



**QUEEN'S  
UNIVERSITY  
BELFAST**

## Interactions between Harmonic Analysis and Operator Theory

Todorov, I. (2015). Interactions between Harmonic Analysis and Operator Theory. *Serdica Mathematical Journal*, 41 (1), 13–34. [http://www.math.bas.bg/serdica/n1\\_15.html](http://www.math.bas.bg/serdica/n1_15.html)

**Published in:**  
Serdica Mathematical Journal

**Document Version:**  
Publisher's PDF, also known as Version of record

**Queen's University Belfast - Research Portal:**  
[Link to publication record in Queen's University Belfast Research Portal](#)

**Publisher rights**  
Copyright 2015 Serdica. This copy is provided for non-commercial research and educational use only.

**General rights**  
Copyright for the publications made accessible via the Queen's University Belfast Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**  
The Research Portal is Queen's institutional repository that provides access to Queen's research output. Every effort has been made to ensure that content in the Research Portal does not infringe any person's rights, or applicable UK laws. If you discover content in the Research Portal that you believe breaches copyright or violates any law, please contact [openaccess@qub.ac.uk](mailto:openaccess@qub.ac.uk).

Provided for non-commercial research and educational use.  
Not for reproduction, distribution or commercial use.

# Serdica

## Mathematical Journal

# Сердика

## Математическо списание

---

The attached copy is furnished for non-commercial research and education use only.  
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.  
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on  
Serdica Mathematical Journal  
which is the new series of  
Serdica Bulgaricae Mathematicae Publicationes  
visit the website of the journal <http://www.math.bas.bg/~serdica>  
or contact: Editorial Office  
Serdica Mathematical Journal  
Institute of Mathematics and Informatics  
Bulgarian Academy of Sciences  
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49  
e-mail: [serdica@math.bas.bg](mailto:serdica@math.bas.bg)

## INTERACTIONS BETWEEN HARMONIC ANALYSIS AND OPERATOR THEORY

I. G. Todorov

*Communicated by M. S. Anoussis*

ABSTRACT. The article is a survey of several topics that have led to fruitful interactions between Operator Theory and Harmonic Analysis, including operator and spectral synthesis, Schur and Herz-Schur multipliers, and reflexivity. Some open questions and directions are included in a separate section.

**1. Introduction.** Functional analytic methods have proved fruitful in Abstract Harmonic Analysis [19], [38], [8]. On the other hand, operator algebras constructed from topological groups have served as important examples in Operator Theory [27]. The aim of this article is to present some aspects of the symbiosis between the two areas which, broadly speaking, starts from Varopoulos' and Arveson's seminal papers [38] and [1]. We will summarise results by a number of authors, which can be grouped under the following headings:

- Spectral and operator synthesis;

---

2010 *Mathematics Subject Classification*: Primary 47L05, 47L35; Secondary 43A45.

*Key words*: locally compact group, spectral synthesis, Schur multiplier, reflexivity.

- Schur and Herz-Schur multipliers;
- Reflexivity and the union problem.

The underlying idea for all of the above connections is to transport notions and properties from Harmonic Analysis to Operator Theory. Harmonic Analysis traditionally studies spaces of functions defined on a topological group; in this sense it can be thought of as a one-variable function theory. Operator Theory, on the other hand, mostly studies operators acting on Hilbert spaces. Since operators can often be identified with matrices, one can heuristically think of Operator Theory as a two-variable (and non-commutative) function theory. The passage from one-variable functions to two-variable ones is given in this context by the following map (first introduced by Varopoulos) which will play a major role in the sequel: given a group  $G$  and a function  $f : G \rightarrow \mathbb{C}$ , let  $Nf : G \times G \rightarrow \mathbb{C}$  be given by  $Nf(s, t) = f(st^{-1})$ . We will see that the function  $N$  carries important objects from Harmonic Analysis to corresponding objects from Operator Theory. In this way, results from Harmonic Analysis can be carried over to Operator Theory and vice versa.

The paper is organised as follows: In Section 2, we recall some basic facts from Harmonic Analysis and Operator Theory that will be needed in the sequel. Section 3 discusses the relation between Schur multipliers and Herz-Schur multipliers. Section 4 is devoted to the notions of spectral and operator synthesis and their interrelations. Finally, Section 5 is centred around the connections of reflexivity and operator synthesis, highlighting some recent applications of reflexivity to the union problem for operator synthesis.

It should be mentioned that many important aspects of the interactions between Harmonic Analysis and Operator Theory are not covered here. For example, we have not touched upon the important developments in the theory of locally compact quantum groups, a rapidly expanding area which can be thought of as a Non-commutative Harmonic Analysis (the repetition is intended). We refer the reader to [13] and the references therein for an overview of the corresponding bibliography. Another area that has not been addressed in this survey is the rich and very fruitful interaction between analytic function theory on the unit disk and Operator Theory. We refer the reader to [29] for an excellent survey of this field.

## 2. Setting the stage.

**2.1. Operator theory.** If  $H, H_1$  and  $H_2$  are Hilbert spaces, we let  $\mathcal{B}(H_1, H_2)$  (resp.  $\mathcal{C}_1(H_1, H_2), \mathcal{C}_2(H_1, H_2)$ ) be the space of all bounded linear (resp. trace class, Hilbert-Schmidt) operators from  $H_1$  into  $H_2$ ; it is a Banach

space, when equipped with the operator (resp. trace, Hilbert-Schmidt) norm. We note that the Banach space dual of  $\mathcal{C}_1(H_2, H_1)$  can be naturally identified with  $\mathcal{B}(H_1, H_2)$  via the pairing  $\langle T, S \rangle = \text{tr}(TS)$ , where  $\text{tr}(A)$  denotes the usual trace of a trace class operator  $A$ . We set  $\mathcal{B}(H) = \mathcal{B}(H, H)$ ,  $\mathcal{C}_1(H) = \mathcal{C}_1(H, H)$  and  $\mathcal{C}_2(H) = \mathcal{C}_2(H, H)$ . The Hilbert spaces appearing in the paper will be assumed to be separable.

Let  $(X, \mu)$  and  $(Y, \nu)$  be standard ( $\sigma$ -finite) measure spaces. A subset of  $X \times Y$  is said to be a *measurable rectangle* (or simply a *rectangle*) if it is of the form  $\alpha \times \beta$ , where  $\alpha \subseteq X$  and  $\beta \subseteq Y$  are measurable subsets. A subset  $E \subseteq X \times Y$  is called *marginally null* if  $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$ , where  $\mu(X_0) = \nu(Y_0) = 0$ . We call two subsets  $E, F \subseteq X \times Y$  *marginally equivalent* (and write  $E \simeq F$ ) if the symmetric difference of  $E$  and  $F$  is marginally null. We say that  $E$  *marginally contains*  $F$  (or  $F$  *is marginally contained in*  $E$ ) if  $F$  is contained in the union of  $E$  and a marginally null set;  $E$  and  $F$  are said to be *marginally disjoint* if  $E \cap F$  is marginally null.

A subset  $E$  of  $X \times Y$  is called  $\omega$ -open if it is marginally equivalent to the union of a countable set of rectangles. The complements of  $\omega$ -open sets are called  $\omega$ -closed. It is clear that the class of all  $\omega$ -open (resp.  $\omega$ -closed) sets is closed under countable unions (resp. intersections) and finite intersections (resp. unions); in other words, the  $\omega$ -open sets form a *pseudo-topology*. A function  $f : X \times Y \rightarrow \mathbb{C}$  is called  $\omega$ -continuous if  $f^{-1}(U)$  is an  $\omega$ -open set for each open set  $U \subseteq \mathbb{C}$ . The set of  $\omega$ -continuous complex valued functions on  $X \times Y$  is an algebra under pointwise addition and multiplication [7]. We would like to note that the theory of pseudo-topologies is becoming increasingly rich; we refer the reader to [7], [20], [31], [32] and [33] for further properties of these structures.

Let  $H_1 = L^2(X, \mu) = L^2(X)$  and  $H_2 = L^2(Y, \nu) = L^2(Y)$ . The predual of  $\mathcal{B}(L^2(X), L^2(Y))$  can be naturally identified with the projective tensor product  $L^2(X) \hat{\otimes} L^2(Y)$ . Suppose that  $h \in L^2(X) \hat{\otimes} L^2(Y)$  and let  $h = \sum_{k=1}^{\infty} f_k \otimes g_k$  be an associated series for  $h$ , where  $\sum_{k=1}^{\infty} \|f_k\|_2^2 < \infty$  and  $\sum_{k=1}^{\infty} \|g_k\|_2^2 < \infty$ . These conditions easily imply that the formula

$$h(x, y) = \sum_{k=1}^{\infty} f_k(x)g_k(y), \quad (x, y) \in X \times Y,$$

defines, up to a marginally null set, a function, which we denote again by  $h$ . One can moreover check that, up to marginal equivalence, the function  $h$  does

not depend on the choice of the representation of the corresponding element of  $L^2(X) \hat{\otimes} L^2(Y)$ . Let  $\Gamma(X, Y)$  be the set of all those functions  $h$ , equipped with the norm

$$\|h\|_{\Gamma} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_2 \|g_k\|_2 : h = \sum_{k=1}^{\infty} f_k \otimes g_k \right\},$$

where  $\sum_{k=1}^{\infty} \|f_k\|_2^2 < \infty$  and  $\sum_{k=1}^{\infty} \|g_k\|_2^2 < \infty$ . The duality between  $\Gamma(X, Y)$  and  $\mathcal{B}(L^2(X), L^2(Y))$  is then given as follows: for  $h = \sum_{k=1}^{\infty} f_k \otimes g_k \in \Gamma(X, Y)$  and  $T \in \mathcal{B}(L^2(X), L^2(Y))$ , one lets

$$\langle T, h \rangle = \sum_{k=1}^{\infty} (Tf_k, g_k).$$

**2.2. Harmonic analysis.** Let  $G$  be a locally compact group which will be assumed throughout to be  $\sigma$ -compact. We let  $L^p(G)$ ,  $p = 1, 2, \infty$ , be the corresponding function spaces with respect to left Haar measure. By  $\lambda : G \rightarrow \mathcal{B}(L^2(G))$  we denote the left regular representation of  $G$ ; thus,  $\lambda_s f(t) = f(s^{-1}t)$ ,  $s, t \in G$ ,  $f \in L^2(G)$ . We recall that the Fourier algebra  $A(G)$  of  $G$  is the space of all “matrix coefficients of  $G$  in its left regular representation”, that is,

$$A(G) = \{s \rightarrow (\lambda_s \xi, \eta) : \xi, \eta \in L^2(G)\}.$$

If  $G$  is commutative, then  $A(G)$  is the image of  $L^1(\hat{G})$  under Fourier transform (where  $\hat{G}$  is the dual group of  $G$ ). The Fourier algebra of general locally compact groups was introduced and studied (along with other objects pertinent to Non-commutative Harmonic Analysis) by Eymard in [8]. It is a commutative regular semi-simple Banach algebra of continuous functions vanishing at infinity and has  $G$  as its spectrum. Moreover, its Banach space dual is isometric to the von Neumann algebra  $\text{VN}(G)$  of  $G$ , that is, to the weakly closed subalgebra of  $\mathcal{B}(L^2(G))$  generated by the operators  $\lambda_s$ ,  $s \in G$ . The duality between these two spaces is given by the formula  $\langle \lambda_x, f \rangle = f(x)$ .

A function  $g \in L^{\infty}(G)$  is called a *multiplier* of  $A(G)$  if  $gf \in A(G)$  for every  $f \in A(G)$ . A standard argument, using the Closed Graph Theorem, shows that if  $g$  is a multiplier of  $A(G)$  then the map  $m_g : A(G) \rightarrow A(G)$  given by  $m_g f = gf$ ,  $f \in A(G)$ , is bounded. The identification of the multipliers of  $A(G)$

(which, as is easy to see, form a function algebra on  $G$ ) has received a considerable attention in the literature. A classical result in this direction (see [28]) states that if  $G$  is abelian then  $g$  is a multiplier of  $A(G)$  if and only if it is the Fourier transform of a regular Borel measure on  $\hat{G}$ . A multiplier  $f$  of  $A(G)$  is called completely bounded [3], if the map  $f \rightarrow gf$  on  $A(G)$  is completely bounded. Here, we equip  $A(G)$  with its canonical operator space structure arising from the identification  $A(G)^* \equiv \text{VN}(G)$  (we refer the reader to [35] for a detailed account of the canonical operator space structures on various function spaces pertinent to Harmonic Analysis, and to [23] for an account of the basic notions from Operator Space Theory that will be used in this paper). We denote by  $M^{\text{cb}}A(G)$  the space of all completely bounded multipliers of  $A(G)$ , which are also known in the literature as *Herz-Schur multipliers* of  $A(G)$ .

**3. Multipliers.** Let  $\mathfrak{S}(X, Y)$  be the multiplier algebra of  $\Gamma(X, Y)$ ; by definition, a function  $\varphi : X \times Y \rightarrow \mathbb{C}$  belongs to  $\mathfrak{S}(X, Y)$  if  $\varphi h$  is marginally equivalent to a function from  $\Gamma(X, Y)$ , for every  $h \in \Gamma(X, Y)$ . If  $\varphi \in \mathfrak{S}(X, Y)$ , one may thus consider the operator  $M_\varphi : \Gamma(X, Y) \rightarrow \Gamma(X, Y)$  given by  $M_\varphi h = \varphi h$ ,  $h \in \Gamma(X, Y)$ . An application of the Closed Graph Theorem shows that  $M_\varphi$  is bounded; indeed, suppose that  $(h_k)_{k \in \mathbb{N}} \subseteq \Gamma(X, Y)$  is a sequence with  $\|h_k\|_\Gamma \rightarrow_{k \rightarrow \infty} 0$  and  $\varphi h_k \rightarrow h$  for some  $h \in \Gamma(X, Y)$ . By [32, Lemma 2.1], there exists a subsequence  $(h_{k_j})_{j \in \mathbb{N}}$  of  $(h_k)_{k \in \mathbb{N}}$  such that  $h_{k_j} \rightarrow_{j \rightarrow \infty} 0$  and  $\varphi h_{k_j} \rightarrow_{j \rightarrow \infty} h$  marginally almost everywhere. It follows that  $h = 0$  marginally almost everywhere, that is,  $h = 0$ .

Taking the dual operator of  $M_\varphi$ , we arrive at an operator

$$S_\varphi : \mathcal{B}(H_1, H_2) \rightarrow \mathcal{B}(H_1, H_2)$$

which has the property

$$\langle S_\varphi(T), h \rangle = \langle T, \varphi h \rangle, \quad h \in \Gamma(X, Y), \quad T \in \mathcal{B}(H_1, H_2).$$

If  $k \in L^2(Y \times X)$ , let  $T_k \in \mathcal{C}_2(H_1, H_2)$  be the operator given by

$$T_k \xi(y) = \int_X k(y, x) \xi(x) d\mu(x), \quad \xi \in H_1, x \in X.$$

One can easily see from the formula given above that if  $k \in L^2(Y \times X)$ , then  $S_\varphi(T_k) = T_{\hat{\varphi}k}$ , where  $\hat{\varphi}(y, x) = \varphi(x, y)$ ,  $x \in X, y \in Y$ .

The functions from  $\mathfrak{S}(X, Y)$  are called *Schur multipliers*. We often identify  $\varphi$  with the corresponding linear transformation  $S_\varphi$ , and speak about Schur

multipliers on  $\mathcal{B}(H_1, H_2)$ . Let  $\mathcal{D}_X$  (resp.  $\mathcal{D}_Y$ ) be the maximal abelian selfadjoint algebra (masa, for short) consisting of all operators on  $L^2(X)$  of multiplication by functions from  $L^\infty(X)$  (resp.  $L^\infty(Y)$ ). It can easily be checked that, if  $\varphi \in \mathfrak{S}(X, Y)$  then

$$S_\varphi(BTA) = BS_\varphi(T)A, \quad T \in \mathcal{B}(H_1, H_2), A \in \mathcal{D}_X, B \in \mathcal{D}_Y.$$

Using a result of R. Smith's [34], one can now see that Schur multipliers are completely bounded. In fact, we have the following fact, first established in this form in [17]:

**Proposition 3.1.** *The transformation  $\varphi \rightarrow S_\varphi$  is an algebra isomorphism from  $\mathfrak{S}(X, Y)$  onto the algebra  $\mathcal{C}$  of all weak\* continuous completely bounded masa-bimodule maps on  $\mathcal{B}(H_1, H_2)$ . Moreover, it is isometric when  $\mathfrak{S}(X, Y)$  is equipped with the multiplier norm, while  $\mathcal{C}$  is equipped with the completely bounded norm.*

A special case of interest arises when  $X = Y = \mathbb{Z}$ , equipped with the counting measure. The elements of  $\mathcal{B}(\ell^2(\mathbb{Z}))$  can be identified with (doubly infinite) matrices. A function  $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$  is in this case a Schur multiplier if and only if, for every  $(x_{i,j})_{i,j \in \mathbb{Z}} \in \mathcal{B}(\ell^2(\mathbb{Z}))$ , the matrix  $(\varphi(i, j)x_{i,j})_{i,j \in \mathbb{Z}}$  defines again a bounded operator on  $\ell^2(\mathbb{Z})$ . A. Grothendieck showed in [11] that the function  $\varphi$  is a Schur multiplier if and only if  $\varphi$  can be represented in the form

$$\varphi(i, j) = \sum_{k=1}^{\infty} a_k(i)b_k(j), \quad i, j \in \mathbb{Z},$$

where  $a_k$  and  $b_k$ ,  $k \in \mathbb{N}$ , are bounded doubly infinite sequences, such that

$$\sup_{i \in \mathbb{Z}} \sum_{k=1}^{\infty} |a_k(i)|^2 < \infty \quad \text{and} \quad \sup_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} |b_k(j)|^2 < \infty.$$

This characterisation was extended by V. Peller [24] (see also [35]) to the general measurable setting; namely, the following result holds true:

**Theorem 3.2.** *Let  $\varphi \in L^\infty(X \times Y)$ . The following are equivalent:*

- (i)  $\varphi \in \mathfrak{S}(X, Y)$ ;
- (ii) *there exist countable families  $(a_k)_{k \in \mathbb{N}} \subseteq L^\infty(X)$  and  $(b_k)_{k \in \mathbb{N}} \subseteq L^\infty(Y)$*



such that  $\operatorname{esssup}_{x \in X} \sum_{k=1}^{\infty} |a_k(x)|^2 < \infty$ ,  $\operatorname{esssup}_{y \in Y} \sum_{k=1}^{\infty} |b_k(y)|^2 < \infty$  and

$$\varphi(x, y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{for almost all } (x, y) \in X \times Y.$$

We refer the reader to [17] for an elegant operator space theoretic proof of this result. As a consequence of Theorem 3.2, one can see the following:

**Corollary 3.3.** *Every Schur multiplier is equivalent to an  $\omega$ -continuous function.*

We now return to the setting of  $\ell^2(\mathbb{Z})$  and suppose that we are interested in the Schur multipliers which leave the space  $\mathcal{T}$  of all Toeplitz operators invariant. Recall that an operator  $(x_{i,j})_{i,j \in \mathbb{Z}} \in \mathcal{B}(\ell^2(\mathbb{Z}))$  is called *Toeplitz* if it has constant diagonals, that is, if, for every  $k \in \mathbb{Z}$ , there exists  $c_k \in \mathbb{C}$  such that  $x_{i,j} = c_k$  for all  $(i, j)$  with  $i - j = k$ . It is easy to note that if  $\varphi \in \mathfrak{S}(\mathbb{Z}, \mathbb{Z})$  leaves  $\mathcal{T}$  invariant then  $\varphi$  is a function of Toeplitz type, that is,  $\varphi = Nf$  for some  $f : \mathbb{Z} \rightarrow \mathbb{C}$ . It is well-known that a function  $\varphi = Nf$  is a Schur multiplier of Toeplitz type if and only if there exists a regular Borel measure  $\mu$  on the unit circle such that  $f(n) = \hat{\mu}(n)$  for every  $n \in \mathbb{Z}$ , where  $\hat{\mu}$  is the Fourier transform of  $\mu$ . In other words:

**Theorem 3.4.** *Let  $f : \mathbb{Z} \rightarrow \mathbb{C}$ . The function  $Nf$  is a Schur multiplier if and only if  $f$  belongs to the Fourier-Stieltjes algebra  $B(\mathbb{Z})$  of  $\mathbb{Z}$ , namely, if and only if it is the Fourier transform of a regular Borel measure on  $\mathbb{T}$ .*

This result remains true for all abelian locally compact groups in the place of  $\mathbb{Z}$ . In general, however, one needs to replace the Fourier-Stieltjes algebra by the algebra of all completely bounded multipliers of  $A(G)$ . This was shown by Bożejko and Fendler in [2]. Alternative proofs were given by Gilbert and Jolissaint (see [16]), while this class of multipliers was also studied by Herz (see [14]).

**Theorem 3.5.** *Let  $G$  be a locally compact group and  $f : G \rightarrow \mathbb{C}$  be a measurable function. Then  $Nf \in \mathfrak{S}(G, G)$  if and only if  $f$  is equivalent to a function from  $M^{\text{cb}}A(G)$ .*

In defining Schur multipliers, our starting point was the space  $\Gamma(X, Y)$ . An equivalent approach is to start with the space  $\mathcal{C}_2(L^2(X), L^2(Y))$  of all Hilbert-Schmidt operators from  $L^2(X)$  into  $L^2(Y)$ , and recall that every such operator

$T$  has the form  $T = T_k$ , for some function  $k \in L^2(Y \times X)$ . Given any measurable complex function  $\varphi$  (note that  $\varphi$  is not assumed any more to be essentially bounded), we can now consider the multiplication operator  $S_\varphi^0$  defined on  $\mathcal{C}_2(L^2(X), L^2(Y))$  and given by  $T_k \rightarrow T_{\hat{\varphi}k}$ ,  $k \in L^2(Y \times X)$ . Here, as before,  $\hat{\varphi} : Y \times X \rightarrow \mathbb{C}$  is the function given by  $\hat{\varphi}(y, x) = \varphi(x, y)$ . The function  $\varphi$  is a Schur multiplier if and only if this operator is bounded in the operator norm of  $\mathcal{C}_2(L^2(X), L^2(Y))$ . If this is the case, it extends by continuity to a bounded operator on  $\mathcal{K}(H_1, H_2)$  and, by taking second duals, to a bounded operator on  $\mathcal{B}(H_1, H_2)$ .

If the operator  $S_\varphi^0$  is not bounded in the operator norm, then one may ask whether it is closable. This line of investigation was taken up in [31]. There are two natural versions of closability that arise in this context:

(a)  $S_\varphi^0$  is called *norm closable* if the conditions  $(T_{k_n})_{n \in \mathbb{N}} \subseteq \mathcal{C}_2(H_1, H_2)$ ,  $\|T_{k_n}\|_{\text{op}} \rightarrow_{n \rightarrow \infty} 0$ ,  $T \in \mathcal{K}(H_1, H_2)$  and  $\|T_{\hat{\varphi}k_n} - T\|_{\text{op}} \rightarrow_{n \rightarrow \infty} 0$  imply that  $T = 0$ ;

(b)  $S_\varphi^0$  is called *weak\* closable* if the conditions  $(T_{k_\alpha})_{\alpha \in \mathbb{A}} \subseteq \mathcal{C}_2(H_1, H_2)$  (where  $\mathbb{A}$  is a directed set),  $T_{k_\alpha} \xrightarrow{w^*}_{n \rightarrow \infty} 0$ ,  $T \in \mathcal{B}(H_1, H_2)$  and  $T_{\hat{\varphi}k_\alpha} \xrightarrow{w^*}_{n \rightarrow \infty} T$  imply that  $T = 0$ .

We call  $\varphi$  a weak\* closable (resp. norm closable) multiplier if the operator  $S_\varphi^0$  is weak\* closable (resp. norm closable). It is easy to notice that weak\* closability implies norm closability, and that if  $H_1 = H_2 = \ell^2(\mathbb{Z})$  then every function  $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$  is a weak\* (and hence norm) closable multiplier. In the case of continuous measure spaces, however, closable multipliers have a rich theory [31]. Here we include a characterisation of weak\* closable multipliers. Let us denote by  $\mathcal{V}(X, Y)$  the space of all measurable functions  $\psi : X \times Y \rightarrow \mathbb{C}$  of the form

$\varphi(x, y) = \sum_{k=1}^{\infty} a_k(x)b_k(y)$ , where  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  are families of measurable

functions on  $X$  and  $Y$ , respectively, satisfying the conditions  $\sum_{k=1}^{\infty} |a_k(x)|^2 < \infty$

for almost all  $x \in X$  and  $\sum_{k=1}^{\infty} |b_k(y)|^2 < \infty$  for almost all  $y \in Y$ . We note that the

difference from the conditions required in Theorem 3.2 is the relaxation of the uniform (essential) boundedness condition.

The following results were established in [31].

**Theorem 3.6.** *Let  $\varphi$  be a complex measurable function on  $X \times Y$ . The following are equivalent:*

- (i) there exists a countable family  $\{\kappa_n\}_{n \in \mathbb{N}}$  of rectangles such that  $\cup_{n \in \mathbb{N}} \kappa_n$  is marginally equivalent to  $X \times Y$  and  $\varphi_{\kappa_n}$  is a Schur multiplier, for every  $n \in \mathbb{N}$ ;
- (ii) there exist increasing sequences  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  of measurable subsets of  $X$  and  $Y$ , respectively, such that  $\mu(X \setminus (\cup_{n \in \mathbb{N}} X_n)) = 0$ ,  $\nu(Y \setminus (\cup_{n \in \mathbb{N}} Y_n)) = 0$  and  $\varphi|_{X_n \times Y_n}$  is a Schur multiplier, for every  $n \in \mathbb{N}$ ;
- (iii)  $\varphi \in \mathcal{V}(X, Y)$ .

**Theorem 3.7.** *Let  $\varphi$  be a complex measurable function on  $X \times Y$ . The following are equivalent:*

- (i)  $\varphi$  is a weak\* closable multiplier;
- (ii) there exist functions  $\psi_1, \psi_2 \in \mathcal{V}(X, Y)$  such that  $\psi_2(x, y) \neq 0$  for marginally almost all  $(x, y)$  and  $\varphi(x, y) = \frac{\psi_1(x, y)}{\psi_2(x, y)}$ , for almost all  $(x, y) \in X \times Y$ .

We note that we lack a complete description of norm closable multipliers; however, various sufficient conditions were found in [31] in terms of *sets of multiplicity*, a notion that arose first in classical Harmonic Analysis on the unit circle [10].

The class of closable multipliers is strictly larger than the class of weak\* closable ones. However, when we restrict attention to the functions of Toeplitz type, these two classes coincide. In this case, there is a convenient description which fits well with Theorem 3.5. In order to formulate it, we recall the definition of the “localised Fourier algebra” of  $G$ :

$$A(G)^{\text{loc}} = \{g : G \rightarrow \mathbb{C} : \text{for every } t \in G \text{ there exists an open } V \subseteq G$$

and  $h \in A(G)$  such that  $g = h$  on  $V\}$ .

**Theorem 3.8.** *Let  $G$  be an abelian locally compact group and  $f : G \rightarrow \mathbb{C}$  be a measurable function. Set  $\varphi = Nf$ . The following are equivalent:*

- (i)  $\varphi$  is a weak\* closable multiplier;
- (ii)  $\varphi$  is a norm closable multiplier;
- (iii)  $f$  is equivalent, with respect to the Haar measure, to a function from the class  $A(G)^{\text{loc}}$ .

**4. Synthesis.** In this section, we review various connections between the notion of spectral synthesis and that of operator synthesis. We start by recalling these two notions.

Let  $G$  be a locally compact group. Eymard has shown [8] that the spectrum of  $A(G)$  can be identified with  $G$  via the natural evaluation map. Associated with a closed set  $E \subseteq G$ , we consider the following two ideals of  $A(G)$ :

$$I(E) = \{f \in A(G) : f(s) = 0 \text{ for all } t \in E\}$$

and

$$J(E) = \overline{\{f \in A(G) : f \text{ has compact support disjoint from } E\}}.$$

We have that  $J(E) \subseteq I(E)$ . On the other hand, given an ideal  $J \subseteq A(G)$ , one can define the *null set* of  $E$  to be the closed subset of  $G$  given by

$$\text{null } J = \{t \in G : f(t) = 0 \text{ for all } f \in J\}.$$

It is well-known that  $\text{null } I(E) = \text{null } J(E)$  and that if  $J$  is a closed ideal of  $A(G)$  such that  $\text{null } J = E$ , then  $J(E) \subseteq J \subseteq I(E)$ . The set  $E$  is called a *set of spectral synthesis* if  $I(E) = J(E)$ , that is, if there exists only one closed ideal of  $A(G)$  whose null set equals  $E$ .

The ‘‘dual picture’’ is often useful: as was pointed out in Section 2, the dual Banach space of  $A(G)$  coincides with the von Neumann algebra  $\text{VN}(G)$  of  $G$ . Moreover,  $\text{VN}(G)$  carries a natural structure of a Banach  $A(G)$ -module: given  $T \in \text{VN}(G)$  and  $u \in A(G)$ , one lets  $u \cdot T$  be the operator in  $\text{VN}(G)$  determined by the identity

$$\langle u \cdot T, v \rangle = \langle T, uv \rangle, \quad v \in A(G).$$

It is easy to see that a closed subspace  $J \subseteq A(G)$  is an ideal if and only if its annihilator  $J^\perp \subseteq \text{VN}(G)$  is a (weak\* closed) submodule of  $\text{VN}(G)$ . We say that an operator  $T \in \text{VN}(G)$  vanishes on an open set  $U \subseteq G$  if  $\langle T, u \rangle = 0$  for all  $u \in A(G)$  with  $\text{supp } u \subseteq U$ . Given a weak\* closed submodule  $\mathcal{U}$  of  $\text{VN}(G)$ , we let  $\text{supp } \mathcal{U}$  be the smallest closed subset  $E$  of  $G$  such that every operator  $T \in \mathcal{U}$  vanishes on  $E^c$ . Via duality, a closed set  $E \subseteq G$  is a set of spectral synthesis if and only if there exists a unique weak\* closed submodule  $\mathcal{U}$  of  $\text{VN}(G)$  with  $\text{supp } \mathcal{U} = E$ . One can easily check that

$$I(E)^\perp = \overline{[\lambda_s : s \in E]},$$

where  $[\cdot]$  denotes linear span. Hence,  $E$  is a set of spectral synthesis if and only if, whenever  $\mathcal{U} \subseteq \text{VN}(G)$  is a weak\* closed submodule with  $\text{supp } \mathcal{U} = E$ , every element of  $\mathcal{U}$  can be approximated in the weak\* topology by linear combinations of translations  $\lambda_s$ , where  $s \in E$ .

The use of the word “synthesis” becomes clear when one considers the case where  $G$  is a locally compact *abelian* group. In this case one has the powerful machinery of the Fourier transform. We recall that the dual group  $\hat{G}$  of  $G$  is the set

$$\hat{G} = \{\gamma : G \rightarrow \mathbb{T} : \text{a continuous group homomorphism}\},$$

equipped with the operation of pointwise product and neutral element the constant homomorphism. The Fourier transform  $\mathcal{F}$  takes a function from  $L^1(G)$  to the function  $\mathcal{F}(f)$  on  $\hat{G}$  given by

$$\mathcal{F}(f)(\gamma) = \int_G f(s) \overline{\gamma(s)} ds, \quad f \in L^1(G), \gamma \in \hat{G}.$$

The group  $\hat{G}$  can be equipped with a natural topology with respect to which it is a locally compact abelian group. The collection  $\{\mathcal{F}(f) : f \in L^1(G)\}$  coincides with the Fourier algebra  $A(\hat{G})$  of  $\hat{G}$ .

We have that  $\mathcal{F}$  is isometric on  $L^1(G) \cap L^2(G)$  with respect to  $\|\cdot\|_2$  and hence has a unique extension to an operator (denoted in the same way) from  $L^2(G)$  into  $L^2(\hat{G})$  which is moreover surjective. We have that  $\mathcal{F}VN(G)\mathcal{F}^*$  equals the algebra  $\mathcal{D}_{\hat{G}}$  of all operators on  $L^2(\hat{G})$  of multiplication by functions from  $L^\infty(\hat{G})$ . The Fourier transform of the operator  $\lambda_s$ , for  $s \in G$ , is easily seen to be equal to the multiplication operator  $M_s$  corresponding to the character  $s$  on  $\hat{G}$ . In classical Harmonic Analysis, a function  $\varphi \in L^\infty(\hat{G})$  is said to admit spectral synthesis if it is in the weak\* closed linear span of its spectrum  $\sigma(\varphi) \stackrel{def}{=} \text{supp } \mathcal{F}^* M_\varphi \mathcal{F}$ .

The characterisation of the sets of spectral synthesis in a locally compact group remains an open problem, even in the special case where  $G = \mathbb{T}$  is the group of the unit circle. We note that in the case where  $G$  is discrete, the problem has a trivial solution, as every set turns out to satisfy spectral synthesis. On the other hand, Malliavin has shown that in every non-discrete locally compact group there exists a closed set which does not satisfy spectral synthesis (see [19]).

We now turn to the notion of operator synthesis. We fix, as in Section 2, two standard measure spaces  $(X, \mu)$  and  $(Y, \nu)$ , and let  $H_1 = L^2(X, \mu)$  and  $H_2 = L^2(Y, \nu)$ . A subspace  $\mathcal{V} \subseteq \Gamma(X, Y)$  will be called  $\mathfrak{S}$ -invariant (or simply *invariant*) if  $\psi h \in \mathcal{V}$  for every  $h \in \mathcal{V}$  and every  $\psi \in \mathfrak{S}(X, Y)$ . Under the identification of  $\Gamma(X, Y)$  with the ideal of all trace class operators, the closed (in the trace norm) invariant subspaces correspond precisely to the closed masa-sub-bimodules, that is, the closed subspaces  $\mathcal{V}$  such that  $BTA \in \mathcal{V}$  for all

$T \in \mathcal{V}$ ,  $A \in \mathcal{D}_X$  and  $B \in \mathcal{D}_Y$ . It is easy to note that the annihilators of the invariant subspaces of  $\Gamma(X, Y)$  are precisely the weak\*-closed masa-bimodules  $\mathcal{U} \subseteq \mathcal{B}(H_1, H_2)$ .

Let  $\kappa \subseteq X \times Y$  be an  $\omega$ -closed subset and  $\chi_\kappa$  be its characteristic function. We let

$$\Phi(\kappa) = \{h \in \Gamma(X, Y) : \chi_\kappa h = 0 \text{ m.a.e.}\}$$

and

$$\Psi(\kappa) = \overline{\{h \in \Gamma(X, Y) : h = 0 \text{ m.a.e. on an } \omega\text{-open nbhd of } \kappa\}}.$$

We clearly have  $\Psi(\kappa) \subseteq \Phi(\kappa)$ . Analogously to the spectral synthesis setting, given an invariant subspace  $\mathcal{V} \subseteq \Gamma(X, Y)$ , we define the null set  $\text{null } \mathcal{V}$  to be the smallest, up to marginal equivalence,  $\omega$ -closed subset  $E \subseteq X \times Y$  such that  $\chi_E h = 0$  m.a.e., for all  $h \in \mathcal{V}$ . The existence of such a set and the following result were established in [32]:

**Theorem 4.1.** *Let  $\kappa \subseteq X \times Y$  be an  $\omega$ -closed set. Then  $\text{null } \Phi(\kappa) = \text{null } \Psi(\kappa)$  and if  $\mathcal{V}$  is a closed invariant subspace of  $\Gamma(X, Y)$  with  $\text{null } \mathcal{V} = \kappa$  then  $\Psi(\kappa) \subseteq \mathcal{V} \subseteq \Phi(\kappa)$ .*

Theorem 4.1 is a subspace version of a deep result of Arveson's [1] concerning commutative subspace lattices. Namely, he associated two canonical weak\* closed operator algebras with every (separably acting) commutative subspace lattice  $\mathcal{L}$  and showed that any weak\* closed operator algebra containing a masa which has  $\mathcal{L}$  as its lattice of closed invariant subspaces lies between these two canonical algebras.

The spaces  $\mathfrak{M}_{\max}(\kappa) \stackrel{def}{=} \Psi(\kappa)^\perp$  and  $\mathfrak{M}_{\min}(\kappa) \stackrel{def}{=} \Phi(\kappa)^\perp$  are weak\* closed masa-bimodules. They can be conveniently described in their own right, without reference to the predual  $\Gamma(X, Y)$ . Let us say that an operator  $T \in \mathcal{B}(H_1, H_2)$  is supported on an  $\omega$ -closed set  $\kappa$  if  $M_{\chi_\beta} T M_{\chi_\alpha} = 0$  for all measurable sets  $\alpha \subseteq X$  and  $\beta \subseteq Y$  such that  $(\alpha \times \beta) \cap \kappa \simeq \emptyset$  [7]. Then

$$\mathfrak{M}_{\max}(\kappa) = \{T \in \mathcal{B}(H_1, H_2) : T \text{ is supported on } \kappa\}.$$

For  $\mathfrak{M}_{\min}(\kappa)$  there are two equivalent descriptions. On one hand, this space coincides with the weak\* closure of the set of all *pseudo-integral operators* supported on  $\kappa$  (these are operators canonically associated with certain measures, and include as a special case all integral operators as well as all composition and multiplication operators, see [1]); the second description of  $\mathfrak{M}_{\min}(\kappa)$  requires the

introduction of some extra notions, such as that of a slice map, and the reader is referred to [32] for its complete statement.

Given a masa-bimodule  $\mathcal{U} \subseteq \mathcal{B}(H_1, H_2)$ , one may define its support  $\text{supp } \mathcal{U}$  to be the smallest, up to marginal equivalence,  $\omega$ -subset of  $X \times Y$  which supports every operator from  $\mathcal{U}$  [7]. Taking duals in Theorem 4.1, we have that if a weak\* closed masa-bimodule has support  $\kappa$  then  $\mathfrak{M}_{\min}(\kappa) \subseteq \mathcal{U} \subseteq \mathfrak{M}_{\max}(\kappa)$ .

We say that the  $\omega$ -closed set  $\kappa$  *satisfies operator synthesis* or is *operator synthetic* if  $\Phi(\kappa) = \Psi(\kappa)$  (equivalently, if  $\mathfrak{M}_{\min}(\kappa) = \mathfrak{M}_{\max}(\kappa)$ ). The next theorem exhibits the largest known single class of operator synthetic sets. An  $\omega$ -closed set  $\kappa \subseteq X \times Y$  is called a set of *finite width* if

$$\kappa = \{(x, y) \in X \times Y : f_i(x) \leq g_i(y), i = 1, \dots, n\},$$

where  $f_i : X \rightarrow \mathbb{R}$  and  $g_i : Y \rightarrow \mathbb{R}$  are real valued measurable functions,  $i = 1, \dots, n$ .

**Theorem 4.2.** *Every set of finite width is operator synthetic.*

This result was established in [1] in the case where  $\mathfrak{M}_{\max}(\kappa)$  is a unital algebra, and in [32] and [37] in the general case. We note that a particular class of sets of finite width consists of the sets of the form

$$\{(x, y) \in X \times Y : f(x) = g(y)\},$$

where  $f : X \rightarrow \mathbb{R}$  and  $g : Y \rightarrow \mathbb{R}$  are measurable functions; these are the supports of masa-bimodules that are also *ternary rings of operators*, and their synthesis was established in [18] and [30].

We now turn to the relation between spectral and operator synthesis. Let  $G$  be a locally compact group. Recall the mapping  $N$  which sends a function  $f : G \rightarrow \mathbb{C}$  to the function  $Nf : G \times G \rightarrow \mathbb{C}$  given by  $Nf(s, t) = f(st^{-1})$ ,  $s, t \in G$ . If  $G$  is compact, then  $N$  maps the Fourier algebra  $A(G)$  of  $G$  into  $\Gamma(G, G)$ . To see this, note that, in this case, the constant function 1 on  $G \times G$  belongs to  $\Gamma(G, G)$  and, since  $Nf$  is a multiplier of  $\Gamma(G, G)$ , we have that  $Nf = (Nf)1 \in \Gamma(G, G)$ .

The following result was established in [36]:

**Theorem 4.3.** *Let  $G$  be a compact group. A closed set  $E \subseteq G$  satisfies spectral synthesis if and only if the subset  $E^*$  of  $G \times G$  satisfies operator synthesis.*

On the other hand, if  $G$  is abelian and locally compact, the same conclusion was established earlier in [9]. The general case of a locally compact group was studied in [22]. We note that in this case the mapping  $N$ , as seen in Section 3, maps  $A(G)$  isometrically into the multiplier algebra  $\mathfrak{S}(G, G)$  of  $\Gamma(G, G)$ .

As closed subset  $E \subseteq G$  is said to satisfy *local spectral synthesis* if  $I^c(E) \subseteq J(E)$ , where

$$I^c(E) = \{f \in I(E) : f \text{ is compactly supported}\}.$$

The following result was established in [22]:

**Theorem 4.4.** *Let  $G$  be a locally compact group. A closed subset  $E \subseteq G$  satisfies local spectral synthesis if and only if  $E^*$  satisfies operator synthesis.*

**5. Reflexivity and sums.** In his seminal work [1], Arveson was interested in questions about *reflexive* operator algebras. To define the notion of reflexivity for operator algebras, let, for a Hilbert space  $H$  and a set of operators  $\mathcal{A} \subseteq \mathcal{B}(H)$ ,

$$\text{Lat } \mathcal{A} = \{P : \text{a projection on } H \text{ with } (I - P)\mathcal{A}P = \{0\}\}.$$

It is trivial to verify that the condition  $(I - P)\mathcal{A}P = \{0\}$  is equivalent to the range  $PH$  of the projection  $P$  being invariant for every operator in  $\mathcal{A}$ . This is the reason why one refers to  $\text{Lat } \mathcal{A}$  as the *invariant subspace lattice* of  $\mathcal{A}$ . We note that  $\text{Lat } \mathcal{A}$  is indeed a lattice (in fact, a complete one) with respect to the operations of intersection and closed linear span.

Dually, given a collection  $\mathcal{L}$  of projections on  $H$ , one defines

$$\text{Alg } \mathcal{L} = \{T \in \mathcal{B}(H) : (I - P)TP = 0 \text{ for all } P \in \mathcal{L}\}.$$

The set  $\text{Alg } \mathcal{L}$  is a unital operator algebra closed in the weak operator topology. For a subset  $\mathcal{A} \subseteq \mathcal{B}(H)$ , we have that  $\mathcal{A} \subseteq \text{Alg Lat } \mathcal{A}$ . A weakly closed unital algebra  $\mathcal{A} \subseteq \mathcal{B}(H)$  is called *reflexive* if  $\mathcal{A} = \text{Alg Lat } \mathcal{A}$ ; in such a case,  $\mathcal{A}$  is completely determined by its invariant subspace lattice. We refer the reader to [4] for a deep analysis of some classes of reflexive algebras and their lattices.

Arveson was interested in the *transitive algebra problem*: Is it true that a unital operator algebra  $\mathcal{A} \subseteq \mathcal{B}(H)$  closed in the weak operator topology and such that  $\text{Lat } \mathcal{A} = \{0, I\}$  must coincide with  $\mathcal{B}(H)$ ? In [1], he showed the following:

**Theorem 5.1.** *If  $\mathcal{A} \subseteq \mathcal{B}(H)$  is a unital algebra containing a masa, such that  $\text{Lat } \mathcal{A} = \{0, I\}$ , then  $\mathcal{A}$  is weak\* dense in  $\mathcal{B}(H)$ .*

The latter result should be compared with the following theorem in Harmonic Analysis which, in the case of the group of the circle, is known as Wiener's Tauberian Theorem:



**Theorem 5.2.** *If  $J \subseteq A(G)$  is an ideal such that  $\text{null}J = \emptyset$  then  $J$  is dense in  $A(G)$ .*

Of course, Theorem 5.2 is just a restatement of the fact that the empty set satisfies spectral synthesis. On the other hand, Theorem 5.1 can be reformulated by saying that every weak\* closed algebra containing a masa such that  $\text{Lat } \mathcal{A} = \{0, I\}$  is automatically reflexive.

We thus see that the notion of reflexivity is intimately related to the notion of synthesis. It was extended from algebras to arbitrary subspaces by Loginov and Shulman [21]: for a subspace  $\mathcal{U} \subseteq \mathcal{B}(H_1, H_2)$  (where  $H_1$  and  $H_2$  are Hilbert spaces), let

$$\text{Ref } \mathcal{U} = \{T \in \mathcal{B}(H_1, H_2) : T\xi \in \overline{\mathcal{U}\xi}, \text{ for all } \xi \in H_1\}$$

be the *reflexive hull* of  $\mathcal{U}$ . The subspace  $\mathcal{U}$  is called *reflexive* if  $\mathcal{U} = \text{Ref } \mathcal{U}$ . It is easy to observe that a unital algebra is reflexive as an algebra if and only if it is reflexive as a subspace. The following result was shown in [7]:

**Theorem 5.3.** *Let  $(X, \mu)$  and  $(Y, \nu)$  be standard measure spaces,  $H_1 = L^2(X, \mu)$ ,  $H_2 = L^2(Y, \nu)$  and let  $\mathcal{U} \subseteq \mathcal{B}(H_1, H_2)$  be a  $\mathcal{D}_Y, \mathcal{D}_X$ -bimodule. The following are equivalent:*

- (i)  $\mathcal{U}$  is reflexive;
- (ii) there exists an  $\omega$ -closed set  $\kappa \subseteq X \times Y$  such that  $\mathcal{U} = \mathfrak{M}_{\max}(\kappa)$ .

We thus see, via Theorem 5.3, that the property of operator synthesis can be expressed as a property of automatic reflexivity: an  $\omega$ -closed set  $\kappa \subseteq X \times Y$  is operator synthetic if and only if every weak\* closed masa-bimodule whose support is (marginally equivalent to)  $\kappa$  is automatically reflexive.

One of the longest standing open questions in Harmonic Analysis is the *union problem*:

**Question 5.4.** *If  $E_1$  and  $E_2$  are closed subsets of a locally compact non-discrete group which satisfy spectral synthesis, does their union  $E_1 \cup E_2$  satisfy spectral synthesis as well?*

For a number of particular cases, Question 5.4 is resolved; for example, it is known that the union of a set satisfying spectral synthesis and a closed subgroup satisfies spectral synthesis. In [6], we initiated the study of the following question:

**Question 5.5.** *Given two reflexive spaces  $\mathcal{S}, \mathcal{T} \subseteq \mathcal{B}(H_1, H_2)$ , when is the weak\* closure  $\overline{\mathcal{S} + \mathcal{T}}^{w*}$  of their sum reflexive?*

Question 5.5 is closely related to the union problem. To see this connection, we note that it can easily be verified that if  $\kappa_1$  and  $\kappa_2$  are  $\omega$ -closed subsets

of  $X \times Y$  then

$$\text{Ref}(\mathfrak{M}_{\max}(\kappa_1) + \mathfrak{M}_{\max}(\kappa_2)) = \mathfrak{M}_{\max}(\kappa_1 \cup \kappa_2).$$

Therefore, an affirmative answer to Question 5.5, in the case  $\mathcal{S} = \mathfrak{M}_{\max}(\kappa_1)$  and  $\mathcal{T} = \mathfrak{M}_{\max}(\kappa_2)$ , with  $\kappa_1$  and  $\kappa_2$  operator synthetic, implies that  $\kappa_1 \cup \kappa_2$  is operator synthetic.

Question 5.5 turns out to be closely related to the class of idempotent Schur multipliers, called simply *Schur idempotents* in the sequel. We denote by  $\mathfrak{I}$  the collection of all Schur idempotents. To explain this connection, recall that  $(X, \mu)$  and  $(Y, \nu)$  are standard measure spaces,  $H_1 = L^2(X, \mu)$  and  $H_2 = L^2(Y, \nu)$ . By Proposition 3.1, Schur idempotents correspond precisely to the Schur multipliers which are characteristic functions of some measurable subsets  $\kappa \subseteq X \times Y$ . By Corollary 3.2, Schur multipliers are  $\omega$ -continuous, and hence such sets  $\kappa$  are necessarily marginally equivalent to sets that are both  $\omega$ -closed and  $\omega$ -open. No characterisation is known of the class of sets  $\kappa$  for which  $\chi_\kappa \in \mathfrak{S}(X, Y)$ . One can, however, describe completely the Schur idempotents of norm one; this was achieved in [17], where it was shown that these are the maps of the form  $S_{\chi_\kappa}$ , where  $\kappa$  is marginally equivalent to a set of the form  $\cup_{i=1}^{\infty} \alpha_i \times \beta_i$ , where  $\{\alpha_i\}_{i=1}^{\infty}$  (resp.  $\{\beta_i\}_{i=1}^{\infty}$ ) is a family of pairwise disjoint measurable subsets of  $X$  (resp.  $Y$ ). The Schur idempotents are easily seen to form a Boolean algebra with respect to the operations  $\Phi \wedge \Psi = \Phi\Psi$  and  $\Phi \vee \Psi = \Phi + \Psi - \Phi\Psi$  with top element the identity and bottom element the zero idempotent. A well-known open problem asks whether the Boolean algebra generated by the Schur idempotents of norm one exhausts all Schur idempotents. An evidence for an affirmative answer is provided by Harmonic Analysis: if  $G$  is an amenable locally compact group then the idempotents in  $M^{\text{cb}}A(G)$ , which in this case coincides with the Fourier-Stieltjes algebra  $B(G)$  of  $G$ , are precisely the elements of the subset ring generated by the cosets of open subgroups of  $G$  [12].

Let  $\Phi$  be a Schur idempotent. It was shown in [6] that the range  $\text{ran } \Phi$  of  $\Phi$  is automatically reflexive; hence, if  $\kappa$  is the  $\omega$ -closed (and, simultaneously,  $\omega$ -open) set such that  $S_{\chi_\kappa} = \Phi$ , then  $\kappa$  satisfies operator synthesis. Moreover, if  $\mathcal{V}$  is any reflexive space, then the algebraic sum  $\text{ran } \Phi + \mathcal{V}$  is automatically reflexive (and hence automatically weakly closed). A similar conclusion remains true in the more general case where  $\text{ran } \Phi$  is replaced with the intersection of a sequence of decreasing ranges of uniformly bounded Schur idempotents. In order to describe the precise generality in which similar results remain true, we need the following definition.

**Definition 5.6** [6]. (i) A subspace  $\mathcal{M} \subseteq \mathcal{B}(H_1, H_2)$  is called  $\mathfrak{I}$ -injective

if  $\mathcal{M} = \text{ran } \Phi$  for some  $\Phi \in \mathfrak{I}$ .

(ii) A subspace  $\mathcal{M} \subseteq \mathcal{B}(H_1, H_2)$  is called approximately  $\mathfrak{I}$ -injective if there exists a sequence  $(\Phi_n)_{n \in \mathbb{N}} \subseteq \mathfrak{I}$  and a constant  $C > 0$  such that  $\|\Phi_n\| \leq C$ ,  $\text{ran } \Phi_{n+1} \subseteq \text{ran } \Phi_n$ ,  $n \in \mathbb{N}$ , and  $\mathcal{M} = \bigcap_{n=1}^{\infty} \text{ran } \Phi_n$ .

(iii) A closed subspace  $\mathcal{V} \subseteq \mathcal{B}(H_1, H_2)$  is called  $\mathfrak{I}$ -decomposable if there exists a sequence  $(\Phi_n)_{n=1}^{\infty} \subseteq \mathfrak{I}$  and a sequence  $(\mathcal{W}_n)_{n=1}^{\infty}$  of  $\mathfrak{I}$ -injective subspaces such that

- (a) there exists  $C > 0$  with  $\|\Phi_n\| \leq C$ ,  $n \in \mathbb{N}$ ;
- (b)  $\mathcal{V} \subseteq \text{ran } \Phi_n + \mathcal{W}_n$  for each  $n$ ;
- (c)  $\mathcal{W}_n \subseteq \mathcal{V}$  for each  $n$ ;
- (d) if  $(T_n)_{n \in \mathbb{N}}$  is a sequence of operators such that  $T_n \in \text{ran } \Phi_n$  for every  $n \in \mathbb{N}$  with a weak\* cluster point  $T$ , then  $T \in \mathcal{V}$ .

Let us point out that, since Schur multipliers are masa-bimodule maps, the three classes of subspaces just defined consist of reflexive masa-bimodules. Clearly, every  $\mathfrak{I}$ -injective masa-bimodule is approximately  $\mathfrak{I}$ -injective, while every approximately  $\mathfrak{I}$ -injective masa-bimodule is  $\mathfrak{I}$ -decomposable. These three classes are however distinct from each other. To see the difference between the classes of  $\mathfrak{I}$ -injective and approximately  $\mathfrak{I}$ -injective subspaces, note that any continuous masa is an approximately  $\mathfrak{I}$ -injective space – it is the intersection of a decreasing sequence of ranges of Schur idempotents of norm one – but such a masa is never the range of a weak\* continuous masa-bimodule projection itself, by a result of Arveson's (see [4]). We note that the class of approximately  $\mathfrak{I}$ -injective masa-bimodules contains the spaces  $\mathfrak{M}_{\max}(\kappa)$ , where  $\kappa$  has the form

$$\kappa = \{(x, y) : f(x) = g(y)\},$$

$f : X \rightarrow \mathbb{R}$  and  $g : Y \rightarrow \mathbb{R}$  being measurable functions.

The class of  $\mathfrak{I}$ -decomposable masa-bimodules is, on the other hand, strictly larger than the class of approximately  $\mathfrak{I}$ -injective masa-bimodules. Indeed, it was shown in [6] that the reflexive masa-bimodules whose support is a subset of the form

$$\{(x, y) : f(x) \leq g(y)\},$$

where  $f$  and  $g$  are real valued measurable functions, are always approximately  $\mathfrak{I}$ -decomposable, but, for example, the Volterra nest algebra, whose support falls into the latter class (see [4]) is not approximately  $\mathfrak{I}$ -injective.

The most general result concerning Question 5.5 proved in [6] is the following:

**Theorem 5.7.** *Let  $\mathcal{U}_i$  be an  $\mathfrak{J}$ -decomposable masa-bimodule,  $i = 1, \dots, n$ , and  $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_i$ . Let  $\mathcal{V}$  be a reflexive masa-bimodule. Then the masa-bimodule  $\overline{\mathcal{U} + \mathcal{V}^{w^*}}$  is reflexive.*

Theorem 5.7 has a number of corollaries concerning operator synthesis. For example, as an immediate consequence we obtain the following:

**Corollary 5.8.** *If  $\kappa$  is a set satisfying operator synthesis and  $\lambda$  is a set of finite width, then the union  $\kappa \cup \lambda$  satisfies operator synthesis.*

Using the results presented in Section 4, one can use Theorem 5.7 to obtain corollaries about spectral synthesis as well; we refer the reader to [6] for details.

**6. Open questions.** In this short section, we collect some open problems centred around the topics described in the previous sections. The first one was raised in [31].

**Question 6.1.** *Is the class of weak\* closable multipliers distinct from the class  $\mathcal{V}(X, Y)$ ?*

**Question 6.2.** *Are the supports of approximately  $\mathfrak{J}$ -injective masa-bimodules necessarily operator synthetic?*

We note that, by [6], an affirmative answer to Question 6.2 would automatically imply that such sets are *operator Ditkin* (see [32] for the definition of this notion).

The characterisation of the subsets  $\kappa \subseteq X \times Y$  with the property that  $\chi_\kappa$  is a Schur multiplier is a well-known open problem in the area. In [5], those sets  $\kappa$  for which all bounded functions supported on  $\kappa$  are Schur multipliers were characterised, in the case  $X$  and  $Y$  are equipped with counting measures, while in [25], a closely related setting for groups was considered. Let us call a subset  $\kappa \subseteq X \times Y$  satisfying the above property a *hereditarily Schur bounded*.

**Question 6.3.** *Let  $(X, \mu)$  and  $(Y, \nu)$  be (standard) continuous measure spaces. Define and study the question of hereditarily Schur bounded sets  $\kappa \subseteq X \times Y$ .*

We note that any developments around Question 6.3 should start with identifying the “right” definition of hereditariness in the continuous setting; it is

easy to see that the naive straightforward translation from the discrete case is not of interest.

For the next question, note that in [31] it was shown that the triangular truncation on  $[0, 1] \times [0, 1]$ , which corresponds to Schur multiplication by the characteristic function of the “triangular” set  $\{(x, y) : x \leq y\}$ , is a closable multiplier.

**Question 6.4.** *For which subsets  $\kappa$  of  $X \times Y$  is  $\chi_\kappa$  a closable multiplier?*

We finally turn to Question 5.5. All known results concerning this question are about masa-bimodules. In view of the connections of the question with synthesis, it would be of interest to study it more generally; this may lead to certain insight concerning possible “non-commutative” versions of synthesis.

**Question 6.5.** *Can one exhibit classes of subspaces, other than masa-bimodules, for which Question 5.5 has an affirmative answer?*

## REFERENCES

- [1] W. B. ARVESON. Operator algebras and invariant subspaces. *Ann. Math.* (2) **100** (1974), 433–532.
- [2] M. BOZEJKO, G. FENDLER. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *Boll. Un. Mat. Ital. A* (6), **3**, 2 (1984), 297–302.
- [3] J. DE CANNIÈRE, U. HAAGERUP. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Am. J. Math.* **107**, 2 (1985), 455–500.
- [4] K. R. DAVIDSON. Nest algebras. Triangular forms for operator algebras on Hilbert space. Pitman Research Notes in Mathematics Series, vol. **191**. Longman Scientific & Technical, Harlow, 1988.
- [5] K. R. DAVIDSON, A. P. DONSIG. Norms of Schur multipliers. *Illinois J. Math.* **51**, 3 (2007), 743–766.
- [6] G. ELEFTHERAKIS, I. G. TODOROV. Ranges of bimodule projections and reflexivity. *J. Funct. Anal.* **262**, 11 (2012), 4891–4915.

- [7] J. A. ERDOS, A. KATAVOLOS, V. S. SHULMAN. Rank one subspaces of bimodules over maximal abelian selfadjoint algebras. *J. Funct. Anal.* **157**, 2 (1998), 554–587.
- [8] P. EYMARD. L’algèbre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France* **92** (1964), 181–236 (in French).
- [9] J. FROELICH. Compact operators, invariant subspaces and spectral synthesis. *J. Funct. Anal.* **81**, 1 (1988), 1–37.
- [10] C. GRAHAM, O. C. MCGEHEE. Essays in Commutative Harmonic Analysis. Fundamental Principles of Mathematical Science, vol. **238**. New York-Berlin, Springer-Verlag, 1979.
- [11] A. GROTHENDIECK. Résumé de la théorie métrique des produits tensoriels topologiques. *Boll. Soc. Mat. São Paulo* **8** (1953), 1–79 (in French).
- [12] B. HOST. Le théorème des idempotents dans  $B(G)$ . *Bull. Soc. Math. France* **114**, 2 (1986), 215–223 (in French).
- [13] Z. HU, M. NEUFANG, Z.-J. RUAN. Completely bounded multipliers over locally compact quantum groups. *Proc. Lond. Math. Soc.* (3) **103**, 1 (2011), 1–39.
- [14] C. HERZ. Une généralisation de la notion de transformée de Fourier-Stieltjes. *Ann. Inst. Fourier (Grenoble)* **24**, 3 (1974), 145–157.
- [15] E. HEWITT, K. A. ROSS. Abstract Harmonic Analysis. Volume I: Structure of Topological Groups, Integration Theory, Group Representations. 2nd ed. Grundlehren der Mathematischen Wissenschaften, vol **115**. Berlin, Springer-Verlag, 1994.
- [16] P. JOLISSAINT. A characterisation of completely bounded multipliers of Fourier algebras. *Colloq. Math.* **63**, 2 (1992), 311–313.
- [17] A. KATAVOLOS, V. PAULSEN. On the ranges of bimodule projections. *Canad. Math. Bull.* **48**, 1 (2005), 97–111.
- [18] A. KATAVOLOS, I. G. TODOROV. Normalizers of operator algebras and reflexivity. *Proc. London Math. Soc.* (3) **86**, 2 (2003), 463–484.
- [19] Y. KATZNELSON. An introduction to Harmonic Analysis. 2nd ed. New York, Dover Publications Inc., 1976.

- [20] E. KISSIN, V. S. SHULMAN. Operator multipliers. *Pacific J. Math.* **227**, 1 (2006), 109–141.
- [21] A. I. LOGINOV, V. S. SHUL'MAN. Hereditary and intermediate reflexivity of  $W^*$ -algebras. *Izv. Akad. Nauk. SSSR Ser. Mat.* **39**, 6 (1975), 1260–1273 (in Russian); English translation in *Math. USSR, Izv.* **9** (1975), 1189–1201.
- [22] J. LUDWIG, L. TUROWSKA. On the connection between sets of operator synthesis and sets of spectral synthesis for locally compact groups. *J. Funct. Anal.* **233**, 1 (2006), 206–227.
- [23] V. PAULSEN. Completely bounded maps and operator algebras. Cambridge Studies in Advanced Mathematics, vol. **78** Cambridge, Cambridge University Press, 2002.
- [24] V.V. PELLER. Hankel operators in the perturbation theory of unitary and selfadjoint operators. *Funkts. Anal. Prilozh.* **19**, 2 (1985), 37–51 (in Russian); English translation in: *Funct. Anal. Appl.* **19** (1985), 111–123.
- [25] G. PISIER. Multipliers and lacunary sets in non-amenable groups. *Am. J. Math.* **117**, 2 (1995), 337–376.
- [26] G. PISIER. Similarity Problems and Completely Bounded Maps. Includes the Solution to “The Halmos problem”. 2nd, expanded ed. Lecture Notes in Mathematics, vol. **1618**. Berlin, Springer-Verlag, 2001.
- [27] G. PISIER. Introduction to Operator Space Theory. London Mathematical Society Lecture Note Series, vol. **294**. Cambridge, Cambridge University Press, 2003.
- [28] W. RUDIN. Fourier Analysis on Groups. Paperback edition. Wiley Classics Library, A Wiley-Interscience Publication. New York etc., John Wiley & Sons, 1990.
- [29] H. S. SHAPIRO. Operator Theory and Harmonic Analysis. Twentieth century harmonic analysis—a celebration. Proceedings of the NATO Advanced Study Institute, Il Ciocco, Italy, July 2–15, 2000. Dordrecht, Kluwer Academic Publishers. NATO Sci. Ser. II, Math. Phys. Chem. **33** (2001), 31–56.
- [30] V. S. SHULMAN. Multiplication operators and spectral synthesis. *Dokl. Akad. Nauk SSSR* **313**, 5 (1990), 1047–1051 (in Russian); English translation in *Soviet Math. Dokl.* **42**, 1 (1991), 133–137.

- [31] V. S. SHULMAN, I. G. TODOROV, L. TUROWSKA. Closable multipliers. *Integral Equations Oper. Theory* **69**, 1 (2011), 29–62.
- [32] V. S. SHULMAN, L. TUROWSKA. Operator synthesis. I: Synthetic sets, bi-lattices and tensor algebras. *J. Funct. Anal.* **209**, 2 (2004), 293–331.
- [33] V. S. SHULMAN, L. TUROWSKA. Operator synthesis. II: Individual synthesis and linear operator equations. *J. Reine Angew. Math.* 590 (2006), 143–187.
- [34] R. R. SMITH. Completely bounded module maps and the Haagerup tensor product. *J. Funct. Anal.* **102**, 1 (1991), 156–175.
- [35] N. SPRONK. Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras. *Proc. London Math. Soc.* (3) **89**, 1 (2004), 161–192.
- [36] N. SPRONK, L. TUROWSKA. Spectral synthesis and operator synthesis for compact groups. *J. London Math. Soc.* (2) **66**, 2 (2002) 361–376.
- [37] I. G. TODOROV. Spectral synthesis and masa-bimodules. *J. London Math. Soc.* (2) **65**, 3 (2002), 733–744.
- [38] N. TH. VAROPOULOS. Tensor algebras and harmonic analysis. *Acta Math.* **119**, 1 (1967), 51–112.

*I. G. Todorov*  
*Pure Mathematics Research Centre*  
*Queen's University Belfast*  
*Belfast BT7 1NN, United Kingdom*  
*e-mail: i.todorov@qub.ac.uk*

*Received April 11, 2013*