{4\pi}{c} j_i = \\
 & = e_{ijk} (h_{k,j} - H_{k,\ell}^\circ u_{\ell,j}), \\
 & d_{i,i} - D_{i,j}^\circ u_{j,i} = 4\pi q, \\
 & \frac{\partial q}{\partial t} - Q_{,j}^\circ v_j + Q_{,j}^\circ V_k^\circ u_{j,k} + j_{i,i} - J_{i,j}^\circ u_{j,i} = 0, \\
 & b_i = h_i + 4\pi\rho^\circ m_i - 4\pi\rho^\circ M_{i,j}^\circ u_{j,j}, \\
 & d_i = e_i + 4\pi\rho^\circ p_i - 4\pi\rho^\circ P_{i,j}^\circ u_{j,j}.
 \end{aligned}$$

The transformation rules I. (38) pass into

$$(42) \quad D_i^{\circ*} = D_i^\circ + \frac{1}{c} e_{ijk} V_j^\circ H_k^\circ, \quad \text{etc.},$$

and

$$(43) \quad d_i^* = d_i + \frac{1}{c} e_{ijk} H_k^\circ v_j + \frac{1}{c} e_{ijk} V_j^\circ h_k, \quad \text{etc.}$$

Considering the mechanical balance equations, we note that we have already employed the balance law of mass for the derivation of (8).

In order to linearize the momentum equation II.(17), we first split up the electromagnetic force $\underline{F}^{(e)}$, according to II.(16), into a part related to the $\underline{\xi}$ -state ($\underline{F}^{(e)\circ}$) plus a small extra term ($\underline{f}^{(e)}$). The mechanical body force will be taken equal to zero. We obtain

$$(44) \quad \underline{F}_i^{(e)\circ} = Q^\circ E_i^\circ + \frac{1}{c} e_{ijk} J_j^\circ B_k^\circ + \rho^\circ P_j^\circ E_{j,i}^\circ + \rho^\circ M_j^\circ H_{j,i}^\circ + \\ + \frac{\rho^\circ}{c} \frac{d}{dt} [e_{ijk} (P_j^\circ B_k^\circ + E_j^\circ M_k^\circ)] - \left(\frac{4\pi\rho^\circ{}^2}{c}\right) e_{jkl} (P_k^\circ M_\ell^\circ V_j^\circ)_{,i} ,$$

and

$$(45) \quad \underline{f}_i^{(e)} = qE_i^\circ + Q^\circ e_i + \frac{1}{c} e_{ijk} J_j^\circ b_k + \frac{1}{c} e_{ijk} B_k^\circ j_j + \rho^\circ P_j^\circ e_{j,i} + \\ + \rho^\circ E_{j,i}^\circ P_j + \rho^\circ M_j^\circ h_{j,i} + \rho^\circ H_{j,i}^\circ m_j + \\ - \rho^\circ [P_j^\circ E_{j,i}^\circ + M_j^\circ H_{j,i}^\circ] u_{k,k} - \rho^\circ [P_j^\circ E_{j,k}^\circ + M_j^\circ H_{j,k}^\circ] u_{k,i} + \\ + \frac{\rho^\circ}{c} \frac{d}{dt} [e_{ijk} (P_j^\circ b_k + B_k^\circ p_j + E_j^\circ m_k + M_k^\circ e_j)] + \\ - \frac{\rho^\circ}{c} \frac{d}{dt} [e_{ijk} (P_j^\circ B_k^\circ + E_j^\circ M_k^\circ)] u_{\ell,\ell} + \\ - \left[\frac{4\pi\rho^\circ{}^2}{c}\right] e_{k\ell m} (-V_k^\circ P_\ell^\circ M_m^\circ u_{j,j} + P_\ell^\circ M_m^\circ v_k + V_k^\circ P_\ell^\circ m_m + V_k^\circ M_m^\circ p_\ell)_{,i} + \\ - \left(\frac{4\pi\rho^\circ{}^2}{c}\right) e_{k\ell m} (V_k^\circ P_\ell^\circ M_m^\circ)_{,j} u_{j,i} .$$

By using the relations (8), (24), (44) and (45), the balance of momentum can be divided into the equation for the $\underline{\xi}$ -state

$$(46) \quad \rho^\circ \ddot{\underline{V}}_i^\circ = T_{ij,j}^\circ + F_i^{(e)\circ} ,$$

plus the linear equation for the disturbances

$$(47) \quad \rho^\circ \ddot{\underline{v}}_i - \rho^\circ \ddot{\underline{V}}_i^\circ u_{j,j} = \tau_{ij,j} - T_{ij,k}^\circ u_{k,j} + f_i^{(e)} .$$

In an analogous way, the balance of moment of momentum II.(31) yields the equations

$$(48) \quad \frac{1}{\Gamma} \dot{M}_i^{**} = e_{ijk} \{ P_j^{**} E_k^{**} + M_j^{**} H_k^{**} - \frac{1}{\rho^0} T_{[jk]}^0 + \frac{1}{\rho^0} (M_j^{**} \Pi_{kl}^0)_{,l} \},$$

and

$$(49) \quad \frac{1}{\Gamma} \dot{m}_i^{**} = e_{ijk} \{ P_j^{**} e_k^{**} + E_k^{**} p_j^{**} + M_j^{**} h_k^{**} + H_k^{**} m_j^{**} - \frac{1}{\rho^0} T_{[jk]}^0 u_{l,l} + \\ - \frac{1}{\rho^0} t_{[ij]} + \frac{1}{\rho^0} (M_j^{**} \Pi_{kl}^0)_{,l} u_{m,m} - \frac{1}{\rho^0} (M_j^{**} \Pi_{kl}^0)_{,m} u_{m,l} + \\ + \frac{1}{\rho^0} (\Pi_{kl}^0 m_{,j}^{**})_{,l} + \frac{1}{\rho^0} (M_j^{**} \pi_{kl}^0)_{,l} \}.$$

Under the absence of heat sources (i.e. $r = 0$), the balance of energy III.(38) can be split up into

$$(50) \quad \rho^0 \dot{\theta}^{\circ\circ} \dot{S}^{\circ\circ} + Q_{i,i}^{\circ\circ} = J_i^{**} E_i^{**},$$

and

$$(51) \quad \rho^0 \dot{\theta}^{\circ\circ} \dot{s}^{\circ\circ} + \rho^0 \dot{S}^{\circ\circ} (\theta - \theta^{\circ\circ} u_{i,i}) + q_{i,i} - Q_{i,j}^{\circ\circ} u_{j,i} = J_i^{**} e_i^{**} + E_i^{**} j_i^{**}.$$

We now have constructed a complete system of linear equations for the disturbances \underline{u} , \underline{m}^* , etc. In the next section the jump conditions will be linearized.

V.5. Linearization of the boundary conditions

The discontinuity conditions, derived in the Sections I.3 and II.5, are given in the deformed configuration. As this configuration is unknown a priori, we wish to refer these jump conditions to the $\underline{\xi}$ -state, which is assumed to be known. In this section, this will be done for the problem of a solid body in vacuum. In an analogous way, the jump conditions on a material discontinuity surface can be transformed to the $\underline{\xi}$ -state.

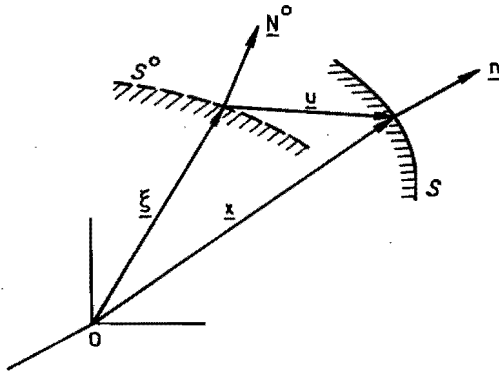


Fig. V.2.

Let us consider a solid body in vacuum with boundary S^0 in the ξ -state and S in the x -state. The unit outward normal on S^0 is named \underline{N}^0 and on S : \underline{n} (cf. Fig. V.2). Moreover, let \underline{u} be the displacement of a material point of the boundary, from its position $\underline{\xi}$ on S^0 to its present position \underline{x} on S . Hence

$$(52) \quad \underline{u} = \underline{x} - \underline{\xi}.$$

According to [1], Section 182, the following relation between \underline{N} and \underline{n} holds

$$(53) \quad n_i = J \xi_{j,i} N_j^0 \frac{dS^0}{dS},$$

where the Jacobian J , for small deformations may be approximated by

$$(54) \quad J = 1 + u_{i,i}.$$

Substituting (52) and (54) into (53), utilizing the fact that \underline{n} is a unit vector, thus

$$(55) \quad n_i n_i = 1,$$

and retaining only linear terms in $u_{i,j}$, yields

$$(56) \quad n_i = N_i^0 + u_{j,k} N_j^0 N_k^0 N_i^0 - u_{j,i} N_j^0.$$

With this relation, we can transform the boundary condition for an arbitrary quantity

$$(57) \quad \Xi = \tilde{\Xi}(\underline{x}, t) = \hat{\Xi}(\underline{\xi}, t) ,$$

holding on S

$$(58) \quad [[\tilde{\Xi}(\underline{x}, t)]n_i] = 0 , \quad \text{on } S$$

to one with respect to S° , reading like

$$(59) \quad [[\hat{\Xi}(\underline{\xi}, t)](N_i^\circ + u_{j,k}N_j^\circ N_k^\circ N_i^\circ - u_{j,i}N_j^\circ)] = 0 , \quad \text{on } S^\circ .$$

By using this result and with the decompositions of the variables defined in the preceding sections, we can split up the boundary conditions into a set for the $\underline{\xi}$ -state plus a set for the \underline{x} -state, both with respect to the surface S° .

We start with the electromagnetic jump conditions I.(35). Note that the velocity W_n , occurring in I.(35), is the velocity of the discontinuity surface. Hence, in our case, it is the velocity of the boundary, so

$$(60) \quad W_n = \bar{V}_i n_i .$$

Because there is no velocity outside the body, as there is a vacuum, we shall drop the upper index $\bar{}$ of \underline{V} in the following boundary conditions. We obtain

$$(61) \quad \begin{aligned} e_{ijk} [[E_j^\circ]N_k^\circ + \frac{1}{c} [[B_i^\circ]V_k^\circ N_k^\circ] &= 0 , & [[D_i^\circ]N_i^\circ] &= 0 , \\ e_{ijk} [[H_j^\circ]N_k^\circ - \frac{1}{c} [[D_i^\circ]V_k^\circ N_k^\circ] &= 0 , & [[B_i^\circ]N_i^\circ] &= 0 , \\ [[J_i^\circ]N_i^\circ - [[Q^\circ]V_i^\circ N_i^\circ] &= 0 , & & \text{on } S^\circ \end{aligned}$$

for the $\underline{\xi}$ -state, and

$$\begin{aligned} e_{ijk} [[e_j]N_k^\circ + \frac{1}{c} [[b_i]V_k^\circ N_k^\circ + \frac{1}{c} [[B_i^\circ]v_k N_k^\circ - e_{ijk} [[E_j^\circ]u_{\ell,k} N_\ell^\circ + \\ - \frac{1}{c} [[B_i^\circ]V_k^\circ u_{\ell,k} N_\ell^\circ] &= 0 , \end{aligned}$$

$$\begin{aligned}
 & [d_i]N_i^\circ - [D_i^\circ]u_{j,i}N_j^\circ = 0, \\
 (62) \quad & e_{ijk} [h_j]N_k^\circ - \frac{1}{c} [d_i]v_k^\circ N_k^\circ - \frac{1}{c} [D_i^\circ]v_k^\circ N_k^\circ - e_{ijk} [H_j^\circ]u_{\ell,k}N_\ell^\circ + \\
 & \quad + \frac{1}{c} [D_i^\circ]v_k^\circ u_{\ell,k}N_\ell^\circ = 0, \\
 & [b_i]N_i^\circ - [B_i^\circ]u_{j,i}N_j^\circ = 0, \\
 & [j_i]N_i^\circ - [q]v_i N_i^\circ - [Q^\circ]v_i N_i^\circ - [J_i^\circ]u_{j,i}N_j^\circ + [Q^\circ]v_i^\circ u_{j,i}N_j^\circ = 0, \\
 & \hspace{25em} \text{on } S^\circ
 \end{aligned}$$

for the disturbances.

Decomposition of the mechanical boundary conditions II. (47)-(49) gives

$$\begin{aligned}
 & T_{ij}^\circ N_j^\circ = 2\pi\{(\rho^\circ M_j^{\circ*} N_j^\circ)^2 + (\rho^\circ P_j^{\circ*} N_j^\circ)^2\} N_i^\circ + T_i^{\circ*}, \\
 (63) \quad & M_{[i]j]k}^\circ N_k^\circ = 0, \\
 & Q_j^\circ N_j^\circ = Q^{\circ*}, \quad \text{on } S^\circ
 \end{aligned}$$

for the ξ -state, and for the disturbances

$$\begin{aligned}
 & t_{ij}^\circ N_j^\circ = T_{ij}^\circ u_{k,j} N_k^\circ - T_{jk}^\circ N_k^\circ u_{j,i} + 2T_{ij}^\circ N_j^\circ (u_{k,\ell} N_\ell^\circ - u_{k,k}) + \\
 & \quad + 4\pi\rho^{\circ 2} M_\ell^{\circ*} N_\ell^\circ (m_j^* - M_k^{\circ*} u_{j,k}) N_j^\circ N_i^\circ + \\
 (64) \quad & \quad + 4\pi\rho^{\circ 2} P_\ell^{\circ*} N_\ell^\circ (p_j^* - P_k^{\circ*} u_{j,k}) N_j^\circ N_i^\circ + t_i^*, \\
 & M_{[i]j]k}^{\circ*} N_k^\circ = M_{[i]j]k}^{\circ*} u_{\ell,k} N_\ell^\circ - m_{[i]j]k}^* N_k^\circ, \\
 & q_j^\circ N_j^\circ = Q_j^\circ u_{k,j} N_k^\circ + q^*, \quad \text{on } S^\circ.
 \end{aligned}$$

Thus, we have supplemented the system of linear constitutive equations and balance equations, obtained in the preceding sections, by a linear set of boundary conditions.

V.6. Simplification of the linearized equations

In the preceding sections, we have decomposed our general nonlinear equations into a nonlinear set for an intermediate state, assumed to be known, and a linear system for the disturbances on this state. In the intermediate state, the medium was interacted by finite electromagnetic fields. However, although the fields are large, the deformation of the body from its initial configuration to the intermediate state is small, as a consequence of the fact that the numerical values of the relevant material coefficients are very large for ferromagnetic materials. The same holds for the changes in the temperature. Therefore, we may use the infinitesimal thermoelastic theory for the determination of the intermediate state.

Let us compare two problems, concerning

- i) a rigid body, with fixed temperature,
- ii) the real thermoelastic body,

both interacted by the same electromagnetic fields.

We assume that for the rigid-body problem the electromagnetic fields, in- and outside the body, can be calculated. These fields will be indicated by a lower pre-index \circ , e.g. $\underline{\circ P}$, $\underline{\circ M}$, $\underline{\circ H}$, etc. In the first problem we have

$$(65) \quad \circ U_{i,j} = 0 \quad \text{and} \quad \circ \theta = \theta_{\circ},$$

where θ_{\circ} is the temperature in the initial state. θ_{\circ} is assumed to be uniform.

In the second problem the body is strained and the temperature of the body is changed. As a consequence, also the electromagnetic fields are altered. However, since the deformations and the changes in temperature are small, the corrections on the fields are small too. Therefore, the values of the electromagnetic fields in the real problem will differ only slightly from those in the rigid-body state. Hence, we may state that

$$(66) \quad |U_{i,j}^{\circ}| \ll 1, \quad \frac{|\theta^{\circ} - \theta_{\circ}|}{\theta_{\circ}} \ll 1, \quad \frac{|\underline{P}^{\circ} - \underline{\circ P}|}{|\underline{\circ P}|} \ll 1, \quad \frac{|\underline{M}^{\circ} - \underline{\circ M}|}{|\underline{\circ M}|} \ll 1,$$

etc.

Moreover, as a consequence of the uniformity of the initial temperature θ_0 , we have

$$(67) \quad \theta_{,j}^{\circ} = (\theta^{\circ} - \theta_0)_{,j},$$

so also the temperature gradient is small.

Let us consider as an example a coefficient of s in (18), say ${}^3\sigma_i$. This coefficient is related to the $\underline{\xi}$ -state. If we denote the same coefficient, but now referred to the rigid-body state, by ${}^3\sigma_i$, hence

$$(68) \quad {}^3\sigma_i = - \left(\frac{\partial \Sigma}{\partial \theta \partial \Lambda_{\alpha}} \right) \delta_{i\alpha},$$

we may put

$$(69) \quad {}^3\sigma_i = {}^3\sigma_i + {}^3\tilde{\sigma}_i.$$

As a consequence of the aforesaid arguments, we may state that ${}^3\tilde{\sigma}_i$ is proportional to $U_{i,j}^{\circ}$; and hence small compared with ${}^3\sigma_i$. Hence, the term ${}^3\sigma_i m_i^*$, occurring in (18), becomes

$$(70) \quad {}^3\sigma_i = {}^3\sigma_i [1 + O(U_{k,l}^{\circ})] m_i^*.$$

At this point we introduce the following approximation: in the sequel, we shall neglect terms that are of the order $O(U_{i,j}^{\circ} u_{i,j})$. Noting that m_i^* is $O(u_{i,j}) M_s$, we then may approximate ${}^3\sigma_i m_i^*$ by

$$(71) \quad {}^3\sigma_i m_i^* = {}^3\sigma_i m_i^*.$$

Similar considerations result in the following simplifications for the equations in the disturbances, derived in the preceding sections:

- i) In the constitutive equations of Section V.3, the coefficients ${}^a\sigma$, ${}^a\eta$, etc. ($a = 1, 2, \dots, 5$), may be approximated by replacing $\xi_{i,\alpha}$ by $\delta_{i\alpha}$, M_i° by ${}^{\circ}M_i$, E_i° by ${}^{\circ}E_i$, θ° by θ_0 and ρ° by ρ_0 , where ρ_0 is the density in the natural state. Furthermore, the derivatives of Σ may be taken with respect to the rigid-body state, instead of to the $\underline{\xi}$ -state. For instance, the coefficient ${}^2\sigma_{ij}$ of (19)³ becomes

$$(72) \quad {}^2\sigma_{ij} = - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial E} \right)_{\alpha \beta} \delta_{i\alpha} \delta_{j\beta} - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Lambda} \right)_{\alpha} M_i^* \delta_{j\alpha} - \left(\frac{\partial^2 \Sigma}{\partial \theta \partial \Delta} \right)_{\alpha} E_i^* \delta_{j\alpha} .$$

ii) Into the linear balance equations (41), (47), (49) and (51) we may substitute for the electromagnetic quantities \underline{B}° , \underline{E}° , etc., their values according to the rigid-body state ${}_0\underline{B}$, ${}_0\underline{E}$, etc., and we may replace ρ° by ρ_0 , θ° by θ_0 and \underline{V}° by ${}_0\underline{V}$. Furthermore, the quantities P_i° , J_i° , T_{ij}° , Π_{ij}° , S° and Q_i° may be taken with respect to the rigid-body state. Hence, for instance, $P_i^{\circ*}$ may be replaced by

$$(73) \quad {}_0P_i^* = - \left(\frac{\partial \Sigma}{\partial \Delta} \right)_{\alpha} \delta_{i\alpha} .$$

iii) Approximations, similar to those mentioned above, may be applied to the boundary conditions (63) and (64). Moreover, we replace in these conditions \underline{N}° by \underline{N} , i.e. the unit normal on the undeformed body, and we refer them to the boundary in the rigid-body state.

Based on the foregoing statements, we can draw the conclusion:

The disturbances on the intermediate state are to determine, in an exactness of the order of $O(u_{i,j}^\circ, u_{i,j})$, without explicit knowledge of this state; we can confine ourselves to solving only the rigid-body problem.

Of course, if we wish to know the complete solution in the same exactness, we have to solve also the equations governing the intermediate state. However, for the solution of these equations we may use an approximated theory, based on the fact that the differences between the values of the fields in the ξ -state and in the rigid-body state are small.

We conclude by noting that it is often more desirable to know the disturbances on an intermediate state than the state itself. To illustrate this remark, we mention the following two examples.

i) Let the intermediate state be a static state, while the disturbances are caused by a small dynamic field. Then, by solving the equations for the disturbances, we can determine, for instance,

the resonance frequencies for the body in the intermediate state. Such a problem will be investigated in one of the next chapters.

- ii) By solving the equations for the disturbances, we can draw a conclusion about the stability of the intermediate state. This method will be employed in the final chapter of this thesis.

A complete elaboration of the linearized equations will be deferred until the next chapter, in which an explicit expression for the thermodynamic functional \mathcal{E} will be given.

VI. MATERIAL COEFFICIENTS

VI.1. Introduction

In the preceding chapter, we have obtained a properly invariant, linearized field theory of electromagnetothermoelasticity, consisting of a system of balance laws and constitutive equations together with a set of boundary conditions. In order to be able to elaborate these relations any further, we need an explicit formula for the thermodynamic functional Σ . In the next section we shall select a specific expression for Σ , in the form of a polynomial approximation in the constitutive variables.

We shall continue with the interpretation of the material coefficients appearing in the formula for Σ . Moreover, we shall give some numerical values for these coefficients. This will be done for the specific case of a single crystal of a magnetically saturated ferro(i)-magnetic material with cubic symmetry. The values for a polycrystalline medium can be obtained by averaging the values for single crystals over a finite volume, under the assumption that the crystals are oriented at random. On the basis of these numerical values, that will be listed in Table VI.1, in each constitutive equation, some material coefficients turn out to be dominant, while the remaining coefficients are negligibly small compared with them. By utilizing this information, and for the special symmetry we are concerned with, we shall, in the final section of this chapter, simplify considerably the set of constitutive equations obtained in the foregoing chapter.

VI.2. An expression for Σ

Underlying our choice of an explicit form for the thermodynamic functional Σ , is the fact that we shall restrict ourselves to problems in which the deformations, the deviations from the initial temperature, the electric field and the gradients of the magnetization are small.

Moreover, the magnetization will be saturated. Therefore, we shall approximate Σ by a polynomial expansion in which only quadratic terms in $E_{\alpha\beta}$, $(\theta - \theta_0)$, Δ_α and their combinations are retained. Since $G_{\alpha\beta}$ is already a second order function of the gradients of the magnetization, we shall retain only linear contributions of this quantity. As for the magnetic interactions, we restrict ourselves to linear terms in $E_{\alpha\beta}$, $(\theta - \theta_0)$ and Δ_α and to quadratic terms in the magnetization (i.e. Λ_α). In the purely magnetic part, we shall include, except a quadratic term, also a fourth order one. As the exchange interaction itself is already a very weak effect, the couplings of this interaction with deformation, temperature, magnetization and electric field are left out of consideration. Argued by these comments, we select for Σ (confer also [13] and [17])

$$\begin{aligned}
 (1) \quad \Sigma &= \Sigma(E_{\alpha\beta}, \Lambda_\alpha, G_{\alpha\beta}, \Delta_\alpha, \theta) = \\
 &= \frac{1}{2} \rho_0 2\chi_{\alpha\beta}^{(m)} \Lambda_\alpha \Lambda_\beta + \frac{1}{4} \rho_0^3 4\chi_{\alpha\beta\gamma\delta}^{(m)} \Lambda_\alpha \Lambda_\beta \Lambda_\gamma \Lambda_\delta - \frac{1}{2\rho_0} \chi_{\alpha\beta}^{(e)} \Delta_\alpha \Delta_\beta + \\
 &+ \psi_{\alpha\beta} \Lambda_\alpha \Lambda_\beta + \frac{1}{2\rho_0} c_{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} + \epsilon_{\alpha\beta\gamma}^{(m)} \Lambda_\alpha E_{\beta\gamma} - \frac{1}{\rho_0} \epsilon_{\alpha\beta\gamma}^{(e)} \Delta_\alpha E_{\beta\gamma} + \\
 &+ \rho_0 b_{\alpha\beta\gamma\delta} E_{\alpha\beta} \Lambda_\gamma \Lambda_\delta + \frac{1}{2} \rho_F \alpha_{\alpha\beta} G_{\alpha\beta} - \frac{1}{2} c(\theta - \theta_0)^2 + \\
 &- v_{\alpha\beta} E_{\alpha\beta} (\theta - \theta_0) - \lambda_\alpha^{(m)} \Lambda_\alpha (\theta - \theta_0) - L_{\alpha\beta}^{(m)} \Lambda_\alpha \Lambda_\beta (\theta - \theta_0) + \\
 &+ \frac{1}{\rho_0} \lambda_\alpha^{(e)} \Delta_\alpha (\theta - \theta_0) \dots
 \end{aligned}$$

The material coefficients occurring in formula (1) are called

- $2\chi_{\alpha\beta}^{(m)}$ and $4\chi_{\alpha\beta\gamma\delta}^{(m)}$: the second- and fourth-order anisotropy constants, respectively,
 $\chi_{\alpha\beta}^{(e)}$: the electric susceptibilities,
 $\psi_{\alpha\beta}$: the magneto-electric constants,
 $c_{\alpha\beta\gamma\delta}$: the coefficients of elasticity,
 $\epsilon_{\alpha\beta\gamma}^{(m)}$: the piezomagnetic constants,
 $b_{\alpha\beta\gamma\delta}$: the magnetostrictive constants,

- $\epsilon_{\alpha\beta\gamma}^{(e)}$: the piezoelectric constants,
 $\alpha_{\alpha\beta}$: the exchange constants,
 c : the reduced thermal constant (per unit of degree),
 $v_{\alpha\beta}$: the thermoelastic constants,
 $\lambda_{\alpha}^{(m)}$: the pyromagnetic constants,
 $L_{\alpha\beta}^{(m)}$: the thermomagnetic constants,
 $\lambda_{\alpha}^{(e)}$: the pyroelectric constants, respectively.

In the next section, we shall give interpretations and numerical values for the coefficients listed above. Since most of the ferro(i)magnetic materials belong to the crystal class Cubic I (cf. [28], Section VII.5) as for instance iron (class $m\bar{3}m$ or $m\bar{3}$), nichel (class $m\bar{3}m$) and the ferromagnetic material yttrium iron garnet (YIG, class $m\bar{3}m$), we shall pay special attention to this class.

The coordinate system, to which the components of the material tensors refer, will be taken along the principal axes of the crystal. For a material of class Cubic I, the following reductions for the arrays of the material coefficients are available (cf. [28], Section VII.5, or [29]).

- i) The first- and third-order coefficients become zero, which means that

$$(2) \quad \epsilon_{\alpha\beta\gamma}^{(m)} = \epsilon_{\alpha\beta\gamma}^{(e)} = \lambda_{\alpha}^{(m)} = \lambda_{\alpha}^{(e)} = 0 .$$

Hence in a crystal of class Cubic I, no piezoelectric or - magnetic and no pyroelectric or - magnetic effects appear.

- ii) The form of the second-order tensors reduces to a unit matrix, thus

$$(3) \quad 2\chi_{\alpha\beta}^{(m)} = 2\chi_{\alpha\beta}^{(m)}\delta_{\alpha\beta}, \quad \chi_{\alpha\beta}^{(e)} = \chi_{\alpha\beta}^{(e)}\delta_{\alpha\beta}, \quad \psi_{\alpha\beta} = \psi_{\alpha\beta}\delta_{\alpha\beta},$$

$$\alpha_{\alpha\beta} = \alpha_1\delta_{\alpha\beta}, \quad v_{\alpha\beta} = v_1\delta_{\alpha\beta}, \quad \text{and} \quad L_{\alpha\beta}^{(m)} = L_{\alpha\beta}^{(m)}\delta_{\alpha\beta} .$$

iii) The fourth-order tensors can, by use of the notation

11 → 1, 22 → 2, 33 → 3, 23 or 32 → 4, 13 or 31 → 5, 12 or 21 → 6 ,

be written in the matrix form

$$(4) \quad c_{\alpha\beta\gamma\delta} = \begin{vmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{vmatrix}, \quad b_{\alpha\beta\gamma\delta} = \begin{vmatrix} b_{11} & b_{12} & b_{12} & 0 & 0 & 0 \\ b_{12} & b_{11} & b_{12} & 0 & 0 & 0 \\ b_{12} & b_{12} & b_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{44} \end{vmatrix}$$

$$4^X_{\alpha\beta\gamma\delta}^{(m)} = \begin{vmatrix} 4^X_{11} & 4^X_{12} & 4^X_{12} & 0 & 0 & 0 \\ 4^X_{12} & 4^X_{11} & 4^X_{12} & 0 & 0 & 0 \\ 4^X_{12} & 4^X_{12} & 4^X_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4^X_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4^X_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4^X_{12} \end{vmatrix}$$

In the next section we shall give interpretations for the remaining material coefficients of (1), specified for a single crystal of class Cubic I.

VI.3. Elastic constants

Let us start with a discussion of the elastic constants. To this end, we consider a purely elastic material, for which (1) becomes

$$(5) \quad \Sigma = \Sigma(E_{\alpha\beta}) = \frac{1}{2\rho_0} c_{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta}.$$

In the infinitesimal elasticity theory (i.e. for small deformations) we get from III.(51)

$$(6) \quad T_{ij} = c_{ijkl} U_{k,l} = c_{ijkl} e_{kl},$$

where

$$(7) \quad e_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}).$$

Writing (6) in components along the principal axes yields, by use of (4)¹,

$$(8) \quad \begin{aligned} T_{11} &= c_{11}e_{11} + c_{12}(e_{22} + e_{33}), \\ T_{12} &= 2c_{44}e_{12}, \text{ etc.} \end{aligned}$$

As a first example, we regard a pure shear in a principal plane. Let the only stress component unequal to zero be T_{12} , then

$$(9) \quad \frac{T_{12}}{2e_{12}} = c_{44}.$$

Next, we consider a uni-axial tension along a principal axis. Taking only T_{11} unequal to zero, it follows from (8) that

$$(10) \quad \frac{T_{11}}{e_{11}} = \frac{(c_{11} - c_{12})(c_{11} + 2c_{12})}{(c_{11} + c_{12})}.$$

We note that the deformations e_{22} and e_{33} are unequal to zero. For these quantities the following relations hold

$$(11) \quad -\frac{e_{22}}{e_{11}} = -\frac{e_{33}}{e_{11}} = \frac{c_{12}}{(c_{11} + c_{12})}.$$

Finally, let us look at a hydrostatic compression. Adding the diagonal terms of the stress tensor, we obtain

$$(12) \quad \frac{T_{kk}}{3e_{kk}} = \frac{1}{3}(c_{11} + 2c_{12}).$$

In case of an isotropic material, the right-hand sides of (9), (10), (11) and (12) define the shear modulus G , Young's modulus E , Poisson's ratio ν and the modulus of compression κ , respectively. The array of the elastic constants (4)¹ alters in this case only in that we must replace c_{44} by

$$(13) \quad c_{44} = \frac{1}{2}(c_{11} - c_{12}).$$

Hence, for an isotropic medium, the following relations between the elastic constants and the moduli that are more common in the technical literature hold

$$(14) \quad G = \frac{1}{2}(c_{11} - c_{12}), \quad E = \frac{(c_{11} - c_{12})(c_{11} + 2c_{12})}{(c_{11} + c_{12})},$$

$$v = \frac{c_{12}}{(c_{11} + c_{12})}, \quad \kappa = \frac{1}{3}(c_{11} + 2c_{12}).$$

It is evident from (14) that only two of these moduli are independent.

VI.4. Magnetic constants

In this section, we shall consider the purely magnetic effects and the magnetoelastic interactions.

We first take a purely magnetic material, in which case the functional Σ reduces to

$$(15) \quad \Sigma = \Sigma(M_i^*) = \frac{1}{2}\rho \sum_{ij} \chi_{ij}^{(m)} M_i^* M_j^* + \frac{1}{4}\rho^3 \sum_{ijkl} \chi_{ijkl}^{(m)} M_i^* M_j^* M_k^* M_l^*,$$

where we must read ρ_s for the density ρ .

Reckoning with the fact that the magnetization is saturated, we introduce the unit vector $\underline{\theta}$, with components θ_i along the principal axes, by

$$(16) \quad \theta_i = \frac{1}{M_s} M_i^*,$$

By use of (3), (4) and (16), the expression (15) can be worked out into the form

$$(17) \quad \rho \Sigma = \left[\frac{1}{2}\rho^2 M_s^2 \sum_{ij} \chi_{ij}^{(m)} + \frac{1}{4}\rho^4 M_s^4 \sum_{ijkl} \chi_{ijkl} \right] + \\ + \frac{1}{4}\rho^4 M_s^4 (3 \sum_{ijkl} \chi_{ijkl} - \sum_{ij} \chi_{ij}^{(m)}) (\theta_1^2 \theta_2^2 + \theta_2^2 \theta_3^2 + \theta_3^2 \theta_1^2).$$

The term between the brackets [] on the right-hand side of (17) constitutes the magnetization energy in case of a saturation magnetization along one of the principal axes. It will turn out that this contribution is insignificant in our ultimate equations. The, more interesting, rest term of (17) represents the so called anisotropy energy, i.e. the amount by which the magnetization energy must be supplemented when the magnetization is not directed along one of the principal axes. The anisotropy coefficient K_1 is defined by (cf. [30], p. 129)

$$(18) \quad K_1 = \frac{1}{2} \rho M_s^4 (3 \chi_{12} - 4 \chi_{11}) .$$

We note that this coefficient is of interest for the constitutive equation of the anti-symmetric part of the stress tensor. This can be shown by substituting into the constitutive equation V.(28) the expression (15), by which these equation reduces to

$$(19) \quad T_{[ij]} = \rho M_{[i}^* \frac{\partial \Sigma}{\partial M_{j]}^*} .$$

By means of (17) and (18) this relation can be worked out into

$$(20) \quad T_{[ij]} = K_1 (\theta_i^3 \theta_j - \theta_i \theta_j^3) .$$

Since $T_{[ij]}$ appears in the angular momentum equation, the coefficient K_1 will also enter this equation.

As a following step, we shall regard the magnetoelastic interactions. To this end we take the functional Σ equal to

$$(21) \quad \Sigma = \Sigma(E_{\alpha\beta}, \Lambda_\alpha) = \frac{1}{2\rho_0} c_{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} + \frac{1}{2} \rho_0 \chi_{\alpha\beta}^{(m)} \Lambda_\alpha \Lambda_\beta + \\ + \frac{1}{4} \rho_0^3 \chi_{\alpha\beta\gamma\delta}^{(m)} \Lambda_\alpha \Lambda_\beta \Lambda_\gamma \Lambda_\delta + \rho_e b_{\alpha\beta\gamma\delta} E_{\alpha\beta} \Lambda_\gamma \Lambda_\delta .$$

Under the restriction of infinitesimal deformations, the constitutive relation III.(51) yields, with the aid of (21)

$$(22) \quad T_{ij} = (\rho^2 b_{ijkl} + \rho^2 \delta_{il} \chi_{jk}^{(m)} + \rho^4 \chi_{jklm}^{(m)} M_m^* M_i^*) M_k^* M_l^* + \\ + \{ c_{ijkl} + \rho^2 b_{kljm} M_m^* M_i^* - \rho^2 \delta_{kl} b_{ijmn} M_m^* M_n^* + \\ + \rho^2 \delta_{ik} b_{jlmn} M_m^* M_n^* + \rho^2 \delta_{jk} b_{ilmn} M_m^* M_n^* + \\ + \rho^2 \delta_{jk} M_i^* M_m^* (\chi_{lm}^{(m)} + \rho^2 \chi_{lmnp}^{(m)} M_n^* M_p^*) + \\ - \rho^2 \delta_{kl} M_i^* M_m^* (\chi_{jm}^{(m)} + \rho^2 \chi_{jimp}^{(m)} M_n^* M_p^*) \} e_{k\ell} ,$$

where, again, $\rho = \rho_0$.

Experimental investigations have established that for most of the ferro(i)-magnetic materials, the values of the terms

$$\rho^2 b_{ijkl} M_k^* M_l^* ,$$

in case of saturation magnetization, are of the order of 10^7 dyne/cm², while the values of

$$(\rho^2 \delta_{il} \chi_{jk}^{(m)} + \rho^4 \chi_{ijklm}^{(m)} M_m^* M_i^*) M_k^* M_l^*$$

are at most of the order of 10^5 dyne/cm². Furthermore, the elastic coefficients are of the order of 10^{12} dyne/cm². Hence, in a very good approximation we may put

$$(23) \quad T_{ij} = \rho^2 b_{ijkl} M_k^* M_l^* + c_{ijkl} e_{kl} .$$

In components, this relation becomes, by use of (4),

$$(24) \quad T_{11} = c_{11} e_{11} + c_{12} (e_{22} + e_{33}) + \rho^2 M_s^2 b_{12} + \rho^2 M_s^2 (b_{11} - b_{12}) \theta_1^2 ,$$

$$T_{12} = 2c_{44} e_{12} + 2\rho^2 M_s^2 b_{44} \theta_1 \theta_2 , \text{ etc.}$$

We write the diagonal terms of the stress tensor still in a somewhat different form

$$(25) \quad T_{11} = c_{11} e_{11} + c_{12} (e_{22} + e_{33}) + \frac{1}{3} \rho^2 M_s^2 (b_{11} + 2b_{12}) +$$

$$\text{etc.} \quad + \rho^2 M_s^2 (b_{11} - b_{12}) \left(\theta_1^2 - \frac{1}{3} \right) ,$$

This is done, because in this form the third term on the right-hand side of (25) represents the volume magnetostriction, while the fourth term stands for the form or length magnetostriction.

We introduce the magnetostriction coefficients B_0 , B_1 and B_2 by (cf. [30], Sections 8.2 and 8.3)

$$B_0 = \frac{1}{3} \rho^2 M_s^2 (b_{11} + 2b_{12}) ,$$

$$(26) \quad B_1 = \rho^2 M_s^2 (b_{11} - b_{12}) ,$$

and

$$B_2 = 2\rho^2 M_s^2 b_{44}.$$

The coefficient B_0 is a measure for the relative change in volume due to magnetostriction, as can be shown in the following way

Let us consider a stressless state ($T_{ij} = 0$). Then it follows from the vanishing of the trace of the stress tensor, with the aid of (25) and (26) that

$$(27) \quad B_0 = -\frac{1}{3}(c_{11} + 2c_{12})e_{kk} = -\frac{1}{3}(c_{11} + 2c_{12})\frac{\partial V}{V},$$

where $(\partial V/V)$ is the relative change in volume.

We note that this volume effect is usually very small, and therefore, it can often be neglected in practical problems. Experimental data, given in [31], p. 641, point out that the volume magnetostriction in saturation for nickel is at most a few percent of the length magnetostriction.

In order to interpret the other two coefficients defined in (26), we again take the stresses in (24) equal to zero. Let us call $(\Delta l/l)$ the relative increase in length along a line with direction cosines (n_1, n_2, n_3) with respect to the principal axes. Then, we have

$$(28) \quad \frac{\Delta l}{l} = e_{11}n_1^2 + e_{22}n_2^2 + e_{33}n_3^2 + 2e_{12}n_1n_2 + 2e_{23}n_2n_3 + 2e_{31}n_3n_1.$$

With (24), under the condition that $T_{ij} = 0$, and with (26) this formula can be rewritten as

$$(29) \quad \frac{\Delta l}{l} = -\frac{B_0}{3(c_{11} + c_{12})} - \frac{B_1}{(c_{11} - c_{12})} (\theta_1^2 n_1^2 + \theta_2^2 n_2^2 + \theta_3^2 n_3^2 - \frac{1}{3}) + \\ - \frac{B_2}{c_{44}} (\theta_1 \theta_2 n_1 n_2 + \theta_2 \theta_3 n_2 n_3 + \theta_3 \theta_1 n_3 n_1).$$

This relation is equal to the corresponding one in [31], p. 650, if in the latter the fourth-order terms in $\underline{\theta}$ are omitted, and if the coefficients

$$(30) \quad \lambda_{100} = -\frac{2B_1}{3(c_{11} - c_{12})}, \quad \lambda_{111} = -\frac{B_2}{3c_{44}},$$

are introduced.

If we neglect B_0 , it is evident from (29) that λ_{100} represents the extension in the $[100]$ -direction ($\eta_1 = 1, \eta_2 = \eta_3 = 0$) due to a saturation magnetization in the same direction ($\theta_1 = 1, \theta_2 = \theta_3 = 0$). The same holds for λ_{111} , but now the extension and the magnetization are in the $[111]$ -direction ($\theta_i = \eta_i = \frac{1}{3}\sqrt{3}$) (cf. also [30], p. 170).

VI.5. Thermal constants

In this section, we shall discuss the thermoelastic and the thermomagnetic interactions. We start with a thermoelastic medium, for which (1) becomes

$$(31) \quad \Sigma = \Sigma(E_{\alpha\beta}, \theta) = \frac{1}{2\rho} c_{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} - \frac{1}{2} c\theta^2 - v_{\alpha\beta} E_{\alpha\beta} \theta,$$

where

$$(32) \quad \rho = \rho_0 \quad \text{and} \quad \theta = \theta - \theta_0.$$

In this case the constitutive equations for the entropy III.(49) and for the stresses III.(51) become, with the aid of (3) and (4) and in a linear approximation,

$$(33) \quad S = c\theta + v_1 u_{i,i},$$

and

$$(34) \quad T_{ij} = c_{ijkl} e_{kl} - \rho v_1 \theta \delta_{ij}.$$

For a cubic material, the tensor of the thermal conduction coefficients κ_{ij} takes the form

$$(35) \quad \kappa_{ij} = \kappa_1 \delta_{ij}.$$

With (35) Fourier's law III.(35) passes into (note that $E_i^* = 0$)

$$(36) \quad Q_i = -\kappa_1 \theta_{,i}.$$

Substitution of (33) and (36) into the energy balance III.(38) with $E_i^* = 0$, yields, after linearization

$$(37) \quad \rho \theta_e (c\dot{\theta} + v_1 \dot{u}_{i,i}) - \kappa_1 \theta_{,ii} = \rho r, \quad \text{where } \kappa_1 = \kappa_1(\theta_0).$$

After comparing (37) with [32], p. 14, eq. (15), we conclude that

$$(38) \quad c\theta_e := c_w: \text{ the specific heat,}$$

that κ_1 is, indeed, the thermal conductivity, and that

$$(39) \quad \nu_1 = \frac{1}{\rho} (c_{11} + 2c_{12})\alpha ,$$

where α is the linear thermal expansion coefficient.

When the stresses are zero, we find from (34) that

$$(40) \quad e_{11} = \frac{\rho \nu_1}{(c_{11} + 2c_{12})} \theta ,$$

hence

$$(41) \quad \alpha = \frac{e_{11}}{\theta} .$$

Substitution of (39) into (34) yields

$$(42) \quad T_{ij} = c_{ijkl} e_{kl} - \delta_{ij} (c_{11} + 2c_{12})\alpha\theta .$$

We note that (42) is in agreement with [32], p. 5, eq. (8), if in the latter, as this relation only holds for isotropic materials, the coefficients $2G(1+\nu)/(1-2\nu) = E/(1-2\nu)$ is replaced by $(c_{11} + 2c_{12})$, in accordance with (14).

The thermomagnetic interaction can be investigated by retaining in (1) the contributions

$$(43) \quad \begin{aligned} \Sigma = \Sigma(M_i^*, \theta) = & \frac{1}{2}\rho \ 2X_{ij}^{(m)} M_i^* M_j^* + \frac{1}{4}\rho^3 \ 4X_{ijkl}^{(m)} M_i^* M_j^* M_k^* M_l^* + \\ & - \frac{1}{2}c(\theta - \theta_0)^2 - L_{ij}^{(m)} M_i^* M_j^* (\theta - \theta_0) . \end{aligned}$$

Substitution of (43) into the constitutive relation III.(49) yields, with (3) and (38)

$$(44) \quad S = \frac{c_w}{\theta_0} \theta + L^{(m)} M_i^* M_i^* .$$

In order to get an estimate of the magnitude of the coefficient $L^{(m)}$, let us consider the following problem. A medium is magnetized from its

initial state (i.e. $\theta = \theta_0$, $M_I^* = 0$) to its saturation point. This process is assumed to be adiabatic. Hence, (44) leads to

$$(45) \quad \delta S = 0 = \frac{c_w}{\theta_p} \theta + L^{(m)} M_S^2,$$

where θ is the increase in temperature due to the magnetization. From (45) it follows that

$$(46) \quad L^{(m)} = - \frac{c_w}{\theta_p M_S^2} \theta.$$

The increase in temperature appears, in general, to be very small. Only in the neighbourhood of the Curie temperature, there occurs an increase in temperature of some importance. For instance, for materials like iron and nickel at room temperature (i.e. 300°K) this increase is of the order of 10^{-3} °K, while at the Curie point it is about 1 à 2°K (cf. [30], pp. 427-431). With a value for c_w of approximately 4×10^6 erg/gr °K (viz. Table VI.1) this gives a value for $L^{(m)}$ of the order of 10^{-2} à 10^{-1} (°K)⁻¹ at $\theta_0 \approx 300$ K and one of 10 à 100 (°K)⁻¹ at the Curie temperature. However, we note that the effects at the Curie point are of a different nature, as at that temperature a phase transition takes place. We do not consider this kind of phenomena.

Another effect, produced by the thermomagnetic interaction, is the decrease of the saturation magnetization with temperature (cf. [30], p. 69). For instance, for iron the value of $4\pi\rho M_S$ decreases from 1735G at 0°K to 1714G at 300°K and for nickel from 509G at 0°K to 484G at 300°K (cf. [31], p. 54 and p. 270, respectively). Assuming that this process is also adiabatic, (45) yields values for $L^{(m)}$ of the order of 1 à 10 (°K)⁻¹.

All these values are so small, especially at room temperature, that in the sequel the thermomagnetic effects will be neglected.

VI.6. Exchange constants

By including in the set of independent variables the gradient of the magnetization, we take into account the macroscopic effect of the quantum mechanical exchange interaction. Since the interactions of the ex-

change - effect with the polarization, magnetization, deformation and temperature are very weak, we leave these out of consideration. In order to describe some typical exchange-actions, it suffices to take the following expression for Σ

$$(47) \quad \Sigma = \Sigma(M_{i,j}^*, M_{i,j}^*) = \frac{1}{2} \rho \chi_{ij}^{(m)} M_i^* M_j^* + \frac{1}{2} \rho^3 \chi_{ijkl}^{(m)} M_i^* M_j^* M_k^* M_l^* + \frac{1}{2} \rho \alpha_1 M_{i,j}^* M_{i,j}^* .$$

Comparing the last term of (47)

$$(48) \quad \frac{1}{2} \rho \alpha_1 M_{i,j}^* M_{i,j}^* = \Sigma_{ex} ,$$

with the expression for the exchange-energy per unit of volume according to [33], p. 61, eq. (11.12)

$$(49) \quad u_{ex} = A(|\nabla\theta_1|^2 + |\nabla\theta_2|^2 + |\nabla\theta_3|^2) ,$$

and bearing in mind that

$$\theta_i = \frac{M_i^*}{M_s} \quad \text{and} \quad u_{ex} = \rho \Sigma_{ex} ,$$

we obtain the following relation between the coefficient α_1 and the so called exchange-stiffness A

$$(50) \quad \alpha_1 = \frac{2A}{\rho^2 M_s^2} .$$

Elimination from the angular momentum equation II.(31) of the quantities $T_{[ij]}$ and Π_{ij} by means of the constitutive relations III.(51) and III.(52), into which (47) is substituted, yields

$$(51) \quad \frac{1}{I} \dot{M}_i^* = e_{ijk} M_j^* (H_k^* + \frac{2K_1}{\rho M_s^4} M_k^{*3} + \rho \alpha_1 M_{k,\ell\ell}^*) .$$

Identifying (51) with the angular momentum law according to [34], eqs. (1) and (5), we again obtain the relation (50).

The numerical values of the coefficient α_i are often determined by measurement of the eigen frequencies of spin waves. This method is used, for instance, in a paper by Le Craw & Walker [35]. They investigated a single crystal of the cubic material YIG, that was magnetized to saturation along a [111]-direction by a static biasing field \underline{H}^0 . Then, small dynamic disturbances of this state are considered. Let the static state be uniform, and let us take the z-axis along the \underline{H}^0 -direction. Thus

$$(52) \quad \begin{aligned} H_i(\underline{x}, t) &= H_i^0 + h_i(\underline{x}, t) , & H_i^0 &= \delta_{i3} H , \\ M_i(\underline{x}, t) &= M_i^0 + m_i(\underline{x}, t) , & M_i^0 &= \delta_{i3} M_s , \end{aligned}$$

where

$$(53) \quad \frac{|h|}{H} \ll 1 \quad \text{and} \quad \frac{|m|}{M_s} \ll 1 .$$

The fact that the magnetization is saturated, implies that (cf. eq. V. (7))

$$(54) \quad M_i^0 m_i = 0 ,$$

hence

$$(55) \quad \underline{m} = (m_x, m_y, 0) .$$

As done in [35], we hereafter neglect in (51) the magnetic anisotropy (i.e. $K_1 = 0$). We substitute (52) - (55) into (51) and linearize this equation with respect to h_i and m_i , yielding

$$(56) \quad \frac{1}{T} \dot{m}_i = e_{ij3} H m_j - e_{ij3} M_s (h_j + \rho \alpha_l m_j) .$$

Under the absence of electric fields, the Maxwell-equations give

$$(57) \quad e_{ijk} h_{k,j} = 0 \quad \text{and} \quad (h_i + 4\pi \rho m_i)_{,i} = 0 .$$

We try to solve the equations (56) - (57) by means of the substitution

$$(58) \quad \underline{h}(\underline{x}, t) = \hat{h} e^{i(\underline{k} \cdot \underline{x} + \omega t)} \quad \text{and} \quad \underline{m}(\underline{x}, t) = \hat{m} e^{i(\underline{k} \cdot \underline{x} + \omega t)}$$

where \underline{k} is the wave vector and ω the angular frequency.

If we restrict ourselves to spin waves in the x-z-plane, we may write the wave vector as

$$(59) \quad \underline{k} = (k_x, 0, k_z) .$$

Let φ be the angle between \underline{k} and the z-axis, i.e. the \underline{H}^0 -direction. Then, it follows from (57)¹ that

$$(60) \quad \hat{h}_y = 0, \quad \frac{\hat{h}_x}{\hat{h}_z} = \frac{k_x}{k_z} = \tan \varphi .$$

Substituting (60) into (57)² and using (55), give

$$(61) \quad \begin{aligned} \hat{h}_x &= -4\pi\rho\hat{m}_x\sin^2\varphi , \\ \hat{h}_y &= 0 , \\ \hat{h}_z &= -4\pi\rho\hat{m}_x\sin\varphi\cos\varphi . \end{aligned}$$

With these results, (56) can be worked out further. Written in components, we obtain

$$(62) \quad \begin{aligned} \frac{i\omega}{\Gamma} \hat{m}_x &= (H + \rho\alpha_1 k^2 M_s) \hat{m}_y , \\ \frac{i\omega}{\Gamma} \hat{m}_y &= -(H + \rho\alpha_1 k^2 M_s + 4\pi\rho M_s \sin^2\varphi) \hat{m}_x . \end{aligned}$$

Solving the characteristic equation of this system, results in the following relation for the angular frequency of the spin waves

$$(63) \quad \frac{\omega^2}{\Gamma^2} = (H + \rho\alpha_1 k^2 M_s)(H + \rho\alpha_1 k^2 M_s + 4\pi\rho M_s \sin^2\varphi) .$$

This relation corresponds with [35], eq. (1), if

$$(64) \quad \alpha_1 = \frac{D}{\rho\hbar\gamma M_s}$$

where D is an exchange parameter, \hbar is Planck's constant, i.e.

$$(65) \quad \hbar = 1.054 \times 10^{-27} \text{ erg-sec.} ,$$

and $\gamma = -\Gamma$: the negative of the gyromagnetic ratio.

On the basis of the experimental results of [35], the coefficient D turns out to be almost insensible to temperature in the range from 100°K to 400°K . At room temperature, a numerical value for D is found equal to

$$(66) \quad D = 0.99 \times 10^{-28} \text{ erg-cm}^2,$$

With the value of \hbar of (65) and the values

$$(67) \quad \rho M_s = 139\text{G} \quad \text{and} \quad \Gamma = -1.76 \times 10^7 (\text{G sec})^{-1},$$

according to [27], we find from (64) a value of

$$(68) \quad \alpha_1 = 3.74 \times 10^{-11} \text{ cm}^2,$$

for the exchange coefficient of YIG.

Since the value of α_1 is very small, the exchange interaction is often negligible. Yet, at very high frequencies ($> 10^9$ Hz.) these effects can have an observable influence.

VI.7. Electric constants

For a discussion of the electric effects in a cubic material, we retain the following terms in (1)

$$(69) \quad \Sigma = \Sigma(M_i^*, E_i^*) = \frac{1}{2} \rho \chi^{(m)} M_i^* M_i^* + \frac{1}{4} \rho^3 \chi_{ijkl}^{(m)} M_i^* M_j^* M_k^* M_l^* + \\ - \frac{1}{2\rho} \chi^{(e)} E_i^* E_i^* + \psi M_i^* E_i^*.$$

Substitution of (69) into the constitutive equation for the polarization III.(16) yields

$$(70) \quad \rho P_i^* = \chi^{(e)} E_i^* - \psi \rho M_i^*.$$

Leaving, for the time being, the magnetoelectric interaction out of consideration, (70) reduces to

$$(71) \quad \rho P_i^* = \chi^{(e)} E_i^*,$$

from which it is evident that $\chi^{(e)}$ represents the electric susceptibility (cf. [4], p. 12).

Substitution of (71) into I.(37) taken in its convective form, gives

$$(72) \quad D_i^* = (1 + 4\pi\chi^{(e)})E_i^* .$$

Introducing the dielectric constant or permittivity ϵ by

$$(73) \quad D_i^* = \epsilon E_i^* ,$$

we get

$$(74) \quad \chi^{(e)} = \frac{1}{4\pi} (\epsilon - 1) .$$

Returning to (70), and comparing this relation with eq. (2.9) of the monograph by T.H. O'Dell [36], we conclude that ψ is the magnetoelectric susceptibility. In [36], the theory and the experiments on the domain of magnetoelectric interactions up to 1970 are reviewed. For the very first time, this interaction is observed by Astrov [37] in 1960, who measured this effect in a single crystal of chromium oxide (Cr_2O_3), and almost simultaneously by Rado & Folen [38] in 1961. A magnetization of about 3 Amp/m ($= 10^{-3}$ G) was measured in a crystal Cr_2O_3 , placed in an electric field of 10^6 V/m.

O'Dell [36] shows that, for the class of crystalline materials with a point of symmetry, to which most of the ferro(i)magnetic materials belong, there can occur no linear magnetoelectric effects. However, a nonlinear phenomenon, the so called induced magnetoelectric effect, is possible (cf. [36], p. 142, and [39]). This effect stems from the lowering of the symmetry of a crystal by the application of a strong electric or magnetic field. So, for instance, a crystal loses its point of symmetry under these circumstances. This nonlinear effect can be described by including in Σ the terms

$${}_3\psi_{ijk}^{(e)} E_i^* E_j^* M_k^* + {}_3\psi_{ijk}^{(m)} M_i^* M_j^* E_k^* .$$

In the papers [39] to [42] the effect, mentioned above, is demonstrated by experimental measurements on YIG. It turns out that the term with ${}_3\psi_{ijk}^{(m)}$ can not appear, so it suffices to retain only the term with ${}_3\psi_{ijk}^{(e)}$, by which the constitutive equation for the polarization becomes

$$(75) \quad \rho P_i^* = [\chi_{ij}^{(e)} - 2 \sum_k \chi_{ijk}^{(e)} M_k^*] E_j^* .$$

From (75), it follows that the electric susceptibility depends on the magnetization. Cardwell [41], found the following relation for the relative change in the electric susceptibility due to a magnetic field along the [111]-direction for a single crystal YIG placed in an electric field along the [110]-direction

$$(76) \quad \frac{\Delta \chi^{(e)}}{\chi^{(e)}} \approx 3 \times 10^{-8} \times B .$$

For a value of B of 1 kG, this gives

$$(77) \quad \frac{\Delta \chi^{(e)}}{\chi^{(e)}} = 3 \times 10^{-5} ,$$

hence, a very small change.

The results found in [41] are, within the limits of experimental error, in agreement with those measured by Lee [42].

The foregoing results show that the effects of the magnetoelectric interactions are as weak as that we may neglect them in the following.

In the foregoing, all electroelastic interactions are vanished. The piezoelectric effect disappears due to the particular material symmetry we have considered. Moreover, we neglected in (1) the term

$$- \frac{1}{\rho_0} d_{\alpha\beta\gamma\delta} \Delta_\alpha \Delta_\beta E_{\gamma\delta} ,$$

representing the electrostriction, because we retained only linear terms in Δ_α in the interactions. It should be noted that the symmetry of the biased crystal is lower than that of the unloaded crystal. Consequently, just as for the magnetoelectric effect, there will appear an induced piezoelectric phenomenon in the biased crystal. However, this effect is a very weak one, and we shall neglect it in the sequel.

The piezoelectric effect is, in a technical sense, more important than the electrostriction, this in contrast with the corresponding magnetic phenomena. There has appeared a vast amount of literature on the subject of piezoelectricity. We refer only to the standard works by Cady

[43] and by Mason [44] and to the more recent book by Tiersten [45]. The theory of piezoelectricity is employed in the analysis and design of crystal oscillators, filters and transducers.

The electrostrictive interaction is of interest in ferroelectric materials, especially in those in which, due to the symmetry of the crystals, no piezoelectric effects occur (cf. [46], p. 51).

VI.8. Coefficients of conductivity

In concluding, we shall regard the coefficients occurring in the two laws of conductivity, i.e. Ohm's law III.(36) and Fourier's law III.(35).

We first consider Ohm's law, in which, until further notice, the coefficients β_{ij} are omitted. Instead of the electric conductivity σ_{ij} , often the reciprocal tensor r_{ij} , the so called electric resistivity, is used. Then, Ohm's law III.(36), with $\beta_{ij} = 0$, becomes

$$(78) \quad \underline{E}_i^* = r_{ij} J_j^* .$$

Since we have assumed a linear relationship between \underline{E}^* and \underline{J}^* , and because only the dependence of r_{ij} on $E_{\alpha\beta}$, Λ_α and θ has any practical importance (the influence of $M_{i\alpha}$ and θ_α will be neglected), we may state that

$$(79) \quad r_{ij} = r_{ij}(E_{\alpha\beta}, \Lambda_\alpha, \theta) .$$

We expand the coefficient r_{ij} in a series in its arguments, about the natural state, and we retain only linear terms in $E_{\alpha\beta}$ and $(\theta - \theta_0) = \theta$ and quadratic terms in Λ_α . For small deformations and for cubic material, this procedure yields

$$(80) \quad r_{ij} = (r_0 + r^{(t)}\theta)\delta_{ij} + \rho_\theta^2 r_{ijkl}^{(m)} M_k^* M_l^* + r_{ijkl}^{(d)} e_{kl} .$$

In this formula, r_0 is the resistivity in the natural state, while $r^{(t)}$, $r_{ijkl}^{(m)}$ and $r_{ijkl}^{(d)}$ are the coefficients of thermo-, magneto- and elastoresistivity, respectively.

First, retaining only thermal effects, we obtain

$$(81) \quad \Delta r = r - r_0 = r^{(t)}\theta .$$

This leads us to the following relation for the thermal coefficient of resistivity $R^{(t)}$ (cf. [31], p. 763)

$$(82) \quad R^{(t)} = \frac{\Delta r}{r_0 \theta} = \frac{r^{(t)}}{r_0}.$$

The arrays of the fourth-order tensors, occurring in (80), have, for crystals with a cubic symmetry, a form identical to (4).

Considering only magnetic effects, and defining the unit vector $\underline{\beta}$ by

$$(83) \quad \beta_i = \frac{J_i^*}{J} \text{ where } J = |\underline{J}^*|,$$

the relations (78) and (80) yield

$$(84) \quad \begin{aligned} E_i^* J_i^* = & r_0 J^2 + \frac{1}{3} J^2 M_s^2 (r_{11}^{(m)} + 2r_{12}^{(m)}) + \\ & + J^2 M_s^2 (r_{11}^{(m)} - r_{12}^{(m)}) (\beta_1^2 \theta_1^2 + \beta_2^2 \theta_2^2 + \beta_3^2 \theta_3^2 - \frac{1}{3}) + \\ & + 4J^2 M_s^2 r_{44}^{(m)} (\beta_1 \beta_2 \theta_1 \theta_2 + \beta_2 \beta_3 \theta_2 \theta_3 + \beta_3 \beta_1 \theta_3 \theta_1). \end{aligned}$$

By writing this relation in the form

$$(85) \quad E_i^* J_i^* = r_0 J^2 \left(1 + \frac{\Delta r}{r_0} \right),$$

we obtain

$$(86) \quad \begin{aligned} \frac{\Delta r}{r_0} = & -\frac{1}{3} k_3 + k_1 (\beta_1^2 \theta_1^2 + \beta_2^2 \theta_2^2 + \beta_3^2 \theta_3^2 - \frac{1}{3}) + \\ & + 2k_2 (\beta_1 \beta_2 \theta_1 \theta_2 + \beta_2 \beta_3 \theta_2 \theta_3 + \beta_3 \beta_1 \theta_3 \theta_1), \end{aligned}$$

where

$$(87) \quad k_1 = \frac{\rho^2 M_s^2 (r_{11}^{(m)} - r_{12}^{(m)})}{r_0}, \quad k_2 = \frac{2\rho^2 M_s^2 r_{44}^{(m)}}{r_0}, \quad k_3 = -\frac{\rho^2 M_s^2 (r_{11}^{(m)} + 2r_{12}^{(m)})}{r_0}.$$

The relation (86) corresponds with the equation for the relative change in resistivity due to magnetization according to [31], p. 764, if in the latter the fourth-order terms in θ_i are omitted. These fourth-order terms can be included in (86) by adding to the expansion (80)

$$r_{ijklmn}^{(m)} M_k^* M_\ell^* M_m^* M_n^* .$$

In an analogous way, we can describe the effects of the deformations on the resistivity. In that case we obtain instead of (84)

$$(88) \quad E_i^* J_i^* = r_w J^2 + J^2 \left\{ \frac{1}{3} (r_{11}^{(d)} + 2r_{12}^{(d)}) e_{kk} + \right. \\ \left. + (r_{11}^{(d)} - r_{12}^{(d)}) (e_{11} \beta_1^2 + e_{22} \beta_2^2 + e_{33} \beta_3^2 - \frac{1}{3} e_{kk}) + \right. \\ \left. + 4r_{44}^{(d)} (e_{12} \beta_1 \beta_2 + e_{23} \beta_2 \beta_3 + e_{31} \beta_3 \beta_1) \right\} .$$

Let us suppose that the deformation is solely due to magnetostriction. According to (23) with $T_{ij} = 0$, we have

$$(89) \quad c_{ijkl} e_{kl} = -p b_{ijkl} M_k^* M_\ell^* .$$

We write (88) in a form similar to (85) and we eliminate from this equation the deformations with the aid of (89). This yields the following relation for the relative change in resistivity due to magnetostriction for a cubic crystal

$$(90) \quad \frac{\Delta r}{r_0} = - \frac{B_0}{(c_{11} + 2c_{12})} \frac{(r_{11}^{(d)} + 2r_{12}^{(d)})}{r_0} + \\ + \frac{3\lambda_{100}}{2} \cdot \frac{(r_{11}^{(d)} - r_{12}^{(d)})}{r_0} (\beta_1^2 \theta_1^2 + \beta_2^2 \theta_2^2 + \beta_3^2 \theta_3^2 - \frac{1}{3}) + \\ + 3\lambda_{111} \cdot \frac{2r_{44}^{(d)}}{r_0} (\beta_1 \beta_2 \theta_1 \theta_2 + \beta_2 \beta_3 \theta_2 \theta_3 + \beta_3 \beta_1 \theta_3 \theta_1) ,$$

where (4) and (30) have been used.

Since the volume magnetostriction is in general negligible, we may omit the term with B_0 in (90). The remaining two terms can be interpreted in the following way

Let us take the magnetization and the electric current along the [100]-direction, i.e. let $\beta_1 = \beta_1 = 1$, $\beta_2 = \beta_3 = \theta_2 = \theta_3 = 0$, then

$$(91) \quad \left(\frac{\Delta r}{r_0}\right)_{[100]} = \lambda_{100} \cdot \frac{(r_{11}^{(d)} - r_{12}^{(d)})}{r_0}.$$

Applying both fields along the $[111]$ -direction, i.e. $\beta_i = \theta_i = \frac{1}{3}\sqrt{3}$, ($i = 1, 2, 3$), gives

$$(92) \quad \left(\frac{\Delta r}{r_0}\right)_{[111]} = \lambda_{111} \cdot \frac{2r_{44}^{(d)}}{r_0}.$$

In this way, we can infer for the coefficients $R_1^{(d)}$ and $R_2^{(d)}$, defined by

$$(93) \quad R_1^{(d)} = \frac{1}{\lambda_{100}} \left(\frac{\Delta r}{r_0}\right)_{[100]} \text{ and } R_2^{(d)} = \frac{1}{\lambda_{111}} \left(\frac{\Delta r}{r_0}\right)_{[111]},$$

the following relations

$$(94) \quad R_1^{(d)} = \frac{r_{11}^{(d)} - r_{12}^{(d)}}{r_0} \text{ and } R_2^{(d)} = \frac{2r_{44}^{(d)}}{r_0}.$$

In principle, from (93) and (94) the coefficients $(r_{11}^{(d)} - r_{12}^{(d)})$ and $r_{44}^{(d)}$ could be determined. However, there are very few experimental data on the elastoresistivity in single crystals available. For an isotropic material (e.g. a polycrystalline medium with a random orientation of the crystals and with the domains in the initial state distributed uniformly over all directions of easy magnetization) we have

$$(95) \quad \lambda_{100} = \lambda_{111} = \lambda \text{ and } 2r_{44}^{(d)} = r_{11}^{(d)} - r_{12}^{(d)},$$

thus then

$$(96) \quad R_1^{(d)} = R_2^{(d)} = R^{(d)} = \frac{1}{\lambda} \left(\frac{\Delta r}{r_0}\right).$$

Numerical values for these coefficients can be found in [31], p. 748, where we can calculate from the figures 16-5 and 16-6, the following global values for $R^{(d)}$

$$(97) \quad R^{(d)} \approx -6.0 \times 10^2, \text{ for nickel,}$$

$$R^{(d)} \approx -5.0 \times 10^2, \text{ for iron.}$$

We note that the elastoresistivity is an effect that is employed for the measurement of deformations on the boundaries of elastic bodies by means of strain gauges.

We have already considered Fourier's law in Section VI.5. We have seen there that for small temperature gradients, and under neglect of the influence of magnetization, deformation etc. on the thermal conductivity, and after omission of the β_{ij} -term, this law reduces to

$$(98) \quad Q_i = -\kappa_1 \theta_{,i} \quad \text{where } \kappa_1 = \kappa_1(\theta_0) .$$

Finally, there remains to discuss the coefficients β_{ij} , which for a cubic material become

$$(99) \quad \beta_{ij} = \beta \delta_{ij} .$$

The two terms with β in III.(35) and III.(36) produce well known thermoelectric phenomena like the Seebeck effect and the Peltier- and Thomson-heat (cf. [12], Chapter 12). These processes occur when electric and thermal conduction interfere with one another. These effects are all due to the fact that, as a result of the β -terms, a gradient in temperature causes an electric current, or an electric field produces a heat flux. For a description of the effects mentioned above, and for an impression of the order of magnitude of the effects we refer to [12], Chapter 12.

Numerical values for a number of the coefficients discussed in the preceding sections, for the materials YIG, nickel and iron, are listed in Table VI.1.

Name	Symbol	Unit	YIG	Ni	Fe
Density	ρ	gr/cm ³	5.17	8.90	7.87
Saturation Magnet.	ρM_s	G	139	38.5	136
Gyromagnetic C.	Γ	(G sec) ⁻¹	-1.76×10^7	-1.70×10^7	-1.76×10^7
Elastic C.	c_{11}	dyne/cm ²	2.69×10^{12}	2.50×10^{12}	2.41×10^{12}
	c_{12}	dyne/cm ²	1.08×10^{12}	1.60×10^{12}	1.46×10^{12}
	c_{44}	dyne/cm ²	0.764×10^{12}	1.18×10^{12}	1.12×10^{12}
Anisotropy C.	K_1	dyne/cm ²	6.28×10^3	-3.4×10^4	4.2×10^5
Magnetostrictive C.	B_1	dyne/cm ²	3.22×10^6	62×10^6	-29.5×10^6
	B_2	dyne/cm ²	6.44×10^6	86×10^6	71.2×10^6
Specific Heat	c_w	erg/gr °K	-	4.6×10^6	4.4×10^6
Thermal Conductivity	κ_1	dyne/sec °K	-	5.81×10^7	5.81×10^7
Lin. Thermal Expansion C.	α	(°K) ⁻¹	-	13×10^{-6}	11.7×10^{-6}
Exchange C.	α_1	cm ²	3.74×10^{-11}	-	2.16×10^{-10}
Electric Suscept.	$\chi^{(e)}$	-	1.50	-	-
Resistivity	r_o	Statohm-cm	-	0.756×10^{-17}	1.078×10^{-17}
Thermal C. of Resist.	$R^{(t)}$	(°K) ⁻¹	-	0.007	0.0065
Magneto-Resist. C.	k_1	-	-	0.063	0.00153
	k_2	-	-	0.029	0.00593
	k_3	-	-	-0.036	0.00194

References: [29], [30], [31] and [41].

TABLE VI.1. Numerical values for the material coefficients

Notes concerning Table VI.1:

- i) The numerical values in the table hold for single crystals.
- ii) YIG is an insulator, and Fe and Ni are not polarizable.
- iii) The values are valid at room temperature ($\theta_0 \approx 300^\circ\text{K}$).
- iv) The value of Γ is calculated from the relation (cf. [30], p. 41)

$$(100) \quad \Gamma = - \frac{ge}{2mc} ,$$

where e and m are the electronic charge and the electronic mass, respectively. In Gaussian units, we have for the quotient

$$(101) \quad \frac{e}{2mc} = 0.88 \times 10^7 \text{ (G sec)}^{-1} .$$

For the g -factor, we have used the results of the gyromagnetic experiments (e.g. Einstein-de Haas method) (cf. [30], p. 47).

- v) The coefficients B_1 and B_2 are calculated from the values of λ_{100} and λ_{111} by use of the relations (30).
- vi) The value of α_1 for iron is calculated from the equation (50). Further, we notice that the coefficient α_{11} employed in [29], is half α_1 .

VI.9. Elaboration of the linearized constitutive equations

For convenience, we recapitulate the linearized constitutive equations, derived in Chapter V, and we elaborate these equations by substituting the expression for Σ according to (1). We consider a material with a cubic symmetry that is magnetized to saturation along a cube edge, that is taken as the x_3 -direction. We assume the magnetization to be uniform, hence

$$(102) \quad M_i^* = \delta_{i3} M_s .$$

Taking allowance of the particular symmetry of the material and neglecting the magnetoelectric and the thermomagnetic interactions as well as the thermoelectric effects, we may take equal to zero the following coefficients

$$(103) \quad \psi_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}^{(m)} = \epsilon_{\alpha\beta\gamma}^{(e)} = \lambda_{\alpha}^{(m)} = L_{\alpha\beta}^{(m)} = \lambda_{\alpha}^{(e)} = \beta_{ij} = 0 .$$

Moreover, we employ the simplifications according to Section V.6, and we neglect, argued by the numerical values of Table VI.1, the terms containing ${}_2X_{ij}^{(m)}$ and ${}_4X_{ijkl}^{(m)}$ with respect to those containing b_{ijkl} , and, in turn, we neglect the terms with b_{ijkl} with respect to those with c_{ijkl} .

Utilizing the above information, we obtain for V.(18)-(34) the following set linear constitutive equations for

i) the entropy, by using (38) and (41),

$$(104) \quad s = -c_w \frac{\theta}{\theta_0} + \frac{(c_{11} + 2c_{12})\alpha}{\rho} u_{i,i} ,$$

(throughout this section, one must read ρ_0 for ρ)

ii) the polarization

$$(105) \quad P_i^* = \frac{\chi^{(e)}}{\rho} {}_0E_j^*(u_{i,j} + u_{j,i}) + \frac{\chi^{(e)}}{\rho} e_i^* ,$$

iii) the couple-stresses

$$(106) \quad \pi_{ij} = \rho^2 \alpha_1 m_{i,j}^* ,$$

iv) the stresses, with the aid of (26) and (41),

$$(107) \quad t_{ij} = -\delta_{ij} (c_{11} + 2c_{12})\alpha\theta + c_{ijkl} u_{k,l} + \\ + \frac{B_2}{M_s} (\delta_{ik} \delta_{j3} + \delta_{jk} \delta_{i3}) m_k^* - \chi^{(e)} ({}_0E_j^* e_i^* + {}_0E_i^* e_j^*) ,$$

v) the antisymmetric part of the stress tensor, with (18) and (26),

$$(108) \quad t_{[ij]} = B_2 \delta_{3[i} (u_{3,j]} + u_{j],3}) + \frac{2K_1}{M_s} \delta_{3[i} m_{j]}^* ,$$

vi) the heat flux, according to (98),

$$(109) \quad Q_i = -\kappa_1 \theta_{,i} ,$$

vii) and, finally, for the electric current, we have the reciprocal version of Ohm's law according to (78) with r_{ij} according to (80).

This relation can be worked out by using

$$(110) \quad M_i^* = \circ M_i^* + m_i^* = \delta_{i3} M_s^* + m_i^* ,$$

where

$$(111) \quad \underline{m}^* = (m_1^*, m_2^*, 0) ,$$

from which we find that

$$(112) \quad \rho^2 r_{ijkl}^{(m)} M_k^* M_l^* = \rho^2 M_s^2 r_{ij33}^{(m)} + 2 \rho M_s r_{ijk3}^{(m)} m_k^* .$$

Substituting (112) and (80) into (78), and decomposing \underline{E}^* and \underline{J}^* , we obtain

$$(113) \quad \circ E_i^* + e_i^* = (r \circ \delta_{ij} + \rho^2 M_s^2 r_{ij33}^{(m)}) (\circ J_j^* + j_j^*) + \\ + (r^{(t)} \circ \delta_{ij} + 2 \rho^2 M_s r_{ijk3}^{(m)} m_k^* + r_{ijkl}^{(d)} u_{k,\ell}) \circ J_j^* ,$$

which yields, with the aid of (82) and (87),

$$(114) \quad e_i^* = r \circ [1 - \frac{1}{3} (k_1 + k_3)] j_i^* + r \circ k_1 \delta_{i3} j_3^* + \\ + r \circ R^{(t)} \circ J_i^* \theta + \frac{r \circ}{M_s} k_2 (\delta_{ij} \circ J_3^* + \delta_{i3} \circ J_j^*) m_j^* + \\ + r_{ijkl}^{(d)} \circ J_j^* u_{k,\ell} .$$

In the balance equations and the boundary conditions, there also appear the field variables referred to the rigid-body state, e.g. $\circ P^*$, $\circ S$ etc. For these variables the following constitutive equations hold

$$\circ S = 0, \quad \circ P_i^* = \frac{\chi^{(e)}}{\rho} \circ E_i^* , \\ \circ \Pi_{ij} = 0, \quad \circ T_{[ij]} = 0 , \\ (115) \quad \circ T_{ij} = (\delta_{i3} \delta_{j3} - \frac{1}{3} \delta_{ij}) B_1 - \chi^{(e)} \circ E_i^* \circ E_j^* , \\ \circ Q_i = 0 , \\ \circ E_i^* = r \circ [1 - \frac{1}{3} (k_1 + k_3)] \circ J_i^* + r \circ k_1 \delta_{i3} \circ J_3^* .$$

On deriving the fifth equation of the above set, the relation (26) has been used and the volume magnetostriction has been neglected. We note that, because the body is rigid, the stress tensor σ_{ij} is not a measurable quantity, but that it is merely a mathematical concept. Moreover, these stresses do satisfy neither the momentum equation, except when the ξ -state is uniform, nor the boundary conditions.

Based on the numerical values of Table VI.1, it follows from (107) and (108), that the antisymmetric part of the stress tensor is much smaller than the symmetric part. Hence, in the equation of motion and in the boundary condition for the stresses, we may approximate t_{ij} by $t_{(ij)}$. However, we may not at all neglect $t_{[ij]}$ in the angular momentum equation.

VII. VIBRATIONS OF A CYLINDER IN A MAGNETIC FIELD

VII.1. Introduction

It is the purpose of the present chapter, to apply the theory of Chapter V and VI to a special problem. To this end, the linearized equations and boundary conditions derived there are employed for the important case of a homogeneous, static intermediate state. Moreover, the equations are specified for the cubic material YIG. We are especially interested in the influence of features as the exchange interaction, the magnetic anisotropy, the gyromagnetic coupling etc.

The linear equations of the preceding chapters are applied to the determination of the solution for the steady-state vibrations of an infinite cylinder, which is magnetized to saturation along a cube edge by a biasing magnetic field in the axial direction. The driving field is perpendicular to the cylinder-axis. We notice that the material of the cylinder is an insulator, thus electric currents are absent. Moreover, thermal effects are left out of consideration. It will be shown that the steady-state vibrations are purely axial.

In some articles dealing with similar subjects, cf. e.g. [29] and [47], the Maxwell-equations are taken a priori in its quasi-static version (i.e. $c^{-1} = 0$). This approach will not be followed here, but we shall establish, that under the conditions of our problem, our results differ with those of the quasi-static version only in terms of $O(v^2/c^2)$. However, we must note that this is no longer true in more general problems.

In the present chapter it is shown, on the basis of the data for YIG listed in Table VI.1, that some effects can be neglected. Among the negligible terms are found the stresses of the rigid-body state (i.e. ${}_0T_{ij}$) and the antisymmetric part of the stress tensor compared with the symmetric part. Moreover, it will turn out that for a special range of the numerical values of the parameters of the problem, the exchange in-

teraction can be neglected. In the latter case, the system governing our problem, simplifies considerably. For this case we have found a relation for the determination of the resonant frequencies and we have calculated some values for the stresses in the cylinder.

VII.2. Statement of the problem

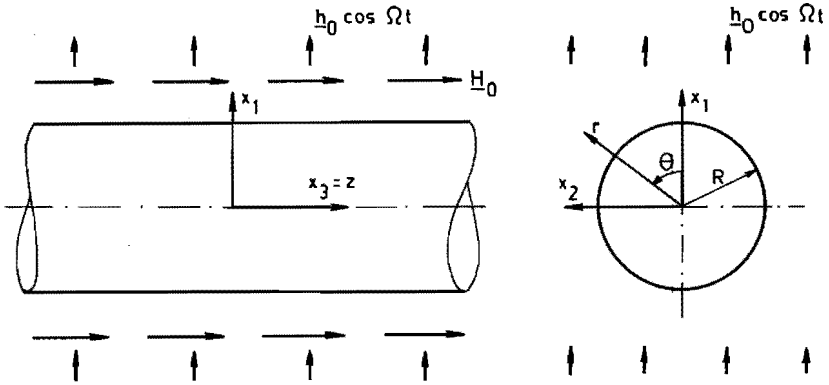


Fig. VII. 1.

Let us consider a homogeneous, nonconducting cylinder of infinite extension, consisting of a single crystal of the cubic material YIG. Along the cube edges of the crystal, a rectangular Cartesian coordinate system (x_1, x_2, x_3) is chosen, with the x_3 -axis in the axial direction (cf. Fig. VII.1). The cylinder is placed in a uniform and static magnetic field H_0 , directed along the x_3 -axis, and as large as that the magnetization is saturated. The space outside the cylinder is vacuum. The cross-section of the cylinder is a circle with radius R .

On the biasing field H_0 a small dynamic field $h_0 \cos \Omega t$, where h_0 is a uniform and constant vector, is superposed. This driving field is directed along the x_1 -axis, hence perpendicular to the axis of the cylin-

der. Thus we have

$$(1) \quad \underline{H}_0 = (0, 0, H_0), \quad \underline{h}_0 = (h_0, 0, 0) \text{ with } \frac{h_0}{H_0} \ll 1.$$

Labeling the electromagnetic quantities outside the cylinder with an upperindex +, and those inside the cylinder with -, the solution of the rigid-body problem reads

$$\begin{aligned} \circ B_i^+ &= \delta_{i3} H_0, \quad \circ H_i^+ = \delta_{i3} H_0, \quad \circ M_i^+ = 0, \\ (2) \quad \circ B_i^- &= \delta_{i3} (H_0 + 4\pi\rho M_s), \quad \circ H_i^- = \delta_{i3} H_0, \quad \circ M_i^- = \delta_{i3} M_s, \\ \circ D_i^\pm &= \circ E_i^\pm = \circ P_i^\pm = \circ J_i^\pm = \circ Q^\pm = 0. \end{aligned}$$

According to VI.(115) the rigid-body stress tensor $\circ T_{ij}$ is equal to

$$(3) \quad \circ T_{ij} = (\delta_{i3} \delta_{j3} - \frac{1}{3} \delta_{ij}) B_1,$$

while the couple stress $\circ \Pi_{ij}$ is equal to zero.

In the forthcoming calculations, it will turn out that the stresses according to (3) do not have any essential influence on the equations of the disturbances.

In the sequel, we shall use instead of the system (x_1, x_2, x_3) the cylindrical coordinates (r, θ, z) defined by

$$(4) \quad x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z.$$

VII.3. Equations for the disturbances

We shall elaborate the balance equations and boundary conditions for the disturbances, outlined in the Sections V.4 and V.5, by taking allowance of the aforementioned restrictions and by employing the simplifications discussed in Section VI.9.

According to VI.(105) and with the results of (2), we obtain for the polarization

$$(5) \quad p_i^* = \frac{\chi}{\rho} e_i^* = \frac{\chi}{\rho} \left[e_i + \frac{(H_0 + 4\pi\rho M_s)}{c} e_{ij3} \dot{u}_j \right],$$

where in the latter step the relations I.(43) are used, which also lead to

$$(6) \quad p_i^* = p_i - \frac{M_s}{c} e_{ij3} \dot{u}_j .$$

From (5) and (6) it follows that

$$(7) \quad p_i = \frac{\chi^{(e)}}{\rho} e_i + \frac{1}{c} \left[\frac{\chi^{(e)} H_0}{\rho} + (1 + 4\pi\chi^{(e)}) M_s \right] e_{ij3} \dot{u}_j .$$

Moreover, the constitutive equations VI.(106)-(108) reduce to

$$(8) \quad \pi_{ij} = \rho^2 \alpha_{1m_{i,j}} ,$$

$$(9) \quad t_{ij} = c_{ijkl} u_{k,l} + \frac{B_2}{M_s} (\delta_{ik} \delta_{j3} + \delta_{jk} \delta_{i3}) m_k ,$$

$$(10) \quad t_{[ij]} = B_2 \delta_{3[i} (u_{3,j]} + u_{j],3}) + \frac{2K_1}{M_s} \delta_{3[i} m_{j]} ,$$

where the fact that

$$(11) \quad m_i^* = m_i ,$$

has been used.

As the material is nonconductive, the quantities q_i and j_i^* are taken equal to zero.

Utilizing the fact that the intermediate state is a uniform and static one and with the results of (2), the balance equations of Section V.4, approximated in the way outlined in Section VI.9, become

$$(12) \quad \begin{aligned} \frac{1}{c} \dot{b}_i &= -e_{ijk} e_{k,j}, & b_{i,i} &= 0, \\ \frac{1}{c} \dot{d}_i &= e_{ijk} h_{k,j}, & d_{i,i} &= 0, \\ b_i &= h_i + 4\pi\rho m_i - 4\pi\rho M_s \delta_{i3} u_{j,j}, \\ d_i &= e_i + 4\pi\rho p_i, \end{aligned}$$

for the electromagnetic quantities, and

$$(13) \quad \rho \ddot{u}_i = t_{ij,j} + f_i^{(e)},$$

where

$$(14) \quad f_i^{(e)} = \rho M_s h_{3,i} + \frac{\rho}{c} e_{ij3} [(H_0 + 4\pi\rho M_s) \dot{p}_j + M_s \dot{a}_j],$$

and, finally

$$(15) \quad \frac{1}{r} \dot{m}_i = e_{ij3} \{-M_s h_j + H_0 m_j\} - \frac{1}{\rho} e_{ijk} t_{[jk]} - \frac{M_s}{\rho} e_{ij3} \pi_{j,kk},$$

$$m_3 = 0, \quad (i = 1, 2)$$

the latter relation according to V.(7).

Analogously, the boundary conditions of Section V.5 reduce to

$$(16) \quad \begin{aligned} [e_{ijk} e_j] N_k &= -\frac{1}{c} \delta_{i3} [{}_a B_3] \dot{u}_{k,N_k} = \delta_{i3} \frac{4\pi\rho M_s}{c} \dot{u}_{k,N_k}, \\ [d_i] N_i &= 0, \\ [e_{ijk} h_j] N_k &= [e_{ijk} \circ H_j] u_{k,N_k} = 0, \\ [b_i] N_i &= [{}_a B_3] u_{j,3} N_j = -4\pi\rho M_s u_{j,3} N_j, \end{aligned}$$

and

$$(17) \quad t_{ij} N_j = {}_a T_{ij} u_{k,j} N_k,$$

and

$$(18) \quad M_s \delta_{i3} [i \pi_j]_k N_k = 0,$$

all holding on the lateral surface of the cylinder ($r = R$), that is assumed to be free of mechanical stresses (i.e. $T_i^* = 0$).

We note that, because the null-stress ${}_a T_{ij}$ is proportional to B_1 and because $B_1 \ll c_{11}$ (or c_{12}, c_{44}) the right-hand side of (17) is negligible compared with the term $c_{ijkl} u_{k,l}$, occurring in t_{ij} . Hence, this stress has no influence on the disturbances.

By elimination of the quantities t_{ij} , $t_{[ij]}$, π_{ij} , p_i , d_i and b_i , the equations listed above can be reduced to a system for the unknowns e_i , h_i , m_i and u_i . The boundary conditions still simplify any further, due to the fact that the normal vector \underline{N} has only a component in the radial

direction. Moreover, we note that the problem is uniform in the z-direction. Therefore, the steady-state solution will be independent of z, so all terms containing $\partial/\partial z$ may be omitted. Thus, we arrive at the following systems

i) outside the cylinder ($r > R$)

$$\begin{aligned}
 \frac{1}{c} \dot{h}_r^+ &= -\frac{1}{r} e_{z,\theta}^+, \quad \frac{1}{c} \dot{h}_\theta^+ = e_{z,r}^+, \\
 \frac{1}{c} \dot{h}_z^+ &= \frac{1}{r} e_{r,\theta}^+ - \frac{1}{r} e_\theta^+ - e_{\theta,r}^+, \\
 h_{r,r}^+ + \frac{1}{r} h_r^+ + \frac{1}{r} h_{\theta,\theta}^+ &= 0, \\
 \frac{1}{c} \dot{e}_r^+ &= \frac{1}{r} h_{z,\theta}^+, \quad \frac{1}{c} \dot{e}_\theta^+ = -h_{z,r}^+, \\
 \frac{1}{c} \dot{e}_z^+ &= -\frac{1}{r} h_{r,\theta}^+ + \frac{1}{r} h_\theta^+ + h_{\theta,r}^+, \\
 e_{r,r}^+ + \frac{1}{r} e_r^+ + \frac{1}{r} e_{\theta,\theta}^+ &= 0, \\
 h_r^+ &\rightarrow h_0 \cos \theta, \quad h_\theta^+ \rightarrow h_0 \sin \theta, \quad h_z^+ \rightarrow 0 \text{ for } r \rightarrow \infty.
 \end{aligned}
 \tag{19}$$

ii) inside the cylinder ($r < R$)

$$\begin{aligned}
 \frac{1}{c} (\dot{h}_r^- + 4\pi\rho\dot{m}_r) &= -\frac{1}{r} e_{z,\theta}^-, \quad \frac{1}{c} (\dot{h}_\theta^- + 4\pi\rho\dot{m}_\theta) = e_{z,r}^-, \\
 \frac{1}{c} [(\dot{h}_z^- - 4\pi\rho M_s (\dot{u}_{r,r} + \frac{1}{r} \dot{u}_r + \frac{1}{r} \dot{u}_{\theta,\theta}))] &= \frac{1}{r} e_{r,\theta}^- - \frac{1}{r} e_\theta^- - e_{\theta,r}^-, \\
 h_{r,r}^- + \frac{1}{r} h_r^- + \frac{1}{r} h_{\theta,\theta}^- + 4\pi\rho (m_{r,r} + \frac{1}{r} m_r + \frac{1}{r} m_{\theta,\theta}) &= 0, \\
 \frac{(1 + 4\pi\chi^{(e)})}{c} \dot{e}_r^- &= \frac{1}{r} h_{z,\theta}^-, \quad \frac{(1 + 4\pi\chi^{(e)})}{c} \dot{e}_\theta^- = -h_{z,r}^-, \\
 \frac{(1 + 4\pi\chi^{(e)})}{c} \dot{e}_z^- &= -\frac{1}{r} h_{r,\theta}^- + \frac{1}{r} h_\theta^- + h_{\theta,r}^-, \\
 (1 + 4\pi\chi^{(e)}) (e_{r,r}^- + \frac{1}{r} e_r^- + \frac{1}{r} e_{\theta,\theta}^-) &= \frac{4\pi\rho M_s}{c} (\frac{1}{r} u_{r,\theta} - \frac{1}{r} u_\theta - u_{\theta,r}),
 \end{aligned}
 \tag{20}$$

$$\begin{aligned}
\rho \ddot{u}_r &= \tau_{rr,r} + \frac{1}{r} \tau_{r\theta,\theta} + \frac{1}{r} (\tau_{rr} - \tau_{\theta\theta}) + \frac{\chi^{(e)} H_0}{c} \dot{e}_\theta^-, \\
\rho \ddot{u}_\theta &= \tau_{\theta r,r} + \frac{1}{r} \tau_{\theta\theta,\theta} + \frac{2}{r} \tau_{\theta r} - \frac{\chi^{(e)} H_0}{c} \dot{e}_r^-, \\
\rho \ddot{u}_z &= \tau_{zr,r} + \frac{1}{r} \tau_{z\theta,\theta} + \frac{1}{r} \tau_{zr} + \frac{B_2}{M_s} (m_{r,r} + \frac{1}{r} m_r + \frac{1}{r} m_{\theta,\theta}), \\
\frac{1}{\Gamma} \dot{m}_r &= H_0 m_\theta - M_s h_\theta^- + \frac{B_2}{\rho} \frac{1}{r} u_{z,\theta} + \frac{2K_1}{\rho M_s} m_\theta + \\
&\quad - \rho M_s \alpha_1 [\Delta m_\theta + \frac{2}{r} m_{r,\theta} - \frac{1}{r^2} m_\theta], \\
-\frac{1}{\Gamma} \dot{m}_\theta &= H_0 m_r - M_s h_r^- + \frac{B_2}{\rho} u_{z,r} + \frac{2K_1}{\rho M_s} m_r + \\
&\quad - \rho M_s \alpha_1 [\Delta m_r - \frac{2}{r} m_{\theta,\theta} - \frac{1}{r^2} m_r], \\
m_z &= 0,
\end{aligned}$$

where

$$(21) \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2},$$

and

$$(22) \quad \tau_{ij} = c_{ijkl} u_{k,l},$$

and

iii) in conclusion, on the lateral surface ($r = R$)

$$\begin{aligned}
e_r^+ - (1 + 4\pi\chi^{(e)}) e_r^- &= - \frac{4\pi\rho M_s}{c} \dot{u}_\theta, \\
e_\theta^+ - e_\theta^- &= - \frac{4\pi\rho M_s}{c} \dot{u}_r, \quad e_z^+ - e_z^- = 0, \\
(23) \quad h_r^+ - h_r^- &= 4\pi\rho m_r, \quad h_\theta^+ - h_\theta^- = 0, \quad h_z^+ - h_z^- = 0,
\end{aligned}$$

$$\tau_{rr} = 0, \tau_{r\theta} = 0, \tau_{rz} = -\frac{B_2}{M_3} m_r,$$

$$m_{r,r} = 0, m_{\theta,r} = 0.$$

We note that the terms τ_{ij} are not yet written out in components; this will be deferred until later on. However, we wish to point out that, due to the particular form of the arrays of the elastic constants for a cubic material, the first two equations of motion (20)^{9,10} and the first two boundary conditions for the stresses (23)^{7,8} contain only the displacement components u_r and u_θ , while the third ones, i.e. (20)¹¹ and (23)⁹, contain only u_z . This can be established more easily by writing (20)^{9,10} and (23)^{7,8} in components with respect to x_1 and x_2 instead of r and θ .

VII.4. General solution

We are searching for the steady-state solution of the system consisting of (19), (20) and (23). Regarding this system we conclude that we can decompose it into a homogeneous system in the unknowns u_r , u_θ , e_r , e_θ and h_z , plus an inhomogeneous system in u_z , e_z , h_r , h_θ , m_r and m_θ . The steady-state solution of the first system reads

$$(24) \quad u_r = u_\theta = e_r = e_\theta = h_z = 0.$$

Moreover, leaded by the particular form of the pertinent equations, it seems reasonable to assume that e_z^\pm is proportional to c^{-1} . This means that we have to put equal to zero the left-hand side of (19)⁷ and of (20)⁷. This assumption can be verified a posteriori by determining e_z^\pm from the relations (19)^{1,2}, (20)^{1,2} and (23)³.

Utilizing this result, the thus obtained equations turn out to be identical with the system that we would have obtained, if we had employed right from the start the quasi-static version (i.e. $c^{-1} = 0$) of the electromagnetic equations. However, we point out that this conclusion does not necessarily hold anymore, if $\epsilon E_i \neq 0$ or if the $\underline{\xi}$ -configuration is no longer a static state. Furthermore, it turns out that, as u_z is the only displacement unequal to zero, we have a purely axial vibration.

We try to solve the remaining system, consisting of the equations (19)^{4,7,9,10}, (20)^{4,7,11,12,13} and (23)^{4,5,9,10,11}, by introducing the scalar potentials ψ and φ by

$$(25) \quad h_r^+ = -\psi_{,r}, \quad h_\theta^+ = -\frac{1}{r} \psi_{,\theta}, \quad h_r^- = -\varphi_{,r}, \quad h_\theta^- = -\frac{1}{r} \varphi_{,\theta},$$

and by a separation of variables according to

$$(26) \quad \begin{aligned} u_z(r, \theta, t) &= \operatorname{Re}[w e^{i\Omega t}], \\ m_r(r, \theta, t) &= \operatorname{Re}[\{\phi_{,r} + \frac{1}{r} \theta_{,\theta}\} e^{i\Omega t}], \\ m_\theta(r, \theta, t) &= \operatorname{Re}[\{\frac{1}{r} \phi_{,\theta} - \theta_{,r}\} e^{i\Omega t}], \\ \varphi(r, \theta, t) &= \operatorname{Re}[\{4\pi\rho\phi + \lambda\} e^{i\Omega t}], \\ \psi(r, \theta, t) &= \operatorname{Re}[\Psi e^{i\Omega t}], \end{aligned}$$

where w , ϕ , θ , λ and Ψ are functions of r and θ .

By the relations (25), the equations (19)⁷ and (20)⁷ are satisfied identically.

The relations (20)¹² and (20)¹³ are transformed into two other equations by means of the operations

$$(27) \quad \begin{aligned} \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)(20)^{12} - \left(\frac{1}{r} \frac{\partial}{\partial \theta}\right)(20)^{13}, \\ \left(\frac{1}{r} \frac{\partial}{\partial \theta}\right)(20)^{12} + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)(20)^{13}. \end{aligned}$$

Substituting (25) and (26) successively into (20)¹¹, the transformed equations (20)¹² and (20)¹³, (20)⁴, (19)⁴ and (19)^{9,10}, we arrive at the following system

$$(28) \quad \begin{aligned} c\Delta w + \rho\Omega^2 w + 2\rho\beta\Delta\phi &= 0, \\ \Delta(\mu\phi - \alpha\Delta\phi + 2\beta M_s w - \frac{i\Omega}{r}\theta) &= 0, \\ \Delta(\kappa\theta - \alpha\Delta\theta + \frac{i\Omega}{r}\phi) &= 0, \\ \Delta\lambda &= 0, \quad (\text{all these for } r < R), \\ \Delta\Psi &= 0, \quad (r > R) \\ \Psi &\rightarrow -h_0 r \cos\theta, \quad \text{for } r \rightarrow \infty. \end{aligned}$$

where

$$(29) \quad c = c_{44}, \quad \beta = \frac{B_2}{2\rho M_s}, \quad \kappa = H_0 + \frac{2K_1}{\rho M_s}, \quad \mu = \kappa + 4\pi\rho M_s,$$

$$\alpha = \rho M_s \alpha_1.$$

By means of the operations (27), we have introduced two extra constants, which can be determined by substituting the general solution of (28) into the original equations (19) and (20).

Elimination from (28)¹ of the variable ϕ by using (28)^{2,3}, yields the following equation for $w(r, \theta)$

$$(30) \quad \Delta\Delta w + a_1\Delta w + a_2w + a_3w = 0,$$

where

$$(31) \quad a_1 = \frac{\rho\Omega^2}{c} + \frac{4\rho M_s\beta^2}{c\alpha} - \frac{(\mu + \kappa)}{\alpha},$$

$$a_2 = \frac{1}{\alpha^2} \left(\mu\kappa - \frac{\Omega^2}{\Gamma^2} \right) - \frac{4\rho M_s\kappa\beta^2}{c\alpha^2} - \frac{\rho(\mu + \kappa)\Omega^2}{c\alpha},$$

$$a_3 = \frac{\rho\Omega^2}{c\alpha} \left(\mu\kappa - \frac{\Omega^2}{\Gamma^2} \right).$$

We note that, as a consequence of (28)⁶, the steady-state solution of (26) must be periodic in θ , with period 2π . Moreover, the function w must remain finite for $r = 0$.

Meeting these conditions, the solution of (30) reads

$$(32) \quad w(r, \theta) = \sum_{n=1}^3 \{ P_n J_1(\lambda_n r) \cos \theta + i Q_n J_1(\lambda_n r) \sin \theta \},$$

where P_n and Q_n are complex constants, $J_1(\lambda r)$ is the first order Bessel function of the first kind, while the numbers λ_n are the roots with $\text{Re}(\lambda_n) > 0$ of

$$(33) \quad \lambda^6 - a_1\lambda^4 + a_2\lambda^2 - a_3 = 0.$$

After substitution of (32) into (28), the latter system can be solved. By once more substituting the then obtained solution into (26), we arrive at the following general solution of the system (19)-(20)

$$\begin{aligned}
 u_z &= \operatorname{Re} \sum_{n=1}^3 \{P_n J_1(\lambda_n r) \cos \theta + i Q_n J_1(\lambda_n r) \sin \theta\} e^{i\Omega t}, \\
 m_r &= \operatorname{Re} \left[E - \sum_{n=1}^3 p_n P_n \frac{dJ_1(\lambda_n r)}{dr} - \sum_{n=1}^3 q_n Q_n \frac{J_1(\lambda_n r)}{r} \right] \cos \theta + \\
 &+ \left\{ F - \sum_{n=1}^3 p_n Q_n \frac{dJ_1(\lambda_n r)}{dr} - \sum_{n=1}^3 q_n P_n \frac{J_1(\lambda_n r)}{r} \right\} i \sin \theta e^{i\Omega t}, \\
 m_\theta &= \operatorname{Re} \left[F - \sum_{n=1}^3 p_n Q_n \frac{J_1(\lambda_n r)}{r} - \sum_{n=1}^3 q_n P_n \frac{dJ_1(\lambda_n r)}{dr} \right] \cos \theta + \\
 (34) \quad &+ \left\{ E - \sum_{n=1}^3 p_n P_n \frac{J_1(\lambda_n r)}{r} - \sum_{n=1}^3 q_n Q_n \frac{dJ_1(\lambda_n r)}{dr} \right\} i \sin \theta e^{i\Omega t}, \\
 \varphi &= \operatorname{Re} \left[\{Cr - 4\pi\rho \sum_{n=1}^3 p_n P_n J_1(\lambda_n r)\} \cos \theta + \right. \\
 &+ \left. \{Dr - 4\pi\rho \sum_{n=1}^3 p_n Q_n J_1(\lambda_n r)\} i \sin \theta e^{i\Omega t} \right], \\
 \psi &= \operatorname{Re} \left[(-h_0 r + \frac{A}{r}) \cos \theta + \frac{B}{r} i \sin \theta e^{i\Omega t} \right],
 \end{aligned}$$

where A, B, C, D, E, F, P_n and Q_n are complex constants and

$$(35) \quad p_n = \frac{(c\lambda_n^2 - \rho\Omega^2)}{2\rho\beta\lambda_n^2}, \quad q_n = \frac{\Gamma}{\Omega} [(\mu + \alpha\lambda_n^2)p_n - 2\beta M_s].$$

The boundary conditions (23) yield ten equations for the twelve unknown constants A, B, C, D, E, F, P_n and Q_n ($n = 1, 2, 3$). As noticed before, two extra constants are introduced by means of the differentiations in (27). By substituting (34) into (20)^{12, 13}, we find the two lacking relations

$$M_s D - \frac{\Omega}{T} E + \kappa F = 0 ,$$

(36)

$$M_s C + \kappa E - \frac{\Omega}{T} F = 0 .$$

At this point, we have derived a complete system from which the twelve coefficients occurring in (34) can be calculated for given values of H_0 , R , Ω and h_0 . However, as this system consists of twelve equations, the solution of it is still a very laborious task. Therefore, in the next section we shall try to approximate this system by a more simple one, holding for a special range of values for R and Ω , by taking into account the numerical values of the material constants for YIG.

Based on the special form of the aforesaid system, we conclude that the fundamental solution of our differential equations are coupled at the stress-free and couple-stress-free lateral surface of the cylinder. However, since the coefficient α_1 is very small, this coupling is very weak. In one of the forthcoming sections we shall show that this coupling disappears when the exchange interaction is neglected (viz. Section VII.6).

VII.5. Elaboration of the solution

We shall elaborate the general solution derived in the last section by using the numerical values for the material coefficients of YIG, as listed in Table VII.1. With these values we find for the pertinent coefficients, with the aid of (29),

$$\begin{aligned} c &= 0.764 \times 10^{12} \text{ (dyne/cm}^2\text{)}, \quad \rho = 5.17 \text{ (gr/cm}^3\text{)} , \\ \rho M_s &= 1.39 \times 10^3 \text{ (G)} , \quad \beta = 2.32 \times 10^4 \text{ (G)} , \\ \alpha &= 5.20 \times 10^{-9} \text{ (G cm}^2\text{)} , \quad \Gamma = -1.76 \times 10^7 \text{ (G}^{-1}\text{sec}^{-1}\text{)} , \\ \kappa &= 0.903 \times 10^2 + H_0 \text{ (G)} , \quad \mu = 1.840 \times 10^3 + H_0 \text{ (G)} . \end{aligned}$$

(37)

We note that for ferromagnetic materials the value of H_0 for which the magnetization of the cylinder is saturated is small (in the order of 1G) compared with M_s (e.g. cf. [31], p. 57, Fig. 3-5, or p. 62, Fig. 3-12). Therefore, within a reasonable exactness, we may replace μ and κ by

$$(38) \quad \kappa = 0.903 \times 10^2 \text{ (G)}, \mu = 1.840 \times 10^3 \text{ (G)} .$$

In this approximation, the solution will be independent of H_0 . In order to simplify the equations of the foregoing section, we impose the following two restrictions on the parameters Ω and R

$$(39) \quad 10^7 \leq \Omega \leq 10^9 \text{ (Hz)} ,$$

and

$$(40) \quad 10^{-3} \leq R \leq 10^{-1} \text{ (cm)} .$$

Underlying the choice for the bounds of Ω according to (39) are the following arguments

- i) The lower bound is chosen as large as that the gyromagnetic effects have a noticeable influence.
- ii) The upper bound is a reasonable limit of the frequencies that are in practice attainable in experiments.

The range for the values of R is selected in such a way that there are resonant frequencies within the reach of (39).

Using the numerical values of (37) and allowing for the restrictions according to (39), it appears that

$$(41) \quad \begin{aligned} a_1 &\approx \frac{\rho\Omega^2}{c} =: \omega^2, \quad (0.678 \times 10^3 \leq \omega^2 \leq 0.678 \times 10^7 \text{ (cm}^{-2}\text{)}) \\ a_2 &\approx \frac{1}{\alpha^2} \left(\mu\kappa - \frac{\Omega^2}{\Gamma^2} \right) =: \eta^4, \quad (6.02 \times 10^{21} \leq \eta^4 \leq 6.14 \times 10^{21} \text{ (cm}^{-4}\text{)}), \\ a_3 &\approx \frac{\rho\Omega^2}{c\alpha^2} \left(\mu\kappa - \frac{\Omega^2}{\Gamma^2} \right) = \omega^2 \eta^4 . \end{aligned}$$

Substitution of (41) into (33) transforms the characteristic equation into

$$(42) \quad \lambda^6 - \omega^2 \lambda^4 + \eta^4 \lambda^2 - \omega^2 \eta^4 = 0 .$$

This equation has the following roots λ_n , with $\text{Re } \lambda_n > 0$,

$$(43) \quad \lambda_1 = \omega, \lambda_2 = i^{\frac{1}{2}} \eta, \lambda_3 = i^{-\frac{1}{2}} \eta, \quad (\omega, \eta > 0) .$$

At this point, a remark has to be made. By substituting the expression for λ_1 according to (43)¹ into (35)¹ we find $p_1 = 0$. However, this equality is not correct, because λ_1 is only approximately equal to ω . To obtain an improved expression for p_1 , we put

$$(44) \quad \lambda_1^2 = \omega^2 + \epsilon \text{ where } \epsilon \ll \omega^2,$$

substitute this relation into (33), with the coefficients a_n ($n=1,2,3$) according to (31), and linearize the thus obtained equation with respect to ϵ . In this way we find, retaining only dominant terms,

$$(45) \quad \lambda_1^2 = \omega^2 \left[1 + \frac{4\rho\kappa\beta^2 M_s}{c(\mu\kappa - \frac{\Omega^2}{\Gamma^2})} \right].$$

Substitution of (45) into (35) yields

$$(46) \quad p_1 = \frac{2\kappa\beta M_s}{(\mu\kappa - \frac{\Omega^2}{\Gamma^2})} \text{ and } q_1 = \frac{2\beta M_s \Omega}{\Gamma(\mu\kappa - \frac{\Omega^2}{\Gamma^2})} = \frac{\Omega}{\kappa\Gamma} p_1.$$

By means of (43), the solution (32) can be written as

$$(47) \quad w(r, \theta) = \{P_1 J_1(\omega r) + P_2 J_1(i^{\frac{1}{2}} \eta r) + P_3 J_1(i^{-\frac{1}{2}} \eta r)\} \cos \theta + \\ + \{Q_1 J_1(\omega r) + Q_2 J_1(i^{\frac{1}{2}} \eta r) + Q_3 J_1(i^{-\frac{1}{2}} \eta r)\} i \sin \theta.$$

Let us note that the arguments of the Bessel functions occurring in this formula are complex. We pass to real arguments by introducing the so called Kelvin functions ber_1 and bei_1 by (cf. [48], Chapter VII)

$$(48) \quad \text{ber}_1(\eta r) = -\frac{1}{2} [J_1(i^{\frac{1}{2}} \eta r) + J_1(i^{-\frac{1}{2}} \eta r)], \\ \text{bei}_1(\eta r) = \frac{1}{2i} [J_1(i^{\frac{1}{2}} \eta r) - J_1(i^{-\frac{1}{2}} \eta r)].$$

By utilizing the definitions

$$(49) \quad \bar{P}_2 = -(P_2 + P_3), \quad \bar{P}_3 = i(P_2 - P_3), \\ \bar{Q}_2 = -(Q_2 + Q_3), \quad \bar{Q}_3 = i(Q_2 - Q_3),$$

we can transform (47) into

$$(50) \quad w(r, \theta) = \{P_1 J_1(\omega r) + \bar{P}_2 \text{ber}_1(\eta r) + \bar{P}_3 \text{bei}_1(\eta r)\} \cos \theta + \\ + \{Q_1 J_1(\omega r) + \bar{Q}_2 \text{ber}_1(\eta r) + \bar{Q}_3 \text{bei}_1(\eta r)\} i \sin \theta .$$

In an analogous way the relations for the other quantities can be re-written. For convenience, we introduce the amplitudes $M_R^{(c)}(r)$, $M_\theta^{(c)}(r)$, $\phi^{(c)}(r)$, $M_R^{(s)}(r)$, $M_\theta^{(s)}(r)$ and $\phi^{(s)}(r)$ by

$$(51) \quad m_R = \text{Re}[M_R^{(c)}(r) \cos \theta + M_R^{(s)}(r) i \sin \theta] e^{i\Omega t} , \\ m_\theta = \text{Re}[M_\theta^{(c)}(r) \cos \theta + M_\theta^{(s)}(r) i \sin \theta] e^{i\Omega t} , \\ \varphi = \text{Re}[\phi^{(c)}(r) \cos \theta + \phi^{(s)}(r) i \sin \theta] e^{i\Omega t} .$$

Before writing out the expressions for these amplitudes, we first make the following intermediate calculations:

It follows from (35)¹, with the aid of (43) and (41), that

$$(52) \quad p_{2,3} = \frac{c}{2\rho\beta} \cdot \frac{-\omega^2 \pm i\eta^2}{\pm i\eta^2} = \frac{c\eta^2}{2\rho\beta} (1 \pm i\omega^2) .$$

By using these relations and the definitions (48) and (49), we can derive

$$(53) \quad p_2 P_2 J_1(i^{\frac{1}{2}} \eta r) + p_3 P_3 J_1(i^{-\frac{1}{2}} \eta r) = \\ = \frac{c}{2\rho\beta} \left[\{\text{ber}_1(\eta r) + \frac{\omega^2}{\eta^2} \text{bei}_1(\eta r)\} \bar{P}_2 + \right. \\ \left. + \{-\frac{\omega^2}{\eta^2} \text{ber}_1(\eta r) + \text{bei}_1(\eta r)\} \bar{P}_3 \right] .$$

Since

$$(54) \quad \frac{\omega^2}{\eta^2} \ll 1 ,$$

the relation (53) may be approximated by

$$(55) \quad p_2 P_2 J_1(i^{\frac{1}{2}} \eta r) + p_3 P_3 J_1(i^{-\frac{1}{2}} \eta r) = \\ = \frac{c}{2\rho\beta} [\text{ber}_1(\eta r) \bar{P}_2 + \text{bei}_1(\eta r) \bar{P}_3] .$$

By a similar procedure, we obtain from (35)

$$(56) \quad q_{2,3} = \frac{\Gamma}{\Omega} [(\mu \pm i\alpha\eta^2) \frac{c}{2\rho\beta} (1 \pm i \frac{\omega^2}{\eta^2}) - 2\beta M_B] = \\ = \frac{c\Gamma}{2\rho\beta\Omega} (\mu \pm i\alpha\eta^2) [1 + O(\frac{\omega^2}{\eta^2})] ,$$

which yields with (41)² and the definition

$$(57) \quad \sigma := \sqrt{\frac{\kappa}{\mu} - (\frac{\Omega}{\mu\Gamma})^2} = \frac{\alpha}{\mu} \eta^2 > 0 ,$$

the relation

$$(58) \quad q_2 P_2 J_1(i^{\frac{1}{2}} \eta r) + q_3 P_3 J_1(i^{-\frac{1}{2}} \eta r) = \\ = \frac{c\mu\Gamma}{2\rho\beta\Omega} [(\text{ber}_1(\eta r) + \sigma \text{bei}_1(\eta r)) \bar{P}_2 + \\ + (-\sigma \text{ber}_1(\eta r) + \text{bei}_1(\eta r)) \bar{P}_3] ,$$

where again terms of $O(\omega^2/\eta^2)$ are neglected,

With the results obtained above, we can derive the following expressions for the coefficients of the solution in the form of (51)

$$M_r^{(c)}(r) = E - p_1 \omega J_1'(\omega r) P_1 - \frac{c\eta}{2\rho\beta} \text{ber}_1'(\eta r) \bar{P}_2 + \\ - \frac{c\eta}{2\rho\beta} \text{bei}_1'(\eta r) \bar{P}_3 - \frac{\Omega p_1}{\kappa\Gamma} \frac{J_1(\omega r)}{r} Q_1 + \\ - \frac{c\mu\Gamma}{2\rho\beta\Omega} \cdot \frac{[\text{ber}_1(\eta r) + \sigma \text{bei}_1(\eta r)]}{r} \bar{Q}_2 + \\ + \frac{c\mu\Gamma}{2\rho\beta\Omega} \cdot \frac{[\sigma \text{ber}_1(\eta r) - \text{bei}_1(\eta r)]}{r} \bar{Q}_3 ,$$

$$\begin{aligned}
 (59) \quad M_{\theta}^{(c)}(r) &= F - \frac{\Omega p_1 \omega}{\kappa \Gamma} J_1'(\omega r) P_1 + \frac{c \mu \Gamma \eta}{2 \rho \beta \Omega} [\text{ber}'_1(\eta r) + \sigma \text{bei}'_1(\eta r)] \bar{P}_2 + \\
 &+ \frac{c \mu \Gamma \eta}{2 \rho \beta \Omega} [\sigma \text{ber}'_1(\eta r) - \text{bei}'_1(\eta r)] \bar{P}_3 - p_1 \frac{J_1(\omega r)}{r} Q_1 + \\
 &- \frac{c}{2 \rho \beta} \frac{\text{ber}_1(\eta r)}{r} \bar{Q}_2 - \frac{c}{2 \rho \beta} \frac{\text{bei}_1(\eta r)}{r} \bar{Q}_3, \\
 \phi^{(c)}(r) &= -\frac{\kappa}{M_s} r E + \frac{\Omega}{\Gamma M_s} r F - 4 \pi \rho p_1 J_1(\omega r) P_1 + \\
 &- \frac{4 \pi \rho c}{2 \rho \beta} [\text{ber}_1(\eta r) \bar{P}_2 + \text{bei}_1(\eta r) \bar{P}_3],
 \end{aligned}$$

where

$$J_1'(z) = \frac{dJ_1(z)}{dz}, \quad \text{ber}'_1(z) = \frac{d \text{ber}_1(z)}{dz} \quad \text{and} \quad \text{bei}'_1(z) = \frac{d \text{bei}_1(z)}{dz}.$$

From (59)³, the constants C and D are eliminated with the aid of (36). The expressions for $M_r^{(s)}(r)$, $M_{\theta}^{(s)}(r)$ and $\phi^{(s)}(r)$ can be obtained from the formulae for $M_r^{(c)}(r)$, $M_{\theta}^{(c)}(r)$ and $\phi^{(c)}(r)$, respectively, by replacing in the latter (E, F, P₁, \bar{P}_2 , \bar{P}_3 , Q₁, \bar{Q}_2 , \bar{Q}_3) by (F, E, Q₁, \bar{Q}_2 , \bar{Q}_3 , P₁, \bar{P}_2 , \bar{P}_3), respectively.

By using (22) and (25), the boundary conditions (23)^{4,5,9,10,11}, holding on the surface $r = R$, can be written in the form

$$\begin{aligned}
 (60) \quad \varphi - \psi &= 0, \quad \varphi_{,r} - \psi_{,r} - 4 \pi \rho m_r = 0, \\
 m_{r,r} &= 0, \quad m_{\theta,r} = 0, \\
 \frac{c}{2 \rho \beta} u_{z,r} + m_r &= 0, \quad \text{on } r = R.
 \end{aligned}$$

The complete solution for our problem can now be determined by substituting the relations (34)⁵, (50) and (51) together with the formulae (59) into the boundary conditions (60), equating to zero separately the coefficients preceding $\cos \theta$ and $\sin \theta$, and calculating from the thus obtained system the ten unknown constants A, B, E, F, P₁, \bar{P}_2 , \bar{P}_3 , Q₁, \bar{Q}_2 , \bar{Q}_3 . Although the formulation of this system is in principle rather trivial, the resulting equations are very extensive. Therefore, we shall not write out these equations explicitly, but we shall confine ourselves to merely indicating how we did arrive at our ultimate result.

By means of (60)¹ and (60)⁵ the constants A, B, E and F are eliminated. Thus, we have reduced the system to a system of six equations for the six constants $P_1, \bar{P}_2, \bar{P}_3, Q_1, \bar{Q}_2$ and \bar{Q}_3 . In order to shorten the formulation, we introduce the dimension-less six-vector \underline{X} by

$$(61) \quad \underline{X} := -\frac{c}{2\beta h_0} \left(\frac{P_1}{R}, \frac{Q_1}{R}, \frac{\Gamma\mu}{\Omega} \frac{\eta^2 R^2}{M_1} \frac{\bar{P}_2}{R}, \frac{\Gamma\mu}{\Omega} \frac{\eta^2 R^2}{M_1} \frac{\bar{P}_3}{R}, \right. \\ \left. \frac{\Gamma\mu}{\Omega} \frac{\eta^2 R^2}{M_1} \frac{\bar{Q}_2}{R}, \frac{\Gamma\mu}{\Omega} \frac{\eta^2 R^2}{M_1} \frac{\bar{Q}_3}{R} \right),$$

where $M_1 = M_1(\eta R)$, the modulus of the Kelvin functions, defined by

$$(62) \quad \text{ber}_\nu(z) = M_\nu(z) \cos \theta_\nu(z), \quad \text{bei}_\nu(z) = M_\nu(z) \sin \theta_\nu(z),$$

for $\nu = 1$. The angle θ_ν is called the phase.

The system can now be written in matrix form

$$(63) \quad A_{ij} X_j = V_i, \quad i, j = 1, 2, \dots, 6,$$

where A_{ij} is the matrix of coefficients, while the inhomogeneous right-hand side \underline{V} is descended from the term $(-h_0 r \cos \theta)$ occurring in the expression for ψ , (34)⁵. It appears that the vector \underline{V} is equal to the unit vector

$$(64) \quad \underline{V} = (1, 0, 0, 0, 0, 0).$$

Our discussion will now be directed to simplifying the system (63), within the same exactness as the foregoing calculations, and to indicating for which values of the parameters ω and η this approximation is valid.

To this end, we introduce the two small numbers δ_1 and δ_2 by

$$(65) \quad \delta_1 := \frac{2\beta\rho M_s}{c} = 0.845 \times 10^{-5}, \quad \delta_2 := \frac{1}{\eta R} \leq 3.60 \times 10^{-3}$$

and we define

$$(66) \quad \varepsilon := \max(\delta_1, \delta_2) = 3.60 \times 10^{-3},$$

where the maximum is taken over the ranges according to (39) and (40).

On elaborating the matrix A_{ij} , it turns out that in each row some terms are dominant compared with the others. This matrix can be written in the form

$$(67) \quad A = \begin{vmatrix} C & O(\delta_2) & O(\delta_2) \\ O(\delta_1) & D_1 & O(\delta_2) \\ O(\delta_1) & O(\delta_2) & D_2 \end{vmatrix},$$

where C , D_1 and D_2 are matrices of second order. By defining

$$(68) \quad J := \omega R J_0(\omega R) - J_1(\omega R),$$

and

$$(69) \quad \text{ber}_1'' := \frac{d^2 \text{ber}_1(z)}{dz^2} \Big|_{z=\eta R} \quad \text{and} \quad \text{bei}_1'' := \frac{d^2 \text{bei}_1(z)}{dz^2} \Big|_{z=\eta R},$$

and after neglectation of terms of $O(\epsilon)$, the matrices C , D_1 and D_2 appear to be equal to

$$(70) \quad C = \begin{vmatrix} \frac{(\kappa + 2\pi\rho M_s)J}{\rho M_s} & -\frac{\Omega J}{\rho M_s \Gamma} \\ -\frac{\Omega J}{\rho M_s \Gamma} & \frac{(\kappa + 2\pi\rho M_s)J}{\rho M_s} \end{vmatrix}$$

$$D_1 = D_2 = D = \begin{vmatrix} \frac{\text{ber}_1''}{M_1} & \frac{\text{bei}_1''}{M_1} \\ \frac{-1}{M_1} (\text{ber}_1'' + \sigma \text{bei}_1'') & \frac{1}{M_1} (\sigma \text{ber}_1'' - \text{bei}_1'') \end{vmatrix}.$$

For the second derivatives of the Kelvin functions, the following relations hold

$$(71) \quad \begin{aligned} \text{ber}_1''(z) &= \frac{2 \text{ber}_1(z)}{z^2} - \text{bei}_1(z) + \frac{\text{ber}_0(z) + \text{bei}_0(z)}{z\sqrt{2}}, \\ \text{bei}_1''(z) &= \frac{2 \text{bei}_1(z)}{z^2} + \text{ber}_1(z) + \frac{-\text{ber}_0(z) + \text{bei}_0(z)}{z\sqrt{2}}. \end{aligned}$$

We note that the numerical value of the argument ηR is very large ($\eta R \geq 0.278 \times 10^3$). For large values of the argument z , the following asymptotic expansions for the moduli and the phases of the Kelvin functions hold

$$\begin{aligned}
 M_0(z) &= \frac{e^{z\sqrt{2}}}{\sqrt{2}\pi z} \left[1 + \frac{1}{8\sqrt{2}} z^{-1} + O(z^{-2}) \right], \\
 M_1(z) &= \frac{e^{z\sqrt{2}}}{\sqrt{2}\pi z} \left[1 - \frac{3}{8\sqrt{2}} z^{-1} + O(z^{-2}) \right], \\
 \theta_0(z) &= \frac{z}{\sqrt{2}} - \frac{\pi}{8} + O(z^{-1}), \\
 \theta_1(z) &= \frac{z}{\sqrt{2}} + \frac{3\pi}{8} + O(z^{-1}).
 \end{aligned}
 \tag{72}$$

Hence, for $z = \eta R$ we have

$$\begin{aligned}
 M_0(\eta R) &= M_1[1 + O(\epsilon)], \quad \sin \theta_0(\eta R) = -\cos \theta_1(\eta R) + O(\epsilon), \\
 \cos \theta_0(\eta R) &= \sin \theta_1(\eta R) + O(\epsilon).
 \end{aligned}
 \tag{73}$$

By defining the angle $\hat{\theta}$ by

$$\hat{\theta} = \frac{\eta R}{\sqrt{2}} + \frac{3\pi}{8} - 2k\pi,
 \tag{74}$$

where k is chosen in such a way that $0 \leq \hat{\theta} \leq 2\pi$, the relations (69) can be written, with the aid of (71), as

$$\text{ber}_1'' = -M_1[\sin \hat{\theta} + O(\epsilon)], \quad \text{bei}_1'' = M_1[\cos \hat{\theta} + O(\epsilon)].
 \tag{75}$$

We now shall establish that, except for some specific values of Ω , the exact solution of our system may be approximated by the solution of the reduced system that is obtained by omission in (63) of all terms of $O(\epsilon)$. It will be proved that the errors of this approximation are of $O(\epsilon)$. The exceptional values of Ω will appear to be equal to the resonant frequencies.

To achieve this purpose, we write the matrix A_{ij} as

$$A_{ij} = A_{ij}^0 + \epsilon B_{ij},
 \tag{76}$$

where A_{ij}° is obtained from the matrix A_{ij} according to (67), when all terms of $O(\epsilon)$ are omitted.

Let \underline{X}° be the solution of

$$(77) \quad A_{ij}^{\circ} X_j^{\circ} = V_i$$

then, the solution of (63) may be decomposed as

$$(78) \quad X_i = X_i^{\circ} + \epsilon Y_i,$$

where \underline{Y} is the solution of

$$(79) \quad (A_{ij}^{\circ} + \epsilon B_{ij}) Y_j = -B_{ij} X_j^{\circ}.$$

It is evident from (78) that the solutions \underline{X} and \underline{X}° differ only in terms of $O(\epsilon)$, provided that the solution \underline{Y} of (79) is bounded.

Let $\|\underline{Y}\|$ represent the norm of \underline{Y} , then it follows from (79) that

$$(80) \quad \|\underline{Y}\| \leq \| (A^{\circ} + \epsilon B)^{-1} \| \| B \| \cdot \|\underline{X}^{\circ}\|.$$

Further, as

$$(81) \quad (A^{\circ} + \epsilon B)^{-1} = (A^{\circ})^{-1} [I + \epsilon B (A^{\circ})^{-1}]^{-1},$$

where I is the unit matrix, the following inequality holds

$$(82) \quad \| (A^{\circ} + \epsilon B)^{-1} \| \leq \frac{\| (A^{\circ})^{-1} \|}{1 - \epsilon \| B \| \| (A^{\circ})^{-1} \|}.$$

We note that the solution \underline{X}° is bounded if $\| (A^{\circ})^{-1} \|$ is bounded, as follows immediately from (77). Furthermore, we state, without writing out B explicitly, that all elements B_{ij} are always bounded. Hence, $\| B \|$ is also bounded. We then conclude from (80) and (82) that $\|\underline{Y}\|$ is bounded, when $\| (A^{\circ})^{-1} \|$ is bounded.

To investigate the boundedness of $(A^{\circ})^{-1}$, we start from (67), where all terms of $O(\epsilon)$ are taken equal to zero, from which we infer

$$(83) \quad (A^{\circ})^{-1} = \begin{vmatrix} C^{-1} & 0 & 0 \\ 0 & D^{-1} & 0 \\ 0 & 0 & D^{-1} \end{vmatrix}.$$

In this relation is, as can be deduced from (70)¹,

$$(84) \quad C^{-1} = \frac{\rho M_s}{J\Delta} \begin{vmatrix} (\kappa + 2\pi\rho M_s) & \frac{\Omega}{\Gamma} \\ \frac{\Omega}{\Gamma} & (\kappa + 2\pi\rho M_s) \end{vmatrix},$$

where

$$(85) \quad \Delta := (\kappa + 2\pi\rho M_s)^2 - \left(\frac{\Omega}{\Gamma}\right)^2 > 0,$$

where the $>$ -sign holds for every value of Ω in the interval (39). Hence, $\|C^{-1}\|$ is bounded, provided that the quantity J , defined by (68), is unequal to zero.

Furthermore, as a consequence of (70)² and (75), we have

$$(86) \quad D^{-1} = \begin{vmatrix} -\sin \theta & \cos \theta \\ (\sin \theta - \sigma \cos \theta) & (-\sigma \sin \theta - \cos \theta) \end{vmatrix}^{-1} = \\ = \frac{1}{\sigma} \begin{vmatrix} (-\sigma \sin \theta - \cos \theta) & (-\sin \theta + \sigma \cos \theta) \\ -\cos \theta & -\sin \theta \end{vmatrix}.$$

Returning to (57), we see that σ is always greater than zero, so (86) shows that $\|D^{-1}\|$ is bounded.

The foregoing considerations result in the following final conclusion:

$\|(A^\circ)^{-1}\|$ and, consequently, $\|\underline{Y}\|$ are bounded, except for those values of (ωR) where $J = 0$. Hence, excluding a small neighbourhood of the points where $J = 0$, we may state that the solution \underline{X}° of the reduced system (77) is, to $O(\varepsilon)$, equal to the exact solution of (63).

The exceptional values for ω are the roots of

$$(87) \quad \omega R J_\circ(\omega R) - J_1(\omega R) = 0.$$

We find for the first three roots of (87)

$$(88) \quad (\omega R)_1 = 1.84, \quad (\omega R)_2 = 5.33, \quad (\omega R)_3 = 8.54,$$

or, expressed in the driving frequency Ω

$$(89) \quad \Omega_1 = \frac{7.07 \times 10^5}{R}, \quad \Omega_2 = \frac{20.5 \times 10^5}{R}, \quad \Omega_3 = \frac{32.8 \times 10^5}{R} \quad (\text{Hz}).$$

Solving (76) leads to

$$P_1 = - \frac{2\beta\rho M_s R(\kappa + 2\pi\rho M_s)}{c\Delta J} h_0 ,$$

$$(90) \quad Q_1 = - \frac{2\beta\rho M_s \Omega R}{c\Delta J} h_0 ,$$

$$\bar{P}_2 = \bar{P}_3 = \bar{Q}_2 = \bar{Q}_3 = 0 .$$

Substitution of (90) into (34)¹ yields the following expression for the displacement in the axial direction

$$(91) \quad u_z(r, \theta, t) = - \frac{2\beta\rho M_s R}{c\Delta J} h_0 J_1(\omega r) [(\kappa + 2\pi\rho M_s) \cos \theta \cos \Omega t +$$

$$- \frac{\Omega}{\Gamma} \sin \theta \sin \Omega t] .$$

As follows from this relation, the displacement u_z tends to infinity when J tends to zero, from which we conclude that for those values of ω that are zero's of J , there occurs resonance. Hence, the roots of (87) are, approximately (in $O(\epsilon)$), the resonant frequencies of our system. A complete description of the solution will be deferred until the next section, in which it will be shown that the solution determined in this section is identical to the solution that is obtained when the exchange interaction is neglected a priori.

VII.6. Neglecton of the exchange interaction

As we have already noted in Section VI.6, the exchange coefficient α_1 is very small. Therefore, it seems interesting to investigate how far the solution alters, when the exchange interaction is neglected a priori. To this end we put α_1 equal to zero in the equations of Section VII.3. It will appear that this reduces the order of the differential equation for $w(r, \theta)$ from six to two.

By taking $\alpha = 0$, the equation (30) reduces to

$$(92) \quad \Delta w + aw = 0 ,$$

where

$$(93) \quad a = \frac{\rho (\kappa\Gamma^2 - \Omega^2)\Omega}{c(\mu\kappa\Gamma^2 - \Omega^2) - 4\rho\kappa\beta\Gamma^2 M_s} \approx \frac{\rho\Omega^2}{c} = \omega^2 .$$

Under the usual restrictions, i.e. periodicity in θ and boundedness for $r = 0$, the solution of (92) reads

$$(94) \quad w(r, \theta) = PJ_1(\omega r)\cos \theta + QJ_1(\omega r)i \sin \theta .$$

We note that this relation corresponds with the solution according to (34)¹, if in the latter the summations are omitted. Analogously, we find from (34) the solutions for m_r , m_θ , φ and ψ . The coefficients p and q can be determined from (35)¹ for $n = 1$, by replacing λ^2 by a . We then obtain

$$(95) \quad p = P_1 = \frac{2\kappa\beta\Gamma^2 M_s}{(\mu\kappa\Gamma^2 - \Omega^2)} , \quad q = Q_1 = \frac{2\Omega\beta\Gamma M_s}{(\mu\kappa\Gamma^2 - \Omega^2)} = \frac{\Omega}{\kappa\Gamma} P .$$

As a consequence of the neglect of the exchange interaction, the couple-stresses π_{ij} are equal to zero. Hence, the boundary conditions (23)^{10,11} are satisfied identically. There are still three boundary conditions left to be met. Since each boundary condition gives two equations, these conditions together with the two relations (36) yield a system of eight equations for the eight unknown constants A, B, C, D, E, F, P and Q.

Calculating from the system mentioned above, the constants P and Q and neglecting in the thus obtained expressions p with respect to $c/(2\rho\beta)$, we arrive at

$$(96) \quad P = P_1 \quad \text{and} \quad Q = Q_1 ,$$

where P_1 and Q_1 are given by (90).

Hence, the solution agrees completely with the one obtained in the preceding section. From this we may conclude that, for values of Ω in the range of (39), the neglect of the exchange interaction is allowed, if the restriction

$$(97) \quad (\eta R)^{-1} \ll 1 ,$$

is met. The factor $(\eta R)^{-1}$ is a measure for the inaccuracies of the ap-

proximated solutions of the last two sections.

We shall now write out explicitly the complete solution of our problem:

$$\begin{aligned}
 u_z(r, \theta, t) &= PJ_1(\omega r) \cos \theta \cos \Omega t - QJ_1(\omega r) \sin \theta \sin \Omega t, \\
 m_r(r, \theta, t) &= [E - pP\{\omega J_0(\omega r) - \frac{1}{r} J_1(\omega r)\} - \frac{\Omega}{\kappa\Gamma} pQ \frac{J_1(\omega r)}{r}] \cdot \\
 &\quad \cdot \cos \theta \cos \Omega t - [F - pQ\{\omega J_0(\omega r) - \frac{1}{r} J_1(\omega r)\} + \\
 &\quad - \frac{\Omega}{\kappa\Gamma} pP \frac{J_1(\omega r)}{r}] \sin \theta \sin \Omega t, \\
 m_\theta(r, \theta, t) &= [-F + pQ \frac{J_1(\omega r)}{r} + \frac{\Omega}{\kappa\Gamma} pP\{\omega J_0(\omega r) + \\
 (98) \quad &\quad - \frac{1}{r} J_1(\omega r)\}] \cos \theta \sin \Omega t + [-E + pP \frac{J_1(\omega r)}{r} + \\
 &\quad + \frac{\Omega}{\kappa\Gamma} pQ\{\omega J_0(\omega r) - \frac{1}{r} J_1(\omega r)\}] \sin \theta \cos \Omega t, \\
 \varphi(r, \theta, t) &= [(-\frac{\kappa}{M_s} E + \frac{\Omega}{\Gamma M_s} F)r - 4\pi\rho pPJ_1(\omega r)] \cos \theta \cos \Omega t + \\
 &\quad - [(\frac{\Omega}{\Gamma M_s} E - \frac{\kappa}{M_s} F)r - 4\pi\rho pQJ_1(\omega r)] \sin \theta \sin \Omega t, \\
 \psi(r, \theta, t) &= (-h_\theta r + \frac{A}{r}) \cos \theta \cos \Omega t - \frac{B}{r} \sin \theta \sin \Omega t,
 \end{aligned}$$

where $P = P_1$ and $Q = Q_1$ are given in (90), p in (95) and further

$$\begin{aligned}
 A &= h_0 R^2 \left\{ 1 + \frac{1}{\Delta} \left[\frac{\Omega^2}{\Gamma^2} - \kappa(\kappa + 2\pi\rho M_s) \right] + \frac{4\pi\rho^2 M_s \beta p}{\kappa\Gamma\Delta} \left[2\kappa(\kappa + 2\pi\rho M_s) - \frac{\Omega^2}{\Gamma^2} \right] \right\}, \\
 B &= \frac{h_\theta R^2}{\Delta} \left\{ \frac{2\pi\rho M_s \Omega}{\Gamma} + \frac{2\rho\beta p \Omega J_1}{\kappa\Gamma J} \left[\mu(\kappa + 2\pi\rho M_s) - \frac{\Omega^2}{\Gamma^2} \right] \right\}, \\
 (99) \quad E &= \frac{M_s h_0}{\Delta} \left\{ (\kappa + 2\pi\rho M_s) - \frac{2\rho\beta}{c} \cdot \frac{p}{\kappa} \cdot \frac{\Omega^2}{\Gamma^2} \cdot \frac{J_1}{J} \right\}, \\
 F &= \frac{M_s h_0}{\Delta} \frac{\Omega}{\Gamma} \left\{ 1 - \frac{2\rho\beta p}{\kappa\Gamma} (\kappa + 2\pi\rho M_s) \frac{J_1}{J} \right\}.
 \end{aligned}$$

In the deduction of the results (98), the relation

$$(100) \quad \frac{dJ_1(\omega r)}{dr} = \omega J_0(\omega r) - \frac{1}{r} J_1(\omega r)$$

is used.

According to (9), the following expressions for the stresses hold

$$(101) \quad \begin{aligned} t_{rz} &= cu_{z,r} + 2\rho\beta m_r, \\ t_{\theta z} &= \frac{c}{r} u_{z,\theta} + 2\rho\beta m_\theta. \end{aligned}$$

By using the results (98), these expressions can be written out into the form

$$(102) \quad \begin{aligned} t_{rz} &= \frac{2\rho\beta M_s R h_0}{J\Delta} \left[\{(\kappa + 2\pi\rho M_s) \left(\frac{J}{R} - \frac{J(\omega r)}{r}\right) + \right. \\ &\quad \left. - \frac{2\rho\beta\Omega}{c\Gamma} \left(\frac{J_1}{R} - \frac{J_1(\omega r)}{r}\right)\} \cos\theta \cos\Omega t - \left\{\frac{\Omega}{\Gamma} \left(\frac{J}{R} - \frac{J(\omega r)}{r}\right) + \right. \\ &\quad \left. - \frac{2\rho\beta}{c} (\kappa + 2\pi\rho M_s) \left(\frac{J_1}{R} - \frac{J_1(\omega r)}{r}\right)\} \sin\theta \sin\Omega t, \right. \\ t_{\theta z} &= - \frac{2\rho\beta R M_s h_0}{J\Delta} \left[\{(\kappa + 2\pi\rho M_s) \frac{J}{R} + \right. \\ &\quad \left. - \frac{2\rho\beta\Omega}{c\Gamma} \left(\frac{J_1}{R} - \frac{J_1(\omega r)}{r} - \omega J_0(\omega r)\right)\} \sin\theta \cos\Omega t + \left\{\frac{\Omega J}{\Gamma R} + \right. \\ &\quad \left. - \frac{2\rho\beta}{c} (\kappa + 2\pi\rho M_s) \left(\frac{J_1}{R} - \frac{J_1(\omega r)}{r} - \omega J_0(\omega r)\right)\} \cos\theta \sin\Omega t. \right. \end{aligned}$$

In order to give an impression of the order of magnitude of the stresses, we have calculated t_{rz} for $r = 0$ and for $R = 10^{-2}$ as a function of Ω . The results are shown in Fig. VII.2, where

$$(103) \quad \begin{aligned} T_r^{(c)} &= t_{rz} (r = 0, \theta = 0, \Omega t = 0), \\ T_r^{(s)} &= t_{rz} (r = 0, \theta = \frac{\pi}{2}, \Omega t = \frac{\pi}{2}). \end{aligned}$$

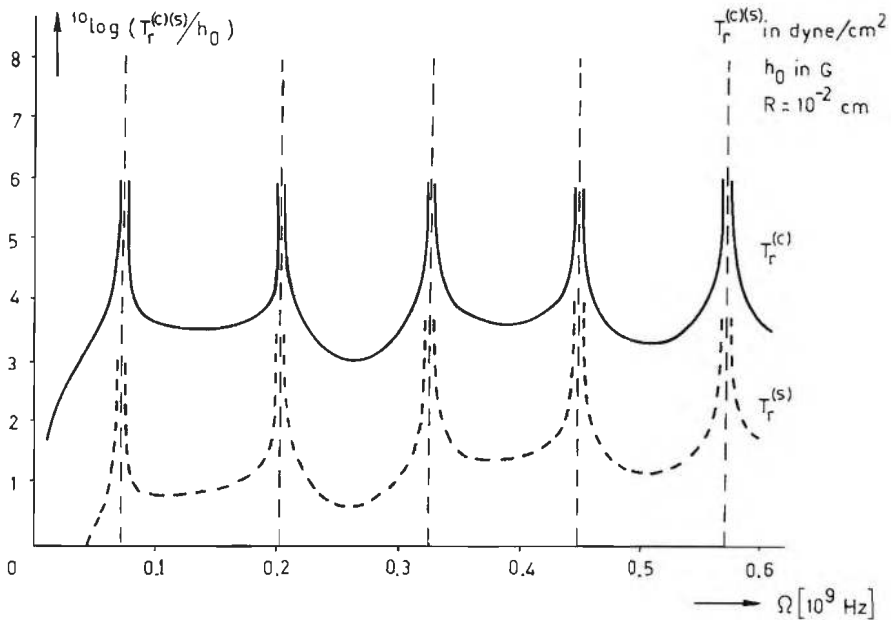


Fig. VII.2.

VII.7. Conclusions

Based on the results of the preceding sections the following conclusions, holding for the problem treated here and under the well-known restrictions, can be drawn:

- i) If terms preceded by a factor c^{-2} are neglected consistently, the electromagnetic equations may be replaced by their quasi-static version. Let us note that this only holds when the intermediate state is a static one in which only magnetic fields are applied.
- ii) In neglecting the exchange interaction, an error is made of the order $(\eta R)^{-1}$. This error is usually very small.
- iii) The resonant frequencies can be determined from an equation (i.e. eq. (87)) that is independent of H_0 . Hence the resonant frequencies are independent of H_0 . It should be noted that this not only holds for $H_0 \ll \kappa$ (or μ) but also for higher values of H_0 , up to values in the order of ρM_S . As follows immediately from (87), the

sonant frequencies depend, in a very good approximation, only on the density ρ , the elasticity constant c and the radius of the cylinder R . Hence, the resonances are purely mechanical.

- iv) In general the solution of the problem is coupled at the stress-free and couple-stress-free surface of the cylinder. However, when the exchange interaction is neglected, this coupling vanishes.

VIII. SOFT-MAGNETOELASTIC MATERIALS

VIII.1. Introduction

In this chapter, we shall derive a dynamic theory of magnetoelastic interactions in soft-magnetic materials. Such a material is characterized by a linear dependence of the magnetization on the magnetic field intensity in the material. For ferromagnetic materials this linear relationship only occurs for very low values for the field intensity. Moreover, we restrict ourselves to linear elasticity, i.e. to infinitesimal deformations. We assume the medium to be nonpolarizable, nonconducting, isotropic and we leave thermal effects out of consideration. Due to the low values for the field intensity, and thus also for the magnetization, the effects as gyromagnetic coupling, exchange interaction and magnetostriction will be neglected.

We shall first derive a general, nonlinear system of balance equations, constitutive equations and boundary conditions. Underlying this derivation is a similar though reduced energy balance as the one employed in Chapter II, and the same procedure as in Chapter II and III will be followed here. The constitutive equations will be simplified considerably, owing to the fact that the set of independent variables is much more restrictive than that of Chapter III. Next, the system will be linearized with respect to a static intermediate state, in a way analogous to the one adopted in Chapter V. These equations will be simplified further by taking the Maxwell-equations in their quasi-static version and by allowing for the fact that the deformations in the intermediate state are small (cf. Section V.6). The results of this chapter will be applied in the determination of the stability of elastic plates interacted by magnetic fields. This will be done in the final chapter of this thesis.

Recently, there appeared a paper of Hutter and Yih-Hsing Pao [49], treating the dynamical theory of soft-magnetic elastic materials that

are thermally and electrically conductive. The general equations of this article are, when applied to our more restrictive case, in a one to one correspondence with the results of the present chapter. However, in the linearization of these general equations some errors are made, for instance in the linearization of the Maxwell-equations (eq. (5.6)²) and of the boundary conditions (e.g. eq. (7.7a)).

VIII.2. General equations

In a way, analogous to the one followed in Chapter II, we shall derive a system of balance equations. Hence, we start with a global energy balance in the form of II.(1), into which II.(2), II.(5), II.(9) and II.(15) are substituted. However, as we neglect the gyromagnetic and exchange interaction and all thermal effects, the terms ρK , ρr , $M_{ikj} \Omega_{ik}$ and Q_j must be omitted. All this has no influence on the local balance equations for the mass and the momentum, which means that the equations II.(12) and II.(14) remain valid. On the contrary, the local balance equation of moment of momentum II.(31) reduces to

$$(1) \quad e_{ijk} \{ M_j^* H_k^* - \frac{1}{\rho} T_{[jk]} \} = 0 .$$

The terms $\frac{1}{\Gamma} \dot{M}_i^*$, $\frac{1}{\rho} (e_{ijk} M_j^* \Pi_{kl}^*)_{,l}$ and $e_{ijk} P_j^* E_k^*$ of II.(31) are chosen equal to zero, because the gyromagnetic and exchange interaction is neglected and the medium is nonpolarizable.

Based on the fact that, in the medium under consideration, polarization, electrical and thermal conduction, exchange interaction and all thermal effects are absent, the quantities $M_{i\alpha}$, E_i^* , θ and θ_α may be left out of the set of constitutive variables according to III.(7). Hence, it suffices to take

$$(2) \quad \Sigma = \Sigma (F_{i\alpha}, M_i^*) .$$

According to III.(15), III.(16), III.(21), III.(35) and III.(36), we then have

$$(3) \quad S = P_i^* = h_{ij} = Q_i = J_i^* = 0 .$$

Moreover, the vector $\underline{\sigma}$ occurring in the entropy inequality is again equal to zero.

Utilizing the above results, we arrive at the following entropy inequality (cf. III.(14))

$$(4) \quad (\rho H_i^* - \rho \frac{\partial \Sigma}{\partial M_i^*}) \dot{M}_i^* + (T_{ij} - \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha}) V_{i,j} \geq 0 .$$

Since the magnetization is no longer saturated, \underline{M}^* is not restrained by a relation like II. (21). Then, according to the principle of Coleman and Noll, the coefficients preceding \dot{M}_i^* and $V_{i,j}$ in (4) must be taken equal to zero. Thus, we arrive at the following constitutive equations

$$(5) \quad H_i^* = \frac{\partial \Sigma}{\partial M_i^*} ,$$

and

$$(6) \quad T_{ij} = \rho \frac{\partial \Sigma}{\partial F_{i\alpha}} F_{j\alpha} .$$

Introducing objective variables according to III.(43)² and III.(45), these constitutive equations transform into

$$(7) \quad H_i^* = \frac{\partial \Sigma}{\partial \Lambda_\alpha} F_{i\alpha} ,$$

and

$$(8) \quad T_{ij} = \rho \frac{\partial \Sigma}{\partial E_{\alpha\beta}} F_{i\alpha} F_{j\beta} + \rho \frac{\partial \Sigma}{\partial \Lambda_\alpha} F_{j\alpha} M_i^* .$$

We note that with (7) and (8) the angular momentum equation (1) is satisfied identically. Hence, this relation is disposed of.

For soft-magnetic, linear elastic materials, an expression quadratic in $E_{\alpha\beta}$ and Λ_α must be taken for the energy functional Σ . Hence, restricting ourselves to isotropic materials, we must take

$$(9) \quad \Sigma = \frac{1}{2\rho_0} c_{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} + \frac{1}{2} \rho_0 \chi \Lambda_\alpha \Lambda_\alpha .$$

Due to the isotropy, the quadratic, piezomagnetic term has disappeared, and the magnetostrictive term is not retained, as this term is of the

third order in $E_{\alpha\beta}$ and Λ_α (cf. eq. VI.(1)). Thus, there is no coupling between $E_{\alpha\beta}$ and Λ_α .

For an isotropic material the tensor of the coefficients of elasticity has the following components

$$(10) \quad c_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta}c_{11} + \frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})c_{12}.$$

Substituting (9) into the constitutive equations (7) and (8) yields

$$(11) \quad H_i^* = \rho_0 \chi \Lambda_\alpha F_{i\alpha} = \rho_0 \chi F_{i\alpha} F_{j\alpha} M_j^*,$$

and

$$(12) \quad T_{ij} = \frac{\rho}{\rho_0} c_{\alpha\beta\gamma\delta} F_{i\alpha} F_{j\beta} E_{\gamma\delta} + \rho \rho_0 \chi F_{j\alpha} F_{i\alpha} M_k^* M_k^*.$$

In a linear theory, i.e. neglecting all terms that are quadratic in $E_{\alpha\beta}$ and M_i^* , (11) and (12) reduces to

$$(13) \quad H_i^* = \rho_0 \chi M_i^*,$$

and

$$(14) \quad T_{ij} = c_{ijkl} U_{k,l}.$$

Defining the magnetic permeability μ by

$$(15) \quad B_i^* = \mu H_i^*,$$

it follows from (13) and I.(36) that

$$(16) \quad \chi = \frac{4\pi}{(\mu - 1)}.$$

By use of (16), the relation (13) can be rewritten as

$$(17) \quad \rho_0 M_i^* = \frac{(\mu - 1)}{4\pi} H_i^*.$$

It is a well known fact that, for ferromagnetic materials, the permeability μ is always large ($\mu = 10^4 \text{ à } 10^6$).

Due to the fact that the convective polarization \underline{P}^* is zero, we have, according to I.(38),

$$(18) \quad M_i = M_i^* \text{ and } P_i = \frac{1}{c} e_{ijk} V_j M_k .$$

Moreover, as the convective current \underline{J}^* is zero, the relation

$$(19) \quad J_i = QV_i$$

holds.

Since thermal effects are left out of consideration, the reduced energy balance (III.(38)) may be disposed of.

As concerns the boundary conditions, we will restrict ourselves to the problem of a body in a vacuum. We note that of all mechanical boundary conditions (II.(47)-(49)) only those for the stresses are retained. Recapitulating, we have the following general equations with boundary conditions for the unknown variables

$$\rho, U_i, E_i, Q, B_i, H_i, M_i, T_{ij} ,$$

holding for a soft-magnetic body, placed in a vacuum

$$(20) \quad \begin{aligned} \frac{1}{c} \frac{\partial B_i}{\partial t} &= -e_{ijk} E_{k,j}, \quad B_{i,i} = 0 , \\ \frac{1}{c} \frac{\partial E_i}{\partial t} + \frac{4\pi}{c} QV_i &= e_{ijk} H_{k,j}, \quad E_{i,i} = 4\pi Q + \left(\frac{4\pi\rho}{c} e_{ijk} M_j V_k\right)_{,i} , \\ \dot{Q} + QV_{i,i} &= 0, \quad B_i = H_i + 4\pi\rho M_i , \\ \dot{\rho} + \rho V_{i,i} &= 0, \\ \rho \dot{V}_i &= T_{ij,j} + \rho F_i^{(m)} + F_i^{(e)} , \end{aligned}$$

where

$$F_i^{(e)} = QE_i^* + \rho M_j H_{j,i} - \frac{\rho}{c} e_{jkl} V_k M_l E_{j,i} + \frac{\rho}{c} \frac{d}{dt} (e_{ijk} E_j M_k) ,$$

and

$$T_{ij} = \rho \frac{\partial \Sigma}{\partial E_{\alpha\beta}} F_{i\alpha} F_{j\beta} + \rho \frac{\partial \Sigma}{\partial \lambda_\alpha} F_{j\alpha} M_i ,$$

and on the surface S of the body

$$\begin{aligned}
& \llbracket e_{ijk} E_j n_k + \frac{1}{c} B_i V_j n_j \rrbracket = 0, \quad \llbracket E_i + \frac{4\pi\rho}{c} e_{ijk} V_j M_k \rrbracket n_i = 0, \\
(21) \quad & \llbracket e_{ijk} H_j n_k - \frac{1}{c} E_i V_j n_j \rrbracket = 0, \quad \llbracket B_i \rrbracket n_i = 0, \\
& T_{ij} n_j = 2\pi(\rho M_j n_j)^2 n_i + T_i^* \quad \text{on } S.
\end{aligned}$$

From these equations, the electric displacement \underline{D} is eliminated with the aid of I. (37) and (18)².

We note that the stress vector \underline{T}^* can be due, for instance, to a support. In this case \underline{T}^* is an unknown, and the condition (21)⁵ on that part of the surface that is supported must be replaced by a condition for the displacements.

VIII.3. Linearization with respect to a static intermediate state

By a method similar to that used in Chapter V, we shall linearize the general equations of the preceding section with respect to the disturbances on an intermediate state $\underline{\xi}$. This state is assumed to be static but not necessarily uniform. The body is in the $\underline{\xi}$ -state only loaded by a magnetic field.

Omitting the standard calculations, we write out at once the ultimate equations:

i) For the $\underline{\xi}$ -state (upper index \circ)

$$\begin{aligned}
(22) \quad & E_i^\circ = D_i^\circ = P_i^\circ = Q^\circ = J_i^\circ = 0, \\
& B_{i,i}^\circ = 0, \quad e_{ijk} H_{k,j}^\circ = 0, \\
& B_i^\circ = H_i^\circ + 4\pi\rho^\circ M_i^\circ, \quad H_i^\circ = \left(\frac{\partial \Sigma}{\partial \Lambda_\alpha}\right)^\circ \xi_{i,\alpha}, \\
& \rho^\circ = \frac{1}{J^\circ} \rho_0 \text{ where } J^\circ = \det\{\xi_{i,\alpha}\}, \\
& T_{ij,j}^\circ + \rho^\circ M_j^\circ H_{j,i}^\circ = 0, \quad (F_i^{(m)})^\circ = 0, \\
& T_{ij}^\circ = \rho^\circ \left(\frac{\partial \Sigma}{\partial E_{\alpha\beta}}\right)^\circ \xi_{i,\alpha} \xi_{j,\beta} + \rho^\circ \left(\frac{\partial \Sigma}{\partial \Lambda_\alpha}\right)^\circ \xi_{j,\alpha} M_i^\circ,
\end{aligned}$$

with the boundary conditions

$$(23) \quad \begin{aligned} & [[e_{ijk} H_{jk}^{\circ}] N_k^{\circ} = 0, [B_i^{\circ}] N_i^{\circ} = 0, \\ & T_{ij}^{\circ} N_j^{\circ} = 2\pi (\rho^{\circ} M_j^{\circ} N_j^{\circ})^2 N_i^{\circ} + T_i^{\circ*}, \text{ on } S^{\circ}, \end{aligned}$$

and

ii) for the disturbances (lower case letters)

$$(24) \quad \begin{aligned} & \frac{1}{c} \frac{\partial b_i}{\partial t} - \frac{1}{c} B_{i,j}^{\circ} v_j = -e_{ijk} e_{k,j}, \\ & b_{i,i} - B_{i,j}^{\circ} u_{j,i} = 0, \\ & \frac{1}{c} \frac{\partial e_i}{\partial t} = e_{ijk} (h_{k,j} - H_{k,l}^{\circ} u_{l,j}), \\ & e_{i,i} = 4\pi q + \frac{4\pi}{c} e_{ijk} (\rho^{\circ} M_j^{\circ} v_{k,i}), \\ & b_i = h_i + 4\pi \rho^{\circ} m_i - 4\pi \rho^{\circ} M_i^{\circ} u_{j,j}, \\ & h_i^* = {}^2\zeta_{ijk} u_{j,k} + {}^3\zeta_{ij} m_j, \end{aligned}$$

where

$$(25) \quad \begin{aligned} {}^2\zeta_{ijk} &= \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right)^{\circ} \delta_{ij} \xi_{k,\alpha} + \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial E_{\beta\gamma}} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} + \\ &+ \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} M_j^{\circ} \xi_{i,\alpha} \xi_{k,\beta}, \\ {}^3\zeta_{ij} &= \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta}, \end{aligned}$$

and

$$(26) \quad \rho = \rho^{\circ} (1 - u_{i,i}),$$

$$(27) \quad \rho^{\circ} \dot{v}_i = t_{ij,j} - T_{ij,k}^{\circ} u_{k,j} + f_i^{(e)},$$

where

$$(28) \quad f_i^{(e)} = \rho^{\circ} M_{j,j,i}^{\circ} h_{j,i} + \rho^{\circ} H_{j,i}^{\circ} m_j - \rho^{\circ} M_{j,j,i}^{\circ} H_{j,i}^{\circ} u_{k,k} - \rho^{\circ} M_{j,j,k}^{\circ} H_{j,k}^{\circ} u_{k,i} + \\ + \frac{\rho^{\circ}}{c} e_{ijk} M_k^{\circ} \dot{e}_j ,$$

and

$$(29) \quad t_{ij} = {}^2\tau_{ijkl} u_{k,l} + {}^3\tau_{ijk} m_k ,$$

where

$$(30) \quad {}^2\tau_{ijkl} = -\delta_{kl} T_{ij}^{\circ} + \delta_{jk} T_{il}^{\circ} + \delta_{ik} T_{jl}^{\circ} + \bar{c}_{ijkl} , \\ \bar{c}_{ijkl} = \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial E_{\gamma\delta}} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} \xi_{l,\delta} + \\ + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \Lambda_{\gamma}} \right)^{\circ} [M_k^{\circ} \xi_{i,\alpha} \xi_{j,\beta} \xi_{l,\gamma} + M_i^{\circ} \xi_{j,\gamma} \xi_{k,\alpha} \xi_{l,\beta}] + \\ + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} M_i^{\circ} M_k^{\circ} \xi_{j,\alpha} \xi_{l,\beta} - \delta_{ik} M_j^{\circ} \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right)^{\circ} \xi_{l,\alpha} , \\ {}^3\tau_{ijk} = \rho^{\circ} \left(\frac{\partial \Sigma}{\partial \Lambda_{\alpha}} \right)^{\circ} \delta_{ik} \xi_{j,\alpha} + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial E_{\alpha\beta} \partial \Lambda_{\gamma}} \right)^{\circ} \xi_{i,\alpha} \xi_{j,\beta} \xi_{k,\gamma} + \\ + \rho^{\circ} \left(\frac{\partial^2 \Sigma}{\partial \Lambda_{\alpha} \partial \Lambda_{\beta}} \right)^{\circ} M_i^{\circ} \xi_{j,\alpha} \xi_{k,\beta} ,$$

supplemented by the boundary conditions

$$(31) \quad [e_{ijk} e_j + \frac{1}{c} B_i^{\circ} v_k] N_k^{\circ} = 0 , \\ [e_i + \frac{4\pi}{c} e_{ijk} \rho^{\circ} M_k^{\circ} v_j] N_i^{\circ} = 0 , \\ [e_{ijk} h_j] N_k^{\circ} - [e_{ijk} H_j^{\circ}] u_{l,k} N_l^{\circ} = 0 , \\ [b_i] N_i^{\circ} - [B_i] u_{j,i} N_j^{\circ} = 0 , \\ t_{ij} N_j^{\circ} = T_{ij}^{\circ} u_{k,j} N_k^{\circ} + \tau_i + t_i^* , \quad \text{on } S^{\circ} ,$$

where

$$\begin{aligned}
 (32) \quad \tau_i = & -2\pi(\rho^{\circ} \underline{M}^{\circ}, \underline{N}^{\circ})^2 u_{k,i} N_k^{\circ} - 4\pi(\rho^{\circ} \underline{M}^{\circ}, \underline{N}^{\circ})^2 u_{k,k} N_i^{\circ} + \\
 & - 4\pi(\rho^{\circ} \underline{M}^{\circ}, \underline{N}^{\circ}) \{ \rho^{\circ} M_j^{\circ} - (\rho^{\circ} \underline{M}^{\circ}, \underline{N}^{\circ}) N_j^{\circ} \} u_{k,j} N_k^{\circ} N_i^{\circ} + \\
 & + 4\pi(\rho^{\circ} \underline{M}^{\circ}, \underline{N}^{\circ}) (\rho^{\circ} \underline{m}, \underline{N}^{\circ}) N_i^{\circ} .
 \end{aligned}$$

VIII.4. Further simplifications

We wish to apply the foregoing equations in a stability problem, especially in the determination of the buckling values of a plate interacted by magnetic fields. In this case, the configuration of the plate just before buckling is chosen as the $\underline{\xi}$ -state, while in the \underline{x} -state the plate is buckled, but with still a very small amplitude.

In this kind of problems the following simplifications will be applied:

- i) We approximate the electromagnetic equations by their quasi-static version, i.e. we put c^{-1} equal to zero.
- ii) We use the expression (9) for the functional Σ and we relate χ to μ by (14).
- iii) We neglect in the equations for the disturbances the deformations in the $\underline{\xi}$ -state, except in those terms where they turn up in combination with c_{ijkl} , as for instance in T_{ij}° and \bar{c}_{ijkl} . This exception is made, motivated by the large numerical values for the elasticity coefficients c_{ijkl} in case of ferromagnetic materials. We note that this is in contrast with the method followed in Section V.6, where T_{ij}° was replaced by ${}_{\circ}T_{ij}$, i.e. the purely electromagnetic stress tensor in a rigid-body. As will appear later on (viz. pag. 176) the stresses T_{ij}° have no essential influence on the buckling values calculated in the forthcoming chapter. The same did hold for the example of Chapter VII.

Using this approximation, we may replace ρ° and \underline{N}° by their initial values ρ_{\circ} and \underline{N} . Furthermore, we may take for the magnetic fields the values according to the rigid-body problem, i.e. \underline{B} , \underline{H} and \underline{M} , being the solutions of

$$\begin{aligned}
 & \circ B_{i,i} = 0, \quad e_{ijk} \circ H_{k,j} = 0, \\
 (33) \quad & \circ B_i = \circ H_i + 4\pi\rho \circ M_i, \quad 4\pi\rho \circ M_i = (\mu - 1) \circ H_i, \quad (\rho = \rho_0), \\
 & \llbracket e_{ijk} \circ H_j \rrbracket N_k = 0, \quad \llbracket \circ B_i \rrbracket N_i = 0 \text{ on } S_0.
 \end{aligned}$$

iv) Owing to the fact that we shall especially consider stability problems, as an extra step we replace \bar{c}_{ijkl} by c_{ijkl} in (30)¹. We are aware of the fact that this is inconsistent with regard to the retaining of the terms T_{ij}° in the coefficients ${}^2\tau_{ijkl}$. However, it can be established, although this will not be done here, that the omitted terms have no appreciable influence on the buckling values. This approximation is a common one in the theory of the stability of elastic bodies. We return to this subject after the derivation of the stability equations (viz. pag. 176).

With the aid of the simplifications listed above, the equations for the disturbances reduce to

$$\begin{aligned}
 & b_{i,i} = \circ B_{i,j} u_{j,i}, \quad e_{ijk} h_{k,j} = e_{ijk} \circ H_{k,\ell} u_{\ell,j}, \\
 & b_i = h_i + 4\pi\rho m_i - 4\pi\rho \circ M_i u_{j,j}, \\
 & h_i = \circ H_j (u_{i,j} + u_{j,i}) + \frac{4\pi\rho}{(\mu - 1)} m_i, \\
 (34) \quad & \rho \dot{v}_i = t_{ij,j} - T_{ij,k}^{\circ} u_{k,j} + \rho \circ M_j h_{j,i} + \rho \circ H_{j,i} m_j + \\
 & \quad - \rho \circ M_j (\circ H_{j,i} u_{k,k} + \circ H_{j,k} u_{k,i}), \\
 & t_{ij} = -T_{ij}^{\circ} u_{k,k} + T_{ik}^{\circ} u_{j,k} + T_{jk}^{\circ} u_{i,k} + c_{ijkl} u_{k,\ell} + \\
 & \quad + \rho \circ H_j m_i + \rho \circ H_i m_j,
 \end{aligned}$$

while the boundary conditions become

$$\begin{aligned}
 & \llbracket e_{ijk} h_j \rrbracket N_k = \llbracket e_{ijk} \circ H_j \rrbracket u_{\ell,k} N_{\ell}, \\
 (35) \quad & \llbracket b_i \rrbracket N_i = \llbracket \circ B_i \rrbracket u_{j,i} N_j,
 \end{aligned}$$

and

$$t_{ij} N_j = T_{ij}^{\circ} u_{k,j} N_k + \tau_i + t_i^* ,$$

holding on the undeformed surface S_0 .

The intermediate stresses T_{ij}° can be calculated from (22) and (23), adapted to the simplifications stated above. This yields

$$T_{ij,j}^{\circ} + \rho_0 M_{j0} H_{j,i} = 0 ,$$

$$(36) \quad T_{ij}^{\circ} = c_{ijkl} U_{k,l}^{\circ} + \rho_0 M_{i0} H_j ,$$

$$T_{ij}^{\circ} N_j = 2\pi (\rho_0 \underline{M}, \underline{N})^2 N_i + T_i^{\circ*} \text{ on } S_0 .$$

IX. BUCKLING OF SOFT-MAGNETOELASTIC PLATES

IX.1. Introduction

The instability of a thin elastic plate under compression in its plane is well known for a long time. A similar phenomenon will be discussed in this chapter, where the buckling of thin soft-magnetoelastic plates caused by magnetic fields is studied. To this end, we consider a plate placed in a uniform magnetic field directed perpendicular to the plate. For small magnitudes of the field, the plate only deforms in its plane. We may expect this state to be stable for all values of the field up to some critical one. When this critical value is reached, this state will become unstable, and the plate will have displacements out of its plane. This phenomenon is called magnetoelastic buckling and the critical magnitude of the field is the buckling value.

The first treatment of this kind of problems was due to Mozniker [50], whose paper was discussed in the book of Panovka and Gubanova [51] on page 17. They treated the problem of a beam placed midway a series of magnetic poles along its length. The magnetic force was assumed transverse to the beam and proportional to the deflection, and the field did not react to this deflection. Recently, there appeared some articles, i.e. [52] to [55], in which, in contrast with the former, the magnetic force was supposed to be proportional to the rotation of the midsurface of the plate. In these papers the response of the magnetic field on the deformations is taken into account. In most of these works, a stress tensor according to the Amperian-current model is used. Only in [52] a stress tensor based on a magnetic dipole model is employed, but Moon has showed that these stresses differ only by a factor of the order μ^{-1} from the former.

To our opinion, in all these articles there lacks a fundamental derivation of the ultimate buckling equations. These equations are set up, without any reference to the pre-buckled state. Moreover, in none of

these papers, a constitutive equation relating the stresses to the deformations and the magnetic field is used. They merely posit global constitutive relations between the bending moments in the plate and the curvature of the midsurface of the plate, similar to those of the classical plate theory. Furthermore, the global equations of equilibrium of this classical theory are taken over.

We hold the view, that it is necessary to go out from the pre-buckled state for the derivation of the buckling equations. We then consider small dynamical disturbances on this state, and we search for the values of the parameters of the problem, for which the disturbances tend, for increasing time, to infinity. When this effect occurs, we pose that the state under consideration becomes unstable. For this procedure, the equations of the preceding chapter will be employed.

We shall start with solving the problem of a finite rigid plate placed transverse in a uniform magnetic field. Next, the equations for the intermediate, i.e. the pre-buckled state are deduced, and, finally, those for the disturbances on this state are inferred. The equations are approximated by neglecting terms that are relatively of the order μ^{-1} , where μ is the magnetic permeability ($\mu \gg 1$). The so-called plate equations will be derived by integrating these general three-dimensional equations in the thickness direction of the plate, whereby the Kirchhoff-Love hypothesis is used. In the plate theory an error of the order $(h/R)^2$ is made, where h is a measure for the thickness and R is a measure for the dimensions of the plate in its plane. The thus obtained equations will be applied in the determination of the buckling values of a circular plate that is either clamped or simply supported at its boundary. It will turn out that, for the particular form of the stresses used here, the magnetic stress vector at the upper and the lower surface of the plate is dominant compared with the magnetic body force, and hence causes the instability.

If the midsurface of the plate is rectangular with one dimension much larger than the other, the plate can be considered as a beam with a rectangular cross section. Just as for the plate, beam equations can be derived by integration of the general three-dimensional equations, but now the integration must be performed over the cross section of the

beam. These equations are inferred too and used in the final section for a comparison with the equations of [52] and [53]. In [52] experiments are reported, which when compared with our results or with those of [53] give a certain confirmation of the remark posed in [54], that the Amperian-current stress model minimizes the influence of the boundary effects.

In conclusion we note that, since the linear equations used here do not hold at the lateral boundaries of a finite plate, the theory derived in the present chapter is only completely motivated for clamped plates. For simply supported or free plates, the boundary conditions at the lateral surfaces are undetermined, as the constitutive equations for the bending moments and the shear forces are not known there, due to the fact that the linear magnetic theory does not hold in the neighbourhood of the boundaries.

IX.2. General three-dimensional equations

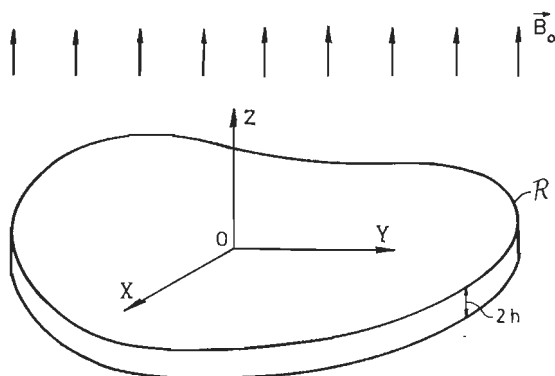


Fig. IX.1.

Let us consider a homogeneous, magnetoelastic plate with thickness $2h$ and with an arbitrary boundary R placed in a uniform magnetic field \vec{B}_0 (viz. Fig. IX.1). The space outside the plate is a vacuum. A coordi-

nate system OXYZ is chosen, with the origin in the centre of mass of the plate, the X- and Y-axes in the midsurface of the plate and the Z-axis perpendicular to it. The basic field \underline{B}_0 is directed along the Z-axis, so

$$(1) \quad B_{0i} = \delta_{i3} B_0,$$

and its magnitude is small enough that the equations for a soft-magnetic medium, as outlined in the foregoing chapter, may be applied. We shall have to deal with three distinct sets of equations, being the equations for

- i) the rigid-body problem,
- ii) the intermediate state or ξ -state, i.e. the pre-buckled state in which the plate only deforms in its plane,
- iii) the final or \underline{x} -state, i.e. the state in which there is also a displacement in the Z-direction.

All deformations are assumed to be small.

The equations for the rigid-body problem follow from VIII.(33). This problem can be solved by introducing two scalar potentials, one for the region outside the plate: $\Psi(\underline{X})$ and one inside the plate: $\Phi(\underline{X})$ defined by (an upperindex + stands for values outside the plate and - for values inside the plate)

$$(2) \quad \circ B_i^+ = \circ H_i^+ = -\Psi_{,i},$$

and

$$(3) \quad \circ H_i^- = -\Phi_{,i}, \quad \circ B_i^- = -\mu \Phi_{,i}, \quad \circ M_i^- = -\frac{(\mu - 1)}{4\pi\rho} \Phi_{,i},$$

where μ is the magnetic permeability.

These potentials have to satisfy the following equations

$$(4) \quad \begin{aligned} \Delta \Psi &= 0, \text{ outside the plate,} \\ \Psi &\Rightarrow -B_0 Z, \text{ for } |\underline{X}| \Rightarrow \infty, \\ \Delta \Phi &= 0, \text{ inside the plate,} \\ \Psi - \Phi &= 0 \text{ and } \frac{d\Psi}{dN} - \mu \frac{d\Phi}{dN} = 0, \text{ on the surface } S \text{ of the plate.} \end{aligned}$$

In order to solve this system, we first introduce a new potential $\chi(\underline{X})$ defined by

$$(5) \quad \chi(\underline{X}) = \begin{cases} \psi(\underline{X}) + B_0 Z, & \underline{X} \text{ outside the plate,} \\ \mu\phi(\underline{X}) + B_0 Z, & \underline{X} \text{ inside the plate.} \end{cases}$$

This potential has to satisfy the following equations

$$(6) \quad \begin{aligned} \text{i)} & \quad \Delta\chi = 0, \text{ in- and outside the plate,} \\ \text{ii)} & \quad \llbracket \frac{d\chi}{dN} \rrbracket = 0, \text{ on } S, \\ \text{iii)} & \quad \chi^+ - \frac{1}{\mu} \chi^- = - \frac{(\mu - 1)}{\mu} B_0 Z, \text{ on } S, \\ \text{iv)} & \quad \chi = O(|\underline{X}|^{-2}), \text{ for } |\underline{X}| \rightarrow \infty. \end{aligned}$$

The last condition is a consequence of the fact that there is no free magnetic charge, and hence $\chi(\underline{X})$ must vanish as the potential of a dipole or a multipole of higher order.

Under these conditions, the general solution for $\chi(\underline{X})$ reads (cf. [56], p. 371, eq. (3))

$$(7) \quad \chi(\underline{X}) = \int_S \sigma(\underline{\zeta}) \frac{\partial}{\partial N_{\underline{\zeta}}} \left(\frac{1}{|\underline{X} - \underline{\zeta}|} \right) dS_{\underline{\zeta}},$$

where $\sigma(\underline{\zeta})$ is a continuous density on S , which can be determined from the condition iii) of (6). The remaining three conditions of (6) are satisfied identically by (7).

By use of the relations (cf. [56], p. 360, eq. (1))

$$(8) \quad \chi^{\pm}(\underline{X}_0) = \lim_{\substack{\underline{X} \rightarrow \underline{X}_0 \\ \underline{X}^{\pm} \in S}} \chi(\underline{X}) = \mp 2\pi\sigma(\underline{X}_0) + \overline{\chi(\underline{X}_0)}, \quad \underline{X}_0 \in S,$$

where $\overline{\chi(\underline{X}_0)}$ is the direct value of the potential at the point \underline{X}_0 , we find from the condition iii) of (6) the following integral equation for $\sigma(\underline{X}_0)$

$$(9) \quad \sigma(\underline{X}_0) = - \frac{(\mu - 1)}{2\pi(\mu + 1)} B_0 Z_0 - \frac{(\mu - 1)}{2\pi(\mu + 1)} \int_S \sigma(\underline{\zeta}) \frac{\partial}{\partial N_{\underline{\zeta}}} \left(\frac{1}{|\underline{X}_0 - \underline{\zeta}|} \right) dS_{\underline{\zeta}}.$$

Replacing $\sigma(\underline{X}_o)$ by a density $\phi^S(\underline{X}_o)$ defined by

$$(10) \quad \phi^S(\underline{X}_o) = \frac{4\pi}{(\mu - 1)} \sigma(\underline{X}_o)$$

we obtain from the equations (5), (7) and (9)

$$(11) \quad \begin{aligned} \phi^S(\underline{X}_o) &= -\frac{2B_o Z_o}{(\mu + 1)} - \frac{(\mu - 1)}{2\pi(\mu + 1)} \oint_S \phi^S(\underline{\zeta}) \frac{\partial}{\partial N_{\underline{\zeta}}} \left(\frac{1}{|\underline{X}_o - \underline{\zeta}|} \right) dS_{\underline{\zeta}} , \\ \phi(\underline{X}) &= -\frac{B_o Z}{\mu} - \frac{(\mu - 1)}{4\pi\mu} \oint_S \phi^S(\underline{\zeta}) \frac{\partial}{\partial N_{\underline{\zeta}}} \left(\frac{1}{|\underline{X} - \underline{\zeta}|} \right) dS_{\underline{\zeta}} , \\ \Psi(\underline{X}) &= -B_o Z - \frac{(\mu - 1)}{4\pi} \oint_S \phi^S(\underline{\zeta}) \frac{\partial}{\partial N_{\underline{\zeta}}} \left(\frac{1}{|\underline{X} - \underline{\zeta}|} \right) dS_{\underline{\zeta}} . \end{aligned}$$

This system of equations is equivalent to the one used by van Bladel ([57], pp. 73-77) for the determination of the electric potential in a cubic dielectric.

By taking the limits for $\underline{X}^{\pm} \rightarrow \underline{X}_o \in S$ of (11)² or (11)³, it turns out that $\phi^S(\underline{X}_o)$ represents the distribution of the potential on the surface S . The solution of the system (4) can now be obtained by first solving for $\phi^S(\underline{X}_o)$ the integral equation (11)¹ and then determining $\phi(\underline{X})$ and $\Psi(\underline{X})$ from (11)² and (11)³, respectively.

For the determination of the displacements U_i° and the stresses T_{ij}° in the $\underline{\xi}$ -state, the equations VIII.(36) will be used. After substitution of (3) they read

$$(12) \quad \begin{aligned} T_{ij}^{\circ} &= c_{ijkl} U_{k,l}^{\circ} + \frac{(\mu - 1)}{4\pi} \phi_{,i} \phi_{,j} , \\ T_{ij,j}^{\circ} &= -\frac{(\mu - 1)}{4\pi} \phi_{,j} \phi_{,ij} , \\ T_{ij}^{\circ} N_j &= \frac{(\mu - 1)^2}{8\pi} \left(\frac{d\phi}{dN} \right)^2 N_i + T_i^{\circ*} \text{ on } S . \end{aligned}$$

As noticed before, the permeability μ for ferromagnetic materials, is always large compared with unity ($\mu = 10^4$ à 10^6). Therefore, the term with ϕ in the boundary condition (12)³, that is of the order μ^2 , is dominant with respect to those in the constitutive equations (12)¹ and

the momentum equation (8)², that both are of the order μ . Retaining only terms with μ^2 , the system (12) reduces to

$$\begin{aligned} T_{ij}^{\circ} &= c_{ijkl} U_{k,\ell}^{\circ}, \\ (13) \quad T_{ij,j}^{\circ} &= 0, \\ T_{ij,N_j}^{\circ} &= \frac{\mu^2}{8\pi} \left(\frac{d\phi}{dN}\right)^2 N_i + T_i^{\circ*} \text{ on } S. \end{aligned}$$

From (12) and (13) we conclude that the magnetic part in the constitutive equations for the stresses and the magnetic body force are negligible compared with the magnetic stress vector at the surface.

Finally, we need the equations for the disturbances. From VIII.(34) - (35), we obtain, with (2) and (3):

outside the plate

$$\begin{aligned} b_i^+ &= h_i^+, m_i^+ = 0, \\ (14) \quad e_{ijk} b_{k,j}^+ &= 0, b_{i,i}^+ = 0, \\ b_i^+ &\Rightarrow 0 \text{ for } r \Rightarrow \infty, \quad (r := |\underline{x}|), \end{aligned}$$

inside the plate, where, for convenience, ρ is written for ρ_0 ,

$$\begin{aligned} b_{i,i}^- &= -\phi_{,ij} u_{j,i}, e_{ijk} h_{k,j}^- = -e_{ijk} \phi_{,k\ell} u_{\ell,j}, \\ b_i^- &= h_i^- + 4\pi\rho m_i^- + (\mu-1)\phi_{,i} u_{j,j}, \\ h_i^- &= -\phi_{,j}(u_{i,j} + u_{j,i}) + \frac{4\pi\rho}{(\mu-1)} m_i^-, \\ (15) \quad \rho \dot{v}_i &= t_{ij,j} - T_{ij,k}^{\circ} u_{k,j} - \frac{(\mu-1)}{4\pi} \phi_{,j} h_{j,i}^- - \rho \phi_{,ij} m_j^- + \\ &\quad + \frac{(\mu-1)}{4\pi} \phi_{,j} (\phi_{,ij} u_{k,k} + \phi_{,jk} u_{k,i}), \\ t_{ij} &= -T_{ij}^{\circ} u_{k,k} + T_{ik}^{\circ} u_{j,k} + T_{jk}^{\circ} u_{i,k} + c_{ijkl} u_{k,\ell} - \rho \phi_{,j} m_i^- + \\ &\quad - \rho \phi_{,i} m_j^-, \end{aligned}$$

and on the surface of the plate

$$\begin{aligned}
 e_{ijk} (h_j^+ - h_j^-) N_k &= -e_{ijk} (\psi_{,j} - \phi_{,j}) u_{k,j} N_k, \\
 (16) \quad (b_i^+ - b_i^-) N_i &= -(\psi_{,i} - \mu \phi_{,i}) u_{j,i} N_j, \\
 t_{ij} N_j &= T_{ij}^0 u_{k,j} N_k - \tau_i + t_i^* \text{ on } S,
 \end{aligned}$$

where τ_i is given by VIII.(34)⁶. From this formula it follows that τ_i is proportional to μ^2 .

The systems (15) and (16) can be simplified by eliminating h_i^- and m_i^- with the aid of (15)³ and (15)⁴, and by retaining only the terms with the highest power of μ . Still one simplification is applied, based on the following considerations:

When a plate is interacted by a uniform magnetic field perpendicular to that plate, the field inside the plate is almost everywhere uniform. Only in a very small region in the neighbourhood of the boundary R , this field is nonuniform. Consequently, in the equations (15), holding in the inner of the plate, the fields \underline{B} , \underline{H} and \underline{M} may assumed to be uniform. This means that we may put the potential ϕ , occurring in (15), equal to

$$(17) \quad \phi = -\frac{Bz}{\mu},$$

where B is a constant. It should be beared in mind, that, due to the finiteness of the plate, the constant B is unequal to B_0 , the magnitude of the basic field. However, in the example for a circular plate, that we shall consider in one of the next sections, the difference between B and B_0 is negligible.

Under these approximations, (15) reduces to

$$\begin{aligned}
 h_i^- &= \frac{1}{\mu} b_i^- + \frac{B}{\mu} (u_{j,j} \delta_{i3} + u_{i,3} + u_{3,i}), \\
 \rho m_i^- &= \frac{1}{4\pi} b_i^- + \frac{B}{4\pi} u_{j,j} \delta_{i3}, \\
 (18) \quad b_{i,i}^- &= 0, \quad e_{ijk} b_{k,j}^- = -B(e_{ij3} u_{k,lj} + e_{ijk} u_{k,j3}),
 \end{aligned}$$

$$\rho \dot{\psi}_i = t_{ij,j} - T_{ij,k}^{\circ} u_{k,j} ,$$

$$t_{ij} = -T_{ij}^{\circ} u_{k,k} + T_{ik}^{\circ} u_{j,k} + T_{jk}^{\circ} u_{i,k} + c_{ijkl} u_{k,l} .$$

The boundary conditions (16) can be decomposed into two parts, being the conditions on the upper and lower surfaces $z = \pm h$, and those on the lateral surface R .

Into the boundary conditions on $z = \pm h$, we may again substitute ϕ according to (17). Neglecting all terms that are relatively of the order μ^{-1} , these boundary conditions become

$$e_{ij3} (\mu b_j^+ - b_j^-) = -\mu B e_{ij3} u_{3,j} , \quad b_3^+ - b_3^- = 0 , \quad (19)$$

$$t_{i3} = T_{ij}^{\circ} u_{3,j} - \frac{B^2}{8\pi} u_{3,i} + \frac{B^2}{4\pi} \delta_{i3} b_3^- \quad \text{on } z = \pm h ,$$

where the upper and lower surfaces of the plate are assumed to be free of mechanical stresses, i.e. $t_i^* = 0$.

Let us note that in the equations for the disturbances also the magnetic part in the constitutive equations for the stresses and the magnetic body force are negligible compared with the magnetic stress vector at the surface. Hence, we conclude that instability must be caused by the magnetic surface stresses. Notice, that this only holds for the particular form of the stresses that are used here (i.e. Maxwell-model I, cf. Section III.6).

As follows from (14), we can introduce for the fields outside the plate a scalar potential $\psi(\underline{x})$, such that

$$(20) \quad b_i^+ = h_i^+ = -\psi_{,i} ,$$

and

$$(21) \quad \begin{aligned} \Delta \psi &= 0, \quad \text{outside the plate} , \\ \psi &\rightarrow 0, \quad \text{for } r \rightarrow \infty . \end{aligned}$$

Substituting (20) into the boundary condition (19)¹ yields

$$(22) \quad e_{ij3} b_j^- = \mu B e_{ij3} u_{3,j} - \mu e_{ij3} \psi_{,j} \quad \text{on } z = \pm h .$$

We note that the right-hand side of (22) is of the order μ . The right hand side of (18)⁴ is of the order μ^0 , and hence negligible compared with the former. Thus, the equations for b_i^- become

$$(23) \quad b_{i,i}^- = 0, \quad e_{ijk} b_{k,j}^- = 0,$$

from which we conclude that the displacement \underline{u} effects only the boundary conditions for b_i^- . We now can also introduce a scalar potential for b_i^- . This will be done by

$$(24) \quad b_i^- = -\mu \varphi_{,i},$$

which leads to

$$(25) \quad \Delta \varphi = 0, \text{ inside the plate.}$$

Finally, we define the tensor τ_{ij} by

$$(26) \quad \tau_{ij} := c_{ijkl} u_{k,l}.$$

By utilizing the foregoing results, we obtain the following system of equations for the unknowns ψ , φ and u_i :
outside the plate

$$(27) \quad \begin{aligned} \Delta \psi &= 0, \\ \psi &\rightarrow 0, \text{ for } r \rightarrow \infty, \end{aligned}$$

inside the plate, with the aid of (13)²

$$(28) \quad \begin{aligned} \Delta \varphi &= 0, \\ \rho \ddot{u}_i &= \tau_{ij,j} + T_{jk}^0 u_{i,jk}, \end{aligned}$$

and on the surface $z = \pm h$

$$(29) \quad \psi_{,3} - \mu \varphi_{,3} = 0, \quad e_{ij3} (\psi_{,j} - \varphi_{,j}) = B e_{ij3} u_{3,j},$$

$$\tau_{i3} = \frac{B^2}{8\pi} (u_{j,j} \delta_{i3} - u_{i,3} - u_{3,i}) - \frac{\mu B}{4\pi} \delta_{i3} \varphi_{,3} \text{ on } z = \pm h,$$

where in the latter equation, use has been made of (13)³, which with (17) and with $T_i^{0*} = 0$ yields

$$(30) \quad T_{i3}^{\circ} = \frac{B^2}{8\pi} \delta_{i3} \text{ on } z = \pm h .$$

We recall that the equations listed above only hold on that part of the plate that is outside a close neighbourhood of the boundary R . At R , the boundary conditions in the form (16) must be retained.

When the boundary R of the plate is either clamped or simply supported, the following relations hold for the displacements of the midsurface in the $\underline{\xi}$ -state

$$(31) \quad U_1^{\circ} = U_2^{\circ} = 0 \text{ on } R .$$

In this case $T_i^{\circ*$ is not equal to zero on R , but an unknown support reaction. Therefore, we must replace the boundary condition for T_{ij}° on R by (31). The system (13) then transforms into

$$(32) \quad \begin{aligned} T_{ij}^{\circ} &= c_{ijkl} U_{k,l}^{\circ}, \quad T_{ij,j}^{\circ} = 0, \\ T_{i3}^{\circ} &= \frac{B^2}{8\pi} \delta_{i3} \text{ on } z = \pm h, \\ U_1^{\circ} &= U_2^{\circ} = 0 \text{ on } R. \end{aligned}$$

The solution of (32) reads

$$(33) \quad U_1^{\circ} = U_2^{\circ} = 0, \quad U_3^{\circ} = \frac{B^2}{8\pi c_{11}} z ,$$

and

$$(34) \quad \begin{aligned} T_{11}^{\circ} &= T_{22}^{\circ} = \frac{c_{12}}{c_{11}} \frac{B^2}{8\pi}, \quad T_{33}^{\circ} = \frac{B^2}{8\pi}, \\ T_{12}^{\circ} &= T_{23}^{\circ} = T_{31}^{\circ} = 0 . \end{aligned}$$

IX.3. Plate equations

In this section, we shall specialize the equations of motion (28)² together with the boundary conditions (29)³ to the case of a thin plate. A plate is said to be thin, if one dimension, i.e. the thickness $2h$, is small compared with the other two dimensions. Let R be a characteristic measure for the plate in the x - y -plane, then

$$(35) \quad \epsilon := \frac{h}{R} \ll 1 ,$$

for a thin plate. For an excellent treatise of the classical theory of thin plates, we refer to [58].

In order to derive the plate equations, the equations of motion will be integrated, eventually after multiplication by z , in the thickness direction.

Before doing so, we first introduce the integral quantities

$$(36) \quad \begin{aligned} Q_x &:= \int_{-h}^h \tau_{xz} dz, & Q_y &:= \int_{-h}^h \tau_{yz} dz, \\ M_{xx} &:= \int_{-h}^h z \tau_{xx} dz, & M_{yy} &:= \int_{-h}^h z \tau_{yy} dz, \\ M_{xy} &:= \int_{-h}^h z \tau_{xy} dz. \end{aligned}$$

Furthermore, we shall use the Kirchhoff-Love hypothesis, which states the following:

Let $\underline{u}(x,t)$ describe the motion of the plate out of its plane, and let w be the displacement of the midsurface in the z -direction, thus

$$(37) \quad w(x,y,t) = u_z(x,y,0,t) ,$$

then, according to the Kirchhoff-Love hypothesis the following relations hold

$$(38) \quad \begin{aligned} u_x(x,y,z,t) &= -zw_{,x} , \\ u_y(x,y,z,t) &= -zw_{,y} . \end{aligned}$$

Based on the approximation, common to the classical theory of plates, in which the normal stress in the z -direction is neglected, i.e. $\tau_{zz} \approx 0$, we deduce from (22) the following relation (viz. Note after eq. (43))

$$(39) \quad u_{z,z} = - \frac{\nu}{(1-\nu)} (u_{x,x} + u_{y,y}) ,$$

where

$$(40) \quad \frac{c_{12}}{c_{11}} = \frac{\nu}{(1 - \nu)}$$

is used,

By means of (37) and (38) it follows from (39) that

$$(41) \quad u_z(x, y, z, t) = w + \frac{\nu}{2(1 - \nu)} z^2 (w_{,xx} + w_{,yy}) .$$

Substitution of (26), (38) and (41) into (36) yields the following set of constitutive equations for the moments in the plate

$$(42) \quad \begin{aligned} M_{xx} &= -D(w_{,xx} + \nu w_{,yy}), \quad M_{yy} = -D(w_{,yy} + \nu w_{,xx}), \\ M_{xy} &= -D(1 - \nu)w_{,xy}, \end{aligned}$$

where D is the plate constant, which for a plate with thickness $2h$ is equal to

$$(43) \quad D = \frac{2Eh^3}{3(1 - \nu^2)} .$$

These constitutive equations are the common ones in the classical theory of thin plates (cf. [54], p. 39). The errors made in this theory are of the order ϵ^2 .

Note. We are aware of the fact that, in the classical theory of thin plates, the assumption $\tau_{zz} = 0$ is based on the condition that this stress component must be zero at the upper and lower surfaces of the plate. However, as can be seen from (29), in our problem τ_{zz} is not equal to zero at $z = \pm h$, so by using the expression (41) for u_z the condition (29) is violated. This can be corrected by adding to (41) an extra term, such that (29) does be satisfied. However, this results in the ultimate equations only in a correction that is $O(\epsilon^2)$ and hence neglectable in the theory of thin plates. This approach is equivalent to the way in which, in the classical plate theory, the problem of a purely elastic plate, loaded by surface loads perpendicular to the surface is treated. In this case, also τ_{zz} is taken equal to zero and the surface loads are distributed over the thickness and considered as global volume forces.

In the sequel, we shall assume that the plate is either clamped or simply supported. Hence, for the intermediate stresses T_{ij}^0 , the formulae (34) hold.

First we integrate the equation of motion (28)² for $i = 3$ in the thickness direction, yielding

$$(44) \quad \int_{-h}^h \rho \ddot{u}_z dz = \int_{-h}^h (\tau_{xz,x} + \tau_{yz,y} + \tau_{zz,z}) dz + \\ + \frac{B^2}{8\pi} \int_{-h}^h \left[\frac{\nu}{(1-\nu)} (u_{z,xx} + u_{z,yy}) + u_{z,zz} \right] dz,$$

where (34) and (40) are used.

Utilizing the definitions (36)^{1,2}, the expressions for the displacements (38) and (41) and the boundary condition (29)³ for $i = 3$, equation (44) can be elaborated, after partial integration and under neglectation of terms of $O(\epsilon^2)$, to give

$$(45) \quad \rho h \ddot{w} = Q_{x,x} + Q_{y,y} + \beta - \frac{(1-2\nu)B^2 h}{4\pi(1-\nu)} \Delta w,$$

where

$$(46) \quad \beta = \beta(x,y,t) := -\frac{\mu B}{4\pi} \varphi_{,z} \Big|_{z=-h} \text{ and } \Delta w = w_{,xx} + w_{,yy}.$$

Next, we integrate, over the thickness, after multiplication by z , equation (28)² for $i = 1$, yielding

$$(47) \quad \int_{-h}^h z \ddot{u}_x dz = \int_{-h}^h z (\tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z}) dz + \\ + \frac{B^2}{8\pi} \int_{-h}^h z \left[\frac{\nu}{(1-\nu)} (u_{x,xx} + u_{x,yy}) + u_{x,zz} \right] dz.$$

After partial integration, this relation becomes

$$(48) \quad -\frac{2h^3}{3} \rho \ddot{w}_{,x} = M_{xx,x} + M_{xy,y} - Q_x.$$

In an analogous way, we obtain from (28)² for $i = 2$, the equation in the y -direction

$$(49) \quad -\frac{2h^3}{3} \rho \ddot{w}_{,y} = M_{xy,x} + M_{yy,y} - Q_y .$$

Differentiating (48) with respect to x and (49) with respect to y and adding the thus obtained equations and (43), we find after neglection of terms of $O(\varepsilon^2)$, the equation

$$(50) \quad \rho h \ddot{w} = M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \beta - \frac{(1-2\nu)B^2h}{4\pi(1-\nu)} \Delta w .$$

Substitution of the constitutive equations (42) into (50) results in the following equation for $w(x,y,t)$

$$(51) \quad Dh\Delta\Delta w - \beta h + \frac{(1-2\nu)B^2h^2}{4\pi(1-\nu)} \Delta w = -\rho h \ddot{w} .$$

For a plate that is clamped at R , (51) must be supplemented by the boundary conditions

$$(52) \quad w = \frac{dw}{dN} = 0 \text{ on } R .$$

In case of a simply supported plate, only the first condition of (52) holds. A second condition is in fact undetermined. We shall return to this subject in the following section.

Note. Regarding (51), we see that the first term of (51) is $O(\varepsilon^4).w$, hence of the same order as the terms that are omitted in the derivation of (51). However, the coefficients of the omitted terms were always proportional to B^2 , while the first term of (51) is proportional to Young's modulus E . Since $E \gg B^2$, the neglection of the other terms that are $O(\varepsilon^4).w$, with respect to the first term of (51), is justified.

IX.4. Circular plate

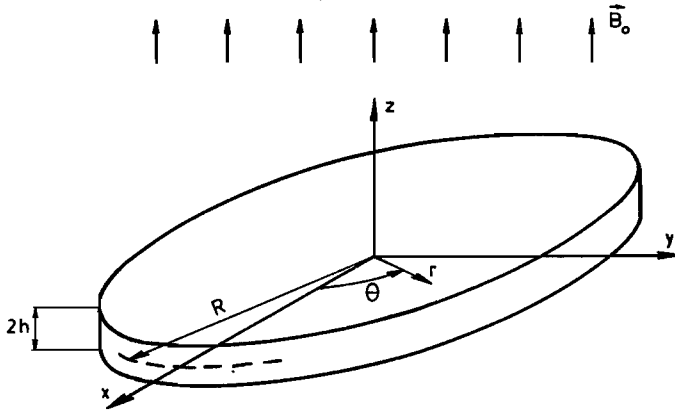


Fig. IX.2.

In this section, we shall apply the plate equations derived in the preceding section, in the example of a circular plate with radius R and thickness $2h$ (Fig. IX.2). We take the boundary of the plate clamped. The quantity B , occurring in (51), can be determined from the integral equations (11). However, it turns out that the difference between B and B_0 is $O(\epsilon)$ ($\epsilon = h/R \ll 1$), and therefore we may take

$$(53) \quad B = B_0 .$$

We assume the deflection of the plate to be rotationally symmetric. Expressed in the polar coordinates r and θ (viz. Fig. IX.2), we then obtain from (27), (28)¹ and (51) the following set of equations for the unknowns ψ , φ and w , holding in the region $r < R$

$$(54) \quad \begin{aligned} \Delta\psi &= 0, \quad |z| > h, \\ \psi &\rightarrow 0, \quad |z| \rightarrow \infty, \\ \Delta\varphi &= 0, \quad |z| < h, \\ \Delta\Delta w - \frac{\beta}{D} + \frac{(1-2\nu)B_0^2 h^2}{4\pi(1-\nu)D} \Delta w &= -\frac{\rho h}{D} \ddot{w}, \quad |z| < h \end{aligned}$$

where

$$(55) \quad w = w(r, t) \text{ and } \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

together with the boundary conditions on the upper and lower surface of the plate, coming from (29)^{2,3}

$$(56) \quad \begin{aligned} \psi_{,r} - \varphi_{,r} &= B_0 w', \quad (w' := \frac{\partial w}{\partial r}), \\ \psi_{,z} - \mu \varphi_{,z} &= 0, \text{ on } |z| = h. \end{aligned}$$

We try as a solution of this system

$$(57) \quad \begin{aligned} w'(r, t) &= A J_1(\alpha r) e^{i\omega t}, \\ \psi(r, z, t) &= B_0 C(z) J_0(\alpha r) e^{i\omega t}, \\ \varphi(r, z, t) &= B_0 D(z) J_0(\alpha r) e^{i\omega t}. \end{aligned}$$

The coefficient α is to determine from the boundary condition (52)², that here becomes

$$(58) \quad w'(R, t) = 0,$$

and that gives

$$(59) \quad J_1(\alpha R) = 0.$$

The lowest root of (59) is

$$(60) \quad \alpha_1 = \frac{3.83}{R}.$$

The frequency ω will be calculated from the equation for $w(r, t)$. For small magnitudes B_0 this will yield a real value for ω . The intermediate state is then stable. With increasing B_0 , the point will be reached where ω becomes complex. At this point the intermediate state becomes unstable. The relevant magnitude of B_0 is the buckling value. Substitution of (57)^{2,3} into (54)^{1,3} yields

$$(61) \quad \begin{aligned} \Delta \psi &= B_0 J_0(\alpha r) e^{i\omega t} [C''(z) - \alpha^2 C(z)] = 0, \\ \Delta \varphi &= B_0 J_0(\alpha r) e^{i\omega t} [D''(z) - \alpha^2 D(z)] = 0. \end{aligned}$$

By means of (54)² and the symmetry condition $\varphi(z) = \varphi(-z)$, following from the particular form of the boundary conditions at $z = \pm h$, we infer from (61) that

$$(62) \quad C(z) = ce^{-\alpha|z|} \quad \text{and} \quad D(z) = d \cosh(\alpha z), \quad (\alpha > 0).$$

The coefficients c and d can be calculated by substituting (62) into the boundary conditions (56). This gives

$$(63) \quad c = -\frac{A}{2h\Lambda} e^{\alpha h} \sinh(\alpha h) \quad \text{and} \quad d = \frac{A}{2\mu\alpha h\Lambda},$$

where

$$(64) \quad \Lambda = \frac{\mu \sinh(\alpha h) + \cosh(\alpha h)}{\mu\alpha h}.$$

Using these results, the function $\beta(r, t)$ defined by (46)¹ becomes

$$(65) \quad \beta(r, t) = -\frac{2B_0^2}{4\pi\Lambda\alpha h} \sinh(\alpha h) AJ_0(\alpha r) e^{i\omega t}.$$

Since

$$(66) \quad \alpha h = O(\varepsilon) \ll 1,$$

the following approximations may be applied

$$(67) \quad \sinh(\alpha h) \approx \alpha h \quad \text{and} \quad \cosh(\alpha h) \approx 1.$$

By using these approximations (61) reduces to

$$(68) \quad \beta(r, t) = -\frac{B_0^2}{2\pi\Lambda} AJ_0(\alpha r) e^{i\omega t}.$$

By first differentiating (54)⁴ with respect to r , then substituting (57)¹ and (68) into this equation and, finally, dividing by $AJ_1(\alpha r)e^{i\omega t}$, we have deduced the following formula for ω

$$(69) \quad \frac{\rho h}{\alpha} \omega^2 = \frac{2E}{3(1-\nu^2)} (\alpha h)^3 - \frac{B_0^2}{2\pi\Lambda} \left[1 + \frac{(1-2\nu)\alpha h}{2(1-\nu)} \right].$$

As mentioned before, buckling occurs when ω becomes complex. As can be seen from (69), the transition from real values of ω to complex values takes place at $\omega^2 = 0$. Hence, the critical value for B_0 follows from

the relation

$$(70) \quad \frac{(B_o^2)_{\text{buc.}}}{2\pi\Lambda} \left[1 + \frac{(1-2\nu)}{2(1-\nu^2)} \alpha h \right] = \frac{2E}{3(1-\nu^2)} (\alpha h)^3 .$$

Based on (66), the second term in the left-hand side of (70) may be neglected with respect to the first one. Since this second term represents the influence of the intermediate stress T_{ij}^o , we can conclude that this stress has no essential influence on the buckling value. Moreover, it follows from the fact that the point of buckling is reached for $\omega = 0$, that the buckling value also can be calculated by means of a static method.

Equation (70) yields, after neglectation of the term of $O(\epsilon)$ and by taking for α the lowest value according to (60)

$$(71) \quad \frac{(B_o^2)_{\text{buc.}}}{4\pi E} = \frac{\Lambda}{3(1-\nu^2)} \left(\frac{3.83h}{R} \right)^3 = \frac{18.7\Lambda}{(1-\nu^2)} \left(\frac{h}{R} \right)^3 .$$

It appears from (71) that the buckling value depends on Λ , which according to (64) and with the approximations (67) is equal to

$$(72) \quad \Lambda = \frac{\mu\alpha h + 1}{\mu\alpha h} .$$

When

$$(73) \quad \mu \gg \alpha h$$

the expression (72) may be approximated by

$$(74) \quad \Lambda = 1 .$$

Hence, in this case the buckling value is independent of the magnetic permeability.

In order to get an impression of the magnitude of B_o , we assume (74) to hold and we use the numerical values

$$(75) \quad E = 2.1 \times 10^{12} \text{ dyne/cm}^2 \text{ and } \nu = 0.3 .$$

This gives

$$(76) \quad (B_o)_{\text{buc.}} = 2.33 \times 10^7 (h/R)^{3/2} \text{ Gauss} ,$$

or

$$(77) \quad (B_0)_{\text{buc.}} = 736 \text{ Gauss, for } h/R = 10^{-3},$$
$$(B_0)_{\text{buc.}} = 23.3 \text{ Gauss, for } h/R = 10^{-4}.$$

So far, we have obtained the buckling value for a clamped, circular plate. We wish to regard also the problem of a circular plate that is simply supported at the boundary. However, in this case the condition (58) no longer holds. It must be replaced by a condition for the bending moment at the boundary of the plate. However, as we do not know at all the values of the fields at the boundary $r = R$, it is not possible to give the exact value of this moment. The only possibility that remains, is to make a, physically reasonable, assumption about the value of this moment, such that the problem can be solved, and to compare the thus obtained results with experimental values.

For the circular plate that is simply supported at its boundary, we will assume that the bending moment M_{rr} at $r = R$ is equal to zero, i.e.

$$(78) \quad M_{rr} = -D(w_{,rr} + \frac{\nu}{r} w_{,r}) = 0 \text{ at } r = R,$$

in accordance with (42).

For a purely elastic, simply supported plate, the relation (78) holds exactly. However, for a plate that is magnetically loaded, this is no longer true, due to the occurrence of magnetic moments at the boundary. We, nevertheless, shall use the condition (78), based on the conjecture that these magnetic moments most likely are of the order (B^2/E) compared with the purely elastic part of these moments, and therefore can be neglected (see also the remark made after eq. (81)).

Under the condition (78), the simply supported plate can be treated in an analogous way as the clamped plate. We merely have to replace the condition (59) by (78). This results in the following relation for α

$$(79) \quad J_0(\alpha R) - \frac{\nu}{\alpha R} J_1(\alpha R) = 0.$$

When we take $\nu = 0.3$, the first root of (79) is

$$(80) \quad \alpha R = 2.27,$$

which gives, analogously to (71),

$$(81) \quad \left(\frac{B^2}{4\pi E}\right)_{\text{buc.}} = \frac{(2.27)^3 \Lambda}{3(1 - \nu^2)} \left(\frac{h}{R}\right)^3 = \frac{3.90\Lambda}{(1 - \nu^2)} \left(\frac{h}{R}\right)^3 .$$

For a further motivation of the assumption (78), we refer to the final section of this chapter, in which it will be shown that the constitutive equations for the bending moments defined on an alternative stress tensor (i.e. the Amperian-current model) do only differ in a negligible amount from those according to (42). Furthermore, the results obtained there for a cantilever do reasonably correspond with experimental results.

The statement, concerning the neglect of the magnetic part of the bending moment, does not hold in a similar form for the shear forces Q_x and Q_y . As a consequence, the consideration of the free plate, based on the assumption that the tangential bending moment and the shear force, as defined in (36), are equal to zero, will not give the correct results. We may conclude that the magnetic shear forces at the boundary are not negligible in our formulation.

IX.5. Beam equations

We have derived in Section IX.3 the equations of motion for a thin plate, which we have specialized in Section IX.4 to the case of the circular plate. In this section, we shall regard a rectangular plate, with dimensions $2a$ and $2b$ in the x - and y -direction, respectively. If the restriction $b \ll a$ holds, this plate may be considered as a beam. We shall derive global equations of motion for a beam, by integrating the local equations over the cross section of the beam. In the sequel, it will always be assumed that the beam bends in the x - z -plane (viz. Fig. IX.3)

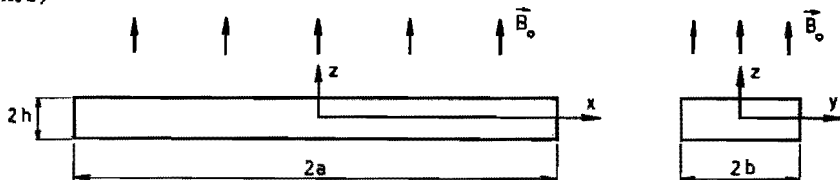


Fig. IX.3.

We shall derive the buckling equations for a beam loaded by a magnetic field, and we shall compare our results with those of the references [52] and [53]. We introduce the shear force Q_z and the bending moment M_y by

$$(82) \quad Q_z := \int_A \tau_{xz} dS \quad \text{and} \quad M_y := \int_A z \tau_{xx} dS ,$$

where A is the area of the cross section.

Underlying the derivation of the global equations for the bending of the beam, is the Bernoulli-hypothesis. According to this hypothesis, the bending in the x - z -plane is described by the displacement field

$$(83) \quad \begin{aligned} u_x(x,y,z,t) &= -zw'(x,t), \quad (w' = w_{,x}) , \\ u_y(x,y,z,t) &= 0 , \\ u_z(x,y,z,t) &= w(x,t) , \end{aligned}$$

and the following constitutive equation for the bending moment M_y holds

$$(84) \quad M_y = -EIw'' ,$$

where, for the rectangular cross section under consideration,

$$(85) \quad I = \frac{4}{3} bh^3 .$$

The errors made in this theory are of the order (b^2/a^2) . We note that the relation (84) can be derived in a similar way as (42).

In this theory, the dependence on the y -coordinate is neglected. This will also be done for the potentials ψ and φ .

Just like in the preceding section, it can here be shown that the intermediate stresses T_{ij}° do not have any essential influence on the buckling values.

By omitting T_{ij}° and by taking all quantities independent of y , the following system for ψ , φ and w can be derived from the equations (27) to (29) and with the aid of (83)

$$\Delta\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad |z| > h,$$

$$\psi \Rightarrow 0, \quad |z| \Rightarrow \infty,$$

$$(86) \quad \Delta\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad |z| < h,$$

$$\rho \ddot{u}_x = -\rho z \ddot{w}' = \tau_{xx,x} + \tau_{xz,z}, \quad |z| < h,$$

$$\rho \ddot{u}_z = \rho \ddot{w} = \tau_{xz,x} + \tau_{zz,z}, \quad |z| < h.$$

Moreover, we also find the boundary conditions on the upper and lower surfaces of the beam

$$(87) \quad \psi_{,x} - \varphi_{,x} = Bw', \quad \psi_{,z} - \mu\varphi_{,z} = 0,$$

$$\tau_{xz} = 0 \text{ and } \tau_{zz} = -\frac{\mu B}{4\pi} \varphi_{,z} \text{ on } |z| = h.$$

By integrating the equations (86)⁵ and (86)⁴, the latter multiplied by z , over the cross section, with

$$(88) \quad \int_A dS = 2b \int_{-h}^h dz,$$

and after partial integration whereby (87)^{3,4} are used, we obtain the global equations of motion

$$(89) \quad 4\rho b h \ddot{w} = Q_{z,x} + \beta,$$

$$-\frac{4}{3} \rho b h^3 \ddot{w}' = M_{y,x} - Q_z,$$

where, now

$$(90) \quad \beta = \beta(x,t) = -\frac{b\mu B}{2\pi} \varphi_{,z} \Big|_{z=-h}^h.$$

Differentiating (89)² with respect to x and adding to (89)¹ and then substituting the constitutive equation (84) yields the following equation for the deflection $w(x,t)$

$$(91) \quad -\frac{4}{3} E b h^3 w'''' + \beta = 4\rho b h \ddot{w},$$

where terms of $O(b^2/a^2)$ are neglected.

For a beam that is clamped at $x = \pm a$, the boundary conditions are

$$(92) \quad w = w' = 0 \text{ for } x = \pm a .$$

If the beam is simply supported at $x = \pm a$, we, just as in the preceding section, assume that the bending moment is equal to zero in the end points. With (84), we then have

$$(93) \quad w = w'' = 0 \text{ for } x = \pm a .$$

Both boundary conditions can be satisfied by taking a displacement field in the form

$$(94) \quad w(x,t) = [A_1 \cos(\alpha x) + A_2] e^{i\omega t} .$$

Substituting (94) successively into (92) and (93), yields the following values for α , belonging to the lowest buckling values

$$(95) \quad \alpha = \frac{\pi}{a} ,$$

for the clamped beam, and

$$(96) \quad \alpha = \frac{\pi}{2a}$$

for the simply supported beam.

In the same way as in the preceding section, we can solve the equations for ψ and φ and thus obtain an expression for the function $\beta(x,t)$ defined in (90). This results in

$$(97) \quad \varphi(x,z,t) = - \frac{BA_1}{\mu a h \Lambda} \cosh(\alpha z) \cos(\alpha x) e^{i\omega t} ,$$

and

$$(98) \quad \beta(x,t) = \frac{bB^2 A_1}{\pi h \Lambda} \sinh(\alpha h) \cos(\alpha x) e^{i\omega t} ,$$

where Λ is given in (64).

By differentiating (91) with respect to x , in order to eliminate the coefficient A_2 occurring in (94), and then substituting (98) and (94) into the thus obtained equation, we find the following formula for ω

$$(99) \quad 4\rho b h \omega^2 = \frac{4}{3} E b h^3 \alpha^4 - \frac{b B^2}{\pi h \Lambda} \sinh(\alpha h) .$$

Again, instability is reached for $\omega = 0$. With

$$(100) \quad \alpha h \ll 1 ,$$

we find the buckling value

$$(101) \quad \left(\frac{B^2}{4\pi E}\right)_{\text{buc.}} = \frac{\Lambda}{3} (\alpha h)^3 .$$

In (101), B is the value of the magnetic field at the upper or lower surface of the plate. According to Wallerstein & Peach [53], who solved the integral equation (11), by means of a discretization of this equation, the following relation between B and the magnitude of the basic field B_0 holds for a large range of values for (a/h)

$$(102) \quad B = 1.86 B_0 .$$

We note that the coefficient B/B_0 is a function of the beam dimensions. However, a variation of as much as 60 percent in a/h produces a variation in B/B_0 of less than 3.5 percent (cf. [53]).

By use of (95) and (96), we then find the following buckling values for B_0 :

For a clamped beam

$$(103) \quad \left(\frac{B_0^2}{4\pi E}\right)_{\text{buc.}} = \frac{\pi^3 \Lambda}{3 \times (1.86)^2} \left(\frac{h}{a}\right)^3 = 2.99 \Lambda \left(\frac{h}{a}\right)^3 ,$$

and for a simply supported beam

$$(104) \quad \left(\frac{B_0^2}{4\pi E}\right)_{\text{buc.}} = \frac{\pi^3 \Lambda}{24 \times (1.86)^2} \left(\frac{h}{a}\right)^3 = 0.374 \Lambda \left(\frac{h}{a}\right)^3 .$$

In the next section a cantilever, i.e. a beam that is clamped at one side and free at the other, is studied.

IX.6. Discussion

Recently, there have appeared a number of articles dealing with the buckling of a cantilever in a transverse magnetic field. Moon & Yih-Hsing Pao, [52], used a stress tensor based on a magnetic dipole model, according to Brown [11]. They assumed that the beam is infinitely wide and infinitely long. Furthermore, several experiments are reported. The correlation between their experimental and theoretical results was not very good, (their theoretical results show a percent excess in field above the experimental results of more than 100%), but this was improved by Wallerstein & Peach [53], who considered plates of finite dimensions. They found about a 20 percent excess of the theoretical buckling values over the experimental values. In [53], just like in the paper of Popelar [54], for the stresses the Amperian-current model was chosen (cf. Section III.5). Popelar used the principle of virtual work and he showed, by developing a postbuckling theory, that the buckling value for a cantilever is very sensible to a misalignment of the field with the normal of the beam. This may explain the discrepancies between theoretical and experimental results.

In none of these papers, there is given a constitutive equation relating the stresses to the deformations and the magnetic field. They only give global constitutive equations, relating the bending moments to the curvature of the plate or beam, that are identical to those of the classical theory of plates and beams. In this section, we shall derive the constitutive equations for the moments and shear forces belonging to an Amperian-current stress model analogically as we have done in a foregoing section for the stresses according to the Maxwell-model. By comparing the equations belonging to the Amperian-stresses with those of the Maxwell-stresses, it will turn out that there are no essential differences in the examples of the clamped or the simply supported beam, but for the problem of the cantilever only the Amperian-model gives results that are in correspondance with the experiments. This is a corroboration of the remark stated in [54], that says that the Amperian-model tends to minimize the influences of end effects.

Let us denote the stresses according to the Amperian-current model by $T_{ij}^{(3)}$. We then have, according to III.(57) under the absence of polari-

zation,

$$(105) \quad T_{ij}^{(3)} = T_{ij} - \rho M_i B_j^- + \frac{1}{2} \delta_{ij} \rho M_k (B_k^- + H_k^-),$$

and this equation reduces, after the neglect of terms of $O(u^{-1})$, to

$$(106) \quad T_{ij}^{(3)} = T_{ij} - \rho M_i B_j^- + \frac{1}{2} \delta_{ij} \rho M_k B_k^-.$$

This relation, applied in the intermediate state $\underline{\xi}$, yields, meeting the usual restrictions, the following expression for the intermediate stresses

$$(107) \quad T_{ij}^{(3)\circ} = c_{ijkl} U_{k,\ell}^\circ - \rho \circ M_i \circ B_j^- + \frac{1}{2} \delta_{ij} \rho \circ M_k \circ B_k^-.$$

In the inner part of the plate, at some distance from the boundary, this gives with (3) and (17)

$$(108) \quad T_{ij}^{(3)\circ} = c_{ijkl} U_{k,\ell}^\circ - \frac{B^2}{4\pi} (\delta_{i3} \delta_{j3} - \frac{1}{2} \delta_{ij}).$$

As is evident from (108), the magnetic part of the constitutive equation for the stresses may not be neglected in the Amperian-current formulation.

Substituting the displacement field according to (33) into (108), and using the relation

$$(109) \quad \frac{c_{11} + c_{12}}{c_{11}} = \frac{1}{1 - \nu},$$

yields

$$(110) \quad \begin{aligned} T_{11}^{(3)\circ} &= T_{22}^{(3)\circ} = \frac{B^2}{8\pi(1-\nu)}, \quad T_{33}^{(3)\circ} = \frac{B^2}{16\pi}, \\ T_{12}^{(3)\circ} &= T_{23}^{(3)\circ} = T_{31}^{(3)\circ} = 0. \end{aligned}$$

These stresses are of the same order of magnitude as those according to (34), and therefore, just like in (70), have no influence on the buckling equations. We note that this result only holds in the inner part of the plate. At the boundary, the stresses T_{ij}° and $T_{ij}^{(3)\circ}$ can still differ considerably.

Expanding (97) about the ξ -state, utilizing the results of the preceding sections and neglecting the intermediate stresses, we arrive at the following expression for the stresses $t_{ij}^{(3)}$, holding inside the plate or beam

$$(111) \quad t_{ij}^{(3)} = \tau_{ij} + \frac{\mu B}{4\pi} (\delta_{i3}\varphi_{,j} + \delta_{j3}\varphi_{,i} - \delta_{ij}\varphi_{,z}) .$$

Again, the magnetic part in this equation is not negligible.

By means of (28) and (29), the following equation of motion with boundary condition can be derived from (111)

$$(112) \quad t_{ij,j}^{(3)} = \tau_{ij,j} = \rho \ddot{u}_i ,$$

and

$$(113) \quad t_{i3}^{(3)} = \frac{\mu B}{4\pi} (-\delta_{i3}\varphi_{,z} + \varphi_{,i}), \text{ for } |z| = h ,$$

where terms that are of the order T_{ij}° are neglected.

We define the shear force and the bending moment belonging to the stress tensor $t_{ij}^{(3)}$ by

$$(114) \quad Q_z^{(3)} := \int_A t_{xz}^{(3)} dA \text{ and } M_y^{(3)} := \int_A z t_{xx}^{(3)} dA .$$

Comparing these quantities with the moment and the force defined in (82), we find that

$$(115) \quad Q_z^{(3)} = Q_z + \frac{\mu b B}{2\pi} \int_{-h}^h \varphi_{,x} dz ,$$

and

$$(116) \quad M_y^{(3)} = M_y - \frac{\mu b B}{2\pi} \int_{-h}^h z \varphi_{,z} dz .$$

By substituting for $\varphi(x,z,t)$ the expression (98) into (115) and (116), and using the approximation $ah \ll 1$, these equations transform into

$$(117) \quad Q_z^{(3)} = Q_z + \frac{bB^2}{\pi \Lambda} A_1 \sin(\alpha x) e^{i\omega t} ,$$

and

$$(118) \quad M_y^{(3)} = M_y - \frac{bB^2}{3\pi\Lambda} (\alpha h)^2 \frac{A_1 \cos(\alpha x)}{\alpha} e^{i\omega t} .$$

As we shall show, the second term in the right-hand side of (118), when substituted into the global equation of motion, gives only rise to a term that is $O((\alpha h)^2)$ with respect to the other terms. Therefore, this term may be neglected, and we may put

$$(119) \quad M_y^{(3)} = M_y = -EIw'' .$$

This statement can be proved as follows:

In the usual way, we infer from (112) and (113) the global equations of motion

$$(120) \quad M_{y,x}^{(3)} - Q_z^{(3)} + \frac{\mu b B}{2\pi} (z\varphi, x) \Big|_{z=-h}^h = -\frac{4}{3} \rho b h^3 \ddot{w} ,$$

and

$$(121) \quad Q_{z,x}^{(3)} = 4\rho b h \ddot{w} .$$

Differentiating (111) with respect to x and adding this equation to (112), yields, after neglectation of terms of $O((\alpha h)^2)$

$$(122) \quad M_{y,xx}^{(3)} + \beta^{(3)} = 4\rho b h \ddot{w} ,$$

where

$$(123) \quad \beta^{(3)} = \frac{\mu b B}{2\pi} (z\varphi, xx) \Big|_{z=-h}^h .$$

By use of (97) and the usual approximations, (123) can be elaborated, to show that

$$(124) \quad \beta^{(3)} = \frac{\alpha b B^2}{\pi\Lambda} A_1 \cos(\alpha x) e^{i\omega t} = \beta$$

with β according to (98).

It follows from (109) that

$$(125) \quad M_{y,xx}^{(3)} = M_{y,xx} + \frac{\alpha b B^2}{3\pi\Lambda} (\alpha h)^2 A_1 \cos(\alpha x) e^{i\omega t} .$$

The second term in the right-hand side of (125) is $O((ah)^2)$ with respect to $\beta^{(3)}$, and hence may be neglected in the equation of motion (112). Thus, the relation (119) is justified. \square

As the second term in the right-hand side of (117) is of the same order as the other terms occurring in (120), this one may not be neglected. Hence, there is an essential difference between the shear force belonging to t_{ij} (or τ_{ij}) and the one belonging to $t_{ij}^{(3)}$.

According to (89)², in which the left-hand side may be neglected, we have the constitutive equation for Q_z

$$(126) \quad Q_z = M_{y,x} = -EIw''',$$

holding in the inner part of the beam.

Substituting this relation into (115) yields

$$(127) \quad Q_z^{(3)} = -EIw'''' + \frac{ubB}{2\pi} \int_{-h}^h \varphi_{,x} dz.$$

This expression is in correspondance with (120).

Based on the foregoing results, we may conclude that, as concerns the bending moment, it does not matter whether the stresses according to the Maxwell- or the Amperian-model are used, but for the shear force these two models give essential different expressions. The ultimate equations of motion are, of course, in both cases identical.

We shall apply the foregoing results in the special example of a cantilever, i.e. a beam that is clamped at one end ($x = 0$) and free at the other ($x = a$). For convenience, we shall restrict ourselves from the beginning to the static problem.

According to (91) we then have

$$(128) \quad -EIw''''(x) + \beta(x) = 0.$$

At the clamped end $x = 0$, the displacement $w(x)$ must satisfy

$$(129) \quad w(0) = w'(0) = 0.$$

As concerns the boundary conditions at the free end $x = a$, we shall make two assumptions:

- i) We assume that in the cross section $x = a$ the bending moment and the shear force are zero.
- ii) We suppose the relations (119), (126) and (127), which in principle only hold in the inner part of the beam, also to be valid in the end section $x = a$.

In the Maxwell-formulation, as used throughout this thesis, this gives the boundary conditions

$$(130) \quad w''(a) = w'''(a) = 0 ,$$

while in the Amperian-formulation we get

$$(131) \quad w''(a) = 0, EIw'''(a) = \gamma(a) ,$$

where

$$(132) \quad \gamma(x) = \frac{\mu b B}{2\pi} \int_{-h}^h \varphi_{,x} dz .$$

Hence, these two formulations lead to different boundary conditions at the free end of the beam.

We will first solve the problem in the Amperian-formulation.

Let us represent the displacement field by

$$(133) \quad w(x) = A_1 \cos(\alpha x) + A_2 + A_3 \sin(\alpha x) + A_4 x .$$

From the boundary conditions (129) and (131) and from the equations for $\varphi(x, z)$ of the preceding section, we obtain in the usual way

$$(134) \quad A_2 = -A_1, A_3 = A_4 = 0 ,$$

$$(135) \quad \beta(x) = \frac{\alpha b B^2}{\pi \Lambda} A_1 \cos(\alpha x) ,$$

$$(136) \quad \gamma(x) = \frac{b B^2}{\pi \Lambda} A_1 \sin(\alpha x) ,$$

while the coefficient α must satisfy

$$(137) \quad A_1 \cos(\alpha a) = 0 \text{ and } [EI\alpha^3 - \frac{bB^2}{\pi\Lambda}]A_1 \sin(\alpha a) = 0 .$$

If

$$(138) \quad EI\alpha^3 - \frac{bB^2}{\pi\Lambda} = 0 ,$$

the differential equation (128) is satisfied identically.

From (137), we conclude that a nontrivial solution is only possible if

$$(139) \quad \alpha = \frac{\pi}{2a} ,$$

what, with (138) results in the following critical value for B

$$(140) \quad \left(\frac{B^2}{4\pi E'}\right)_{\text{buc.}} = \frac{\pi^3 \Lambda}{24} \left(\frac{h}{a}\right)^3 .$$

We note that this value is identical to the buckling value of a beam of length $2a$, that is simply supported in its end points. Moreover, this value is in agreement with the results of the papers [52] and [53], in which Λ is taken equal to one. However, it should be noted that eq. (40) of [53] contains a misprint, i.e. the coefficient $\left(\frac{\pi}{3}\right)^{\frac{1}{2}}$ must be replaced by $\left(\frac{\pi}{24}\right)^{\frac{1}{2}}$.

Next, going out from the Maxwell-formulation, the displacement field (133) has to satisfy the conditions (129) and (130), which yields

$$(141) \quad \begin{aligned} A_1 + A_2 = 0, \quad \alpha A_3 + A_4 = 0 \\ A_1 \cos \alpha a + A_3 \sin \alpha a = 0, \quad -A_1 \sin \alpha a + A_3 \cos \alpha a = 0 . \end{aligned}$$

For no value of α , this system is to satisfy otherwise than by

$$(142) \quad A_1 = A_2 = A_3 = A_4 = 0 .$$

Hence, in the Maxwell-formulation, we do not find a static buckling value.

We confine ourselves to merely stating that a dynamical treatment based on the Maxwell-formulation would give a much higher buckling value than the one according to (140).

As stated before, the result (140) is in a good agreement with the experimental values of [52]. This holds certainly when the results of [54] are taken into account. In this paper, Popelar established that a small misalignment of the basic field, consistent with the experiments, can produce errors up to 50 percent.

We conclude, by noting that the result found above is positively a confirmation of the remark posed in [54], that says that the Amperian-formulation minimizes the influences of the end effects. However, a complete proof of this statement can only be given by bringing into account all boundary effects, by which one must be aware of the fact that in the neighbourhood of the boundaries the magnetization is no longer linear dependent on the magnetic field intensity. Since, according to (119), the bending moments do not differ in the two stress-formulations discussed here, the results of the sections IX.4 and IX.5 are independent of the choice of the stress tensor.

APPENDIX I

Units

Throughout this thesis Gaussian units are used. This system mostly occurs in theoretical literature. In the technical literature the Giorgi-system is preferred. In the following table the conversion from the Giorgi-system to the Gaussian system is given.

Quantity	Symbol	Giorgi Unit	multiply by	Gaussian Unit
Mass	m	kilogram (kg)	10^3	gram (g)
Length	L	meter (m)	10^2	centimeter (cm)
Time	t	second (sec)	1	second (sec)
Force	\underline{F}	newton	10^5	dyne
Energy	E	joule	10^7	erg
Charge density	\underline{Q}	coulomb/m ³	3×10^3	statcoulomb/cm ³
Electric current density	\underline{J}	ampère/m ²	3×10^5	statampère/cm ²
Electric field intensity	\underline{E}	volt/m	$\frac{1}{3} \times 10^{-4}$	statvolt/cm
Electric displacement	\underline{D}	coulomb/m ²	$12\pi \times 10^5$	dyne/statcoulomb
Polarization per unit of volume	$\rho \underline{P}$	coulomb/m ²	3×10^5	statcoulomb/cm ²
Resistivity	r	ohm-m	$\frac{1}{9} \times 10^{-9}$	statohm-cm
Conductivity	σ	mho/m	9×10^9	statmho/cm
Magnetic field intensity	\underline{H}	ampère-turn/m	$4\pi \times 10^{-3}$	oersted
Magnetic induction	\underline{B}	weber/m ²	10^4	gauss
Magnetization per unit of volume	$\rho \underline{M}$	weber/m ²	$\frac{1}{4\pi} \times 10^4$	gauss

Moreover, in Giorgi-units the relations I.(36) and I.(37) become

$$\frac{1}{\mu_0} \underline{B}_i = \underline{H}_i + \rho \underline{M}_i \quad \text{and} \quad \underline{D}_i = \epsilon_0 \underline{E}_i + \rho \underline{P}_i ,$$

where

$$\mu_0 = 4\pi \times 10^{-7} \text{ henry/m} \quad \text{and} \quad \epsilon_0 = \frac{1}{36\pi} \times 10^{-9} \text{ farad/m},$$

are the permeability and the permittivity of free space, respectively.

APPENDIX II

Proof of equation II.(37)

We first show with the aid of I.(35)^{1,3} that

$$\begin{aligned}
 (1) \quad & -\frac{c}{4\pi} [e_{jkl} E_k H_l] n_j = -\frac{c}{4\pi} \left\{ \frac{1}{2} (H_l^+ + H_l^-) [e_{jkl} E_k] n_j + \right. \\
 & \left. + \frac{1}{2} (E_k^+ + E_k^-) [e_{jkl} H_l] n_j \right\} = \\
 & = -\frac{1}{8\pi} \{ (H_i^+ + H_i^-) [B_i] W_n + (E_i^+ + E_i^-) [D_i] W_n \} = \\
 & = -\frac{1}{8\pi} \{ 2[H_i B_i + E_i D_i] W_n - (B_i^+ + B_i^-) [H_i] W_n - (D_i^+ + D_i^-) [E_i] W_n \} = \\
 & = -\frac{1}{4\pi} [H_i B_i + E_i D_i] W_n + \frac{1}{8\pi} [H_i H_i + E_i E_i] W_n + \\
 & + \frac{1}{2} \{ (\rho M_i)^+ + (\rho M_i)^- \} [H_i] W_n + \frac{1}{2} \{ (\rho P_i)^+ + (\rho P_i)^- \} [E_i] W_n .
 \end{aligned}$$

With (1) and with I.(38) we obtain for the left-hand side of II.(37)

$$\begin{aligned}
 (2) \quad & \left[-\frac{c}{4\pi} e_{jkl} E_k H_l - \frac{1}{8\pi} (E_i^* E_i^* + H_i^* H_i^*) V_j + \frac{1}{4\pi} (E_i^* D_i^* + H_i^* B_i^*) V_j + \right. \\
 & \left. + \frac{2}{4\pi c} e_{ikl} E_k H_l V_j V_i + \frac{\rho^2}{c} e_{ikl} (P_k B_l + E_k M_l) V_j V_i \right] n_j = \\
 & = \left[-\frac{c}{4\pi} e_{jkl} E_k H_l - \frac{1}{8\pi} (E_i E_i + H_i H_i) V_j + \frac{1}{4\pi} (E_i D_i + H_i B_i) V_j + \right. \\
 & \left. - \frac{4\pi \rho^2}{c} e_{ikl} P_k M_l V_j V_i \right] n_j = \\
 & = \frac{1}{2} \{ (\rho M_i)^+ + (\rho M_i)^- \} [H_i] W_n + \frac{1}{2} \{ (\rho P_i)^+ + (\rho P_i)^- \} [E_i] W_n + \\
 & - \left[\frac{4\pi \rho^2}{c} e_{ikl} P_k M_l V_j V_i \right] n_i - \left[\left\{ \frac{1}{8\pi} (E_i E_i + H_i H_i) + \right. \right. \\
 & \left. \left. - \frac{1}{4\pi} (E_i D_i + H_i B_i) \right\} (V_j n_j - W_n) \right] .
 \end{aligned}$$

It follows from I.(35)^{3,4} and I.(36) that

$$\begin{aligned}
 (3) \quad [H_i] &= [H_j]n_j n_i + e_{ijk} e_{klm} [H_l]n_m n_j = \\
 &= -4\pi[\rho M_j]n_j n_i + \frac{1}{c} e_{ijk} [D_k]W_n n_j = \\
 &= -4\pi[\rho M_j]n_j n_i + \frac{4\pi}{c} e_{ijk} [\rho P_k]W_n n_j,
 \end{aligned}$$

and, analogously, from I.(35)^{1,2} and I.(37) that

$$(4) \quad [E_i] = -4\pi[\rho P_j]n_j n_i - \frac{4\pi}{c} e_{ijk} [\rho M_k]W_n n_j.$$

By using (3) and (4) we find that

$$\begin{aligned}
 (5) \quad &\frac{1}{2} \{(\rho M_i)^+ + (\rho M_i)^-\} [H_i]W_n + \frac{1}{2} \{(\rho P_i)^+ + (\rho P_i)^-\} [E_i]W_n = \\
 &= -2\pi[(\rho M_i n_i)^2 + (\rho P_i n_i)^2]W_n + \frac{4\pi}{c} e_{ijk} [\rho^2 P_j M_k]W_n^2 n_i = \\
 &= -2\pi[(\rho M_i^* n_i)^2 + (\rho P_i^* n_i)^2]W_n + \\
 &+ \frac{4\pi}{c} e_{ijk} [\rho^2 V_j (P_k M_\ell - P_\ell M_k) n_\ell] n_i W_n + \frac{4\pi}{c} e_{ijk} [\rho^2 P_j M_k]W_n^2 n_i = \\
 &= -2\pi[(\rho M_i^* n_i)^2 + (\rho P_i^* n_i)^2]W_n + \\
 &+ \frac{4\pi}{c} e_{ijk} [\rho^2 P_j M_k (V_i - V_\ell n_\ell n_i)]W_n + \frac{4\pi}{c} e_{ijk} [\rho^2 P_j M_k]W_n^2 n_i.
 \end{aligned}$$

After substitution of (5) into (2), the proof of II.(37) is completed. \square

APPENDIX III

Conversion from Chu-formulation to Minkowski-formulation

In this thesis, the electromagnetic equations are written in the Minkowski-formulation. This implies that the fundamental electromagnetic quantities are \underline{D} , \underline{E} , \underline{B} and \underline{H} , and that for the Maxwell-equations the form as given in Section I.3 holds (cf. also [25], p. 196). An alternative notation is the Chu-formulation, used in [25]. Here, the basic concepts are \underline{E} , \underline{H} , \underline{P} and \underline{M} , and the Maxwell-equations read, in Gaussian units (cf. [25], p. 190)

$$\begin{aligned}
 (1) \quad & e_{ijk} H_{k,j} - \frac{1}{c} \frac{\partial E_i}{\partial t} = \frac{4\pi}{c} \left[-\frac{\partial}{\partial t} (\rho P_i) + e_{ijk} (\rho e_{k\ell m} P_\ell V_m)_{,j} \right] + \frac{4\pi}{c} J_i, \\
 & e_{ijk} E_{k,j} + \frac{1}{c} \frac{\partial H_i}{\partial t} = -\frac{4\pi}{c} \left[\frac{\partial}{\partial t} (\rho M_i) + e_{ijk} (\rho e_{k\ell m} M_\ell V_m)_{,j} \right], \\
 & E_{i,i} = -4\pi (\rho P_i)_{,i} + 4\pi Q, \\
 & H_{i,i} = -4\pi (\rho M_i)_{,i}.
 \end{aligned}$$

In this notation, the electromagnetic volume force takes the form (cf. [25], p. 99)

$$\begin{aligned}
 (2) \quad F_i^{(e)} = & Q E_i + \frac{1}{c} e_{ijk} J_j H_k + \rho P_j E_{i,j} + \rho M_j H_{i,j} + \\
 & + \frac{\rho}{c} e_{ijk} V_j H_{k,\ell} P_\ell + \frac{1}{c} e_{ijk} H_k \left[\frac{\partial}{\partial t} (\rho P_j) + (\rho P_j V_\ell)_{,\ell} \right] + \\
 & - \frac{\rho}{c} e_{ijk} V_j E_{k,\ell} M_\ell - \frac{1}{c} e_{ijk} E_k \left[\frac{\partial}{\partial t} (\rho M_j) + (\rho M_j V_\ell)_{,\ell} \right].
 \end{aligned}$$

The Minkowski variables and the Chu variables are related by (in the following equations, the Minkowski variables have an upper index M and the Chu variables have an upper index C) (cf. [25], p. 197)

$$\begin{aligned}
 (3) \quad E_i^C = & E_i^M + \frac{1}{c} e_{ijk} V_j (B_k^M - H_k^M), \\
 H_i^C = & H_i^M - \frac{1}{c} e_{ijk} V_j (D_k^M - E_k^M),
 \end{aligned}$$

$$4\pi\rho P_i^C = D_i^M - E_i^M - \frac{1}{c} e_{ijk} V_j (B_k^M - H_k^M) ,$$

$$4\pi\rho M_i^C = B_i^M - H_i^M + \frac{1}{c} e_{ijk} V_j (D_k^M - E_k^M) .$$

In the foregoing relations, terms that are of the order (v^2/c^2) are neglected.

After elimination of Q and J by means of (1), the expression (2) can be transformed into (still in Chu-notation)

$$(4) \quad F_i^{(e)} = \frac{1}{4\pi} E_i [E_{j,j} + (4\pi\rho P_j)_{,j}] + \rho P_j E_{i,j} + \frac{1}{4\pi} (E_{i,j} - E_{j,i}) E_j + \\ + \frac{1}{4\pi} H_i [H_{j,j} + (4\pi\rho M_j)_{,j}] + \rho M_j H_{i,j} + \frac{1}{4\pi} (H_{i,j} - H_{j,i}) H_j + \\ - \frac{1}{4\pi c} \frac{\partial}{\partial t} (e_{ijk} E_j H_k) + \left[\frac{\rho}{c} e_{ijk} V_j (H_k^P - E_k^M) \right]_{,l} .$$

With the aid of (3) this formula can be rewritten in Minkowski-notation, yielding

$$(5) \quad F_i^{(e)} = - \frac{1}{4\pi c} \frac{\partial}{\partial t} (e_{ijk} E_j H_k) + \\ + \frac{1}{4\pi} [E_i D_j + H_i B_j - \frac{1}{2} \delta_{ij} (E_k E_k + H_k H_k)]_{,j} + \\ + \frac{1}{4\pi c} \{ [e_{ijk} V_k (D_j B_l - D_j H_l - B_j D_l + B_j E_l)]_{,j} + \\ + [e_{jkl} V_k (-E_j B_l + E_j H_l + H_j D_l - H_j E_l)]_{,i} + \\ + [e_{ijk} V_j (H_k D_l - H_k E_l - E_k B_l + E_k H_l)]_{,l} \} .$$

By using the relations

$$(6) \quad [e_{ikl} V_k (D_j B_l - B_j D_l)]_{,j} = [e_{ikl} D_k B_l V_j]_{,j} - [e_{jkl} D_k B_l V_j]_{,i} , \\ [e_{jkl} V_k (E_j H_l - H_j E_l)]_{,i} + [e_{ijk} V_j (E_k H_l - H_k E_l)]_{,l} = \\ = - [e_{jkl} V_j E_k H_l]_{,i} - [e_{ikl} E_k H_l V_j]_{,j} , \\ [e_{ikl} V_k (B_j E_l - D_j H_l)]_{,j} + [e_{ijk} V_j (H_k D_l - E_k B_l)]_{,l} = 0 ,$$

and the equations I.(36), I.(37) and I.(38), (5) can be elaborated to the form

$$\begin{aligned}
 (7) \quad F_i^{(e)} = & -\frac{1}{4\pi c} \frac{\partial}{\partial t} (e_{ijk} E_j H_k) + \\
 & + \frac{1}{4\pi} [E_i^* D_j^* + H_i^* B_j^* - \frac{1}{2} \delta_{ij} (E_k^* E_k^* + H_k^* H_k^*)]_{,j} + \\
 & - [\frac{1}{4\pi c} e_{jkl} E_k H_l V_i]_{,j} ,
 \end{aligned}$$

an expression that is identical to the one according to II.(18). Thus, we have showed that the expression for the electromagnetic volume force derived in Chapter II is in a one to one correspondence to that found by Penfield and Haus [25].

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SAMENVATTING

In het eerste deel van dit proefschrift worden de algemene, niet-lineaire vergelijkingen afgeleid, die de interacties tussen electromagnetische en elastische velden in een thermoelastisch lichaam beschrijven. Deze afleiding is gebaseerd op een stelsel postulaten, zoals de eerste en tweede hoofdwet van de thermodynamica en een invariantieprincipe. Het aldus verkregen systeem bestaat uit een stelsel electromagnetische vergelijkingen, locale mechanische balansvergelijkingen voor de dichtheid, de impuls, het impulsmoment en de energie, met de bijbehorende discontinuïteits- of randvoorwaarden, plus een set constitutieve vergelijkingen voor de entropie, de polarisatie, de spanningen, de koppelspanningen, de entropieflux, de warmteflux en de elektrische stroomdichtheid. Dit werk houdt zich speciaal bezig met magnetisch verzadigde, ferromagnetische media en fysische verschijnselen zoals o.a. magnetostrictie, gyromagnetische actie, exchange interactie, thermoelectrische effecten, etc. worden besproken.

Hetzelfde stelsel vergelijkingen als hierboven genoemd, zal ook worden afgeleid op een alternatieve manier, namelijk met behulp van het principe van Hamilton.

De algemene, niet-lineaire vergelijkingen zullen worden gelineariseerd naar de storingen op een, bekend veronderstelde, tussentoestand. Deze gelineariseerde vergelijkingen worden nog verder vereenvoudigd door gebruik te maken van het feit dat de deformaties in een ferromagnetisch lichaam ten gevolge van electromagnetische velden klein zijn. Vervolgens wordt een expliciete uitdrukking voor de energiefunctionaal gekozen en worden de coëfficiënten die in deze uitdrukking voorkomen geïnterpreteerd in termen van bekende verschijnselen zoals magneto-anisotropie, magnetostrictie, thermo-electrische effecten, etc. Verder wordt een tabel gegeven met de numerieke waarden van verschillende van deze coëfficiënten voor de materialen ijzer, nikkel en YIG. Aan de hand van deze numerieke waarden zien we dat verschillende coëfficiënten in de ge-

lineariseerde vergelijkingen te verwaarlozen zijn ten opzichte van enkele die overheersen. Door het toepassen van deze verwaarlozingen worden de gelineariseerde vergelijkingen nog verder vereenvoudigd. De uiteindelijke vergelijkingen worden toegepast op een tweetal voorbeelden, te weten:

- i) De trillingen van een magnetisch verzadigde cirkelcylinder belast door een groot, statisch magnetisch veld in axiale richting plus een klein dynamisch veld loodrecht op de as van de cylinder.
- ii) De knik van magneto-elastische platen. Voor dit laatste voorbeeld worden de vergelijkingen aangepast voor het geval van een zogenaamd soft-magnetisch lichaam, dit is een lichaam met lineaire magnetisatie.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 14 augustus 1942 geboren in Valkenswaard. In 1959 behaalde hij aan het Hertog Jan College te Valkenswaard het HBS-B diploma. Daarna studeerde hij tot 1965 voor werktuigbouwkundig ingenieur, afstudeerrichting mechanica, aan de Technische Hogeschool te Eindhoven. Sinds 1 mei 1965 is hij als wetenschappelijk medewerker werkzaam bij Prof.dr. J.B. Alblas, sectie mechanica van de onderafdeling der Wiskunde van de Technische Hogeschool te Eindhoven.

STELLINGEN

I

Voor een lichaam, met elastische en electromagnetische wisselwerkingen, dat zich in een toestand bevindt waarin het alleen belast wordt door electromagnetische velden, waarbij de deformaties klein zijn, kan men de storingen ten opzichte van deze toestand, welke worden veroorzaakt door een kleine verandering van de belasting, bepalen met behulp van de oplossing van het starre-lichaams-probleem voor de ongestoorde belasting, dus zonder expliciet de ongestoorde toestand te kennen, indien tweede-orde termen in de deformaties worden verwaarloosd.

Literatuur: Dit proefschrift, pag. 89.

II

De knikwaarde van een lineair-magnetische, elastische, dunne plaat, belast door een magnetisch veld loodrecht op de plaat, is onafhankelijk van de voorspanningen, d.w.z. de spanningen in de niet-uitgeknikte toestand.

Literatuur: Dit proefschrift, pag. 176.

III

De uitdrukking voor de electromagnetische volumekracht volgens Vlasov en Ishmukhametov is incorrect: er ontbreekt een term.

Literatuur: K.B. Vlasov and B.Kh. Ishmukhametov, Equations of Motion and State for Magnetoelastic Media, Sovjet Physics JETP, 19 (1964), 142-148 (formule (30)).

IV

De uitdrukking voor de grootheid $\underline{G}(\underline{y}, \underline{\bar{E}})$, zoals die gegeven is door Hutter en Pao, is incorrect. De uitdrukking kan worden gecorrigeerd door in rekening te brengen dat de partiële afgeleide naar de tijd in de algemene, niet-lineaire vergelijkingen, betrokken is op de eindtoestand en niet op de referantietoestand. Daarmee wordt de, volgens deze

auteurs inconsistente, betrekking staande onder formule (5.6) een identiteit.

Literatuur: K. Hutter and Yih-Hsing Pao, A Dynamic Theory for Magnetizable Elastic Solids with Thermal and Electrical Conduction, J. of Elasticity, 4 (1974), 89-114 (formule (5.6)).

V

Een lineaire differentiaalvergelijking van de tweede orde met homogene beginvoorwaarden, waarin de demping en het rechterlid een stochastisch karakter van het type witte ruis bezitten en waarbij de verwachtingswaarde van het rechterlid gelijk aan nul is, heeft een oplossing waarvan de verwachtingswaarde naar een stationaire waarde ongelijk aan nul gaat, mits de spectraaldichtheid van het stochastische deel van de dempingscoëfficiënt kleiner is dan de halve waarde van het deterministische deel van deze coëfficiënt. Deze stationaire waarde is evenredig met de amplitude van de correlatie-functie tussen de dempingscoëfficiënt en het rechterlid.

Literatuur: A.A.F. v.d. Ven, Random-trillingen met behulp van Markov-processen, WSK-notitie, nr. 1, januari 1967, Technische Hogeschool Eindhoven.

VI

De oplossing van een systeem eerste-orde, lineaire, stochastische differentiaalvergelijkingen, waarin de stochastische termen stationaire Gauss-processen met verwachtingswaarden nul en met correlatie-functies, welke begrensd kunnen worden door een negatieve e-macht, zijn, terwijl het deterministische deel van het systeem asymptotisch stabiel is, is begrensd in de eerste- en tweede-orde momenten (d.w.z. in het gemiddelde en in de spreiding) indien het maximum van de spectraaldichtheid van de stochastische coëfficiënten kleiner is dan een constante welke volledig bepaald is door het deterministische deel van het systeem.

Literatuur: A.A.F. v.d. Ven, On the Boundedness in the Mean Square of the Forced Oscillations of Linear Systems with Stochastic Coefficients, Appl. Sci. Res. 20 (1969), 233-245.

VII

Voor slanke balken is de, in de stabiliteitstheorie voor elastische lichamen gebruikelijke, benadering om de uitbuigingsvergelijking en de elasticiteitsconstanten te betrekken op de ongedeformeerde toestand (in plaats van op de niet-uitgeknikte voorspanningstoestand) consistent met de klassieke theorie van slanke balken.

Literatuur: A.A.F. v.d. Ven, Knik van Rechte Balken, WSK-notitie, Werkbespreking Sectie Mechanica, november 1973, Technische Hogeschool Eindhoven.

VIII

Terwijl in een enkelvoudige magnetoelastische stof, voor een omkeerbaar proces, de entropie flux wordt verkregen door de warmte flux te delen door de absolute temperatuur, geldt deze eenvoudige betrekking niet meer voor een mengsel van twee magnetoelastische materialen. Indien beide componenten gelijke temperaturen hebben en indien er geen chemische reacties optreden, moet de uitdrukking voor de entropie flux worden aangevuld met een bijdrage, welke afhankelijk is van het snelheidsverschil der individuele componenten.

Literatuur: A.A.F. v.d. Ven, Magneto-elastische Mengsels, WSK-notitie, Werkbespreking Sectie Mechanica, oktober 1970, Technische Hogeschool Eindhoven.

IX

De zinvolheid van een steeds meer gedetailleerde beschrijving, vanuit de continuumsmechanica, van homogene materialen, welke is te verkrijgen door het in rekening brengen van hogere afgeleiden van de deformatietensor, is begrensd als gevolg van het optreden van spanningsfluctuaties in de macroscopisch spanningsvrije toestand.

X

Als

$$p_{nv}(x) := \binom{n}{v} x^v (1-x)^{n-v}, \quad (n = 1, 2, \dots; v = 0, 1, \dots, n)$$

dan geldt voor $x \in [0, 1]$ en voor $\delta \geq 0$ de ongelijkheid

$$\sum_{|v/n-x| \geq \delta} p_{nv}(x) \leq 2e^{-\delta^2 n}.$$

Literatuur: R.K.S. Rathore, Approximation of Unbounded Functions with Linear Positive Operators, Proefschrift, Delftse Universitaire Pers, Delft, 1974 (pp. 122-123).

XI

In woongebieden, waarvan de vormgeving dusdanig is dat het gebruik van de auto wordt teruggedrongen ten gunste van voetgangers en fietsers, dient de verkeerswetgeving te worden aangepast.

XII

De correlatie tussen de termen bier en gerstenat is een strikt monotoon dalende functie van de tijd over de laatste twee decennia.