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Interactions Between the Flow Field, Combustion, and Wave Motions in Rocket Motors

by

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ABSTRACT

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FOREWORD

This report presents analyses of unsteady gas flow in solid propellant combustors, with particular reference to axial mode oscillations. Of primary concern are the effects of deviations from one-dimensionality, and of the role of the mean flow in the oscillatory behavior. The results provide a guide to design and interpretation of laboratory combustor experiments, which, together with analysis, provide the means for valid calculation of stability of new motor designs.

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CONTENTS

NOMENCLATURE

A_b	Admittance function for pressure coupling (Eq. 9.6)
a	Speed of sound
$\vec{C}$	Eq. 5.38
C_v	Specific heat at constant volume
E_N^2	Eq. 5.35
$E_{1\ell}^2$	Eq. 5.27
E_ℓ^2	Eq. 5.14
e	Internal energy per unit volume $\int C_v dT$ of gases
e_a	Acoustic energy density (Eq. 5.2)
e_o	Stagnation internal energy, $e + u^2/2$
e_p	Total internal energy of particulate matter (per unit volume of chamber)
e_{po}	Stagnation internal energy of particulate matter, $e_p + u_p^2/2$
e_s	Internal energy of gases entering at the surface
$\mathcal{E}_\ell$	Time-averaged acoustic energy (Eq. 7.6)
$\mathcal{E}_N$	Time-averaged acoustic energy (Eq. 7.4)
F	Force of interaction between gases and particles (Eq. 2.8)
$\vec{F}$	Eq. 4.5
f	Eq. 4.16
f_1	Eq. 2.21
h	Enthalpy of gas per unit volume, $e + p/\rho$; also Eq. 4.15

h_o Stagnation enthalpy

h_{os} Stagnation enthalpy of gases entering at the surface

h_1 Eq. 2.19

h_{11} Eq. 3.8

K_N Time-averaged density of acoustic kinetic energy (Eq. 7.3)

k Complex wavenumber, $k = (\omega - i\alpha)/\bar{a}$

k_ℓ Wavenumber for one-dimensional modes

k_N Wavenumber for three-dimensional modes

L Length of chamber

M_b Mach number of gases entering at burning surface, u_b/a

$\vec{m}$ Vector mass flux, $\rho\vec{u}$, of gases

m_b Mass flux of gas entering at the burning surface

$m_b^{(p)}$ Mass flux of particulate matter entering at the burning surface

n Coordinate normal to the surface (Section 6)

$\hat{n}$ Unit normal vector, positive outward from the surface

P_1 Eq. 2.17

P_{11} Eq. 3.6

P' Eq. 4.2

P_N Time-averaged potential energy (Eq. 7.3)

p Pressure

$\hat{p}_\ell$ Mode shape for one-dimensional oscillation

$\hat{p}_N$ Mode shape for three-dimensional oscillations

Q Heat release, per unit volume, in homogeneous reactions

Q_p Heat release, per unit volume of chamber, associated with combustion of particulate matter (Eq. 2.12)

q Perimeter of chamber

R	Gas constant
R_b	Response function for pressure coupling (Eq. 9.2)
r	Radial coordinate
r_c	Radius of cylindrical chamber
S_{be}	Area of burning surface at the end of a chamber
S_c	Cross-sectional area of chamber
S_{co}	Cross-sectional area of uniform chamber
S_{bs}	Area of burning surface on the lateral surface of a chamber
s	Coordinate parallel to the surface (Section 6)
T	Temperature
T_o	Stagnation temperature
ΔT	Nonisentropic change of temperature
t	Time
$\vec{u}$	Velocity
$\vec{u}_{\parallel}$	Velocity of gases entering the chamber, parallel to the surface
u_b	Velocity of gases normal to the burning surface
$\vec{u}_p$	Velocity of particulate matter (mass-averaged)
$\vec{u}_{p\parallel}$	Velocity of particulate matter entering the chamber parallel to the surface
u_s	Value of $\vec{u}_{\parallel}$ for one-dimensional motions
u_{ps}	Value of $\vec{u}_{p\parallel}$ for one-dimensional motions
dV	Volume element
w_p	Combustion of particulate matter to gas, mass per unit volume per second
z	Coordinate along axis of the chamber
α	Growth constant (positive for growth of waves)
γ	Ratio of specific heats

$\delta_{\parallel}$	Eq. 6.19
$\delta_{\perp}$	Eq. 6.20
$\delta_{n\parallel}, \delta_{n\perp}$	Values of $\delta_{\parallel}$, $\delta_{\perp}$ at entrance to the nozzle
$\varkappa_{mn}$	Roots of $dJ_m(\varkappa_{mn}r)/dr = 0$ for $r = r_c$
ρ	Density
ρ_p	Density of particulate matter, mass per unit volume of chamber
ϕ	Azimuthal angle
$\nabla_{\perp}$	Gradient in direction normal to the surface
$\nabla_{\parallel}$	Gradient parallel to the surface
$\Delta_{m\ell}$	Eq. 10..
$(\ )'$	Fluctuation
$\bar{(\ )}$	Mean value; also time-averaged value
$\hat{(\ )}$	Amplitude of fluctuation, without time dependence
$(\ )^{(r)}$	Real part
$(\ )^{(i)}$	Imaginary part
$\langle\ \rangle$	Time average (Section 8)

1. INTRODUCTION

It has long been recognized (Ref. 1-4, for example) that a proper description of unstable motions in a rocket motor must account for interactions between the average and fluctuating flow fields. This is an important feature not generally included in problems of classical acoustics. The associated exchange of energy may be either a loss or a gain for small amplitude motions. Moreover, the contributions may be large enough in some cases to be dominant influences for the stability of oscillations.

Since residual combustion and the presence of liquid and solid material suspended in the gas phase are accounted for, the work covered here is valid for any kind of combustion chamber. However, most of the discussion will be concerned with unstable oscillations in solid propellant rocket motors; other applications may be extracted as special cases. For a solid propellant motor, the flow of mass into the chamber has some particularly interesting consequences. This has become more apparent with recent work on "axial" or "longitudinal" waves in a chamber. The main reason is that when there are wave motions parallel to the surface, the mass entering normal to the surface undergoes inelastic processes of acceleration to acquire the parallel motions. As a result of work on the axial modes, earlier computations (Ref. 2 and 3) of acoustic modes have been reexamined; it appears that contributions associated with the mass flux at the boundary were incompletely taken into account.

Since the configurations of experimental burners and rocket motors quite naturally fall into several classes, the treatment here is also split. At the present time, there is considerable interest in the axial modes, which are best handled in a one-dimensional or modified one-dimensional formulation. However, not only are there three-dimensional modes having motions predominantly parallel to the boundary, but also the axial modes may occur in very complicated configurations. It is therefore necessary to have at hand the complete formulation for three-dimensional problems.

The emphasis of the present work is mainly on linear problems, and particularly those aspects related to the interactions between average and fluctuating quantities. However, the important influences of particles in the flow, burning within the volume of the chamber, and the exhaust nozzle are included. The first part of this report (Sections 2-4) cover the formulation, including linearization of three classes of problems: one-dimensional, modified one-dimensional, and the familiar three-dimensional problem. The modified one-dimensional problem takes into account, in the unperturbed problem, variations of cross-sectional area along the axis; the result are therefore more accurate than those obtained from the purely one-dimensional formulation. A restriction which should be particularly noted is that for all cases treated here, linearization is carried out according to a limiting process discussed in Ref. 3; the governing equations used throughout are valid only if the Mach number of the mean flow is small. When the Mach number becomes "large" (i.e., greater than, roughly, 0.3), both the mode structure and frequency are affected strongly, in respects which cannot be ignored.

For each of the three classes of problems, a wave equation is formed. Solution for the case of harmonic motions, covered in Section 5, produces formulas for the complex wave number and hence the growth constant. Comparison of the formulas for the modified one-dimensional and the three-dimensional problems shows significant differences which lead to the principal new results of the present work.

1. If the interactions, or coupling, between the wave motions and surface combustion are nonisentropic, then the formal representations of the coupling are different for the two limiting cases; no acoustic motion parallel to the surface, and all acoustic motion parallel to the surface.

2. There are energy losses, for the waves in the chamber, when mass flows in at an element of surface where there is a component of acoustic motion parallel to the surface.

Incorporation of these features in the three-dimensional problem is accomplished in Section 6, and leads to what is here referred to as the modified three-dimensional problem. The physical interpretation of these new contributions to the growth constant--that is, to the energy balance for acoustic waves--is covered in Section 7.

Perhaps the most important practical consequence of this work is that it is clear now that in general two functions must be known to characterize completely the coupling between transient motions and combustion. These are conventionally introduced as the admittance and response functions; other definitions are of course possible. The physical origin of the need for two independent pieces of information is that the coupling itself comprises two processes. One is mechanical, which may be interpreted grossly as p-v work associated with velocity or mass fluctuations of gas leaving the surface. The other is due to nonisentropic temperature fluctuations at the edge of the combustion zone. Since the temperature fluctuations in the acoustic field are very closely isentropic--exactly so within the analysis covered here--a nonisentropic fluctuation at the boundary will generate a temperature or entropy wave which propagates through the chamber with the speed of a mean flow. The energy required to generate such waves must ultimately be accounted for in the energy balance of the acoustic waves within the chamber.

Several years ago, Cantrell and Hart (Ref. 2) proposed that the growth constant for acoustic waves in a chamber, with mean flow and combustion at the surface, be computed by examining the total energy in the chamber. This is clearly a sound idea, but care is required in the formulation. The analysis of Ref. 2 is very briefly summarized in Section 8 to show an important restriction: it is not possible, without modifications which are presently unknown, to determine the effects of mass and energy sources distributed within the volume of the chamber. This seems to be a consequence of treating the total energy of the system. In Section 8.2, the linearized equations are used to form an equation for the balance of acoustic energy only. This produces directly a formula for the growth constant which is identical with the earlier computation of the complex wave number.

The last two sections are concerned with applications of the analysis. Very briefly in Section 9 a simple configuration of the T-burner is examined, to indicate how experimentally the response and admittance functions might be determined. Those functions must be known for the analysis of the stability of oscillations in a motor, a subject treated in Section 10. The measurements of Brownlee and Marble (Ref. 5), which have been discussed in previous works, are reexamined. A conclusion of some recent work by Perry (Ref. 7) was that use of measurements taken in a T-burner, and the analysis existing at that time led to significant differences between the predicted and observed stability boundaries. Although it cannot be established definitely at present, owing to the lack of sufficient measurements, it appears that the new results obtained here may provide better agreement.

So far as the formal analysis of oscillations in a combustion chamber are concerned, the most important subjects not covered here are the influence of high Mach number, of the mean flow, and nonlinear behavior.

2. ONE-DIMENSIONAL PROBLEM

This section is based largely on Ref. 4. The formulation is well-known in fluid mechanics: a one-dimensional flow with mass addition at the lateral boundary. What is new ultimately concerns mainly problems of waves in such a flow, with the principal source of driving located in the combustion zone at the lateral boundary.

2.1. GOVERNING EQUATIONS

The conservation equations are

mass (gas)
$$\frac{\partial}{\partial t}(\rho S_c) + \frac{\partial}{\partial z}(\rho u S_c) = \int m_b dq + w_p S_c \qquad (2.1)$$

mass (particles)
$$\frac{\partial}{\partial t}(\rho_p S_c) + \frac{\partial}{\partial z}(\rho_p u_p S_c) = \int m_b^{(p)} dq - w_p S_c \qquad (2.2)$$

momentum
$$\frac{\partial}{\partial t}[(\rho_p u_p + \rho u)S_c] + \frac{\partial}{\partial z}[(\rho u^2 + \rho_p u_p^2)S_c] + S_c \frac{\partial p}{\partial z} =$$
$$= u_s \int m_b dq + u_{ps} \int m_b^{(p)} dq \qquad (2.3)$$

energy
$$\frac{\partial}{\partial t}[(\rho e_o + \rho_p e_{po})S_c] + \frac{\partial}{\partial z}[(\rho u e_o + \rho_p u_p e_{po})S_c] + \frac{\partial}{\partial z}(p u S_c) \qquad (2.4)$$

The first two terms on the right-hand side of Eq. 2.3 represent momentum added to the flow when the axial speeds of the gases and particles entering from the lateral boundary are non-zero. The two terms on the right-hand side of Eq. 2.4 represent corresponding contributions to the energy balance. Heat released by chemical reactions in the gas phase is represented by Q; hence, e and h represent only thermal energy and enthalpy, respectively.

In this section, variations of area will be treated as perturbations from the case of a uniform chamber, so the equations for conservation of mass are written

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = \frac{1}{S_c}\int m_b dq - \frac{\rho u}{S_c}\frac{dS_c}{dz} + w_p \tag{2.5}$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial z}(\rho_p u_p) = \frac{1}{S_c}\int m_b^{(p)} dq - \frac{\rho_p u_p}{S_c}\frac{dS_c}{\partial z} - w_p \tag{2.6}$$

The momentum equation can be rewritten in the form

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial z} + \frac{\partial p}{\partial z} = F - \sigma + \mu \tag{2.7}$$

with

$$F = -\left[\rho_p \frac{\partial u_p}{\partial t} + \rho_p u_p \frac{\partial u_p}{\partial z}\right] \tag{2.8}$$

$$\sigma = \frac{1}{S_c}\left[u\int m_b dq + u_p \int m_b^{(p)} dq\right] + (u - u_p) w_p \tag{2.9}$$

$$\mu = \frac{1}{S_c}\left[u_s \int m_b dq + u_{p_s} \int m_b^{(p)} dq\right] \tag{2.10}$$

Thus, F represents the force of the particles on the gas, averaged over the cross-section, and σ represents a momentum exchange between the chamber gases and two sources of mass: the flow-in at the lateral boundary, and the mass of gas produced by burning of particles. The influence of differences between the average momenta of the channel flow and the mass added at the boundary is contained in μ.

The energy equation can be written for the temperature in the usual way, first by expanding the left-hand side and using the continuity in Eq. 2.1 and 2.2.

$$\rho \frac{De_o}{Dt} + \rho_p \frac{De_{po}}{Dt} + \frac{\partial}{\partial z}(pu) = \frac{1}{S_c}(h_{os} - e_o)\int m_b dq + \frac{1}{S_c}(e_{pos} - e_{po})\int m_b^{(p)} dq$$

$$-(e_o - e_{po}) w_p + Q - \frac{pu}{S_c}\frac{dS_c}{dz} .$$

Now remove the kinetic energies from e_o and e_{po} by subtracting u times Eq. 2.3 and up times Eq. 2.8:

$$\rho C_v \frac{\partial T}{\partial t} + \rho C_v u \frac{\partial T}{\partial z} + p \frac{\partial u}{\partial z} = \frac{1}{S_c} (h_{os} - e_o) \int m_b dq + \frac{1}{S_c} (e_{pos} - e_{po}) \int m_b^{(p)} dq$$

$$+ (u_p - u) F + u(\sigma - \mu) + (e_{po} - e_o) w_p - \frac{pu}{S_c} \frac{dS_c}{dz} + (Q + Q_p) \qquad (2.11)$$

where Q_p is the energy released by combustion of particles:

$$Q_p = - \left[\rho_p \frac{\partial e_p}{\partial t} + \rho_p u_p \frac{\partial e_p}{\partial z} \right] \qquad (2.12)$$

For reasons covered later, the total energy and enthalpy of the gases issuing from the boundary will not be assumed equal to the average values in the chamber. Thus, while $e_{pos} \simeq e_p$,[1] $h_{os} \neq h_o$ and $e_{os} \neq e_o$. Define the difference between e_{os} and e_o to be Δe_o:

$$e_{os} = e_o + \Delta e_o \qquad (2.13)$$

so that

$$h_{os} - e_o = (e_o + \Delta e_o) + p/\rho - e_o$$

$$= a^2/\gamma + \Delta e_o \;.$$

With the equation of state, $p = \rho RT$, for a perfect gas, Eq. 2.11 provides an important equation for the pressure:

$$\frac{\partial p}{\partial t} + \gamma p \frac{\partial u}{\partial z} + u \frac{\partial p}{\partial z} = \frac{1}{S_c} (a^2 + R \Delta T_o) \int m_b dq + \frac{R}{C_v} [(u_p - u) F + u(\sigma - \mu)] - \frac{\gamma p u}{S_c} \frac{dS_c}{dz} +$$

$$+ \frac{R}{C_v} \left[(e_{po} - \frac{u^2}{2}) w_p + (Q + Q_p) \right] \qquad (2.14)$$

2.2. LINEARIZED PERTURBATION EQUATIONS

For simplicity, the mean pressure and temperature are assumed constant. All quantities are written as sums of mean values, $\overline{(\;)}$, and fluctuations, $(\;)'$, and squares of fluctuations are dropped.[2] Then Eq. 2.7 and 2.14 lead to

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[1] The kinetic energy of the particles is assumed to be small compared with the internal energy.

[2] Throughout this report, the primed $(\;)'$ quantities contain dependence on time, while later quantities signified by $\hat{(\;)}$ do not. Thus, for linear harmonic motions, the fluctuation of pressure is $p' = \hat{p} \exp(i\bar{a}kt)$.

$$\bar{\rho}\frac{\partial u'}{\partial t}+\frac{\partial p'}{\partial z} = -\bar{\rho}\frac{\partial}{\partial z}(\bar{u}u')+F'-\sigma'+\mu' \quad (2.15)$$

$$\frac{\partial p'}{\partial t}+\gamma\bar{p}\frac{\partial u'}{\partial z} = -\bar{u}\frac{\partial p'}{\partial z}-\gamma p'\frac{d\bar{u}}{dz}+P_1' \quad (2.16)$$

where P_1 is the perturbation of the right-hand side of Eq. 2.14,

$$P_1=\frac{1}{S_c}(a^2+R\Delta T_o)\int m_b dq+\frac{R}{C_v}[(u_p-u)F+u(\sigma-\mu)]-\frac{\gamma pu}{S_c}\frac{dS_c}{dz}+$$
$$+\frac{R}{C_v}\left[(e_{po}-\frac{u^2}{2})w_p+(Q+Q_p)\right] . \quad (2.17)$$

The common linearized problems encountered in motors are most conveniently analyzed by working with the wave equation for pressure fluctuations. This is constructed by differentiating Eq. 2.16 with respect to time and inserting Eq. 2.15 for $\partial u'/\partial t$ to give:

$$\frac{\partial^2 p'}{\partial z^2}-\frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h_1 \quad (2.18)$$

with

$$h_1 = -\bar{\rho}\frac{\partial^2}{\partial z^2}(\bar{u}u')+\frac{\partial}{\partial z}(F'-\sigma'+\mu')+\frac{\bar{u}}{\bar{a}^2}\frac{\partial^2 p'}{\partial z\partial t}+\frac{\gamma}{\bar{a}^2}\frac{\partial p'}{\partial t}\frac{d\bar{u}}{dz}-\frac{1}{\bar{a}^2}\frac{\partial P_1'}{\partial t} . \quad (2.19)$$

Solution to Eq. 2.18 will eventually require a boundary condition for p', set according to Eq. 2.15,

$$\frac{\partial p'}{\partial z} = -f_1 \quad (2.20)$$

with

$$f_1 = \bar{\rho}\frac{\partial u'}{\partial t}+\bar{\rho}\frac{\partial}{\partial z}(\bar{u}u')-(F'-\sigma'+\mu') . \quad (2.21)$$

In order to obtain quantitative results, the expressions for F', σ', μ', and P' are required. What one uses for F' depends on the law of force between the particles and the gas: Eq. 2.8 is merely the definition of F; the force of interaction is required in order to determine u_p and hence u_p' and F'. That problem will not be discussed here. On the other hand, σ, and to a lesser extent μ, are central to later discussion. It is a straightforward matter to find σ', and μ':

$$\sigma' = \frac{1}{S_c}[u'\int\bar{m}_b dq+u_p'\int\bar{m}_b^{(p)}dq]+(u'-u_p')\bar{w}_p \quad (2.22)$$

$$u' = \frac{1}{S_c}[(u_s')\int\bar{m}_b dq+(u_{ps}')\int\bar{m}_b dq] \quad (2.23)$$

The terms

$$\bar{u}\int m_b' dq \quad , \quad \bar{u}_p\int m_b^{(p)\prime} dq \quad , \quad (\bar{u}-\bar{u}_p)w_p'$$

and similar terms in μ' are dropped as higher order contributions. The reasons are : m_b' and $m_b^{(p)'}$ are of order $\bar{u}$, as shown by computations of the combustion processes at the surface, and $(\bar{u}-\bar{u}_p)$ is a small quantity because the particles may be assumed, usually, to be sufficiently small that in the steady state they very nearly follow the gas motions, so that $(\bar{u}-\bar{u}_p)w'$ is in practice a negligibly small term.

The terms

$$(u_p-u)F \ , \quad u(\sigma-\mu) \ , \quad (e_{po} - \frac{u^2}{2})w_p$$

also produce fluctuations of higher order (see Ref. 4 for a discussion of the first two) and may safely be ignored in linear problems. In the term $a^2/\gamma + \Delta e_o$, the kinetic energy produces a second order term, and with $a^2 = \gamma RT$, one finds the perturbation

$$[(a^2+R\Delta T_o)\int m_b dq]' = \gamma R(\bar{T}+\frac{\Delta\bar{T}}{\gamma})\int m_b' dq + \gamma R(T'+\frac{\Delta T'}{\gamma})\int \bar{m}_b dq \ . \tag{2.24}$$

It is assumed that the average temperature of the gas leaving the surface is equal to the average chamber temperature, so $\Delta\bar{T} = 0$, but differences in the fluctuations will still be allowed; one then has for P_1' :

$$P_1' = \frac{\gamma R}{S_c}\left[\bar{T}\int m_b' dq + (T'+\frac{\Delta T'}{\gamma})\int \bar{m}_b dq\right] - \frac{\gamma}{S_c}(p'\bar{u}+\bar{p}u')\frac{dS_c}{dz} + \frac{R}{C_v}[e_p'\bar{w}_p+\bar{e}_p w_p' + (Q+Q_p')] \ . \tag{2.25}$$

The fluctuation T' is the isentropic value associated with acoustic waves in the chamber, while $\Delta T'$ is a nonisentropic fluctuation associated with the unsteady combustion processes (see Sections 6 and 8). This completes the preparations required for calculations of strictly one-dimensional problems.

3. MODIFIED ONE-DIMENSIONAL PROBLEM

When calculations are based on the formulation of Section 2, variations of area are treated as perturbations, a point which will become more apparent later in Section 5.1. It is more accurate if the influence of nonuniform cross-section can be accounted for in "zeroth order", and in certain practically interesting situations this can be done without excessive difficulty. To do so requires only relatively minor alterations of the formulation just covered. Formally, the difference consists mainly in retaining the area S_c on the left-hand side of the governing equations.

3.1. GOVERNING EQUATIONS

The continuity equations for the gases and particles are retained in the form in Eq. 2.1 and 2.2; the momentum in Eq. 2.2 is written (cf., Eq. 2.7):

$$\rho S_c \frac{\partial u}{\partial t} + \rho u S_c \frac{\partial u}{\partial z} + S_c \frac{\partial p}{\partial z} = S_c(F-\sigma+\mu) \quad . \qquad (3.1)$$

Similarly, the energy equation 2.11 is written as:

$$\rho C_v S_c \frac{\partial T}{\partial t} + \rho C_v u S_c \frac{\partial T}{\partial z} + p \frac{\partial}{\partial z}(uS_c) = (\frac{a^2}{\gamma} + \Delta e_o)\int m_b dq + S_c(u_p - u)F + u(\sigma-\mu)S_c$$

$$+ S_c w_p(e_{po} - e_o) + (Q+Q_p)S_c \quad . \qquad (3.2)$$

One then finds that, corresponding to Eq. 2.12, the equation for the pressure is

$$\frac{\partial}{\partial t}(pS_c) + \gamma p \frac{\partial}{\partial z}(uS_c) + uS_c \frac{\partial p}{\partial z} = (a^2 + R\Delta T_o)\int m_b dq + \frac{R}{C_v} S_c[(u_p - u)F + u(\sigma-\mu)]$$

$$+ \frac{R}{C_v} S_c [(e_{po} - \frac{u^2}{2}) w_p + (Q+Q_p)] \quad . \qquad (3.3)$$

3.2. LINEARIZED PERTURBATIOז EQUATIONS

The perturbation equations for velocity and pressure are nov

$$\bar{\rho} \frac{\partial}{\partial t}(S_c u') + S_c \frac{\partial p'}{\partial z} = -\bar{\rho} S_c \frac{\partial}{\partial z}(\bar{u} u') + S_c(F' - \sigma' + \mu') \qquad (3.4)$$

$$\frac{\partial}{\partial t}(p'S_c)+\gamma\bar{p}\frac{\partial}{\partial z}(u'S_c) = -\bar{u}S_c\frac{\partial p'}{\partial z} - \gamma p'\frac{d}{dz}(\bar{u}S_c)+S_cP'_{11} \tag{3.5}$$

where $S_cP'_{11}$ is the fluctuation of

$$S_cP_{11} = (a^2+\Delta T_o)\int m_b dq + \frac{R}{C_v}S_c[(u_p-u)F+u(\sigma-\mu)] +$$

$$+\frac{R}{C_v}S_c\left[(e_{po}-\frac{u^2}{2})w_p+(Q+Q_p)\right] . \tag{3.6}$$

Hence, the wave equation for the pressure fluctuations is found to be

$$\frac{1}{S_c}\frac{\partial}{\partial z}(S_c\frac{\partial p'}{\partial z}) - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h_{11} \tag{3.7}$$

with

$$h_{11} = -\bar{\rho}\frac{\partial^2}{\partial z^2}(\bar{u}u') + \frac{1}{S_c}\frac{\partial}{\partial z}[S_c(F'-\sigma'+\mu')] + \frac{\bar{u}}{\bar{a}^2}\frac{\partial^2 p'}{\partial z\partial t} + \frac{\gamma}{\bar{a}^2}\frac{\partial p'}{\partial t}\frac{1}{S_c}\frac{d}{dz}(\bar{u}S_c) -$$

$$-\bar{\rho}\frac{\partial}{\partial z}(\bar{u}u')\frac{d\,\ell n S_c}{dz} - \frac{1}{\bar{a}^2}\frac{\partial P'_{11}}{\partial t} \tag{3.8}$$

The associated boundary condition on p' is exactly (Eq. 2.20) still, since the momentum equation is unchanged.

While F' is still given as the perturbation of the right-hand side of Eq. 2.8, and σ' is correctly given by Eq. 2.22, P'_{11} differs from P'_1 because the term $\gamma p u d\ell n S_c/dz$ in Eq. 2.15 does not appear in Eq. 3.6. Thus, P'_{11} is given by

$$P'_{11} = \frac{1}{S_c}[\bar{a}^2\int m'_b dq + (\gamma RT' + R\Delta T')\int \bar{m}_b dq] + \frac{R}{C_v}[(e'_p\bar{w}_p+\bar{e}_p w'_p)+(Q'+Q'_p)] . \tag{3.9}$$

4. THREE-DIMENSIONAL PROBLEM: CLASSICAL ACOUSTICS

The analyses of Ref. 2 and 3 which are, so far as the discussion here is concerned, the most general linear treatments available, are based on modified forms of classical acoustics theory. The principal modifications are due to the presence of a nonuniform mean flow. In Ref. 3, the calculations were based on integrals, of time-averaged quantities, over the chamber and its boundary; the developments of Ref. 4 were founded on the differential equations. The final results, when comparable, agree exactly; but for present purposes, it is preferable to work with the differential equations. Comparisons of the two approaches will arise in subsequent discussion.

4.1. GOVERNING EQUATIONS

In the differential equations for three-dimensional problems, no boundary sources appear explicitly, so one has for the conservation laws, instead of Eq. 2.1-2.4,

mass (gas)
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = w_p \tag{4.1}$$

mass (particles)
$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot (\rho_p \vec{u}_p) = -w_p \tag{4.2}$$

momentum
$$\frac{\partial}{\partial t}(\rho\vec{u} + \rho_p \vec{u}_p) + \nabla \cdot (\rho \vec{u}\vec{u} + \rho \vec{u}_p \vec{u}_p) + \nabla p = 0 \tag{4.3}$$

energy
$$\frac{\partial}{\partial t}(\rho e_o + \rho_p e_{po}) + \nabla \cdot (\rho \vec{u} e_o + \rho_p \vec{u}_p e_{po}) + \nabla \cdot (p\vec{u}) = Q \tag{4.4}$$

The momentum equation for the gas phase is

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} + \nabla p = \vec{F} - \vec{\sigma} \tag{4.5}$$

where

$$\vec{F} = -\left[\rho_p \frac{\partial \vec{u}_p}{\partial t} + \rho_p \vec{u}_p \cdot \nabla \vec{u}_p\right] \tag{4.6}$$

and

$$\vec{\sigma} = (\vec{u} - \vec{u}_p)w_p \tag{4.7}$$

Consequently, the energy equation with temperature as the independent variable is

$$\rho C_v \frac{\partial T}{\partial t} + \rho C_v \vec{u} \cdot \nabla T + p\nabla \cdot \vec{u} = (\vec{u}_p - \vec{u}) \cdot \vec{F} + \vec{u} \cdot \vec{\sigma} + (e_{po} - e_o)w_p + (Q + Q_p) \tag{4.8}$$

and the equation for the pressure is

$$\frac{\partial p}{\partial t} + \gamma p \nabla \cdot \vec{u} + \vec{u} \cdot \nabla p = \frac{R}{C_v}[(\vec{u}_p - \vec{u}) \cdot \vec{F} + \vec{u} \cdot \vec{\sigma} + (e_{po} - e_o)w_p + (Q + Q_p)] + RTw_p \tag{4.9}$$

Variations of cross-sectional area of course do not appear explicitly; they enter the problem when the three-dimensional differential equations are solved in a specified geometry.

4.2. LINEARIZED PERTURBATION EQUATIONS

In Ref. 3 the limiting process for finding the linearized equations has been described. This will produce the equations usually used, in which terms of first order in the wave amplitude and mean flow Mach number are retained. The sets of equations already found in Sections 2 and 3 are essentially based on that procedure, although they have not been so rigorously deduced here.

From Eq. 4.5 and 4.9, one finds the linearized equations for $\vec{u}'$ and p':

$$\bar{\rho}\frac{\partial \vec{u}'}{\partial t} + \nabla p' = -\bar{\rho}(\vec{\bar{u}} \cdot \nabla \vec{u}' + \vec{u}' \cdot \nabla \vec{\bar{u}}) + \vec{F}' - \vec{\sigma}' \tag{4.10}$$

$$\frac{\partial p'}{\partial t} + \gamma \bar{p} \nabla \cdot \vec{u}' = -\vec{\bar{u}} \cdot \nabla p' - \gamma p' \nabla \cdot \vec{\bar{u}} + P' \qquad (4.11)^3$$

with

$$P' = \frac{R}{C_v}[e_p' \bar{w}_p + \bar{e}_p w_p' + (Q' + Q_p')]. \tag{4.12}$$

The linearized wave equation for pressure fluctuations is

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h \tag{4.13}$$

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[3] Note that the assumption of incompressible mean flow, $\nabla \cdot \vec{\bar{u}} = 0$, is not made. This is obviously necessary if there is residual combustion ($w_p \neq 0$), but it is also a formal convenience for later comparison with the one-dimensional results.

subject to the boundary condition, deduced from Eq. 4.10, for the normal gradient at the boundary:

$$\hat{n} \cdot \nabla p' = -f \tag{4.14}$$

The functions h and f are now

$$h = -\bar{\rho}\nabla\cdot(\bar{\vec{u}}\cdot\nabla\vec{u}'+\vec{u}'\cdot\nabla\bar{\vec{u}})+\frac{1}{\bar{a}^2}\bar{\vec{u}}\cdot\nabla\frac{\partial p'}{\partial t}+\nabla\cdot(\vec{F}'-\vec{\sigma}')+\frac{\gamma}{\bar{a}^2}\frac{\partial p'}{\partial t}\nabla\cdot\bar{\vec{u}}-\frac{1}{\bar{a}^2}\frac{\partial P'}{\partial t}, \tag{4.15}$$

$$f = \bar{\rho}\frac{\partial\vec{u}'}{\partial t}\cdot\hat{n}+\bar{\rho}(\bar{\vec{u}}\cdot\nabla\vec{u}'+\vec{u}'\cdot\nabla\bar{\vec{u}})\cdot\hat{n}-(\vec{F}'-\vec{\sigma}')\cdot\hat{n} \quad . \tag{4.16}$$

As formulated here, the three-dimensional problem does not account totally for all interactions between the wave motions and the flow issuing from the surface. This can be seen most easily by examining the special case of one-dimensional motions. The left-hand side of Eq. 4.13 is then identical to the left-hand side of Eq. 2.18 but the right-hand sides and the boundary conditions in Eq. 2.20 and 4.14 do <u>not</u> agree.

With variations only in the z-direction, Eq. 4.15, 4.16 and 4.12 give

$$h = -\bar{\rho}\frac{\partial^2}{\partial z^2}(\bar{u}u')+\frac{1}{\bar{a}^2}\bar{u}\frac{\partial^2 p'}{\partial z\partial t}+\frac{\partial}{\partial z}(F'-\sigma')+\frac{\gamma}{\bar{a}^2}\frac{\partial p'}{\partial t}\frac{d\bar{u}}{dz}-\frac{1}{\bar{a}^2}\frac{\partial P'}{\partial t}$$

$$f = 0$$

$$P' = \frac{R}{C_v}[e_p'\bar{w}_p+\bar{e}_p w_p'+(Q'+Q_p')]$$

These will obviously not reproduce the one-dimensional problem discussed in Section 2. Indeed, there is no possibility here of representing any coupling with the burning surface! The reason, of course, is that in inviscid classical acoustics the only boundary condition is placed on the velocity fluctuation normal to the boundary--this is simply zero, always, for one-dimensional problems. It is mainly this difficulty which motivated the work of Ref. 4. Modifications of the inviscid three-dimensional problem consist of approximations to the processes occurring in boundary layers present in the real problem.

However, the case of truly three-dimensional motions with significant components parallel to the boundary surface has not yet been treated. This is referred to here as the "modified three-dimensional problem"; it is treated in Section 6. As an aid to that discussion, it is useful first to examine some more detailed results for the stability of harmonic motions.

5. CALCULATION OF THE COMPLEX WAVE NUMBER FOR THE LINEARIZED NATURAL MODES

The normal or natural modes for a linear system are characterized by harmonic time dependence; all fluctuations vary as $\exp(i\bar{a}kt)$ where

$$\bar{a}k = \omega - i\alpha \tag{5.1}$$

is the complex wave number. If the growth constant α is positive, then the amplitude of the waves grows with time. Now for a linear acoustic wave, the energy density is the sum of potential and kinetic energies:

$$e_a = \frac{(\delta p)^2}{2\bar{\rho}\bar{a}^2} + \tfrac{1}{2}\bar{\rho}(\delta\vec{u})^2 . \tag{5.2}$$

Here, δp, $\delta\vec{u}$ are written since they must be real quantities to make the energy real.[4]

All phases in time are referred to the pressure oscillation; for harmonic motions, then

$$p' = \hat{p}\, e^{\alpha t}\, e^{i\omega t} \tag{5.3a}$$

$$\vec{u}' = \hat{\vec{u}}\, e^{\alpha t}\, e^{i(\omega t+\phi)} \tag{5.3b}$$

where ϕ is the phase of the velocity fluctuation relative to the pressure fluctuation. In general, both $\hat{p}$ and $\hat{u}$ may be complex functions of position, but suppose for the present that they are real. Then the real parts of Eq. 5.3a, b may be used for δp and $\delta\vec{u}$ in Eq. 5.2 to give

$$e_a = \left[\frac{1}{2}\frac{\hat{p}^2}{\bar{\rho}\bar{a}^2}\cos^2\omega t + \frac{1}{2}\bar{\rho}(\hat{\vec{u}})^2\cos^2(\omega t+\phi)\right]e^{2\alpha t} \tag{5.4}$$

and the time-averaged value is approximately

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[4] One could of course introduce complex conjugates and write

$$e_a = \frac{1}{2\bar{\rho}\bar{a}^2}\, p'p'^* + \tfrac{1}{2}\bar{\rho}\,\vec{u}'\cdot\vec{u}'^* .$$

$$\bar{e}_a = \frac{1}{4}\left(\frac{\hat{p}^2}{\bar{\rho}\,\bar{a}^2} + \bar{\rho}\,\hat{\bar{u}}^2\right) e^{2\alpha t} \quad . \tag{5.5}$$

The time average here is over one cycle of the oscillation and is valid only if the amplitude of the wave doesn't vary rapidly: $\alpha \ll \omega$. It follows from Eq. 5.5 that the growth constant is related to the rate of change of energy density according to

$$2\alpha = \frac{1}{\bar{e}_a}\frac{d\bar{e}_a}{dt} \quad . \tag{5.6}$$

For interpretation of later results, it should be noted that Eq. 5.6 can be integrated over the entire volume of the cavity, showing that 2α equals the fractional rate of change of the total time-averaged acoustic energy in the cavity.

The linear stability of the normal modes depends on α. It is easy, following the development first given in Ref. 3, to find a formula for k^2 which can be evaluated using the classical acoustic mode shapes. The wave number for each mode is known to first order in the perturbations (i.e., to first order in the mean flow Mach number) when the modes are known to zeroth order. Those are the modes for a closed chamber with no combustion and having rigid walls. This is a well-known characteristic of perturbation expansions of the sort used here.

On the other hand, the actual mode shapes in the chamber will be distorted from the classical modes by terms of first order in the mean flow Mach number. Moreover, they will be complex quantities of position, which means that the actual pressure and velocity fluctuations will have phases varying with position in the chamber. All phases are measured relative to the unperturbed (classical) acoustic pressure oscillation. Thus, for example, Eq. 5.3a then becomes

$$p' = |\hat{p}|\, e^{\alpha t}\, e^{i(\omega t + \phi_p)}$$

with ϕ_p a function of position. The actual energy of the wave, Eq. 5.2, will also depend to first order on the mean flow Mach number. These refinements are of no consequence for the first approximation to linear stability.

5.1. ONE-DIMENSIONAL PROBLEM

With the exponential time dependence, all fluctuations can be written in the form $p' = \hat{p}\,\exp(i\bar{a}kt)$, for example. Hence, Eq. 2.18 becomes

$$\frac{d^2\hat{p}}{dz^2} + k^2\hat{p} = \hat{h}_1 \tag{5.7}$$

with the boundary condition at the ends set by Eq. 2.20:

$$\frac{d\hat{p}}{dz} = -\hat{f}_1 \; . \qquad (5.8)$$

In the absence of combustion, Eq. 5.7 and 5.8 become

$$\frac{d^2\hat{p}}{dz^2} + k_\ell^2\hat{p} = -i\gamma k_\ell \bar{p}\hat{u}\frac{1}{S_c}\frac{dS_c}{dz} \qquad (5.9)$$

$$\frac{d\hat{p}}{dz} = 0 \qquad (z = 0, L) \qquad (5.10)$$

Equation 5.9 exhibits explicitly the contribution of nonuniform area alone, when there is no combustion or average flow.

The procedure for finding an expression for the wave number involves essentially a comparison of the problem of interest, including perturbations, with the problem having no perturbations. The latter is simply the classical acoustic problem for a straight tube, governed by the differential equations

$$\frac{d^2\hat{p}_\ell}{dz^2} + k_\ell^2\,\hat{p}_\ell = 0 \qquad (5.11)$$

$$\frac{d\hat{p}_\ell}{dz} = 0 \qquad (z = 0, L) \qquad (5.12)$$

To find the first order formula for k^2, multiply Eq. 5.7 by $\hat{p}_\ell$, Eq. 5.11 by $\hat{p}$, substract the equations and integrate over $0 \leq z \leq L$. After use of the boundary conditions Eq. 5.8 and 5.12 one finds

$$k^2 = k_\ell^2 + \frac{1}{E_\ell^2}\left\{\int_0^L \hat{h}_1\hat{p}_\ell dz + [\hat{f}_1\hat{p}_\ell]_0^L\right\} \qquad (5.13)$$

where

$$E_\ell^2 = \int_0^L \hat{p}^2 dz \approx \int_0^L \hat{p}_\ell^{\,2} dz \; . \qquad (5.14)$$

All terms in $\hat{h}_1$ and $\hat{f}_1$ contain the Mach number of the mean flow as a factor (see Ref. 3 and 4 for further details). That is, all perturbations of the classical problem are small, as necessary for the calculation to be valid. Hence, it is sufficiently accurate (i.e., to first order) to replace $\hat{p}$, $\hat{u}$ everywhere in $\hat{h}_1$, $\hat{f}_1$ and E_ℓ^2 by $\hat{p}_\ell$ and $\hat{u}_\ell$. The right-hand side of Eq. 5.13 may therefore be evaluated. To do so, it will be necessary to use the relation for the acoustic velocity,

$$\hat{u}_\ell = \frac{i}{\bar{\rho}\,\bar{a}k_\ell}\,\frac{d\hat{p}_\ell}{dz} \,. \tag{5.15}$$

With (2.19) for $h_1 = \hat{h}_1 \exp(ik\bar{a}t) \approx \hat{h}_1 \exp(ik_\ell \bar{a}t)$,

$$\int_0^L \hat{h}_1 \hat{p}_\ell dz = -\frac{i}{\bar{a}k_\ell}\left[\frac{S_{be}}{S_{co}}\hat{p}_\ell \frac{d}{dz}\left(\bar{u}\frac{d\hat{p}_\ell}{dz}\right)\right]_0^L + 2\frac{ik_\ell}{\bar{a}}\int_0^L \bar{u}\hat{p}_\ell \frac{d\hat{p}_\ell}{dz}dz + i\gamma\frac{k_\ell}{\bar{a}}\int_0^L \hat{p}_\ell^2 \frac{d\bar{u}}{dz}dz - i\frac{k_\ell}{\bar{a}}\int_0^L \hat{P}_1 \hat{p}_\ell dz - \int_0^L (\hat{F} - \hat{\sigma}' + \hat{\mu})\frac{d\hat{p}_\ell}{dz}dz + \left[\frac{S_{be}}{S_{co}}(\hat{F} - \hat{\sigma} + \hat{\mu})\hat{p}_\ell\right]_0^L \tag{5.16}$$

and with (2.21) for $f_1 = \hat{f}_1 \exp(ik_\ell \bar{a}t)$,

$$[\hat{f}_1 \hat{p}_\ell]_0^L = i\bar{\rho}\,\bar{a}k_\ell\left[\hat{u}\,\hat{p}_\ell \frac{S_{be}}{S_{co}}\right]_0^L + \frac{i}{\bar{a}k_\ell}\left[\frac{S_{be}}{S_{co}}\hat{p}_\ell \frac{d}{dz}\left(\bar{u}\frac{d\hat{p}_\ell}{dz}\right)\right]_0^L - \left[\frac{S_{be}}{S_{co}}(\hat{F} - \hat{\sigma} + \hat{\mu})\hat{p}_\ell\right]_0^L \tag{5.17}$$

The area S_{be} has been introduced as the area at the end of the chamber over which u' differs from zero; this may occur usually at a burning surface or at the entrance to a nozzle. Addition of Eq. 5.16 and 5.17 produces, after some further integration by parts:

$$(k^2 - k_\ell^2)E_\ell^2 = i\bar{\rho}\,\bar{a}k_\ell\left[\left(\hat{u}\,\hat{p}_\ell + \frac{\bar{u}\hat{p}_\ell^2}{\bar{\rho}\,\bar{a}^2}\right)\frac{S_{be}}{S_{co}}\right]_0^L + i(\gamma - 1)\frac{k_\ell}{\bar{a}}\int_0^L \hat{p}_\ell^2 \frac{d\bar{u}}{dz}dz - \int_0^L (\hat{F} - \hat{\sigma} + \hat{\mu}')\frac{d\hat{p}_\ell}{dz}dz - i\frac{k_\ell}{\bar{a}}\int_0^L \hat{P}_1 \hat{p}_\ell dz \,. \tag{5.18}$$

With Eq. 2.20 and 2.21 one finds

$$\int_0^L \hat{\sigma}\frac{d\hat{p}_\ell}{dz}dz = \frac{i}{\bar{\rho}\,\bar{a}k_\ell}\int_0^L \left(\frac{d\hat{p}_\ell}{dz}\right)^2 \int \bar{m}_b dq \frac{dz}{S_c} + \int_0^L \hat{u}_p \frac{d\hat{p}_\ell}{dz}\int \bar{m}_b^{(p)} dq \frac{dz}{S_c} + \int_0^L (\hat{u} - \hat{u}_p)\bar{w}_p \frac{d\hat{p}_\ell}{dz}dz \tag{5.19}$$

$$-i\frac{k_\ell}{\bar{a}}\int_0^L \hat{P}_1\hat{P}_\ell dz = -i\bar{a}k_\ell\int_0^L \hat{p}_\ell\int\hat{m}_b dq\,\frac{dz}{S_c} - i(\gamma-1)\frac{k_\ell}{\bar{a}}\int_0^L \hat{p}_\ell^{\,2}\,\frac{1}{\bar{\rho}S_c}\int\bar{m}_b dq dz$$

$$-\frac{1}{2}\int_0^L \frac{d\hat{p}_\ell^2}{dz}\,\frac{dS_c}{dz}\,\frac{dz}{S_c} + i\gamma\,\frac{k_\ell}{\bar{a}}\int_0^L \hat{p}_\ell^{\,2}\bar{u}\,\frac{dS_c}{dz}\,\frac{dz}{S_c}$$

$$-i\frac{k_\ell}{\bar{a}}\,R\int_0^L \Delta\hat{T}\,\hat{p}_\ell\int\bar{m}_b dq\,\frac{dz}{S_c}$$

$$-i\frac{k_\ell}{\bar{a}}\cdot\frac{R}{C_v}\int_0^L \hat{p}_\ell\left[(\hat{e}_p'\bar{w}_p + \bar{e}_p\hat{w}_p') + (\hat{Q}' + \hat{Q}_p')\right] dz \quad . \tag{5.20}$$

The -1 part of $(\gamma-1)$ times the second integral is combined with the first integral in Eq. 5.19, while the γ part is rewritten by use of the continuity equation for steady flow:

$$\frac{1}{\bar{\rho}S_c}\int\bar{m}_b dq = \frac{d\bar{u}}{dz} + \frac{\bar{u}}{S_c}\,\frac{dS_c}{dz} - \frac{\bar{w}_p}{\bar{\rho}} \tag{5.21}$$

Then the formula for k^2 has the final form

(I)

$$(k^2 - k_\ell^2)E_\ell^2 = \left\{ i\bar{\rho}\,\bar{a}k_\ell\left[\left(\hat{u}\hat{p}_\ell + \frac{\bar{u}\hat{p}_\ell^2}{\bar{\rho}\,\bar{a}^2}\right)\frac{S_{be}}{S_{co}}\right]_0^L - i\bar{a}k_\ell\int_0^L \hat{p}_\ell\int\hat{m}_b dq\,\frac{dz}{S_c} \right\}$$

(II)

$$\left\{ i\frac{k_\ell}{\bar{a}}\left[\frac{1}{\bar{\rho}}\int_0^L\left[\hat{p}_\ell^{\,2} + \frac{1}{k_\ell^2}\left(\frac{d\hat{p}_\ell}{dz}\right)^2\right]\int\bar{m}_b dq\,\frac{dz}{S_c} - \int_0^L \hat{p}_\ell^{\,2}\,\frac{d\bar{u}}{dz}\,dz - \frac{1}{2}\int_0^L \frac{d\hat{p}_\ell^2}{dz}\,\frac{dS_c}{dz}\,\frac{dz}{S_c}\right] \right\}$$

(III)

$$+\left\{ \int_0^L \hat{u}_p\,\frac{d\hat{p}_\ell}{dz}\int\bar{m}_b^{(p)} dq\,\frac{dz}{S_c} - \int_0^L \hat{F}\,\frac{d\hat{p}_\ell}{dz}\,dz \right\}$$

(IV)

$$+\left\{ -i\frac{k_\ell}{\bar{a}}\,\frac{R}{C_v}\int_0^L \hat{p}_\ell\left[(\hat{e}_p\bar{w}_p + \bar{e}_p\hat{w}_p) + (\hat{Q} + \hat{Q}_p)\right]\frac{dz}{S_c} + i\frac{\gamma k_\ell}{\bar{\rho}\,\bar{a}}\int_0^L \hat{p}_\ell^{\,2}\bar{w}_p dz + \int_0^L (\hat{u} - \hat{u}_p)\bar{w}_p\,\frac{d\hat{p}_\ell}{dz}\,dz \right\}$$

$$\overset{\text{(V)}}{-i\frac{k_\ell}{\bar{a}} R\int_0^L \Delta\hat{T}\hat{p}_\ell \int \bar{m}_b dq \frac{dz}{S_c}} - \overset{\text{(VI)}}{\int_0^L \hat{\mu}\frac{d\hat{p}_\ell}{dz} dz} \tag{5.22}$$

The various contributions have been grouped as shown according to the following general interpretations:

(I.) These contain all terms representing the response of burning surfaces ($\hat{u}$ is related to the admittance function and $\hat{m}_b$ to the response function). In addition, the influence of a nozzle at the end of the chamber appears through its admittance function (the $\hat{u}$ term). The term $\bar{u}\hat{p}_\ell^2$ represents the convection of acoustic energy, by the mean flow, through the end walls. It is an energy gain to the wave system because it represents the addition of energy by the inward flow after the flow has acquired the acoustical motions.

(II) These represent interactions between the mean gas flow and the acoustic field. The first term represents an energy loss for the waves in the chamber. It is due to the inelastic process by which the incoming flow acquires acoustic energy. Note that $\hat{p}_\ell^2 + k_\ell^{-2}(d\hat{p}_\ell/dz)^2$ is proportional to the volume density of acoustic energy. The second term is the net (averaged) work done by the acoustic pressure against the mean flow. The third term shifts the natural frequency. Its interpretation is not obvious it arises from the term $\gamma p'\bar{u}\, d\ell n S_c/dz$ in Eq. 2.25.

(III). The first term is associated with the addition of particles from the burning surface, and the fact that they must acquire the local fluctuating motions. The second term represents the influence of particles distributed in the volume of the chamber and hence contains, partly, the attenuation of acoustic waves by the particle/gas interactions.

(IV) All these terms vanish if there is no residual burning within the volume. Those containing $\bar{w}_p$ or $\hat{w}_p$ as a factor are present only when there is burning of solid (or liquid) particles. The first term is associated with energy release and the second term represents a source of acoustic energy due to the distributed source of mass.

(V.) This represents the effect of temperature fluctuations at the combustion zone, which are nonisentropic. Hence, it arises due to the production of entropy, and part of it is an energy loss for the acoustic waves.

(VI) The last term represents the effect of momentum fluctuations at the boundary layer when the axial components of momenta of the gases and particles entering differ from those of the gases and particles in the bulk channel flow.

Some of the above will be interpreted in more detail later.

5.2. MODIFIED ONE-DIMENSIONAL PROBLEM

The calculation of the wave number goes through in the same way: the only major difference is that the unperturbed or zeroth order equation must now correspond to Eq. 3.7, which, for harmonic motions, is

$$\frac{1}{S_c}\frac{d}{dz}\left(S_c\frac{d\hat{p}}{dz}\right) + k^2\hat{p} = \hat{h}_{11} \tag{5.23}$$

The boundary condition is still Eq. 5.8, but the unperturbed problem is represented by

$$\frac{1}{S_c}\frac{d}{dz}\left(S_c\frac{d\hat{p}_\ell}{dz}\right) + k_\ell^2\hat{p}_\ell = 0 \tag{5.24}$$

$$\frac{d\hat{p}_\ell}{dz} = 0 \tag{5.25}$$

Now multiply Eq. 5.22 by $S_c\hat{p}_\ell$, Eq. 5.23 by $S_c\hat{p}$ and follow the prescription outlined above to find

$$k^2 = k_\ell^2 + \frac{1}{E_{1\ell}^2}\left\{\int_0^L \hat{h}_{11}\hat{p}_\ell S_c dz + \left[\hat{f}_1\hat{p}_\ell S_c\right]_0^L\right\} \tag{5.26}$$

where now

$$E_{1\ell}^2 = \int_0^L \hat{p}_\ell^2 S_c dz \ . \tag{5.27}$$

The definitions in Eq. 2.21 and 3.8 of f_1 and h_{11} lead to

$$(k^2-k_\ell^2)E_{1\ell}^2 = i\bar{\rho}\,\bar{a}k_\ell\left[\left(\hat{u}\hat{p}_\ell + \frac{\bar{u}\hat{p}_\ell^2}{\bar{\rho}\,\bar{a}^2}\right)S_{be}\right]_0^L + i(\gamma-1)\frac{k_\ell}{\bar{a}}\int_0^L \hat{p}_\ell^2\frac{d}{dz}(\bar{u}S_c)dz$$

$$-\int_0^L (\hat{F}-\hat{\sigma}+\hat{\mu})\frac{d\hat{p}_\ell}{dz}S_c dz - i\frac{k_\ell}{\bar{a}}\int_0^L \hat{P}_{11}\hat{p}_\ell S_c dz \tag{5.28}$$

The integral over $\hat{\sigma}$ is like Eq. 5.19 except that dz/S_c is replaced by dz; that involving $\hat{P}_{11}$ is

$$-i\frac{k_\ell}{\bar{a}}\int_0^L \hat{P}_{11}\hat{p}_\ell S_c dz = -i\bar{a}k_\ell\int_0^L \hat{p}_\ell \int m_b dqdz - i(\gamma-1)\frac{k_\ell}{\bar{\rho}\bar{a}}\int_0^L \hat{p}_\ell^2 \int \bar{m}_b dqdz$$

$$-i\frac{k_\ell}{\bar{a}}\frac{R}{C_v}\int_0^L \hat{p}_\ell[\hat{e}_p\bar{w}_p + \bar{e}_p\hat{w}_p + (\hat{Q} + \hat{Q}_p)]dz$$

$$-i\frac{k_\ell}{\bar{a}}R\int_0^L \Delta\hat{T}\,\hat{p}_\ell \int \bar{m}_b dqdz \quad . \tag{5.29}$$

Thus, Eq. 5.28 becomes

$$(k^2 - k_\ell^2)E_{1\ell}^2 = \overset{\text{(I)}}{\left\{i\bar{\rho}\bar{a}k_\ell\left[\left(\bar{u}\hat{p}_\ell + \frac{\bar{u}\hat{p}_\ell^2}{\bar{\rho}\bar{a}^2}\right)S_{be}\right]_0^L\right.} \overset{\text{(1)}}{- i\bar{a}k_\ell \int_0^L \hat{p}_\ell \int \hat{m}_b dqdz\Big\}} +$$

$$\overset{\text{(2)}}{\left\{i\frac{k_\ell}{\bar{a}}\frac{1}{\bar{\rho}}\int_0^L[\hat{p}_\ell^2 + \frac{1}{k_\ell^2}(\frac{d\hat{p}_\ell}{dz})^2]\int \bar{m}_b dqdz\right.} \overset{\text{(II)}}{- i\frac{k_\ell}{\bar{a}}\int_0^L \hat{p}_\ell^2 \frac{1}{S_c}\frac{d}{dz}(S_c\bar{u})S_c dz\Big\}}$$

$$+\Big\{\overset{\text{(3)}}{\int_0^L \hat{u}_p \frac{d\hat{p}_\ell}{dz}\int \bar{m}_b^{(p)} dqdz} \overset{\text{(III)}}{- \int_0^L \hat{F}\frac{d\hat{p}_\ell}{dz}S_c dz}\Big\}$$

$$+\Big\{\overset{\text{(IV)}}{-i\frac{k_\ell}{\bar{a}}\frac{R}{C_v}\int_0^L \hat{p}_\ell[\hat{e}_p\bar{w}_p + \bar{e}_p\hat{w}_p + (\hat{Q} + \hat{Q}_p)]S_c dz} + i\frac{k_\ell\gamma}{\bar{\rho}\bar{a}}\int_0^L \bar{w}_p p_\ell^2 S_c dz + \int_0^L(\hat{u} - \hat{u}_p)\bar{w}_p\frac{d\hat{p}_\ell}{dz}S_c dz\Big\}$$

$$\overset{\text{(V)}}{-i\frac{k_\ell}{\bar{a}}R\int_0^L \Delta\hat{T}\hat{p}_\ell \int \bar{m}_b dqdz} \overset{\text{(VI)}}{- \int_0^L \hat{\mu}\frac{d\hat{p}_\ell}{dz}S_c dz} \tag{5.30}$$

The interpretations of the various terms are the same as those listed above.

5.3. THREE-DIMENSIONAL PROBLEM

The calculation now centers on solution to Eq. 4.13. The unperturbed problem is described by

$$\nabla^2 \hat{p}_N + k_\ell^2 \hat{p}_N = 0 \tag{5.31a}$$

$$\hat{n} \cdot \nabla \hat{p}_N = 0 \tag{5.31b}$$

where N stands for three indices, one for each dimension.

For harmonic motions, Eq. 4.13 is

$$\nabla^2 \hat{p} + k^2 \hat{p} = \hat{h} \tag{5.32}$$

with the boundary condition

$$\hat{n} \cdot \nabla \hat{p} = -\hat{f} \quad . \tag{5.33}$$

Now multiply Eq. 5.32 by $\hat{p}_N$, Eq. 5.31a by $\hat{p}$, subtract, and integrate over the volume of the chamber. After use of Green's theorem and the boundary conditions in Eq. 5.31b and 5.33, one finds

$$k^2 = k_N^2 + \frac{1}{E_N^2} \left\{ \int \hat{p}_N \hat{h}\, dV + \oiint \hat{f}\, \hat{p}_N dS \right\} \quad . \tag{5.34}$$

Again, of course, this result is valid only to first order in the Mach number of the mean flow. Thus, in $\hat{h}$ and $\hat{f}$ one must use the zeroth order approximation, $\hat{p}_N$, to the mode shape (cf. remarks following Eq. 5.6). Here, E_N^2 is

$$E_N^2 = \int \hat{p}_N^2 dV \quad . \tag{5.35}$$

The acoustic velocity is now given by the formula

$$\hat{u}_N = \frac{i}{\bar{\rho}\, \bar{a} k_N} \nabla \hat{p}_N \quad . \tag{5.36}$$

Hence, with $\hat{n}$ given by Eq. 4.15, and $\hat{f}$ by Eq. 4.16,

$$\int \hat{h}\, \hat{p}_N dV = -\frac{i}{\bar{a} k_N} \int \hat{p}_N \nabla \cdot \vec{C}\, dV + i \frac{k_N}{2\bar{a}} \int \bar{u} \cdot \nabla \hat{p}_N^2 dV + i \frac{k_N \gamma}{\bar{a}} \int \hat{p}_N^2 \nabla \cdot \bar{\vec{u}}\, dV + \int \hat{p}_N \nabla \cdot (\hat{\vec{F}} - \hat{\vec{\sigma}}) dV$$
$$- i \frac{k_N}{\bar{a}} \int \hat{P} \hat{p}_N dV \tag{5.37}$$

$$\oiint \hat{f} \hat{p}_N dS = i \bar{\rho}\, \bar{a} k_N \oiint \hat{u} \cdot \hat{n}\, \hat{p}_N dS + \frac{i}{\bar{a} k_N} \oiint \hat{n} \cdot \vec{C}\, \hat{p}_N dS - \oiint \hat{n} \cdot (\hat{\vec{F}} - \hat{\vec{\sigma}}) \hat{p}_N dS$$

where

$$\vec{C} = \vec{\bar{u}} \cdot \nabla(\nabla\hat{p}_N) + \nabla\hat{p}_N \cdot \nabla\vec{\bar{u}} \tag{5.38}$$

All terms in $\vec{C}$ can be combined to give an interesting simpler result as follows; since in Eq. 5.34 the integrals over $\hat{h}$ and $\hat{f}$ must be summed, one encounters the following combination:

$$-\frac{i}{\bar{a}k_N}\int \hat{p}_N \nabla\cdot \vec{C}dV + \frac{i}{\bar{a}k_N}\oiint \hat{n}\cdot \vec{C}p_N dS$$

$$= -\frac{i}{\bar{a}k_N}\int \nabla\cdot(\hat{p}_N\vec{C})dV + \frac{i}{\bar{a}k_N}\int \vec{C}\cdot \nabla\hat{p}_N dV + \frac{i}{\bar{a}k_N}\oiint \hat{n}\cdot \vec{C}p_N dS$$

$$= -\frac{i}{\bar{a}k_N}\oiint \hat{p}_N\vec{C}\cdot \hat{n}dS + \frac{i}{\bar{a}k_N}\int \vec{C}\cdot \nabla\hat{p}_N dV + \frac{i}{\bar{a}k_N}\oiint \hat{n}\cdot \vec{C}p_N dS \quad .$$

The first and last terms cancel, while the second one can be rewritten.

$$\vec{C}\cdot\nabla\hat{p}_N = [\vec{\bar{u}}\cdot \nabla(\nabla\hat{p}_N) + \nabla\hat{p}_N\cdot\nabla\vec{\bar{u}}]\cdot\nabla\hat{p}_N$$

$$= [\nabla(\vec{\bar{u}}\cdot\nabla\hat{p}_N) - \nabla\hat{p}_N \times \nabla \times \vec{\bar{u}}]\cdot\nabla\hat{p}_N$$

$$= \nabla\hat{p}_N\cdot\nabla(\vec{\bar{u}}\cdot\nabla\hat{p}_N)$$

$$= \nabla\cdot[(\vec{\bar{u}}\cdot\nabla\hat{p}_N)\nabla\hat{p}_N] - (\vec{\bar{u}}\cdot\nabla\hat{p}_N)\nabla^2\hat{p}_N$$

$$= \nabla\cdot[(\vec{\bar{u}}\cdot\nabla\hat{p}_N)\nabla\hat{p}_N] + k_N^2 p_N \vec{\bar{u}}\cdot\nabla\hat{p}_N \quad .$$

Consequently,

$$\frac{i}{\bar{a}k_N}\int\vec{C}\cdot\nabla\hat{p}_N dV = \frac{i}{\bar{a}k_N}\int\nabla\cdot[(\vec{\bar{u}}\cdot\nabla\hat{p}_N)\nabla\hat{p}_N]dV + i\frac{k_N}{2\bar{a}}\int\vec{\bar{u}}\cdot\nabla\hat{p}_N^2 dV$$

$$= \frac{i}{\bar{a}k_N}\iint(\vec{\bar{u}}\cdot\nabla\hat{p}_N)\nabla\hat{p}_N\cdot\hat{n}dS + i\frac{k_N}{2\bar{a}}\int\vec{\bar{u}}\cdot\nabla\hat{p}_N^2 dV \quad .$$

The first integral vanishes because of the boundary condition in Eq. 5.31b, while the second is exactly equal to the second term in Eq. 5.37. Then since the mean flow field is not assumed to be free of sources,

$$i\frac{k_N}{\bar{a}}\int \bar{\vec{u}}\cdot\nabla\hat{p}_N^2 dV = i\frac{k_N}{\bar{a}}\oiint \hat{p}_N^2\bar{\vec{u}}\cdot\hat{n}dS - i\frac{k_N}{\bar{a}}\int \hat{p}_N^2\nabla\cdot\bar{\vec{u}}dV \ .$$

Hence, after a few other simple manipulations, the formula in Eq. 5.34 for the complex wave number can be written explicitly as

$$\begin{aligned}
(k^2-k_N^2)E_N^2 = &\Big\{\overset{\text{I}}{i\bar{\rho}\bar{a}k_N \oiint\Big[\hat{\vec{u}}\,\hat{p}_N + \frac{\bar{\vec{u}}\hat{p}_N^2}{\bar{\rho}\bar{a}^2}\Big]\cdot\hat{n}dS} - \underset{①}{\rule{4cm}{0.4pt}}\Big\} \\
&+\Big\{\underset{②}{\rule{3cm}{0.4pt}} - \overset{\text{II}}{i\frac{k_N}{\bar{a}}\int \hat{p}_N^2\nabla\cdot\bar{\vec{u}}dV}\Big\} \\
&+\Big\{\underset{③}{\rule{3cm}{0.4pt}} \overset{\text{III}}{-\int\hat{\vec{F}}\cdot\nabla\hat{p}_N dV} \\
&+\Big\{\overset{\text{IV}}{-i\frac{k_N}{\bar{a}}\frac{R}{C_v}\int\hat{p}_N[\hat{e}_p\bar{w}_p+\bar{e}_p\hat{w}_p)+(\hat{Q}+\hat{Q}_p)]dV + i\frac{k_\ell\gamma}{\bar{\rho}\bar{a}}\int\hat{p}_N^2\bar{w}_p dV^{5}} \\
&\qquad + \int\bar{w}_p(\hat{u}-\hat{u}_p)\cdot\nabla p_N dV\Big\} \\
&- \underset{④}{\overset{\text{V}}{\rule{4cm}{0.4pt}}} - \underset{⑤}{\overset{\text{VI}}{\rule{4cm}{0.4pt}}}
\end{aligned} \qquad (5.39)$$

The various contributions have again been labelled; a few blank spaces are purposely indicated for comparison with the one-dimensional results. Apart from the inclusion of residual burning in the chamber, Eq. 5.39 is the same as a result obtained in Ref. 3.

[5] In this term only, $\nabla\cdot\bar{\vec{u}}$ has been replaced by $\bar{w}_p/\bar{\rho}$, by use of the continuity in Eq. 4.1 written for the mean flow. This substitution aids comparison with the one-dimensional result.

These results for the complex wave numbers will be discussed further in later sections. First, however, they will be used to motivate a fourth formulation of the analysis of instabilities.

6. THE MODIFIED THREE-DIMENSIONAL PROBLEM

At the end of Section 4, it was noted that the classical three-dimensional formulation does not reduce to the one-dimensional results covered in Sections 2 and 3. Moreover, it is not possible, within the strict three-dimensional analysis, to handle coupling at the boundary when the main wave motions are parallel to the boundary. Both of these difficulties are removed by patching the three-dimensional analysis.

The essential point is that the one-dimensional formulation[6] implicitly accounts for viscous processes occurring in the flow adjacent to the lateral boundaries. If those terms correctly representing boundary layer effects can be identified in the one-dimensional ayalysis, then they may be incorporated in certain results of the three-dimensional analysis.

Since the differential equations for the one-dimensional problem contain these boundary terms (e.g., those terms involving m_b and $m_b^{(p)}$ in Eq. 2.1-2.4) while the partial differential equations for three-dimensional problems do not, it is necessary to establish a connection elsewhere). In the present work, it is convenient to compare the formulas for the complex wave numbers in Eq. 5.30 and 5.39.

The problem comes down to determining what terms of Eq. 5.30 represent genuine boundary effects and should therefore be included as surface terms in the three-dimensional result in Eq. 5.39. Resolution rests on correct interpretation of the various integrals. Comparison shows that the one-dimensional result in Eq. 5.30 contains six terms more than the three-dimensional result in Eq. 5.39. These are numbered in Eq. 5.30 and will be considered in turn. It must be emphasized that the right-hand side of Eq. 5.30 represents the various contributions to rates of energy addition to acoustical motions entirely parallel to the lateral boundary, while the mass addition ($\hat{m}_b$, $\bar{m}_b$, etc.) is occurring mainly normal to the surface. Deviations at the surface from motion normal to the surface are represented by $\hat{\bar{m}}$ and $\hat{u}$. It should also be noted that in all terms, one factor $\hat{p}_\ell$, or in some cases $d\hat{p}_\ell/dz$, is present because of the spatial structure of the acoustic mode. Thus, for example, a process which couples to the pressure fluctuation is most effective where $\hat{p}_\ell$ is large. The numbered terms in Eq. 5.30 have the following interpretations.

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[6] The difference between the formulations of Sections 1 and 2 lies solely in the way in which variations of area are handled. This is of no consequence for the present discussion, so the two analyses will be referred to together as the one-dimensional formulation. It should, however, be emphasized that the modified one-dimensional formulation produces more accurate results when the cross-sectional area varies along the axis of the chamber. The results of the two formulations of course agree for the case of constant area.

① and ④ . This represents the direct influence of fluctuations of the mass flux from the lateral boundary. It arises, of course, due to the coupling between the acoustic field and the combustion processes at the boundary; it should be compared with the term $i\bar{p}\bar{a}k_\ell[\hat{u}\hat{p}_\ell S_{be}]_0^L$ which represents the corresponding interaction at the end surfaces where there is no component of acoustic velocity parallel to the surface. In the latter case, the driving of the waves is proportional to the velocity fluctuation.

The difference between the two situations may be interpreted as follows. When the surface is oriented normal to the wave motion, motion of either the surface or of the gases leaving the surface produces p-v work: it is the rate of p-v work, proportional to the velocity, which matters. But suppose that the surface is oriented parallel to the wave motion. The influence of fluctuations of the flow leaving the surface is not now as an agent doing work on the waves, but as a mechanism for locally increasing the pressure of the acoustics wave. The origin of this term is $a^2 \int m_b dq$ on the right-hand side of Eq. 3.3. It may be recalled that Eq. 3.3 is the result of adding R/C_v times the energy equation (Eq. 3.2) to RT times the continuity equation (Eq. 2.1); that step produces the source term

$$\left[\frac{R}{C_v}(h_{os}-e_o)+RT\right]\int m_b dq \equiv [\gamma RT+R\Delta T_o)\int m_b dq \ .$$

There are, in fact, two perturbation terms arising here, for

$$(a^2\int m_b dq)' = \bar{a}^2\int m_b' dq+(a^2)'\int \bar{m}_b dq = \bar{a}^2\int m_b' dq+\gamma RT'\int \bar{m}_b dq$$

$$= \bar{a}^2\int m_b' dq+\bar{a}^2\left(\frac{\gamma-1}{\gamma}\right)\frac{p'}{\bar{p}}\int \bar{m}_b dq \ . \qquad (6.1)$$

If only the first term is taken into account on the right-hand side of Eq. 3.5, one has

$$\frac{\partial}{\partial t}(p'S_c) + \gamma\bar{p}\frac{\partial}{\partial z}(u'S_c) = \bar{a}^2\int m_b' dq \ . \qquad (6.2)$$

Thus, the term in question appears as a source of pressure changes.

As an aside, it is instructive to examine an approximate means of determining the contribution from α from Eq. 6.2; differentiate Eq. 6.2 with respect to time and replace $\partial u'/\partial t$ by the approximation

$$\frac{\partial u'}{\partial t} \approx -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} \ ,$$

so

$$\frac{\partial^2}{\partial t^2}(p'S_c)-\bar{a}^2\frac{\partial^2(p'S_c)}{\partial z^2} = \bar{a}^2\int\frac{\partial m_b'}{\partial t}dq \approx i\bar{a}^3k_\ell\int m_b' dq \ . \qquad (6.3)$$

Now suppose S_c is constant and note that for standing waves,

$$\frac{\partial^2 p'}{\partial z^2} \approx -k_\ell^2 p'$$

$$\frac{\partial^2 p'}{\partial t^2} \approx -\bar{a}^2 k^2 p' \; ,$$

so Eq. 6.3 gives

$$(k^2-k_\ell^2)\hat{p} \approx - \frac{i\bar{a}k_\ell}{S_c} \int \hat{m}_b dq \; .$$

Now multiply by p' and integrate over the chamber to find

$$(k^2-k_\ell^2)E_\ell^2 \approx \frac{-i\bar{a}k_\ell}{S_c} \int_0^L \hat{p}_\ell \int \hat{m}_b dqdz \; ,$$

which is exactly that exhibited in Eq. 5.30 for S_c constant.

The point is that this term involving $\hat{m}_b$ is a source term associated with turning of the flow at the boundary; it should therefore also appear in the three-dimensional case. However, it should be present only when there is a component of the acoustic velocity parallel to the surface. To see what should be used in the general three-dimensional problem, it is helpful to contrast the two cases which have arisen in the one-dimensional problem. The contributions to $(k^2-k_\ell^2)E_\ell^2$, per unit area, for the direct coupling between the acoustics field and combustion are

no acoustic velocity parallel to the surface	$-i\bar{\rho}\,\bar{a}k_\ell\hat{p}_\ell\hat{u}-i\frac{k_\ell}{\bar{a}}\bar{u}\hat{p}_\ell^2$,	(6.5a)[7]
acoustic velocity only parallel to the surface	$-i\bar{a}k_\ell\hat{p}_\ell\hat{m}_b-i\frac{k_\ell}{\bar{a}}R\hat{p}_\ell\Delta\hat{T}\bar{m}_b$.	(6.5b)

It should be recalled that the acoustic velocity referred to here is that of the classical unperturbed acoustic modes. At the surface itself, the normal component is always zero for those motions; the $\hat{u}$ in Eq. 6.5a and $\hat{m}_b$ in Eq. 6.5b are the small fluctuations normal to the surface due to coupling between the acoustic field and combustion. Term (4) has also been included in Eq. 6.5b since it is also, obviously, associated with coupling at the boundary; it too comes from the source term $(a^2 + R\Delta T_o) \int m_b dq$.

[7] The minus sign arises when the first term in Eq. 5.30 is written out; e.g., $[\hat{u}\hat{p}_\ell S_{be}]_0^L = (S_{be}\hat{p}_\ell\hat{u})_L - (S_{be}\hat{p}\hat{u})_o$. At z = 0, $\hat{u}$ is positive for flow outward from the surface, which is the convention followed in Eq. 6.5a and 6.5b.

The question which must be faced here concerns the correct form of the coupling terms when the unperturbed acoustic velocity in the vicinity of the surface is neither purely normal to the surface [case (Eq. 6.5a)] nor purely parallel to the surface [case (Eq. 6.5b)]. Several different arguments have been examined, but it appears that the desired result can best be obtained in the following way. By definition, the fluctuation of mass flux, $m = \rho u$, is

$$\frac{\hat{m}_b}{\bar{m}_b} = \frac{\hat{u}}{\bar{u}_b} + \frac{\hat{\rho}}{\bar{\rho}_b} \tag{6.6}$$

so the velocity fluctuation is

$$\hat{u} = \frac{\bar{u}_b}{\bar{m}_b}\hat{m}_b - \bar{u}_b\frac{\hat{\rho}}{\bar{\rho}} = \frac{1}{\bar{\rho}}\left(\hat{m}_b - \bar{m}_b\frac{\hat{\rho}}{\bar{\rho}}\right) . \tag{6.7}$$

By use of the perfect gas law, the density fluctuation--which here is at the edge of the combustion zone--is

$$\frac{\hat{\rho}}{\bar{\rho}} = \frac{\hat{p}}{\bar{p}} - \frac{\hat{T}}{\bar{T}} . \tag{6.8}$$

The temperature fluctuation can be broken, as earlier, into two parts: that associated with the wave motion outside the combustion zone--the isentropic fluctuation--and the nonisentropic part:

$$\frac{\hat{T}}{\bar{T}} = \left(\frac{\hat{T}}{\bar{T}}\right)_{is.} + \frac{\Delta\hat{T}}{\bar{T}} = \frac{\gamma-1}{\gamma}\frac{\hat{p}}{\bar{p}} + \frac{\Delta\hat{\hat{T}}}{\bar{T}} . \tag{6.9}$$

Thus, the density fluctuation is

$$\frac{\hat{\rho}}{\bar{\rho}} = \frac{1}{\gamma}\frac{\hat{p}}{\bar{p}} - \frac{\Delta\hat{T}}{\bar{T}} , \tag{6.10}$$

and Eq. 6.7 becomes

$$\hat{u} = \frac{1}{\bar{\rho}}\hat{m}_b - \bar{u}_b\frac{1}{\gamma}\frac{\hat{p}}{\bar{p}} + \frac{\bar{m}_b}{\bar{\rho}\bar{T}}\Delta\hat{T} . \tag{6.11}$$

The last result is, of course, always true, irrespective of the character of the acoustic mode. Now substitute this into Eq. 6.5a to find

$$-i\bar{\rho}\bar{a}k_\ell\hat{p}_\ell\left[\frac{\hat{m}_b}{\bar{\rho}} - \frac{\bar{u}_b}{\gamma\bar{p}}\hat{p}_\ell + \frac{\bar{m}_b}{\bar{\rho}\bar{T}}\Delta\dot{T}\right] - i\frac{k_\ell}{\bar{a}}\bar{u}\hat{p}_\ell^2 .$$

The convection terms cancel, and the result is

no acoustic velocity parallel to the surface	$-i\bar{a}k_\ell \hat{p}_\ell \hat{m}_b - i\gamma \frac{k_\ell}{\bar{a}} R\hat{p}_\ell \Delta\hat{T}\bar{m}_b$	(6.12)

which should be compared with Eq. 6.5b. Thus, the only difference between the two cases is the factor γ in the nonisentropic term in Eq. 6.12.

Now for the general case, the acoustic velocity of the unperturbed acoustic field near the surface is partly parallel and partly normal. The problem of representing the surface response in this case is discussed in Appendix A, where it is shown that Eq. 6.5b and 6.12 should be weighted respectively by factors $\delta_\parallel$, $\delta_\perp$, such that $\delta_\parallel + \delta_\perp = 1$; in the limit of no parallel velocity, $\delta_\parallel = 0$, $\delta_\perp = 1$, while if the acoustic velocity is entirely parallel, $\delta_\parallel = 1$, $\delta_\perp = 0$. These weighting factors are related to the mode structure as shown in Appendix A. The discussion there breaks essentially into two pieces: a detailed examination of a general acoustic mode in the vicinity of a rigid surface; and a "proof" for the representation of the coupling in the general case (Eq. A-15).

The conclusion of the above argument is that the missing terms ① and ④ in Eq. 5.39 are accounted for and combine with the surface combustion terms already present, in the surface integral

$$-i\bar{a}k_N \oint\!\!\oint \hat{p}_N [\hat{m}_b + \frac{R\bar{m}_b}{\bar{a}^2} \Delta\hat{T}(\delta_\parallel + \gamma\delta_\perp)] dS \quad . \tag{6.13}$$

The remaining boundary terms in Eq. 5.30 will be incorporated in the three-dimensional problem in a similar way.

2 . This term represents an energy exchange associated with the acquisition, by the average inflow, of the local acoustical energy. Again it appears as a source term in the one-dimensional problem and is present only if there is a component of acoustic velocity parallel to the surface. Hence, the corresponding term for the three-dimensional problem is

$$i\frac{k_N}{\bar{\rho}\bar{a}} \oint\!\!\oint [\hat{p}_N^2 + \frac{1}{k_N^2} (\nabla\hat{p}_N)^2] \delta_\parallel \bar{m}_b dS \quad . \tag{6.14}$$

③. This term represents an energy loss associated with turning of the average flow of particles into the chamber from the boundary. The corresponding term in three dimensions is evidently

$$\int_0^L \hat{u}_p \frac{d\hat{p}_\ell}{dz} \int \bar{m}_b^{(p)} dq\,dz \rightarrow \oint\!\!\oint \hat{\vec{u}}_p \cdot \nabla\hat{p}_N \bar{m}_b^{(p)} \delta_\parallel dS \quad . \tag{6.15}$$

④. This term involves fluctuations of the inflow or outflow itself, parallel to the boundary. It can therefore be taken over directly to the three-dimensional problem. At the burning surface, both $\bar{\mu}$ and $\hat{\mu}$ may reasonably be taken to be zero--i.e., the flow is assumed to depart the surface in the normal direction.

All of the preceding arguments lead to

$$\begin{aligned}
(k^2-k_N^2)E_N^2 = & \left\{-i\bar{a}k_N \oint\!\!\oint \hat{p}_N [\hat{m}_b + \frac{R\bar{m}_b}{\bar{a}^2} \Delta\hat{T}(\delta_\parallel+\gamma\delta_\perp)]dS\right\} \\
& + \left\{i\frac{k_N}{\bar{\rho}\bar{a}} \oint\!\!\oint [\hat{p}_N^2 + \frac{1}{k_N^2}(\nabla\hat{p}_N)^2]\delta_\parallel \bar{m}_b dS - i\frac{k_N}{\bar{a}} \int \hat{p}_N^2 \nabla\cdot\bar{\vec{u}}\,dV\right\} \\
& + \left\{\oint\!\!\oint \hat{\vec{u}}_p \cdot \nabla\hat{p}_N \delta_\parallel \bar{m}_b dS - \int \hat{\vec{F}} \cdot \nabla\hat{p}_N dV\right\} \\
& + \left\{-i\frac{k_N}{\bar{a}}\frac{R}{C_V}\int \hat{p}_N[\hat{e}_p\bar{w}_p + \bar{e}_p\hat{w}_p + (\hat{Q}+\hat{Q}_p)]dV\right. \\
& \qquad \left. + i\frac{k_N\gamma}{\bar{a}}\int \hat{p}_N^2 \bar{w}_p dV + \int \bar{w}_p(\hat{\vec{u}}-\hat{\vec{u}}_p)\cdot\nabla\hat{p}_N dV\right\} \\
& - \oint\!\!\oint \left[\hat{\vec{u}}_\parallel \bar{m}_b + \hat{\vec{u}}_{p\parallel}\bar{m}_b^{(p)}\right]\cdot \nabla\hat{p}_N dS \quad .
\end{aligned} \tag{6.16}$$

This is a new result; in later sections it will be applied to some specific problems. It should be noted first that because of the way in which Eq. 6.16 has been constructed, it reduces exactly to the modified one-dimensional result in Eq. 5.30. Also, it should be recalled that some of the boundary terms have been combined and written in a different way. The modified one-dimensional result corresponding exactly to Eq. 6.16--i.e., if Eq. 6.2 is used in place of Eq. 6.5--is:

$$(k^2-k_\ell^2)E_{1\ell}^2 = \left\{i\bar{a}k_\ell\left[\left(\hat{p}_\ell\hat{m}_b + \gamma\frac{R\bar{m}_b}{\bar{a}^2}\Delta\hat{T}\,\hat{p}_\ell\right)S_{be}\right]_0^L\right.$$

$$-i\bar{a}k_\ell \int_0^L \hat{p}_\ell [\int \hat{m}_b dq + \frac{R}{\bar{a}^2} \int \Delta \hat{T} \bar{m}_b dq] dS \Big\}$$

$$+ \Big\{ i \frac{k_\ell}{\bar{\rho}\bar{a}} \int \left[\hat{p}_\ell^2 + \frac{1}{k_\ell^2} \left(\frac{d\hat{p}_\ell}{dz} \right)^2 \right] \int \bar{m}_b dq dz$$

$$-i \frac{k_\ell}{\bar{a}} \int_0^L \hat{p}_\ell^2 \frac{1}{S_c} \frac{d}{dz} (\bar{u} S_c) S_c dz$$

$$+ \Big\{ \int_0^L \hat{u}_p \frac{d\hat{p}_\ell}{dz} \int \bar{m}_b^{(p)} dq dz - \int_0^L \hat{F} \frac{d\hat{p}_\ell}{dz} S_c dz \Big\}$$

$$+ \Big\{ -i \frac{k_\ell}{\bar{a}} \frac{R}{C_V} \int_0^L \hat{p}_\ell [\hat{e}_p \bar{w}_p + \bar{e}_p \hat{w}_p + (\hat{Q} + \hat{Q}_p)] S_c dz + i \frac{k_\ell \gamma}{\bar{\rho}\bar{a}} \int_0^L \hat{p}_\ell^2 \bar{w}_p S_c dz$$

$$+ \int_0^L (\hat{u} - \hat{u}_p) \bar{w}_p \frac{d\hat{p}_\ell}{dz} S_c dz \Big\} - \int_0^L \hat{\mu} \frac{d\hat{p}_\ell}{dz} S_c dz \quad . \qquad (6.17)$$

In summary, it is perhaps helpful to emphasize what has been accomplished. What is eventually required is the formula for the wave number of three-dimensional oscillations. But the purely three-dimensional formulation, for inviscid flow, fails to contain certain contributions arising essentially from the flow in a boundary layer. Strictly, those contributions can be found only by actually solving the boundary layer problems. However, the one-dimensional formulation affords a means of determining approximately the desired results. The accuracy of the results cannot be established using the procedure followed here.

The idea is usefully illustrated by a more familiar example, the attenuation of waves due to viscous forces in the acoustic boundary layer on the wall of a tube. This is usually found by computing the energy dissipation in the (linearized) acoustic boundary layer. For a standing wave in a circular tube of diameter D, the contribution to α is

$$\alpha = -\frac{\sqrt{2}}{D} \sqrt{\nu\omega} \left(1 + \frac{\gamma - 1}{\sqrt{Pr}}\right) \qquad (6.18)$$

where ν is the kinematic viscosity and Pr is the Prandtl number. This result is based on the assumption that the boundary layer is locally one-dimensional, with the velocity at the outer edge equal to the total acoustic velocity.

On the other hand, there is a displacement effect of the boundary layer, which can be taken as the basis of a different calculation which has been worked out by Chester. Because the fluctuation of velocity varies along the axis of the tube, $\partial u/\partial z \neq 0$, preservation of continuity in the boundary layer requires that the velocity v, normal to the axis, at the edge of the boundary layer, be non-zero. If the acoustic equations are written in one-dimensional form for the cross-section of the tube bounded by the edge of the boundary layer, then one has Eq. 2.1-2.4 except that there are now no particles, no combustion of any sort, and the cross-sectional area is fixed. The only difference from the usual one-dimensional equations is a source term in the continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = \frac{1}{S_c}\int \rho v_b dq = \frac{q}{S_c}\rho v_b \ . \tag{6.19}$$

There is, of course, no average flow inward, so v_b has only a fluctuating part. Chester has computed v_b by analyzing the boundary layer flow, and the linearized form of Eq. 6.19 is

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\frac{\partial u'}{\partial z} = \bar{\rho}\,\frac{4}{D}\sqrt{\frac{\bar{\nu}}{\pi}}\left(1 + \frac{\gamma - 1}{\sqrt{Pr}}\right)\int_0^\infty \frac{\partial}{\partial z} u(z, t-\xi)\frac{d\xi}{\sqrt{\xi}} \equiv w \tag{6.20}$$

while the momentum and energy equations are simply

$$\bar{\rho}\frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial z} = 0 \tag{6.21}$$

$$\bar{\rho}\frac{\partial T'}{\partial t} + \frac{\bar{p}}{C_v}\frac{\partial u'}{\partial z} = 0 \tag{6.22}$$

It follows that the wave equation for the pressure fluctuation is

$$\frac{\partial^2 p'}{\partial z^2} - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = -\frac{1}{\gamma}\frac{\partial w}{\partial t} \ . \tag{6.23}$$

For harmonic motions having complex wave number k, the procedure used in Section 5 then produces the formula for a standing wave in a tube,

$$k^2 = k_\ell^2 - \frac{i\omega}{\gamma}\frac{\int_0^L w\hat{p}_\ell dz}{\int_0^L \hat{p}_\ell^2 dz} = k_\ell^2 - i\frac{2\omega}{\gamma L}\int_0^L w\hat{p}_\ell dz \ . \tag{6.24}$$

The imaginary part gives the formula for α,

$$\alpha = \frac{\bar{a}^2}{\gamma L}\int_0^L w^{(r)}\hat{p}_\ell dz \ . \tag{6.25}$$

Now for a standing wave,

$$\hat{u}_\ell = -\frac{ik_\ell}{\rho a} e^{i\omega_\ell t} \sin(k_\ell z) \; .$$

$$\frac{\partial u}{\partial z} = -i\frac{\omega_\ell}{\rho a^2} e^{i\omega_\ell t} \cos(k_\ell z)$$

and

$$w = \hat{w}e^{i\omega_\ell t} = -\frac{4\rho}{D}\sqrt{\frac{\nu}{\pi}}\left(1 + \frac{\gamma - 1}{\sqrt{Pr}}\right)\left(\frac{i\omega_\ell}{\rho a^2}\right)\cos(k_\ell z)e^{i\omega_\ell t}\int_0^\infty e^{-i\omega\xi}\frac{d\xi}{\sqrt{\xi}} \; ,$$

of which the real part of $\hat{w}$ is

$$\hat{w}^{(r)} = -\frac{4\omega_\ell}{a^2 D}\sqrt{\frac{\nu}{\pi}}\left(1 + \frac{\gamma - 1}{\sqrt{Pr}}\right)\left(\sqrt{\frac{\pi}{2\omega_\ell}}\right)\cos(k_\ell z) \; .$$

Since $\hat{p}_\ell = \cos(k_\ell z)$, Eq. 6.18 gives

$$\alpha = -\frac{2/\gamma}{D}\sqrt{\nu\omega_\ell}\left(1 + \frac{\gamma - 1}{\sqrt{Pr}}\right) \; . \qquad (6.26)$$

This differs from Eq. 6.18 only in the replacement of $\sqrt{2}$ by $2/\gamma$. For air ($\gamma = 1.4$) the error is very small indeed--much less than can be measured, in fact.

Although a boundary layer analysis was used to evaluate v_b in Eq. 6.19, this example does show how a one-dimensional formulation with sources at the lateral boundary may reproduce a result found wholly by a boundary layer analysis, i.e., that give Eq. 6.18.

7. A FEW REMARKS ON INTERPRETATION

In Section 5, Eq. 5.6, the growth constant α was related to the rate of change of the time-averaged total energy in the chamber. It is helpful to interpret the formulas in Section 6 in that way. The real and imaginary parts of Eq. 6.16 are

$$
\begin{aligned}
\left[\left(\frac{\omega}{\bar{a}}\right)^2 - k_N^2\right]E_N^2 = &\left\{\bar{a}k_N \oiint \hat{p}_N\left[\hat{m}_b^{(i)} + \frac{R\bar{m}_b}{\bar{a}^2}\Delta\hat{T}^{(i)}(\delta_\parallel + \gamma\delta_\perp)\right]dS\right\} \\
&+\left\{\oiint \hat{\vec{u}}_p^{(r)}\cdot\nabla\hat{p}_N\delta_\parallel\bar{m}_b^{(p)}dS - \int \hat{F}^{(r)}\cdot\nabla\hat{p}_N dV\right\} \\
&+\left\{\frac{k_N}{\bar{a}}\frac{R}{C_v}\int\hat{p}_N\left[\hat{e}_p^{(i)}\bar{w}_p + \bar{e}_p\hat{w}_p^{(r)} + (\hat{Q}^{(i)} + \hat{Q}_p^{(i)})\right]dV \right. \\
&\left. + \int \bar{w}_p(\hat{\vec{u}} - \hat{\vec{u}}_p)^{(r)}\cdot\nabla\hat{p}_N dV\right\} \\
&- \oiint\left[\hat{\vec{u}}_\parallel^{(r)}\bar{m}_b + \hat{\vec{u}}_p^{(r)}\bar{m}_b^{(p)}\right]\cdot\nabla\hat{p}_N dS
\end{aligned}
\qquad (7.1)
$$

$$
\begin{aligned}
-2\alpha\frac{\omega}{\bar{a}^2}E_N^2 = &-\left\{\bar{a}k_N \oiint \hat{p}_N\left[\hat{m}_b^{(r)} + \frac{R\bar{m}_b}{\bar{a}^2}\Delta\hat{T}^{(r)}(\delta_\parallel + \gamma\delta_\perp)\right]dS\right\} \\
&+\left\{\frac{k_N}{\rho\bar{a}^2}\oiint\left[\hat{p}_N^2 + \frac{1}{k_N^2}(\nabla\hat{p}_N)^2\right]\delta_\parallel\bar{m}_b dS - \frac{k_N}{\bar{a}}\int\hat{p}_N^2\nabla\cdot\bar{\vec{u}}dV\right\} \\
&+\left\{\oiint\hat{\vec{u}}^{(i)}\cdot\nabla\hat{p}_N\delta_\parallel\bar{m}_b dS - \int\hat{F}^{(i)}\cdot\nabla\hat{p}_N dV\right\}
\end{aligned}
$$

$$+\left\{-\frac{k_N}{\bar{a}}\frac{R}{C_v}\int \hat{p}_N'[\hat{e}_p^{(r)}\bar{w}_p+\bar{e}_p\hat{w}_p^{(r)}+(\hat{Q}^{(r)}+\hat{Q}_p^{(r)})]\,dV\right.$$

$$\left.+\frac{k_N\gamma}{\bar{a}}\int \hat{p}_N^2\bar{w}_p\,dV+\int \bar{w}_p(\hat{\vec{u}}-\hat{\vec{u}}_p)^{(i)}\cdot\nabla\hat{p}_N\,dV\right\}$$

$$-\oiint [\hat{\vec{u}}_{\parallel}^{(i)}\bar{m}_b+\hat{\vec{u}}_{p\parallel}^{(i)}\bar{m}_b^{(p)}]\cdot\nabla\hat{p}_N\,dS \tag{7.2}$$

The fact that $\hat{\vec{u}}_N$ is pure imaginary (see Eq. 5.36) has been used in Eq. 7.1 and 7.2.

It is mainly Eq. 7.2 which is of practical interest, and especially the first five terms representing the coupling between wave motions and combustion, and the mean flow/acoustics interactions. In order to clarify the meaning of these terms, it is helpful to express α as the ratio of the rate of change of acoustic energy to the total acoustic energy in the chamber, as in Eq. 5.6. First note that E_N^2 is proportional to the total time-averaged acoustic energy in the chamber. The instantaneous time-averaged energy density, with the time factor dropped, is:

$$\bar{e}_N=\frac{1}{4}\left(\frac{\hat{p}_N^2}{\bar{\rho}\bar{a}^2}+\bar{\rho}\hat{\vec{u}}_N^2\right)=P_N+K_N \tag{7.3}$$

where P_N and K_N are the time-averaged densities of potential and kinetic energy. Hence, the total time-averaged energy in the chamber is

$$\mathcal{E}_N=\frac{1}{4}\int\left[\frac{\hat{p}_N^2}{\bar{\rho}\bar{a}^2}+\bar{\rho}\hat{\vec{u}}_N^2\right]dV$$

$$=\frac{1}{4\bar{\rho}\bar{a}^2}\int\left[\hat{p}_N^2+\frac{1}{k_N^2}(\nabla\hat{p}_N)^2\right]dV=\int(P_N+K_N)\,dV\ . \tag{7.4}$$

Now by use of the wave equation for $\hat{p}_N$,

$$\int(\nabla\hat{p}_N)^2dV=\int\nabla\cdot(\hat{p}_N\nabla\hat{p}_N)dV-\int\hat{p}_N\nabla^2\hat{p}_N\,dV$$

$$=\oiint\hat{p}_N\nabla\hat{p}_N\cdot\hat{n}\,dS+k_N^2\int\hat{p}_N^2dV\ .$$

The first integral vanishes according to the boundary condition on $\hat{p}_N$, so that the normalization integral E_N^2 is related to the time-averaged total energy in the chamber according to

$$\int\hat{p}_N^2dV=E_N^2=2\bar{\rho}\bar{a}^2\mathcal{E}_N \tag{7.5}$$

The formulas for the one-dimensional formulation are entirely analogous, giving, for example,

$$\int_0^L \hat{p}_\ell^2 S_c dz = E_{1\ell}^2 = 2\bar{\rho}\bar{a}^2 \mathcal{E}_\ell \tag{7.6}$$

with the time-averaged energy density

$$\bar{e}_\ell = \tfrac{1}{4}\left(\frac{\hat{p}_\ell^{\,2}}{\bar{\rho}\bar{a}^2} + \bar{\rho}\,\hat{u}_\ell^{\,2}\right) \tag{7.7}$$

By taking the imaginary part of Eq. 6.17, one finds for the growth constant in the one-dimensional problem

$$-2\alpha \frac{\omega E_{1\ell}^2}{\bar{a}^2} =$$

$$\bar{a}k_\ell\left\{\left[\left(\hat{p}_\ell \hat{m}_b^{(r)} + \gamma \frac{R\bar{m}_b}{\bar{a}^2}\Delta\hat{T}^{(r)}\hat{p}_\ell\right]_0^L - \int_0^L \hat{p}_\ell\left[\int \hat{m}_b^{(r)} dq + \frac{R}{\bar{a}^2}\int \Delta\hat{T}^{(r)}\bar{m}_b dq\right] dz\right\}$$

$$+\frac{k_\ell}{\bar{a}}\left\{\frac{1}{\bar{\rho}}\int_0^L\left[\hat{p}_\ell^2 + \frac{1}{k_\ell^2}\left(\frac{d\hat{p}_\ell}{dz}\right)^2\right]\int \bar{m}_b dq dz\right.$$

$$-\int_0^L \hat{p}_\ell^{\,2}\,\frac{1}{S_c}\frac{d}{dz}(\bar{u}S_c)S_c dz$$

$$+\left\{\int_0^L \hat{u}_p^{(i)}\frac{d\hat{p}_\ell}{dz}\int \bar{m}_b^{(p)} dq dz - \int_0^L \hat{F}^{(i)}\frac{d\hat{p}_\ell}{dz}S_c dz\right\}$$

$$+\frac{k_\ell}{\bar{a}}\left\{-\frac{R}{C_v}\int_0^L \hat{p}_\ell[\hat{e}_p^{(r)}\bar{w}_p + \bar{e}_p\hat{w}_p^{(r)} + (Q^{(r)} + Q_p^{(r)})]S_c dz\right.$$

$$\left.+\gamma\int_0^L \hat{p}_\ell^2 \bar{w}_p S_c dz + \int_0^L \bar{w}_p\left(\hat{u}^{(r)} - \hat{u}_p^{(r)}\right)\frac{d\hat{p}_\ell}{dz}S_c dz\right\}$$

$$.-\int_0^L [\hat{u}_{\parallel}^{(i)}\int \bar{m}_b dq + \hat{u}_{p\parallel}^{(i)}\int \bar{m}_b^{(p)} dq]\frac{d\hat{p}_\ell}{dz}dz \tag{7.8}$$

To see what the various pieces mean, it is best to replace E_N^2 by use of Eq. 7.4, E_{1e}^2 by Eq. 7.6, and use the fact that -2α is the fractional rate of change of time-averaged energy in the chamber (cf., Eq. 5.6):

$$2\alpha = \frac{1}{\mathcal{E}}\frac{d\mathcal{E}}{dt} \quad . \tag{7.9}$$

For $-2\alpha\xi_N$ and $-2\alpha\xi_\ell$, one finds the sum of terms shown in the following table.

TABLE 1.

	$-2\alpha\mathcal{E}_\ell$	$-2\alpha\mathcal{E}_N$
①	$\frac{1}{2\bar\rho}\left[\hat p_\ell\left(\hat m_b^{(r)}+\gamma\frac{R\bar m_b}{\bar a^2}\Delta\hat T^{(r)}\right)S_{be}\right]_0^L$	$-\frac{1}{2\bar\rho}\oiint \hat p_N\left[\hat m_b^{(r)}+\gamma\frac{R\bar m_b}{\bar a^2}\Delta\hat T^{(r)}\right]\delta_\perp dS$
②	$-\frac{1}{2\bar\rho}\int_0^L \hat p_\ell\left[\int \hat m_b^{(r)}dq+\frac{R}{\bar a^2}\int\Delta\hat T^{(r)}\bar m_b\,dq\right]dz$	$-\frac{1}{2\bar\rho}\oiint \hat p_N\left[\hat m_b^{(r)}+\frac{R\bar m_b}{\bar a^2}\Delta\hat T^{(r)}\right]\delta_\parallel dS$
③	$\frac{2}{\bar\rho}\int_0^L(P_\ell+K_\ell)\int\bar m_b\,dq\,dz$	$\frac{2}{\bar\rho}\oiint(P_N+K_N)\delta_\parallel\bar m_b\,dS$
④	$-\frac{1}{2}\int_0^L\hat p_\ell\left[\frac{\hat p_\ell}{\bar\rho\bar a^2}\frac{1}{S_c}\frac{d}{dz}(\bar u S_c)\right]S_c\,dz$	$-\frac{1}{2}\int\hat p_N\left(\frac{\hat p_N}{\bar\rho\bar a^2}\nabla\cdot\bar{\vec u}\right)dV$
⑤	———	———
⑥	$\frac{1}{2\bar\rho\bar a k_\ell}\int_0^L\hat u_p^{(i)}\frac{d\hat p_\ell}{dz}\int\bar m_b^{(p)}dq\,dz$	$\frac{1}{2\bar\rho\bar a k_N}\oiint\hat{\vec u}_p^{(i)}\cdot\nabla\hat p_N\delta_\parallel\bar m_b\,dS$
⑦	$\frac{-1}{2\bar\rho\bar a k_\ell}\int_0^L\hat F^{(i)}\frac{d\hat p_\ell}{dz}S_c\,dz$	$\frac{-1}{2\bar\rho\bar a k_N}\int\hat{\vec F}^{(i)}\cdot\nabla\hat p_N\,dV$
⑧	$\frac{-1}{2\bar\rho\bar a^2}\frac{R}{C_v}\int_0^L\hat p_\ell[\hat e_p^{(r)}\bar w_p+\bar e_p\hat w_p^{(r)}+(Q^{(r)}+Q_p^{(r)})]S_c\,dz$	$\frac{-1}{2\bar\rho\bar a^2}\frac{R}{C_v}\int\hat p_N[\hat e_p^{(r)}\bar w_p+\bar e_p\hat w_p^{(r)}+(Q^{(r)}+Q_p^{(r)})]dV$

TABLE 1. (Cont'd.)

(9)	$2\gamma \int_0^L P_\ell \bar{w}_p S_c dz$	$2\gamma \int P_N \bar{w}_p dV$
(10)	$\frac{1}{2\bar{\rho}\bar{a}k_\ell} \int_0^L \bar{w}_p(\hat{u}^{(r)} - \hat{u}_p^{(r)}) \frac{d\hat{p}_\ell}{dz} S_c dz$	$\frac{1}{2\bar{\rho}\bar{a}k_N} \int \bar{w}_p(\hat{\vec{u}}^{(r)} - \hat{\vec{u}}_p^{(r)}) \cdot \nabla\hat{p}_N dV$
(11)	$\frac{-1}{2\bar{\rho}\bar{a}k_\ell} \int_0^L [\hat{u}_\parallel^{(i)} \int \bar{m}_b dq + u_{p\parallel}^{(i)} \int \bar{m}_b^{(p)} dq] dz$	$\frac{-1}{2\bar{\rho}\bar{a}k_N} \oint [\hat{\vec{u}}_\parallel^{(i)} \bar{m}_b + \hat{\vec{u}}_{p\parallel}^{(i)} \bar{m}_b^{(p)}] \cdot \nabla\hat{p}_N dS$

A factor ½ appears explicitly in terms (1) (2) (4) (6) (8) (10), and (11) because of the time averaging. For example, consider the first term in (1). If Eq. 6.12 is used to eliminate $\hat{m}_b^{(r)}$ in favor of $\hat{u}_b^{(r)}$ and $\Delta T^{(r)}$, a term

$$(1/2)[\hat{p}_\ell \ \hat{u}^{(r)} \ S_{be}]_C^L$$

arises. The product $\hat{p}_\ell \hat{u}^{(r)}$ is the rate at which work is done per unit area of surface, and the ½ is the average of cos ωt cos (ωt + φ) where φ is the phase difference between p_ℓ' and u' at the surface. The remaining pieces of the groups (1) and (2) are also rates of doing work, associated essentially with fluctuations of density. Similarly, all other terms containing the ½ are various kinds of time-averaged rates of doing work.

The term (4) shows that there is a time-averaged energy loss at the surface, equal (per unit area) to

$$\frac{2}{\bar{\rho}} (P_N + K_N) \bar{m}_b \delta_\parallel \equiv 2\left(\frac{\hat{p}_N^2}{4\bar{\rho}\bar{a}^2} + \bar{\rho}\frac{\hat{\vec{u}}_N^2}{4}\right)\bar{u}_b \delta_\parallel$$

The factor $\delta_\parallel$, as argued above, assures that this term arises only if there are acoustical motions parallel to the surface. As emphasized already, the process represented by this term is the inelastic acceleration of the gases leaving the surface normally, so that they acquire the acoustical motions parallel to the surface. The time-averaged energy possessed by the gas is ultimately $(P_N + K_N)$. The factor 2 arises here because the amount of work done in such a process is twice the mechanical energy finally gained. Half of the work done appears as the mechanical energy, and half is dissipated as heat. The total work is of course done by the waves already in the chamber and hence is obviously an energy loss.

Similarly, the term (10) represents an inelastic process; the gases evolved in the combustion of particulate matter, at the rate $\bar{w}_p$, must acquire the local potential energy, P_N. Once again the work done is twice the energy finally possessed by the gas.

Finally, term ④ should be interpreted as a time-averaged work per unit volume and time. For the acoustic field, $\hat{p}_N/\bar{p} = (\hat{\rho}_N/\bar{\rho})$, so $\hat{p}_N/\bar{\rho}\bar{a}^2 = \hat{\rho}_N/\bar{\rho}$. Hence, the factor in parenthesis is just $\hat{\rho}_N(\nabla\cdot\bar{u})/\bar{\rho}$ and represents a rate of change of volume due to the density fluctuation and dilatation by the mean flow. Thus, the whole term does indeed represent a rate of work by the pressure fluctuation.

In summary, it is possible to provide a physical interpretation for all the terms in the growth constant, although their appearance is not obvious in all cases.

8. AN ALTERNATE METHOD OF COMPUTING THE GROWTH RATE

In Ref. 2, Cantrell and Hart have proposed a means of computing the growth rate for acoustic waves in a rocket chamber. Part of their results are identical with part of the results found here, but their formula for the growth constant α is incomplete. Moreover, if their procedure is followed exactly, it is not possible to determine, among others, the influence of residual combustion. Their technique is therefore not applicable to liquid rockets, for example, whereas the present analysis is valid for any combustion chamber.

It is the purpose of this section to summarize the calculation proposed in Ref. 2; to identify the respects in which it differs from the present work; and finally to show how the general idea proposed there can, when correctly developed; produce results in accord with those found here. However, it should be emphasized that since the analysis of Ref. 3 is for an inviscid, three-dimensional flow field, it does not and cannot contain the boundary processes, discussed in Section 6, which have been incorporated in the modified three-dimensional problem.

8.1. SUMMARY OF THE ANALYSIS OF REFERENCE 2

The main idea proposed by Cantrell and Hart is that the stability of waves in a chamber can be studied by examining the time rate of change of energy in the chamber. This is computed by integrating the energy equation over the volume of the chamber, and making use also of the equations of conservation of mass and momentum. Residual combustion and particulate matter were not accounted for in Ref. 2, but they will be included here. Then the inviscid equations of motion are Eq. 4.1, 4.4, and 4.5. In order to clarify the correspondence between this work and Ref. 2, Eq. 4.4 and 4.5 are slightly modified to introduce the enthalpy $h = \int c_p dT$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{m}) = w_p \tag{8.1}$$

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(h + \frac{u^2}{2}\right) = \frac{1}{\rho}\left(\vec{F} - \vec{\sigma}\right) \tag{8.2}$$

$$\frac{\partial}{\partial t}\, \rho\left(C_v T + \frac{u^2}{2}\right) = -\nabla \cdot \left[\vec{m}\left(h + \frac{u^2}{2}\right)\right] + \vec{u}_p \cdot \vec{F} + e_{po} w_p + (Q + Q_p) \tag{8.3}$$

where $\vec{m} = \rho\vec{u}$ is the vector mass flux of gases. Equations 8.1-8.3 are equations (2)-(4) of Ref. 2 except for the additional terms due to residual combustion and the presence of particles.

Now the left-hand side of Eq. 8.3 is the time rate of change of the total energy density for the flow field: it includes not only the energy of acoustic waves but also the energy associated with the mean flow. As noted below, this is a crucial observation. Thus, integration of Eq. 8.3 over the mean volume of the chamber gives the time rate of change of total energy within the chamber:

$$\frac{\partial}{\partial t} \int \rho\left(C_v T + \frac{u^2}{2}\right)dV = -\int \nabla\cdot\left[\vec{m}\left(h + \frac{u^2}{2}\right)\right]dV + \int \left[\vec{u}_p\cdot\vec{F} + e_{po}w_p + (Q+Q_p)\right]dV. \qquad (8.4)$$

It is necessary to extract from this equation an equation for the balance of acoustic energy only.

To do this, the variables are as usual expressed as sums of mean values and fluctuations. Cantrell and Hart show that the left-hand side and the first volume integral on the right-hand side can be rewritten and partially combined to yield their equation (16). Let ⟨ ⟩ denote time averages, and one has from Eq. 8.4 with their result written using the notation defined here:

$$\left\langle \frac{d}{dt}\int dV\left[\frac{p'^2}{2\bar{\rho}\bar{a}^2} + \tfrac{1}{2}\bar{\rho}\vec{u}'^2 + \frac{(\vec{\bar{u}}\cdot\vec{u}')}{\bar{a}^2}p'\right]\right\rangle =$$

$$= -\left\langle \oiint dS\hat{n}\cdot\left[\vec{u}'p' + \frac{\vec{\bar{u}}p'^2}{\bar{\rho}\bar{a}^2} + \bar{\rho}(\vec{\bar{u}}\cdot\vec{u}')\vec{u}' + \frac{p'}{\bar{a}^2}(\vec{\bar{u}}\cdot\vec{u}')\vec{\bar{u}}\right]\right\rangle$$

$$+ \left\langle \int\left[\vec{u}_p\cdot\vec{F} + e_{po}w_{po} + (Q+Q_p)\right]dV\right\rangle \qquad (8.5)$$

There are now two important points to be made: (1) the last term on the left-hand side of Eq. 8.5 and the last two terms in the surface integral on the right-hand side should be dropped under the circumstances for which the analysis is useful; (2) the last set of terms in Eq. 8.5 will not produce the influences of residual combustion and particulate matter found in the present work, for a reason given below.

As Cantrell and Hart have argued, the left-hand side might be interpreted as the expression for the time-averaged rate of change of total acoustic energy in the chamber. Hence, without the differentiation, it is the time-averaged acoustic energy, and the right-hand side represents the time-averaged rate of change of energy due to the mean flow/acoustics interactions, residual combustion, and particulate matter. Therefore, by the definition in Eq. 7.9 of the growth constant α,

$$-2\alpha = \frac{\langle \oiint d\vec{S}\hat{n}\cdot[p'\vec{u}' + \frac{\vec{\bar{u}}p'^2}{\bar{\rho}\bar{a}^2} + \bar{\rho}(\vec{\bar{u}}\cdot\vec{u}')\vec{u}' + \frac{p'}{\bar{\rho}\bar{a}^2}(\vec{\bar{u}}\cdot\vec{u}')\vec{\bar{u}}]\rangle}{\langle \int dV[\frac{1}{2}\bar{\rho}\vec{u}'^2 + \frac{p'^2}{2\bar{\rho}\bar{a}^2} + \frac{(\vec{\bar{u}}\cdot\vec{u}')}{\bar{a}^2}p']\rangle}$$

$$- \frac{\langle \int dV[\vec{u}_p\cdot\vec{F} + e_{po}w_p + (Q+Q_p)]\rangle}{\langle \int dV[\frac{1}{2}\bar{\rho}\vec{u}'^2 + \frac{p'^2}{2\bar{\rho}\bar{a}^2} + \frac{(\vec{\bar{u}}\cdot\vec{u}')}{\bar{a}^2}p']\rangle} \quad (8.6)$$

Unlike the calculations given in the preceding sections of this report, Eq. 8.6 is formally valid to any order of the mean flow Mach number. Consequently, p' and $\vec{u}'$ represent acoustic quantities which also may differ from the classical unperturbed values by terms of first and higher order in the Mach number. However, for practical purposes, it is certainly a convenience, quite likely adequate, and virtually a necessity if relatively simple results are obtained, to consider perturbations only to first order in the Mach number.

At the burning surface, the first approximation to $\vec{u}'$ (which is of course zero there for the unperturbed field) is of the order of the mean flow Mach number, as for example shown in the definitions in Eq. 9.1 and 9.2 for pressure coupling; $|A_b|$ and $|R_b|$ are around 1 - 7 or so. If the amplitude of the oscillation is taken to be of order unity, then $p'\vec{u}'$ is of order $\bar{M}_b$, as is $\vec{\bar{u}}p'^2/\bar{\rho}\bar{a}^2$ in Eq. 8.6. But $(\bar{u}\cdot\vec{u}')\vec{u}'$ and $(\bar{u}\cdot\vec{u}')\bar{u}$ are no greater than $\bar{M}_b^3$. Hence, if those terms are retained, higher order terms in p' and $\vec{u}'$ must also be retained. In order to find those, one must return to the inhomogeneous differential equations and solve them, a step which has never been taken in practice, and obviously must involve a great deal of effort. (Formulas for the first correction to the unperturbed values have been given in Ref. 3.) For the same reason, the term $(\vec{\bar{u}}\cdot\vec{u}')p'/\bar{a}^2$ should be dropped from the denominator of Eq. 9.6, since it is of order $\bar{M}_b$ and therefore adds a correction term to α of order $\bar{M}_b^2$. Thus, the first part of Eq. 9.6 is correctly

$$\frac{\langle \oiint d\vec{S}\hat{n}\cdot\left[p'\vec{u}' + \frac{\vec{\bar{u}}p'^2}{\bar{\rho}\bar{a}^2}\right]\rangle}{\langle \int dV\left[\frac{1}{2}\bar{\rho}\vec{u}'^2 + \frac{p'^2}{2\bar{\rho}\bar{a}^2}\right]\rangle} \quad (8.7)$$

The denominator is precisely $E_N^2/2\bar{\rho}\bar{a}^2$ and this result is identical to that found from the real part of the first term in Eq. 5.39.

It is obviously desirable to avoid computing the acoustic field to higher orders; numerical procedures are available for computing the classical field in any geometry, but not with perturbations due to the mean flow. This is the main reason why regions of high Mach number, such as exhaust nozzles and the vent of a T-burner, should be treated separately; they are then introduced in the acoustics problem as boundary conditions, represented by admittance functions.

Now consider the second part of Eq. 8.6. It is necessary to expand the quantities to second order, in whatever small parameters are present; for example,

$$\vec{F} = \vec{\bar{F}} + \vec{F}_1' + \vec{F}_2' + \ldots$$

The first order terms are associated with the unperturbed acoustic field. Consider the term $\vec{u}_p \cdot \vec{F}$

$$\vec{u}_p \cdot \vec{F} = (\vec{\bar{u}}_p + \vec{u}_{p1}' + \vec{u}_{p2}' + \ldots) \cdot (\vec{\bar{F}} + \vec{F}_1' + \vec{F}_2' + \ldots)$$

$$= \vec{\bar{u}}_p \cdot \vec{\bar{F}} + (\vec{\bar{u}}_p \cdot \vec{F}_1' + \vec{u}_{p1}' \cdot \vec{\bar{F}}) + (\vec{u}_{p1}' \cdot \vec{F}_1' + \vec{u}_{p2}' \cdot \vec{\bar{F}} + \vec{\bar{u}}_p \cdot \vec{F}_2') + \ldots$$

The first term is associated with the average flow and should be dropped. The time average of the first order terms is zero. In this way, the denominator of the ratio gives

$$\langle \int dV[(\vec{u}_{p1}' \cdot \vec{F}_1' + \vec{u}_{p2}' \cdot \vec{\bar{F}} + \vec{\bar{u}}_p \cdot \vec{F}_2') + (e_{po1}' w_{p1}' + e_{po2}' \bar{w}_p + \bar{e}_{po} w_{p2}') + (Q_2' + Q_{p2}')] \rangle \quad (8.8)$$

Not only are the contributions from particulate matter in disagreement with those found here, and used elsewhere, but the influence of heat release appears only in second order. The second result is clearly wrong; for example, such terms must arise to first order if the behavior of liquid and gas rockets is to be described correctly. The first order effect does appear in the result in Eq. 6.16 found here.

The question, then, is: why does the analysis of Ref. 2 produce the correct form for the mean flow/acoustics interactions (apart from the boundary terms) but fail to represent correctly the influence of particulate matter and residual combustion? The answer follows from the fact that the analysis is based on Eq. 8.4 for the time rate-of-change of the total energy in the chamber, rather than on the energy balance for the acoustic field alone, although the latter is apparently obtained eventually.

For by using the total energy, one cannot in any obvious way show exchanges of energy occurring totally within the system. Thus, for example, heat lost from the mean flow by combustion and added to the acoustic waves appears twice; once with a + and once with a −, thus cancelling. The debit and credit accounts are not shown in detail, but only their net result. A net result does appear at the boundary, as shown by the first term of Eq. 9.6.

8.2. THE EQUATION OF BALANCE FOR THE TOTAL ENERGY OF THE ACOUSTIC FIELD

The idea of Ref. 2, that the growth constant can be found from considerations of the total acoustic energy, can be developed to produce results in exact agreement with those found here. The starting point is the linearized equations in Eq. 4.10 and 4.11 for the three-dimensional acoustic field. Take the scalar product of $\vec{u}'$ with Eq. 4.10 and add the result to $p'/\bar{\rho}\bar{a}^2$ times Eq. 4.11 to find

$$\frac{\partial}{\partial t}\left(\bar{\rho}\,\frac{\vec{u}'^2}{2} + \frac{p'^2}{2\bar{\rho}\bar{a}^2}\right) + \nabla\cdot(\vec{u}'p') = -\bar{\rho}\,\vec{u}'\cdot(\vec{\bar{u}}\cdot\nabla\vec{u}' + \vec{u}'\cdot\nabla\vec{\bar{u}}) + \frac{p'}{\bar{\rho}\bar{a}^2}\,\vec{\bar{u}}\cdot\nabla p' + \vec{u}'\cdot(\vec{F}' - \vec{\sigma}') + p'P' \tag{8.9}$$

Now integrate over the mean volume of the chamber and convert the integral of $\nabla\cdot(\vec{u}'p')$ to a surface integral:

$$\frac{d\mathcal{E}}{dt} + \oiint dS\hat{n}\cdot(\vec{u}'p') = -\int dV\left[\bar{\rho}\,\vec{u}'\cdot(\vec{\bar{u}}\cdot\nabla\vec{u}' + \vec{u}'\cdot\nabla\vec{\bar{u}}) + \frac{p'}{\bar{\rho}\bar{a}^2}\,\vec{\bar{u}}\cdot\nabla p'\right] + \int dV\vec{u}'\cdot(\vec{F}' - \vec{\sigma}') + \int dVp'P' \tag{8.10}$$

This, of course, is valid only to first order in the average Mach number, and one can use unperturbed values for $\vec{u}'$ and p' on the right-hand side.

The terms in the first integral on the right-hand side can be combined in essentially the same way as those following Eq. 5.38 were treated. By use of some vector identities,

$$\vec{u}'\cdot(\vec{\bar{u}}\cdot\nabla\vec{u}' + \vec{u}'\cdot\nabla\vec{\bar{u}}) = \vec{u}'\cdot[\nabla(\vec{\bar{u}}\cdot\vec{u}') - \vec{u}'\times\nabla\times\vec{\bar{u}}'] = \vec{u}'\cdot\nabla(\vec{\bar{u}}\cdot\vec{u}')$$

$$= \nabla\cdot[(\vec{\bar{u}}\cdot\vec{u}')\vec{u}'] - (\vec{\bar{u}}\cdot\vec{u}')\nabla\cdot\vec{u}' \ .$$

Thus, the volume integral is

$$\int dV\left[\bar{\rho}\nabla\cdot\{(\vec{\bar{u}}\cdot\vec{u}')\vec{u}'\} - \bar{\rho}(\vec{\bar{u}}\cdot\vec{u}')\nabla\cdot\vec{u}' + \frac{p'}{\bar{\rho}\bar{a}^2}\,\vec{\bar{u}}\cdot\nabla p'\right]$$

$$= \oiint dS\hat{n}\cdot\vec{u}'\bar{\rho}\,\vec{\bar{u}}\cdot\vec{u}' + \int dV\left[-\bar{\rho}\,\vec{\bar{u}}\cdot(\vec{u}'\nabla\cdot\vec{u}) + \frac{p'^2}{\bar{\rho}\bar{a}^2}\,\vec{\bar{u}}\cdot\nabla p'\right] \ .$$

The integrand of the surface integral is of order $\bar{M}_b^3$ and hence must be dropped. For harmonic motions, the first term in the second integral can be written by use of the acoustic equations

$$\vec{u}' = \frac{1}{i\bar{\rho}\bar{a}k}\nabla p' \quad , \qquad \nabla\cdot\vec{u}' = -\frac{i\bar{a}k}{\gamma\bar{p}}p'$$

so

$$-\overline{\rho}\,\vec{\overline{u}}\cdot(\vec{u}'\nabla\cdot\vec{u}') = \vec{\overline{u}}\cdot\left(\frac{p'}{\gamma\overline{p}}\nabla p'\right) = \frac{p'}{\overline{\rho}\,\overline{a}^2}\,\vec{\overline{u}}\cdot\nabla p'$$

which equals the second term in the second integral, as demonstrated in the argument following Eq. 5.38. Thus, the volume integral reduces to

$$\int dV[\overline{\rho}\,\vec{u}'\cdot(\vec{\overline{u}}\cdot\nabla\vec{u}'+\vec{u}'\cdot\nabla\vec{\overline{u}})+\frac{p'}{\overline{\rho}\,\overline{a}^2}\vec{\overline{u}}\cdot\nabla p'] = \int dV\vec{\overline{u}}\cdot\nabla\left(\frac{p'^2}{\overline{\rho}\,\overline{a}^2}\right)$$

$$= \int dV\left[\nabla\cdot\left(\frac{\vec{\overline{u}}p'^2}{\overline{\rho}\,\overline{a}^2}\right)-\frac{p'^2}{\overline{\rho}\,\overline{a}^2}\nabla\cdot\vec{\overline{u}}\right]$$

$$= \oiint dS\hat{n}\cdot\vec{\overline{u}}\left(\frac{p'^2}{\overline{\rho}\,\overline{a}^2}\right)-\int dV\frac{p'^2}{\overline{\rho}\,\overline{a}^2}\nabla\cdot\vec{\overline{u}}$$

which is the result found just preceding Eq. 5.39.

Now to obtain this result in general, it is necessary to show that

$$-\int dV\overline{\rho}\,\vec{\overline{u}}\cdot(\vec{u}'\nabla\cdot\vec{u}') = \int dV\frac{p'}{\overline{\rho}\,\overline{a}^2}\,\vec{\overline{u}}\cdot\nabla p'$$

or

$$\int dV\vec{\overline{u}}\cdot\left[\frac{p'}{\overline{\rho}\,\overline{a}^2}\nabla p' + \vec{u}'\nabla\cdot\vec{u}'\right] = 0\ .$$

Again by use of the classical acoustics equations, this can be written

$$\int dV\vec{\overline{u}}\cdot\left[p'\frac{\partial\vec{u}'}{\partial t} - \vec{u}'\frac{\partial p'}{\partial t}\right] = 0$$

and by differentiating with respect to time,[8]

$$\int dV\vec{\overline{u}}\cdot\left[p'\frac{\partial^2\vec{u}'}{\partial t^2} - \vec{u}'\frac{\partial^2 p'}{\partial t^2}\right] = 0\ .$$

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[8] Because of this step, the end result in Eq. 8.11 has been shown to be true only to within a constant which can clearly be taken to be zero with no loss of generality.

But both $\vec{u}'$ and p' satisfy the unperturbed wave equation, since terms of order $\overline{M}_b$ must be neglected; hence

$$p' \frac{\partial^2 \vec{u}'}{\partial t^2} - \vec{u}' \cdot \frac{\partial^2 p'}{\partial t^2} \equiv -p' \bar{a}^2 \vec{u}' + \vec{u}' \bar{a}^2 p' = 0 .$$

Therefore, the desired combination is obtained, i.e.,

$$\int dV[-\bar{\rho}\vec{\bar{u}} \cdot (\vec{u}' \nabla \cdot \vec{u}') + \frac{p'}{\bar{\rho}\bar{a}^2} \vec{\bar{u}} \cdot \nabla p'] = \int dV \frac{2p'}{\bar{\rho}\bar{a}^2} \vec{\bar{u}} \cdot \nabla p' = \int dV \vec{\bar{u}} \cdot \nabla (\frac{p'^2}{\bar{\rho}\bar{a}^2})$$

$$= \oiint dS \hat{n} \cdot \vec{\bar{u}} (\frac{p'^2}{\bar{\rho}\bar{a}^2}) - \int dV \frac{p'^2}{\bar{\rho}\bar{a}^2} \nabla \cdot \vec{\bar{u}} \quad (8.11)$$

and Eq. 8.10 can be put in the form

$$\frac{d\mathcal{E}}{dt} = - \oiint dS\hat{n} \cdot [\vec{u}'p' + \vec{\bar{u}} \frac{p'^2}{\bar{\rho}\bar{a}^2}] + \int dV \vec{u}' \cdot (\vec{F}' - \vec{\sigma}') - \int dV \frac{p'^2}{\bar{\rho}\bar{a}^2} \nabla \cdot \vec{\bar{u}} + \int dV p'P' . \quad (8.12)$$

The corresponding value of α is given by the formula obtained after taking time averages of Eq. 8.12

$$-2\alpha\mathcal{E} = \langle \oiint dS\hat{n} \cdot [\vec{u}'p' + \frac{\vec{\bar{u}} p'^2}{\bar{\rho}\bar{a}^2}] \rangle - \langle \int dV \frac{p'^2}{\bar{\rho}\bar{a}^2} \nabla \cdot \vec{\bar{u}} \rangle + \langle \int dV \vec{u}' \cdot (\vec{F}' - \vec{\sigma}') \rangle$$

$$+ \langle \int dV p'P' \rangle \quad (8.13)$$

It is a simple matter to verify that for harmonic motions Eq. 8.13 is identical with the real part of Eq. 5.39, which of course was obtained by an entirely different computation.

Hence, it has been shown that by considering the balance of acoustic energy one can obtain the same result for the growth constant as that found by an approximate solution to the linearized equations of motion. However, it is important to note that both analyses cannot provide the surface terms, essentially associated with viscous boundary layer effects, which have been incorporated in Section 6 in the modified three-dimensional analysis.

9. A SIMPLE APPLICATION TO T-BURNERS

It is an important result of the analysis developed here that not only is the response of a burning surface characterized by two functions, but both necessarily appear in a stability analysis. Consider the case of pure pressure coupling; then the admittance (A_b) and response (R_b) functions are defined by the formulas

$$\frac{\hat{u}}{\bar{a}} = A_b \frac{\hat{p}}{\gamma \bar{p}} \tag{9.1}$$

$$\frac{\hat{m}_b}{\bar{m}} = R_b \frac{\hat{p}}{\gamma \bar{p}} \tag{9.2}$$

These relationships apply, of course, at the edge of the combustion zone. The signs are set so that if A_b and R_b are real and positive, $\hat{u}$ and $\hat{m}_b$ are real and positive outward from the surface. After substitution of these definitions into Eq. 6.11, one finds an expression for $\Delta\hat{T}$ in terms of A_b and R_b:

$$\frac{\Delta\hat{T}}{\bar{T}} = \left[\frac{A_b}{\bar{M}_b} + 1 - R_b\right] \frac{\hat{p}}{\gamma \bar{p}} \quad . \tag{9.3}$$

The combination in brackets might also be called the entropy response function for pressure coupling, since $\Delta\hat{T}/\bar{T} = \Delta\hat{s}/C_p$ where $\Delta\hat{s}$ is the amplitude of entropy fluctuations at the edge of the combustion zone. To see this, begin with the combined expressions of the First and Second Laws of Thermodynamics

$$Tds = de + pdv = de - \frac{p}{\rho}\frac{d\rho}{\rho} \quad .$$

But for a perfect gas, $de = C_v dT$ and $d\rho/\rho = dp/p - dT/T$, so one has

$$Tds = C_p dT - \frac{dp}{\rho} = C_p dT - RT\frac{dp}{p} \quad .$$

Now apply this to small fluctuations of an element of gas, and split dT into a sum of isentropic and nonisentropic parts:

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma}\frac{dp}{p} + \frac{\Delta T}{T} \quad .$$

Substitution in the previous equation leads to

$$Tds = C_p \Delta T + [\frac{\gamma-1}{\gamma} C_p - R] \frac{T}{p} dp = C_p \Delta T \ . \quad (9.4)$$

Since $(\gamma-1)/\gamma C_p = R$, the second term vanishes, leaving the result quoted.

The implications of these results may be most easily seen by examining the simple case of a T-burner having samples of propellant at the ends, normal to the axis, and short grains also at the ends, but extending along the lateral walls. It will be assumed that the latter are approximately flush with the chamber wall so that the cross-sectional area of the chamber, S_c, is constant. Moreover, residual combustion and the presence of particulate matter will be ignored so that Eq. 7.8 simplifies to

$$-2\alpha \frac{\omega E_{1\ell}^2}{\bar{a}^2} = \bar{\rho}\bar{a}k_\ell \Big\{ \Big[(\hat{p}_\ell \hat{m}_b^{(r)} + \gamma \frac{R\bar{m}_b}{\bar{a}^2} \Delta \hat{T}^{(r)} \hat{p}_\ell) S_{be} \Big]_0^L - q \int_0^L \hat{p}_\ell [\hat{m}_b^{(r)} + \frac{R}{\bar{a}^2} \Delta \hat{T}^{(r)} \bar{m}_b] dz \Big\}$$

$$+ \frac{k_\ell q}{\bar{\rho}\bar{a}} \int_0^L [\hat{p}_\ell^2 + \frac{1}{k_\ell^2} (\frac{d\hat{p}_\ell}{dz})^2] \bar{m}_b dz - \frac{k_\ell S_c}{\bar{a}} \int_0^L \hat{p}_\ell^2 \frac{d\bar{u}}{dz} dz \quad (9.5)$$

The perimeter of the chamber is q, so that if the length of the lateral grains is L_b, the area of the burning surface on the sidewalls is $S_{bs} = qL_b$.

With the definitions in Eq. 8.1 and 8.2, the combustion terms in Eq. 8.5 are:

$$\bar{\rho}\bar{a}k_\ell [(\hat{p}_\ell \hat{m}_b^{(r)} + \gamma \frac{R\bar{m}_b}{\bar{a}^2} \Delta \hat{T}^{(r)} \hat{p}_\ell) S_{be}]_0^L \equiv -2k_\ell S_{be}(A_b^{(r)} + \bar{M}_b) \quad (9.6)$$

and

$$-\bar{\rho}\bar{a}k_\ell q \int_0^L \hat{p}_\ell [\hat{m}_b^{(r)} + \frac{R}{\bar{a}^2} \Delta \hat{T}^{(r)} \bar{m}_b] dz \equiv -k_\ell q \int_0^L [R_b^{(r)} + \frac{1}{\gamma}(A_b^{(r)} + \bar{M}_b - \bar{M}_b R_b^{(r)})] dz \ .$$

For short grains, $\hat{p}_\ell^2 \approx 1$ under the integral, and with the assumption that the response functions are constant over all the burning surface, the second contribution is simply

$$-2k_\ell S_{bs} [R_b^{(r)} + \frac{1}{\gamma} (A_b^{(r)} + \bar{M}_b - \bar{M}_b R_b^{(r)})] \ . \quad (9.7)$$

The factor 2 arises in both Eq. 9.5 and 9.6 because grains are placed at both ends of the chamber; the burning area of one lateral grain is $S_{bs} = qL_b$, where L_b is the length of the grain.

Since $d\hat{p}_\ell/dz \approx 0$ near the ends, the surface interaction terms, evaluated for the lateral burning surfaces only, are

$$\frac{k_\ell q}{\rho a}\int_0^L [\hat{p}_\ell^2 + \frac{1}{k_\ell^2}(\frac{d\hat{p}_\ell}{dz})^2]\bar{m}_b dq \approx 2\frac{k_\ell \bar{m}_b S_{bs}}{\rho a} \approx 2k_\ell S_{bs}\bar{M}_b \quad . \tag{9.8}$$

This acoustics/mean flow interaction term also contributes at the vent. For the fundamental (and all odd modes), $\hat{p}_\ell \approx 0$ there, while $d\hat{p}_\ell/dz = -k_\ell \sin(k_\ell L/2) = -k_\ell$. Hence, at the vent,

$$\frac{k_\ell}{\rho a}\int_0^L \frac{1}{k_\ell^2}\left(\frac{d\hat{p}_\ell}{dz}\right)^2 \int \bar{m}_b dq dz \approx -\frac{k_\ell}{\rho a}\bar{m}_v A_v$$

where A_v is the flow area of the vent, and $\bar{m}_v$ is the magnitude of the mass flux through the entrance of the vent. The sign change occurs because the flow is outward. Now by continuity,

$$m_v A_v = 2\bar{m}_b(S_{bs}+S_{be})$$

so

$$\left[\frac{k_\ell}{\rho a}\int_0^L \frac{1}{k_\ell^2}\left(\frac{d\hat{p}_\ell}{dz}\right)^2 \int \bar{m}_b dq\right]_{vent} \approx -2k_\ell \bar{M}_b(S_{be}+S_{bs}) \quad . \tag{9.9}$$

This holds for all modes in the simple uniform burner, since for the even modes, $\hat{p}_\ell \approx 1$ at the center of the burner, but $d\hat{p}_\ell/dz \approx 0$, and one finds again Eq. 9.9.

Note that there is a gain of energy associated with the flow out the vent. It arises because as the flow exhausts, there is an inelastic process at the boundary in which the fluid originally participating in the acoustical motions flows out the vent with no acoustic energy. The energy it loses is given up to the field in the chamber, thereby constituting a gain. This process is distinct from the radiation of acoustic energy, a loss which is associated with pressure fluctuations at the vent, and is therefore present only for the odd modes. It should be emphasized that this contribution from the vent is a formal result of the one-dimensional analysis and in no way constitutes a detailed analysis of the vent. Indeed, one should anticipate that the one-dimensional approximation is much less accurate in the vicinity of the vent, so that the results are suspect. Moreover, while it is relatively easy to understand that there should be an energy loss associated with the influx of mass at the surface, it is difficult to understand how the inverse process actually takes place at the vent. This question is an important one for the interpretation of data taken from T-burner firings, but it will not be examined here in detail. A careful consideration of the vent should probably produce a response or admittance function which appears as an additional contribution to the energy balance and hence the growth constant.

The last term in Eq. 9.5 can be rewritten by using the continuity equation for the mean flow,

$$\frac{d\bar{u}}{dz} = \frac{1}{\bar{\rho} S_c} \int \bar{m}_b \, dq = \frac{q}{\bar{\rho} S_c} \bar{m}_b \; .$$

Over the burning surfaces one finds

$$\frac{k_\ell S_c}{\bar{a}} \int_0^L \hat{p}_\ell^2 \frac{d\bar{u}}{dz} dz \approx 2 \frac{k_\ell S_{bs} \bar{m}_b}{\bar{\rho}\bar{a}} = 2k_\ell S_{bs} \bar{M}_b \; . \tag{9.10}$$

This term also provides a contribution at the vent, but for the even modes only. Since $\hat{p}_\ell^{\,2} \approx 1$ there,

$$\left[\frac{k_\ell S_c}{\bar{a}} \int_0^L \hat{p}_\ell^2 \frac{d\bar{u}}{dz} dz\right]_{vent} \approx \frac{k_\ell S_c}{\bar{a}} \int_{0-}^{0+} d\bar{u} \approx 2k_\ell S_c \frac{\bar{u}}{\bar{a}} \; .$$

Here $\bar{u}$ is the axial velocity in the chamber, and again by continuity,

$$S_c \bar{u} = (S_{bs} + S_{be}) \bar{u}_b$$

so that

$$\left[\frac{k_\ell S_c}{\bar{a}} \int_0^L \hat{p}_\ell^2 \frac{d\bar{u}}{dz} dz\right] \approx 2k_\ell (S_{be} + S_{bs}) \bar{M}_b \tag{9.11}$$

<u>for even modes only</u>.

Finally, for this special case,

$$\frac{\omega}{\bar{a}^2} E_{1\ell}^2 = \frac{\omega}{\bar{a}} \int_0^L \hat{p}_\ell^2 S_c dz \approx \frac{L}{2} k_\ell S_c \; . \tag{9.12}$$

Substitution of Eq. 9.6-9.12 into Eq. 9.5, and some rearrangement, gives

$$\frac{\alpha L}{\bar{a}} = \left\{2(A_b^{(r)}+\bar{M}_b)\frac{S_{be}}{S_c} + 2[\bar{M}_b R_b^{(r)} + \frac{1}{\gamma}(A_b^{(r)}+\bar{M}_b - \bar{M}_b R_b^{(r)})]\frac{S_{bs}}{S_c}\right\}$$

$$+\left\{2\bar{M}_b\left(\frac{S_{be}+S_{bs}}{S_c}\right) - \left[2\bar{M}_b\left(\frac{S_{be}+S_{bs}}{S_c}\right)\right]_{\substack{\text{even}\\ \text{modes}\\ \text{only}}}\right\} - \frac{\alpha_d L}{\bar{a}} \tag{9.13}$$

The term $\alpha_d L/\bar{a}$ represents other possible sources of attenuation, such as wall friction. The first set of brackets contains all contributions from the coupling of pressure fluctuations with combustion; the second set contains the mean flow/ acoustics interactions at the burning surface and at the vent.

An interesting special case is that for odd modes (really only the fundamental is of practical interest); if the coupling should happen to be isentropic, then $\bar{M}_b R_b = A_b + \bar{M}_b$ and Eq. 9.13 becomes

$$\frac{\alpha\ L}{\bar{a}} = 2(A_b^{(r)}+2\bar{M}_b)\left(\frac{S_{be}+S_{bs}}{S_c}\right) - \frac{\alpha_d L}{\bar{a}} \quad . \tag{9.14}$$

The 2 multiplying $\bar{M}_b$ is half due to the end discs and half due to the vent. This result has motivated testing with the variable area T-burner. Measurements of the growth constant, α, are made for different area ratios, $(S_{be}+S_{bs})/S_c$. Then if α is plotted versus the area ratio, the slope is $2(a_b^{(r)}+2\bar{M}_b)$, and the intercept is α_d. This procedure is based on the crucial assumption that the attenuation α_d is independent of area ratio, a circumstance which is likely not to be true.

More importantly, however, there is no justification for assuming that the coupling is isentropic. But, if that assumption fails, Eq. 9.13 contains essentially three unknowns: $A_b^{(r)}$, $R_b^{(r)}$ and α_d. Again for odd modes, one has

$$\frac{\alpha L}{\bar{a}} = 2(A_b^{(r)}+\bar{M}_b)\left[\frac{1}{\gamma}\frac{S_{bs}}{S_c} + \frac{S_{be}}{S_c}\right] + 2\bar{M}_b R_b^{(r)}\left[\frac{\gamma-1}{\gamma}\frac{S_{bs}}{S_c}\right]$$

$$+2\bar{M}_b\left(\frac{S_{be}+S_{bs}}{S_c}\right) - \frac{\alpha_d L}{\bar{a}} \quad . \tag{9.15}$$

In principle, a sequence of three tests (at the same frequency, say) in which S_{be} and S_{bs} are varied will provide three values of α, and hence Eq. 9.15 leads to three equations in the three unknowns $A_b^{(r)}+\bar{M}_b$, $R_b^{(r)}$, and α_d. It remains to be seen whether such a strategy may be useful.

Real T-burners in which the grains are not flush with the lateral walls require more elaborate analysis outside the scope of this work.

10. APPLICATION OF THE MODIFIED THREE-DIMENSIONAL RESULT TO MODES IN A CYLINDRICAL CHAMBER AND THE DATA OF BROWNLEE AND MARBLE

The most extensive study of a stability boundary in solid propellant rockets is that due to Brownlee and Marble (Ref. 5). In Ref. 6 and later in Ref. 3, the classical linearized stability analysis was applied. Although attenuation by particles was suggested as the dominant damping mechanism in Ref. 6, viscous stresses at the head end were supposed to be more important in the discussion of Ref. 3. However, not until much later was the admittance function of the propellant measured in a T-burner (Ref. 7).

Midway in the experimental program reported in Ref. 7, it appeared that the analysis of Ref. 3 had been verified (Ref. 8). Subsequently, this turned out not to be the case. The propellant used in the experiments of Ref. 5 contains no metal, so it is unlikely that attenuation by particles in the product gases is important. Indeed, motion pictures show that the products are in fact very clean. There appear now to be at least two possible reasons why the analysis and measurements do not agree--the discrepancy is quite large (Ref. 7):

(1) The T-burner measurements of Ref. 7 provide only values of the admittance function;

(2) The various boundary terms introduced in the modified three-dimensional analysis have until now not been taken into account.

Unfortunately, there is still no data for the response function. Hence, the question of the nonisentropic behavior associated with the difference between the response and admittance function cannot be explored. The purpose here is simply to examine the importance of the new terms introduced in the modified three-dimensional analysis.

All influences of condensed particles and residual combustion will be ignored, but the possibility of nonisentropic processes at the boundary will be included. Only the stability of waves will be treated here, so the problem is simply to evaluate the right-hand side of Eq. 7.2:

$$2\frac{\alpha}{\bar{a}}E_N^2 = \left\{\bar{a}\oint\hat{p}_N[\hat{m}_b^{(r)} + \frac{R\bar{m}_b}{\bar{a}^2}\Delta\hat{T}^{(r)}(\delta_\parallel + \gamma\delta_\perp)]dS\right\}$$

$$-\left\{\frac{1}{\bar{\rho}\bar{a}}\oint[\hat{p}_N^2 + \frac{1}{k_N^2}(\nabla\hat{p}_N)^2]\delta_\parallel\bar{m}_b dS\right\}$$

$$+\frac{1}{k_N}\oint\bar{m}_b\hat{\vec{u}}_\parallel^{(i)}\cdot\nabla\hat{p}_N dS - 2\frac{\alpha_d}{\bar{a}}E_N^2 \quad . \tag{10.1}$$

As earlier, all other processes of attenuation have been represented by a decay constant, α_d.

In the results reported by Brownlee and Marble, a particular mode was observed, but there is no added work here in computing the growth constant for all modes in a cylindrical chamber and later specializing the results as required. The classical acoustic modes are given by the formula:

$$\hat{p}_N = \cos(k_\ell z)\cos(m\phi)J_m(\varkappa_{mn}r) \tag{10.2}$$

$$\hat{\vec{u}}_N = \frac{i}{\bar{\rho}\bar{a}k_N}\nabla\hat{p}_N = \frac{i}{\bar{\rho}\bar{a}k_N}\left\{\hat{r}\frac{dJ_m}{dr}\cos(k_\ell z)\cos(m\phi) - \hat{\phi}\frac{m}{r}J_m\cos(k_\ell z)\sin(m\phi)\right.$$

$$\left. -\hat{z}k_\ell J_m\sin(k_\ell z)\cos(m\phi)\right\} \tag{10.3}$$

$$k_N^2 = k_\ell^2 + \varkappa_{mn}^2 \tag{10.4}$$

In order to satisfy the boundary condition on p_N, $\varkappa_{mn}$ are the roots of

$$\frac{dJ_m}{dr}(\varkappa_{mn}r) = 0 \qquad (r = r_c)$$

where r_c is the radius of the chamber. As usual, $m = 0, 1, 2, \ldots$ and $k_\ell = \ell\pi/L$ with $\ell = 0, 1, 2 \ldots$ The mean velocity field for incompressible potential flow is

$$\bar{\vec{u}} = \bar{u}_b\left[2\frac{z}{r_c}\hat{z} - \frac{r}{r_c}\hat{r}\right] . \tag{10.5}$$

It is simply a matter of arithmetic to compute $\nabla \cdot \hat{\vec{u}}_N$ and hence $\nabla_\perp \cdot \hat{\vec{u}}_N$, $\nabla_\parallel \cdot \hat{\vec{u}}_N$, $\delta_\perp$, and $\delta_\parallel$. For the cylindrical chamber,

$$\nabla_\perp \cdot \hat{\vec{u}}_N = \frac{\partial \hat{u}_{Nr}}{\partial r} \tag{10.6}$$

$$\nabla_\parallel \cdot \hat{\vec{u}}_N = \frac{1}{r}\frac{\partial u_{N\phi}}{\partial r} + \frac{\partial u_{Nz}}{\partial z} \tag{10.7}$$

In these computations, some use is made of the equation satisfied by the Bessel function,

$$\frac{d^2 J_m}{dr^2} + \frac{1}{r}\frac{dJ_m}{dr} + (\varkappa_{mn}^2 - \frac{m^2}{r^2})J_m = 0 \quad . \tag{10.8}$$

At the surface ($r = r_c$) one then finds

$$\nabla_\perp \cdot \hat{\vec{u}}_N = \frac{i}{\bar{\rho}\, \bar{a} k_N r_c^2} (m^2 - \varkappa_{mn}^2 r_c^2) J_m \cos(k_\ell z) \cos(m\phi) \tag{10.9}$$

$$\nabla_\parallel \cdot \hat{\vec{u}}_N = \frac{i}{\bar{\rho}\, \bar{a} k_N r_c^2} (m^2 - k_\ell^2 r_c^2) J_m \cos(k_\ell z) \cos(m\phi) \tag{10.10}$$

and therefore, by substitution into Eq. 6.19 and 6.20,

$$\delta_\perp = \left[\frac{|m^2 - \varkappa_{mn}^2 r_c^2|}{|m^2 - \varkappa_{mn}^2 r_c^2| + |m^2 - k_\ell^2 r_c^2|} \right] \tag{10.11}$$

$$\delta_\parallel = \left[\frac{|m^2 - k_\ell^2 r_c^2|}{|m^2 - \varkappa_{mn}^2 r_c^2| + |m^2 - k_\ell^2 r_c^2|} \right] \tag{10.12}$$

These formulas show, as one would anticipate from the symmetry of the situation, that the weighting factors $\delta_\parallel$ and $\delta_\perp$ are independent of position on the surface. Note that only for purely radial modes ($m = k_\ell = 0$) does $\delta_\parallel$ vanish.

With these preliminary calculations, it is not difficult to evaluate the contributions to α.

10.1. COUPLING AT THE BURNING SURFACE.

It will be assumed that only pressure coupling is important. With $\Delta\hat{T}$ given by Eq. 9.3, the coupling terms are

$$\frac{\overline{a}\overline{m}_b}{\gamma\overline{p}}\left[R_b^{(r)}(\delta_\parallel+\delta_\perp)+\frac{1}{\gamma}\left(\frac{A_b^{(r)}}{\overline{M}_b}+1-R_b^{(r)}\right)\right]J_m^2(\varkappa_{mn}r_c)\int_0^{2\pi}\cos^2(m\phi)d\phi\int_0^L\cos^2(k_\ell z)dz$$

$$= \overline{M}_b S_{bs}\left[R_b^{(r)}(\delta_\parallel+\delta_\perp)+\frac{1}{\gamma}\left(\frac{A_b^{(r)}}{\overline{M}_b}+1-R_b^{(r)}\right)\right]J_m^2(\varkappa_{mn}r_c)\Delta_{m\ell} \qquad (10.13)$$

where $S_{bs} = 2\pi r_c L$ is the area of the burning surface, and

$$\Delta_{m\ell} = \frac{1}{4}(1+\delta_{mo})(1+\delta_{\ell o}) \ . \qquad (10.14)$$

The Kronecker deltas, δ_{mo}, $\delta_{\ell o}$, are unity if $m = 0$, $\ell = 0$, respectively and zero otherwise.

10.2. ACOUSTICS/MEAN FLOW INTERACTIONS AT THE BURNING SURFACE

This term is non-zero only along the burning surface. Although $\delta_\parallel$ is non-zero at the entrance plane of the nozzle, and of course there is a mean flow there, there is no contribution. The reason is that this term, by construction, is present only if the acoustic field is zero[9] outside the boundary considered; for only then must there be a transition through a boundary layer. The energy loss is

$$\frac{1}{\overline{\rho}\,\overline{a}}\iint\left[\hat{p}_N^2+\frac{1}{k_N^2}(\nabla\hat{p}_N)^2\right]\delta_\parallel\overline{m}_b dS =$$

$$= \overline{M}_b S_{bs}\delta_\parallel J_m^2(\varkappa_{mn}r_c)\left[1+\frac{m^2+k_\ell^2 r_c^2}{r_c^2 k_N^2}\right]\Delta_{m\ell} \ . \qquad (10.15)$$

10.3. INFLUENCE OF THE NOZZLE

The first Surface integral of Eq. 10.1 also applies at the entrance to the nozzle. Relatively little is known about the nozzle losses, but formally at least, admittance and response functions can be introduced in the same way as

[9] Or, more generally, discontinuous. But non-zero values just outside the boundary different from those just inside will not occur in practice.

for the burning surface. The values of $\delta_\perp$ and $\delta_\parallel$ are of course different; for example, $\nabla_\perp \cdot \vec{\hat{u}}_N$ is now $\partial \hat{u}_{Nz}/\partial z$. One then finds

$$\delta_{n\perp} = \left[\frac{k_\ell^2}{|2m^2 - \varkappa_{mn}^2 r_c^2| + k_\ell^2 r_c^2} \right] \tag{10.16}$$

$$\delta_{n\parallel} = \left[\frac{|2m^2 - \varkappa_{mn}^2 r_c^2|}{|2m^2 - \varkappa_{mn}^2 r_c^2| + k_\ell^2 r_c^2} \right] \tag{10.17}$$

The signs on the mass fluxes are changed, $\bar{m}_b \to -\bar{m}_e$ and $\hat{m}_b \to \hat{m}_e$ because the flow is outward; then the response function for the nozzle is, for example,

$$\frac{\hat{m}_e}{\bar{m}_e} = R_n \frac{\hat{p}}{\gamma \bar{p}} \quad . \tag{10.18}$$

The contribution of the nozzle is then

$$\pi \bar{M}_e \left[R_n^{(r)} + \frac{1}{\gamma}(\delta_{n\parallel} + \gamma \delta_{n\perp}) \left(\frac{A_n^{(r)}}{\bar{M}_b} + 1 - R_n^{(r)} \right) \right] (1+\delta_{mo}) \int_0^{r_c} J_m^2(\varkappa_{mn} r) r dr \quad . \tag{10.19}$$

It is likely that the admittance function (in particular the real part) is very small unless the mode has an axial component of velocity. If, in addition, the coupling is isentropic, Eq. 10.19 reduces simply to

$$\pi \bar{M}_e \; (1+\delta_{mo}) \int_0^{r_c} J_m^2(\varkappa_{mn} r) r dr \tag{10.20}$$

which represents a loss of wave energy due to convection by the mean flow exhausting from the chamber.

The next to last term in Eq. 10.1, depending on $\vec{\hat{u}}^{(i)}$, contributes nothing. At the burning surface it is zero because the flow leaving the surface may be assumed to have initially no motion parallel to the surface. At the nozzle entrance, there is no contribution because the parallel motions (i.e., normal to the axis) are continuous through the entrance plane.

Finally, the values of E_N^2 are given by

$$E_N^2 = 2\pi L \Delta_{m\ell} \int_0^{r_c} J_m^2(\varkappa_{mn} r) r dr = S_{bs} r_c \frac{\Delta_{m\ell}}{2} \left(1 - \frac{m^2}{\varkappa_{mn}^2 r_c^2} \right) J_m^2(\varkappa_{mn} r_c) \tag{10.21}$$

Bringing all of the above results together, one has for α:

$$2\alpha \frac{E_N^2}{\bar{a}} = +\bar{M}_b S_{bs}\left[R_b^{(r)} + \frac{1}{\gamma}(\delta_\parallel + \gamma\delta_\perp)\left(\frac{A_b^{(r)}}{\bar{M}_b} + 1 - R_b^{(r)}\right)\right]\Delta_{m\ell} J_m^2(\varkappa_{mn} r_c)$$

(combustion coupling)

$$-\bar{M}_b S_{bs}\,\delta_\parallel \Delta_{m\ell}\left[1 + \frac{m^2 + k_\ell^2 r_c^2}{r_c^2 k_N^2}\right] J_m^2(\varkappa_{mn} r_c)$$

(turning of flow at the burning surface)

$$-\pi\bar{M}_e\left[R_n^{(r)} + \frac{1}{\gamma}(\delta_{n\parallel} + \gamma\delta_{n\perp})\left(\frac{A_n^{(r)}}{\bar{M}_e} + 1 - R_n^{(r)}\right)\right] \times$$

$$\times\ (1+\delta_{mo})\frac{r_c^2}{2}\left(1 - \frac{m^2}{\varkappa_{mn}^2 r_c}\right) J_m^2(\varkappa_{mn} r_c)$$

(nozzle)

$$-2\alpha_d \frac{E_N^2}{\bar{a}} \tag{10.22}$$

(other losses)

Consider the mode reportedly observed by Brownlee and Marble, namely that for $m = 1$ and $\ell = n = 0$. For the nozzle, $\delta_{n\parallel} = 0$ and $\delta_{n\perp} = 1$, while along the burning surface both $\delta_\parallel$ and $\delta_\perp$ are non-zero. The Mach number at the nozzle entrance is found from Eq. 10.5 to be $\bar{M}_e = 2(L/r_c)\bar{M}_b$. Eventually, Eq. 10.22 can be reduced to

$$\frac{\alpha r_c}{\bar{a}\bar{M}_b} = \frac{1}{\left(1 - \frac{1}{\varkappa_{ol}^2 r_c^2}\right)}\left[R_b^{(r)} + \frac{1}{\gamma}(\delta_\parallel + \gamma\delta_\perp)\left(\frac{A_b^{(r)}}{\bar{M}_b} + 1 - R_b^{(r)}\right)\right]$$

$$-\delta_\parallel \frac{(\varkappa_{ol}^2 r_c^2 + 1)}{(\varkappa_{ol}^2 r_c^2 - 1)} - \left(\frac{A_n^{(r)}}{\bar{M}_n} + 1\right)\left(\frac{2}{1+\delta_{\ell o}}\right) - \frac{\alpha_d r_c}{\bar{a}\bar{M}_b} \tag{10.23}$$

On the stability boundary, $\alpha = 0$, and Eq. 10.23 can be rewritten as

$$R_b^{(r)} + \frac{1}{\gamma}(\delta_\parallel + \gamma\delta_\perp)\left(\frac{A_b^{(r)}}{\bar{M}_b} + 1 - R_b^{(r)}\right) = \left(1 - \frac{1}{\varkappa_{ol}^2 r_c^2}\right)\left\{\frac{\alpha_d r_c}{a\bar{M}_b} + \left(\frac{A_n^{(r)}}{\bar{M}_e} + 1\right)\left(\frac{2}{1+\delta_{\ell o}}\right) - \delta_\parallel \frac{(\varkappa_{ol}^2 r_c^2 + 1)}{(\varkappa_{ol}^2 r_c^2 - 1)}\right\} \quad . \tag{10.24}$$

For isentropic flow, $R_b^{(r)} = A_b^{(r)}/\bar{M}_b + 1$, and if both the nozzle admittance function and the loss associated with mass addition at the burning surface are neglected, the Eq. 10.24 reduces to Eq. 24 of Ref. 6; for $\ell = 0$,

$$\frac{A_b^{(r)}}{\bar{M}_b} = \left(1 - \frac{1}{\varkappa_{ol}^2 r_c^2}\right)\frac{\alpha_d r_c}{a\bar{M}_b} - \frac{1}{\varkappa_{ol}^2 r_c^2} \quad . \tag{10.25}$$

Now $\varkappa_{ol} r_c = 1.84$, so the second term on the right-hand side of Eq. 10.25 is -.295. As argued above, and in Ref. 6, losses within the volume are apparently negligible; but that leaves only viscous damping on the head end as the source of loss. If one assumes simple laminar shear losses, it turns out that the first term in Eq. 10.25 is about one-tenth that of the second. Hence, specification of the stability boundary comes down to

$$\frac{A_b^{(r)}}{\bar{M}_b} = -.295 \quad , \tag{10.26}$$

which represents a simple balance between acoustics/combustion coupling and acoustics/mean flow interactions. Measured values of $A_b^{(r)}/\bar{M}_b$, taken from T-burner data by using Eq. 8.13 for end discs only ($S_{be} = S_c$) are shown in Fig. 10.1 taken from Ref. 7, for p = 300 psia.

Note that $A_b^{(r)} < 0$ for frequencies greater than approximately 2000 Hz, which means that the combustion processes are absorbing energy. The -.295 is the net driving effect by the mean flow, shown as the dashed line in Fig. 10.1.

By taking data for several mean pressures, a plot of the intersections of $A_b^{(r)}/\bar{M}_b$ with -.295 gives, according to Eq. 10.26, the stability boundary shown in Fig. 10.2. There is clearly a very large difference between this result and the observations of Brownlee and Marble.

Now suppose that, while $A_n^{(r)}$ is still zero, the coupling is not isentropic, and the loss due to turning of the flow is taken into account. Define ΔR_b as

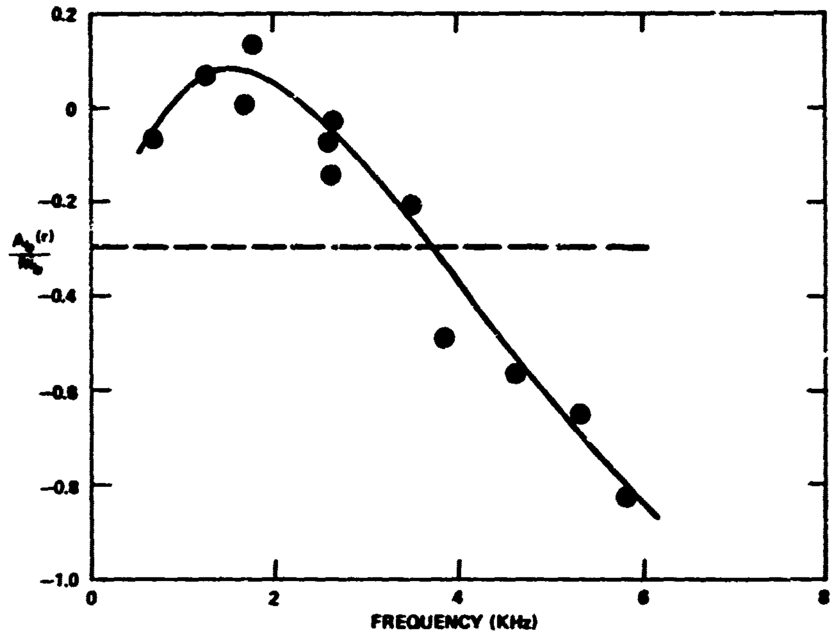

FIG. 10.1. The Ratio $A_b^{(r)}/M_b$ for T-17 Propellant at 300 psig.

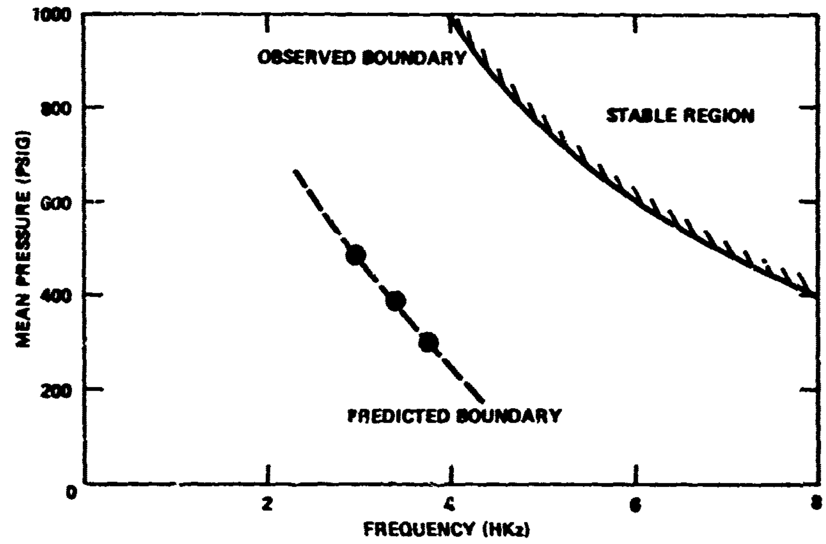

FIG. 10.2. Predicted and Observed Stability Boundaries Based on the Date of Ref. 5 and 8.

$$\Delta R_b = \left(\frac{A_b^{(r)}}{\bar{M}_b} + 1\right) - R_b^{(r)} \quad . \tag{10.27}$$

Then Eq. 10.24 leads to

$$\frac{A_b^{(r)}}{\bar{M}_b} + \left(\frac{\gamma-1}{\gamma}\right) \delta_{\parallel} \Delta R_b = \left(1 - \frac{1}{\varkappa_{o1}^2 r_c^2}\right) \frac{\alpha_d r_c}{a\bar{M}_b} - \frac{1}{\varkappa_{o1}^2 r_c^2} - \delta_{\parallel}\left(1 + \frac{1}{\varkappa_{1o}^2 r_c^2}\right) \tag{10.28}$$

If $\Delta R_b \neq 0$, this may provide a different result for the "predicted" sta ility boundary. To the present time, measurements of R_b (i.e., T-burner tests with grains on the lateral boundary) have not been taken, so it is not possible to determine whether or not the explanation proposed here is adequate to account for the poor agreement shown in Fig. 10.2. Indeed, another very strong possibility is that the response of the propellant to the azimuthal acoustic velocity parallel to the burning surface--i.e., velocity coupling--may be significant.

APPENDIX A

The problem discussed here seems to be one which has not previously been treated. It is concerned with the question of representing in general the coupling between a surface and an unsteady flow, accounting for the fact that the representation depends on the flow-field itself. From the analysis of the stability problems, one finds that the interaction between an acoustic field and a burning surface depends on the component of velocity parallel to the surface. When Eq. 6.5b and 6.12 are inserted in the expressions for the complex wavenumber, one finds

$$(k^2 - k_\ell^2)E_\ell^2 = \begin{cases} -i\bar{a}k_\ell \iint [\hat{m}_b + \frac{R}{\bar{a}^2}\bar{m}_b \Delta\hat{T}]\hat{p}_\ell dS & \text{all parallel acoustic velocity} \quad \text{(A-1a)} \\ -i\bar{a}k_\ell \iint [\hat{m}_b + \gamma\frac{R}{\bar{a}^2}\bar{m}_b \Delta\hat{T}]\hat{p}_\ell dS & \text{no parallel acoustic velocity} \quad \text{(A-1b)} \end{cases}$$

when only these coupling terms are considered.

To motivate the argument, take the imaginary part of Eq. A-1 to find for the growth constant

$$2\alpha = \frac{1}{\mathcal{E}_\ell}\frac{d\mathcal{E}_\ell}{dt} = \begin{cases} \frac{\bar{\rho}\bar{a}^2}{E_\ell^2} \iint [\frac{1}{\bar{\rho}}\hat{m}_b^{(r)} + \bar{u}_b R\Delta\hat{T}^{(r)}]\hat{p}_\ell dS & \text{(A-2a)} \\ \frac{\bar{\rho}\bar{a}^2}{E_\ell^2} \iint [\frac{1}{\bar{\rho}}\hat{m}_b^{(r)} + \bar{u}_b \gamma R\Delta\hat{T}^{(r)}]\hat{p}_\ell dS & \text{(A-2b)} \end{cases}$$

The integrands have the units of velocity; write them respectively as

$$\hat{u}_{\parallel}^{(r)} = \frac{1}{\bar{\rho}} \hat{m}_b^{(r)} + \bar{u}_b R \Delta \hat{T}^{(r)} \qquad \text{(A-3a)}$$

$$\hat{u}_{\perp}^{(r)} = \frac{1}{\bar{\rho}} \hat{m}_b^{(r)} + \bar{u}_b \gamma R \Delta \hat{T}^{(r)} \qquad \text{(A-3b)}$$

As shown in Section 7, $E_\ell^2 = 2\bar{p}\bar{a}^{-2}\,\mathcal{E}_\ell$ where $\mathcal{E}_\ell$ is the time-averaged total acoustic energy in the chamber; hence, Eq. A-2a,b can be written

$$2\alpha = \begin{cases} \dfrac{1}{2\mathcal{E}_\ell} \iint \hat{p}_\ell \hat{u}_{\parallel}^{(r)} dS & \text{(A-4a)} \\[2ex] \dfrac{1}{2\mathcal{E}_\ell} \iint \hat{p}_\ell \hat{u}_{\perp}^{(r)} dS & \text{(A-4b)} \end{cases}$$

Obviously, then, the time-averaged rate of work done on the waves by the interactions at an element of surface dS is

$$\langle \text{rate of work} \rangle = \begin{cases} \frac{1}{2}\, \hat{p}_\ell \hat{u}_{\parallel}^{(r)} dS & \text{(A-5a)} \\[2ex] \frac{1}{2}\, \hat{p}_\ell \hat{u}_{\perp}^{(r)} dS & \text{(A-5b)} \end{cases}$$

It should be emphasized that for each case, the fluctuations $\hat{u}_{\parallel}^{(r)}$, $\hat{u}_{\perp}^{(r)}$ are normal to the surface; the subscripts denote only that feature, of the unperturbed acoustic field, used to distinguish the two limiting cases.

That these different results have been obtained for the coupling in the two limiting cases implies that the coupling for the general case somehow depends on the relative orientation of the surface element and the local acoustic field, for this determines what the parallel velocity will be for a given pressure amplitude. To develop this idea, observe first that however complicated the mode structure may be, the field in the vicinity of a surface may be approximated locally by a superposition of incident and reflected waves. Let θ be the angle between the normal to the surface and the directions of those waves. It is, of course, the unperturbed acoustic field near a rigid surface which is of concern here. Schematically, the limiting cases and an arbitrary acoustic mode may be represented as shown in Fig. A-1. For all parallel velocity, $\theta = \pi/2$ and for no parallel velocity, $\theta = 0$. Note that for the unperturbed field, the normal component $\vec{u} \cdot \hat{n}$ of the velocity is zero.

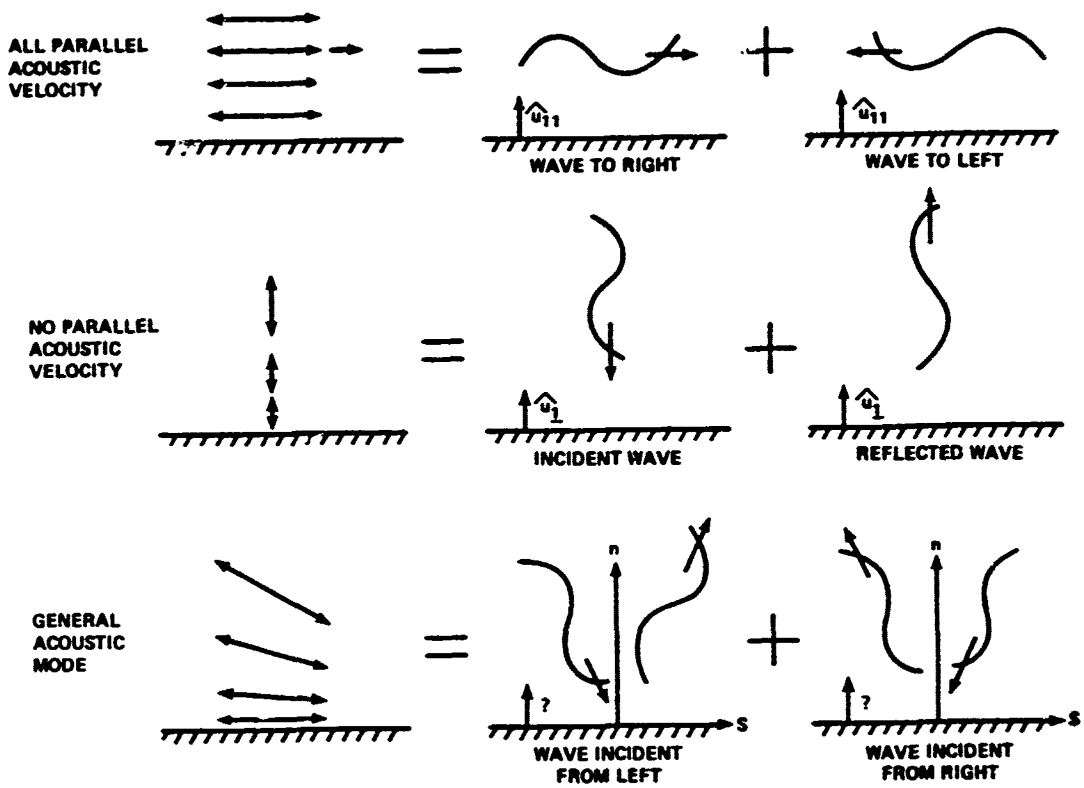

FIG. A-1. Sketch of Acoustic Modes in the Vicinity of a Rigid Surface.

Let n and s denote coordinates normal and parallel to the surface as shown. Then for the wave incident from the left, the incident and reflected waves may be represented as

$$\left\{\begin{aligned} p_t^{(\ell)} &= -\overline{\rho}\,\overline{a}\varphi'(s \sin\theta - n\cos\theta - \overline{a}t) \\ u_{is}^{(\ell)} &= -\sin\theta\varphi'(s\sin\theta - n\cos\theta - \overline{a}t) \\ u_{in}^{(\ell)} &= \cos\theta\varphi'(s\sin\theta - n\cos\theta - \overline{a}t) \end{aligned}\right. \qquad \text{(A-6)}$$

$$\left\{\begin{aligned} p_r^{(\ell)} &= -\overline{\rho}\,\overline{a}\varphi'(s\sin\theta + n\cos\theta - \overline{a}t) \\ u_{rs}^{(\ell)} &= -\sin\theta\varphi'(s\sin\theta + n\cos\theta - \overline{a}t) \\ u_{rn}^{(\ell)} &= -\cos\theta\varphi'(s\sin\theta + n\cos\theta - \overline{a}t) \end{aligned}\right. \qquad \text{(A-7)}$$

Here, subscript i denotes incident, r denotes reflected, and n, s denote the norma and tangential components of the velocity. There are corresponding formulas for the wave incident from the right:

$$\left\{\begin{aligned} p_i^{(r)} &= -\bar{\rho}\,\bar{a}\,\psi'(-s \sin\theta - n\cos\theta - \bar{a}t) \\ u_{is}^{(r)} &= \sin\theta\,\psi'(-s\sin\theta - n\cos\theta - \bar{a}t) \\ u_{in}^{(r)} &= \cos\theta\,\psi'(-s\sin\theta - n\cos\theta - \bar{a}t) \end{aligned}\right. \tag{A-8}$$

$$\left\{\begin{aligned} p_r^{(r)} &= -\bar{\rho}\,\bar{a}\,\psi'(-s\sin\theta + n\cos\theta - \bar{a}t) \\ u_{rs}^{(r)} &= \sin\theta\,\psi'(-s\sin\theta + n\cos\theta - \bar{a}t) \\ u_{rn}^{(r)} &= -\cos\theta\,\psi'(-s\sin\theta + n\cos\theta - \bar{a}t) \end{aligned}\right. \tag{A-9}$$

The functions φ and ψ are velocity potentials; the primes denote differentiation with respect to the argument: e.g., $\varphi' = d\varphi(z)/dz$. (See, for example, Ref. 9, p. 266.) Note that Eq. A-6, A-7, A-8 and A-9 represent solutions to the unperturbed wave equation, satisfying the condition that the velocities normal to the surface, $u_{in}^{(\ell)} + u_{rn}^{(\ell)}$ and $u_{in}^{(r)} + u_{rn}^{(r)}$ vanish at the surface ($n = 0$).

Now expand $u_n^{(\ell)} = u_{in}^{(\ell)} + u_{rn}^{(\ell)}$ in Taylor series about the origin, along the n axis, and retain only the first term:

$$\left(u_n^{(\ell)}\right)_{s=0} = \left(u_{in}^{(\ell)} + u_{rn}^{(\ell)}\right)_{s=0} = \cos\theta\,[\varphi(-n\cos\theta - \bar{a}t) - \varphi(n\cos\theta - \bar{a}t)]$$

$$\approx \cos\theta\, n\,\frac{\partial}{\partial n}\,[\varphi(-n\cos\theta - \bar{a}t) - \varphi(n\cos\theta - \bar{a}t)]_{n=0} \approx -2\cos^2 n\varphi'(-at) \; .$$

Similarly, for the wave incident from the right,

$$\left(u_n^{(r)}\right)_{s=0} \approx -2\cos^2\theta n\psi'(-\bar{a}t) \; .$$

Thus, the sum of these gives

$$\cos^2\theta \approx -\frac{1}{2(\varphi' + \psi')}\left(\frac{u_n}{n}\right)_{n\to 0} = -\frac{\nabla_\perp \cdot \bar{u}}{2(\varphi' + \psi')} \; . \tag{A-10}$$

Here, u_n/n can be interpreted, since $u_n = 0$ at $n = 0$, as $\nabla_\perp \cdot \hat{\vec{u}}$, the divergence of the velocity taken in the direction normal to the surface. To see this, note that $\vec{u} = \vec{u}_n + \vec{u}_s$ and ∇ operates only on $\vec{u}_n$.

Similarly, expand the velocity parallel to the surface, $u_s^{(\ell)} = u_{is}^{(\ell)} + u_{rs}^{(\ell)}$, and $u_s^{(r)}$ to find

$$\left(u_s\right)_{n=0} - \left(u_s\right)_{\substack{s=0\\n=0}} \approx -2\sin^2 s(\varphi'+\psi') \quad .$$

In the limit $s \to 0$, $[n_s(s)-u_s(0)]/s$ can be interpreted as $\nabla_\parallel \cdot \hat{\vec{u}}$, so

$$\sin^2\theta = -\frac{\nabla_\parallel \cdot \hat{\vec{u}}}{2(\varphi'+\psi')} \quad . \tag{A-11}$$

It follows from Eq. A-10 and A-11 that

$$\tan\theta = \left(\frac{|\nabla_\parallel \cdot \hat{\vec{u}}|}{|\nabla_\perp \cdot \hat{\vec{u}}|}\right)^{\frac{1}{2}} \tag{A-12}$$

where absolute values have been taken to assure, as necessary, that only positive numbers arise. Then Eq. A-12 implies

$$\sqrt{\delta_\parallel} = \sin\theta = \left[\frac{|\nabla_\parallel \cdot \hat{\vec{u}}|}{|\nabla_\parallel \cdot \hat{\vec{u}}| + |\nabla_\perp \cdot \hat{\vec{u}}|}\right]^{\frac{1}{2}} \tag{A-13}$$

$$\sqrt{\delta_\perp} = \cos\theta = \left[\frac{|\nabla_\perp \cdot \hat{\vec{u}}|}{|\nabla_\parallel \cdot \hat{\vec{u}}| + |\nabla_\perp \cdot \hat{\vec{u}}|}\right]^{\frac{1}{2}} \tag{A-14}$$

It may be noted that Eq. A-13 and A-14 are likely not to be the most useful forms for practical geometries, since generally the pressure field is computed numerically. However, by using the momentum equation, $\nabla p' = -\bar{\rho}\, \partial\vec{u}'/\partial t$, one can write, for example,

$$\nabla_\perp \cdot \hat{\vec{u}} \approx \frac{1}{i\rho\omega}\, \frac{p(\delta n)-p(0)}{(\delta n)^2}$$

$$\nabla_\parallel \cdot \hat{\vec{u}} \approx \frac{1}{i\rho\omega}\, \frac{p(s+\delta s)-p(s)}{(\delta s)^2}$$

where δn, δs are the incremental distances measured in the normal and tangential directions from the point $n = 0,s$. In this way, the results of a numerical calculation can be used directly to compute $\delta_{\parallel}$ and $\delta_{\perp}$.

But now the question to answer is, "how should the limiting cases shown in Eq. A-1 be combined to give the correct representation for the coupling when θ is neither 0 nor $\pi/2$?" An appealing guess is that the coupling for no parallel velocity should be multiplied by $\cos^2\theta$, and that for all parallel velocity by $\sin^2\theta$. That this is correct may be demonstrated as follows. First, owing to the way in which the one-dimensional analysis of Sections 2 and 3 is formulated, it is clear that it is the component of velocity normal to the velocity in the chamber which matters. For $\theta \neq 0$, the proportion $\sin\theta$ of the mass flux leaving the surface is normal to the acoustic velocity, and the proportion $\cos\theta$ is parallel to the acoustic velocity, as shown in Fig. A-2. Hence, a factor $\sin\theta$ should multiply the fluctuation $\hat{u}$ (of which the real part appears in Eq. A-3a) or Eq. 6.5b, for the coupling due with flow parallel to the surface. And, a factor $\cos\theta$ must multiply $\hat{u}_{\parallel}$, or Eq. 6.12.

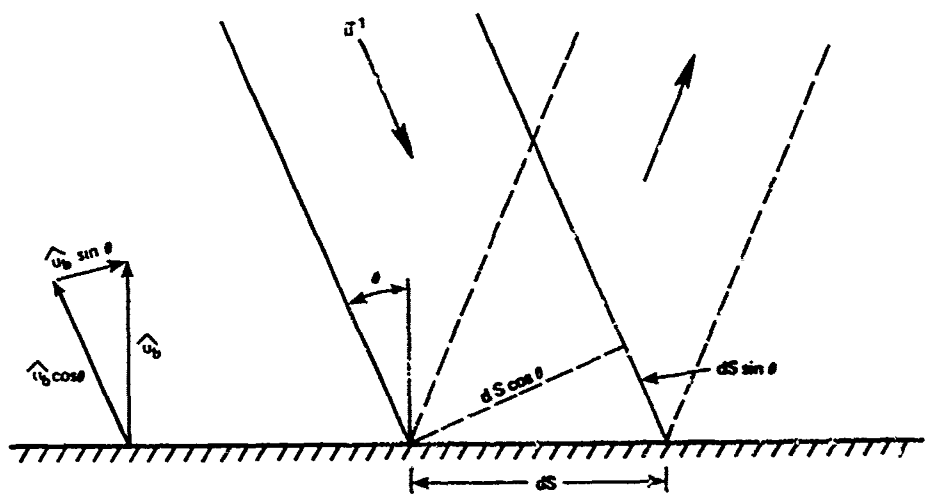

FIG. A-2. Projections of Velocity Fluctuations Associated With Mass Addition, and an Element of Surface Area.

But in addition to weighting the velocity components, the surface areas must be weighted. For the work done due to coupling with the parallel flow, only the area dS $\sin\theta$ parallel to the direction of the incident waves is effective, as indicated in the above sketch. And for the normal flow, the projected area dS $\cos\theta$ normal to the direction of propagation is involved. Consequently, the total rate at which work is done is

$$\hat{p}(\hat{u}_{\parallel}^{(r)} \sin^2\theta + \hat{u}_{\perp}^{(r)} \cos^2\theta)dS$$

It follows that the two cases shown in Fig. A-1 should be combined as

$$(k^2-k_\ell^2)E_\ell^2 = -i\bar{a}k_\ell \iint [\hat{m}_b(\cos^2\theta+\sin^2\theta)+\frac{R}{\bar{a}^2}\bar{m}_b\Delta\hat{T}(\sin^2\theta+\gamma\cos^2\theta)]dS$$

$$= -i\bar{a}k_\ell \iint [\hat{m}_b+\frac{R}{\bar{a}^2}\bar{m}_b\Delta\hat{T}(\delta_\parallel+\gamma\delta_\perp)]dS \qquad \text{(A-15)}$$

with $\delta_\parallel = \sin^2\theta$, $\delta_\perp = \cos^2\theta$.

It must be emphasized that $\hat{m}_b$ and $\bar{m}_b\Delta\hat{T}$ are normal to the surface. In practice, these quantities are likely to be expressed as response functions defined for the limiting cases. For purposes of illustration, consider pressure coupling only, and define $R_{b\parallel}$, $R_{b\perp}$ by

$$\frac{1}{\bar{m}_b}[\hat{m}_b+\frac{R}{\bar{a}^2}\bar{m}_b\Delta\hat{T}] = R_{b\parallel}\frac{\hat{p}}{\bar{p}} \qquad \text{all parallel acoustic velocity (A-16)}$$

$$\frac{1}{\bar{m}_b}[\hat{m}_b+\gamma\frac{R}{\bar{a}^2}\bar{m}_b\Delta\hat{T}] = R_{b\perp}\frac{\hat{p}}{\bar{p}} \qquad \text{no parallel acoustic velocity (A-17)}$$

The preceding argument shows that the coupling in general is represented by

$$R_{b\parallel}\sin\theta\frac{\hat{p}}{\bar{p}} + R_{b\perp}\cos\theta\frac{\hat{p}}{\bar{p}} , \qquad \text{(A-18)}$$

and the rate of work done by processes at the element of area dS is

$$[R_{b\parallel}\sin^2\theta + R_{b\perp}\cos^2\theta]\left(\frac{\hat{p}}{\bar{p}}\right)^2 dS . \qquad \text{(A-19)}$$

The numerical values for $R_{b\parallel}$, $R_{b\perp}$ are those either calculated or measured for the appropriate limiting cases. For example, data taken with an end-burning T-burner should in principle provide $R_{b\perp}$; and from measurements with a T-burner having lateral grains, one may be able to determine $R_{b\parallel}$. See Section 9 for additional discussions.

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