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INTERACTIVE MULTIPLE GOAL PROGRAMMING⁺

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1. Introduction

In literature on management science, operations research and various fields of economics more and more attention is paid to multi-dimensional optimization (M.O.) methods as a tool in modern decision-making. Although this is a relatively new field of academic study, decision-makers in business and government are already quite attentive. This may be declared by the fact that most M.O.methods are able to include a wide variety of relevant decision aspects without translating them into monetary units or any other common denominator. These methods are also able to integrate intangibles normally falling outside the realm of the traditional price and market system.

In an earlier typology (Nijkamp and Spronk [1978]) we distinguished between discrete and continuous M.O.models. Discrete M.O.models (or multi-criteria models) are used in decision problems in which the numbers of feasible alternatives is finite, whereas continuous M.O.models (or multi-objective programming models) are based on an infinite number of possible values for the decision arguments and hence for the objective functions.

In this report we discuss a new Interactive variant of Multiple Goal programming (I.M.G.P.). Multiple goal programming, devised and further developed by Charnes & Cooper, was one of the earliest practicable techniques in multi-objective programming. In section 2.1 and 2.2 a short overview of multiple goal programming will be given. We believe multiple goal programming still to be one of the stronger methods available. Its use of aspiration levels and preemptive priorities closely corresponds to decision-making in practice. Furthermore, goal programming problems can be solved by the present standard linear programming routines. However, an important drawback of multiple goal programming is that it requires a considerable amount of a priori information on the decision-maker's preferences. We will try to side-step this handicap by proposing an interactive variant of multiple goal programming.

Recently, interactive methods have become rather popular in decision analyses. These methods are based on a mutual and successive interplay between a decision-maker and an expert (or analyst). These methods neither require an explicit representation or specification of the decision-maker's preference function nor an explicit quantitative representation of trade-offs among conflicting objectives. Obviously, the

solution of a decision problem requires that the decision-maker provides information about his priorities regarding alternative feasible states, but in normal interactive procedures only a set of achievement levels (or 'satisficing' levels) for the various objectives have to be specified in a stepwise manner. The task of the analyst is to provide all relevant information especially concerning admissible values of the criteria and concerning reasonable compromise solutions.

By means of interactive decision-methods a decision-maker may get more closely involved in evaluation problems, while he also obtains more insight in the trade-offs among different criteria. The feed-back process inherent in interactive decision-methods leads to a closer co-operation between decision-maker and analyst. Therefore, interactive decision-methods may be regarded as an operational application of learning theory (cf. also Atkinson et al [1965], Golledge [1969], and Hilgard and Bower [1966]).

Interactive decision-methods have also been applied in the field of goal programming, although the number of its applications is rather limited so far. In section 2.3. a sample of interactive goal programming methods will be presented and discussed. In section 2.4. we briefly discuss two methods which, just like multiple goal programming, can be characterized as sequential optimization methods. Such methods treat the goals or goal variables sequentially in decreasing order of importance.

In section 3 we present a new interactive variant of multiple goal programming. This presentation is preceded by an enumeration of the prerequisites of I.M.G.P. and is followed by a simple example. In the fourth section we turn to the technical elaboration of I.M.G.P. Special attention is paid to the linear variant of I.M.G.P. Furthermore, we focus on the convergence properties of I.M.G.P., together with the existence, feasibility and uniqueness of the ultimate and the intermediate solutions. An evaluation of I.M.G.P. is given in the final section.

2. Multiple Goal Programming

In this section a brief survey of multiple goal programming will be presented. A more extensive survey is given by Nijkamp and Spronk [1977]. Subsection 2.1. deals with the general formulation of the multiple goal program. In subsection 2.2. we discuss the advantages and disadvantages of goal programming. Because we are advocating an interactive variant of multiple goal programming, a brief overview of interactive variants which are suggested by other authors is presented in subsection 2.3. Finally, in subsection 2.4., we show a number of methods treating the goal variables in decreasing order of importance - which in fact is also done by the multiple goal programming procedure.

2.1. General Formulation

In goal programming a set of 'goals' is assumed which are defined by the decision-maker. In this sense, goals are aspired levels (targets) of certain 'goal variables'. Each goal variable is a function of a number of 'instrumental (policy) variables'. The set of combinations of the instrumental variables which are admissible, is called the 'feasible region'. Within this region a solution must be identified that meets the decision-makers preferences in an optimal way. It is assumed that the decision-maker's preferences for the various outcomes can be represented by a 'preference function', expressed in terms of the goal variables or by a 'dispreference function' which is expressed in terms of the deviations from the aspired goal levels. The mathematical function which is optimized in the goal program is called the objective function. Depending on the specific problem formulation this function may (but need not) coincide with the (dis)preference function.

In its most general form the multiple goal program can be formulated as

$$(2.1.) \left\{ \begin{array}{l} \text{Minimize } f(\underline{y}^+, \underline{y}^-) \\ \text{subject to} \\ \underline{g}(\underline{x}) - \underline{y}^+ + \underline{y}^- = \underline{b} \quad , \\ \underline{x} \in R, R = \{ \underline{x} \mid \underline{h}(\underline{x}) \leq \underline{h} \} \quad , \\ \underline{y}^+, \underline{y}^- \geq \underline{0} \quad , \\ \text{and } y_i^+ \cdot y_i^- = 0 \quad \text{for } i=1, \dots, m \end{array} \right.$$

where f is the (dispreference) function having as arguments the positive (y_i^+) and the negative (y_i^-) deviations from the aspired levels (b_i) of the goal variables $g_i(\underline{x})$, with $i=1, \dots, m$. The feasible region R of the instrumental variables \underline{x} is bounded by the set of constraints $\underline{h}(\underline{x})$. In general, the function to be minimized, f , is assumed to be convex. The feasible region is also assumed to be convex. In many cases, both the goal variables $\underline{g}(\underline{x})$ and the constraining relations $\underline{h}(\underline{x})$ are assumed to be linear in \underline{x} . We then have:

$$(2.2.) \quad \begin{cases} \underline{g}(\underline{x}) = A \cdot \underline{x} \\ \underline{h}(\underline{x}) = B \cdot \underline{x} \end{cases}$$

where A is a matrix of order $(m \times n)$, B is a matrix of order $(k \times n)$, and \underline{x} is a n -dimensional vector. As shown in Nijkamp and Spronk [1977, pp.7-9], the following general form for the function f can be deduced from the Minkovski metric:

$$(2.3.) \quad f(y^+, y^-) = \left\{ \sum_{i=1}^m \alpha_i^+ \cdot \left(\frac{y_i^+}{b_i} \right)^p + \sum_{i=1}^m \alpha_i^- \cdot \left(\frac{y_i^-}{b_i} \right)^p \right\}^{1/p}$$

It is easily seen, that (2.3.) is a weighed (by α_i^+ and α_i^-) and standardized form of the \mathcal{L}_p metric. That is, for $p=1$ we get an absolute value metric¹⁾, for $p=2$ we get a Euclidean metric, and for $p \rightarrow \infty$ the Chebychev (minimax) metric is approached. In multiple goal programming, the weighing factors α_i^+ and α_i^- may be replaced by preemptive priority factors, by which lexicographic ordenings can be handled (Ibid, p.16).

In minimizing the function f , mathematical programming techniques may give good approximations. For $p=1$ and $p \rightarrow \infty$ even an exact solution can be attained when the problem is formulated as a linear program. The same holds true for $p=2$ by using quadratic programming or by means of generalized inverses (Ibid, section 4). As shown for the \mathcal{L}_1 -case (Ibid, pp.15-28), some modifications are needed to include the abovementioned preemptive priority factors. For instance, an aberrant form of the simplex procedure may be used.

1) In the fourth section of Nijkamp and Spronk [1977] a catalogue of possible objective functions is given for this case.

2.2. Advantages and Disadvantages of Goal Programming

In our opinion, goal programming is still to be one of the stronger methods available. It has a close correspondence with decision-making in practice. Furthermore, it has some attractive technical properties. Several empirical findings from decision-making practice are, in our opinion, rather convincing to demonstrate the practical usefulness of multiple goal programming. As mentioned by several writers, the method corresponds fairly well to the results of the behavioral theory of the firm. In practice, decision-makers are aiming at various goals, formulated as aspiration levels. The intensity with which the goals are strived for may vary from goal to goal; in other words, different 'weights' may be assigned to different goals²⁾. The use of aspiration levels in decision-making is also reported by scientists from other fields, like for instance psychology (see for a short overview Monarchi et al [1976]). In the same way, also preemptive priorities are known in real life problems. Support for this in fact lexicographic viewpoint is provided by Fishburn [1974] and Monarchi et al [1976]. A more concrete example of the correspondence of multiple goal programming and practice is provided by Ijiri [1965], who views multiple goal programming as an extension of break-even analysis, which is widely used in business practice.

The above plea for multiple goal programming is of a somewhat theoretical nature. Of course, the operational usefulness of multiple goal programming can only be shown in practice. Although it is a relatively 'young' method, many applications have been reported in literature. To give an idea, we have listed some of these applications, especially in the field of business and managerial economics (see Nijkamp and Spronk [1977]).

One of the technical advantages of multiple goal programming is that there is always a solution to the problem, even if some goals are conflicting, provided that the feasible region R is non-empty. This is due to the inclusion of the deviational variables y_i^+ and y_i^- . These variables show whether the goals are attained or not, and in the latter case they measure the distance between the realized and aspired goal levels. Another advantage of multiple goal programming is that it does not require very sophisticated solution procedures. Especially the linear goal programming problems can be solved by easily available linear programming routines.

An important drawback of multiple goal programming is its need for fairly detailed a priori information on the decision-maker's preferences.

2) As shown by Lane [1970], the correspondence of the behavioral theory and multiple goal programming is not complete, because the latter gives a specific interpretation of 'satisfying goals as close as possible' (See Lane, pp.57-60).

Goal programming requires the definition of aspiration levels, the division in preemptive priority classes and the assessment of weights within these classes. We agree with those scholars advocating interactive approaches to the multiple goal problem (cf. section 1). Unfortunately, most of the usual interactive approaches lack some of the advantages of 'traditional' multiple goal programming, such as for instance the possibility to include preemptive priorities. Furthermore multiple goal programming can handle situations of satisficing behaviour in contrast with most existing interactive methods. This situation, combined with the repeatedly shown power of the traditional approach to include piecewise linear functions (cf. Charnes & Cooper [1977]), justifies the effort to seek for an interactive variant of the traditional approach. In subsection 2.3. we discuss some of such variants mentioned in literature and in sections 3 and 4 we present an own effort to construct a new interactive multiple goal approach. This method can include all advantages of multiple goal programming. For instance, preemptive priorities and piecewise linear functions can be handled in a straightforward way. Furthermore, the interactive process imitates practice in formulating aspiration levels, assessing priorities, seeking for a solution and readjustment of the aspiration levels. The method needs no more a priori information on the decision-maker's preference structure than other interactive multi-objective programming models. However, all available a priori information can be incorporated within the procedure.

2.3. Interactive Goal Programming Methods

One of the first interactive goal programming methods was proposed by Dyer [1972]. He developed an interactive algorithm for the optimization of the multiple criteria problems by interacting with the decision-maker and solving a series of goal programming problems.

A central element in Dyer's approach is the correspondence between the one-sided goal programming model and a subproblem of the Frank-Wolfe algorithm (see Frank and Wolfe [1956]). According to Dyer one may assume that the various decision criteria may be included in a utility function U . This utility function as such is unknown, but the decision-maker is assumed to be able to provide information on his trade-offs among criteria at all points. The trade-offs among the decision criteria are, in general, not constant, but depend on the values of these criteria. Then the proper selection of the trade-offs at a certain point $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$ is calculated as:

$$(2.4.) \quad W_i = \frac{\delta U(\bar{y})}{\delta y_i} / \frac{\delta U(\bar{y})}{\delta y_1} \quad i=2, \dots, m. \quad ,$$

which is the marginal rate of substitution of criterion i for criterion 1 at the point \bar{y} . The next step of the analysis is to operationalize this relationship. By means of questionnaires this problem may be solved. These weights are used to find a solution that solves a weighted goal programming model. In view of the resemblance of the traditional goal programming model and the Frank-Wolfe gradient method the latter method is proposed as an auxiliary tool. After the selection of an initial feasible point \underline{y}_1 one may calculate (2.4.) by interacting with the decision-maker. Then the trade-offs are included as weights in the objective function of the one-sided goal programming model in which the weighed sum of the deviations from a set of prespecified goal levels is minimized. Next, one may define a directional vector $\underline{d}_1 = \bar{z}_1 - \underline{y}_1$, where \bar{z}_1 denotes the combination of the instruments which minimizes the weighed sum of deviations. By denoting the corresponding step length by t , the next initial point \bar{y}_2 is equal to:

$$(2.5.) \quad \bar{y}_2 = \underline{y}_1 + t \cdot \underline{d}_1$$

Therefore, by interacting with the decision-maker an approximation of the step length $t (0 \leq t \leq 1)$ can be determined which maximizes $U(\bar{y}_2) = U(\underline{y}_1 + t \cdot \underline{d}_1)$.

By repeating this procedure a final compromise solution can be achieved, for which indeed convergence properties can be proved.

Dyer's approach is very intriguing, because it relates interactive methods to gradient methods³⁾. On the other hand, there is as such no reason to apply a complicated solution algorithm in the case of a linear goal programming model. Another problem is that the determination of the preference function U is not quite clear, while also the precise nature of the interactive procedure with the decision-maker is somewhat obscure.

Another contribution to interactive goal programming methods was provided by Fichet [1976]. Fichet links an iterative method (called STEM) to the solution of goal programming (GP) problems, hence the name GPSTEM. The STEM method is an interactive decision method composed of a calculation phase and a decision phase. During the calculation phase the following successive programs are solved:

$$(2.6.) \quad \left\{ \begin{array}{l} \text{Min } v \\ \text{subject to} \\ v \geq w_i (y_i^0 - y_i) \\ v \geq 0 \end{array} \right.$$

where:

$$(2.7.) \quad y_i = \sum_{j=1}^n c_j^i x_j \quad \forall_i$$

and:

$$(2.8.) \quad \begin{array}{l} y_i^0 = \max \sum_{j=1}^n c_j^i x_j \quad \forall_i \\ \text{subject to } x_j \in R \quad \forall_j \end{array}$$

with R the feasible region for the decision variables x_j . The relative weights w_i are then calculated as:

$$(2.9.) \quad w_i = \frac{p_i}{\sum_{i'=1}^m p_{i'}} \quad , \quad p_i = \frac{y_i^0 - y_i^{\min}}{y_i^0} \left/ \left\{ \sum_{j=1}^n (c_j^i)^2 \right\}^{\frac{1}{2}} \right.$$

³⁾ See for a general discussion of gradient methods also Nijkamp [1972].

where y_i^{\min} is the minimum feasible value of y_i during the individual optimization of all separate m objectives.

The solutions y_i^* of (2.6.) are proposed to the decision-maker. If some y_i^* values are satisfactory and others not, the decision-maker must accept a certain amount of relaxation Δy_k for objectives k which are already satisfactory. Then the new feasible area is restricted as follows:

$$(2.10.) \left\{ \begin{array}{l} y_k \geq y_k^* - \Delta y_k \\ y_n \geq y_n^* \end{array} \right.$$

where the subscript n refers to the remaining subset of original objectives. These side-conditions are introduced in the next step of the iteration and so forth.

The GPSTEM procedure is based on a similar strategy. The decision-maker has to specify in advance a best satisfactory level \bar{y}_i for each objective. Then step (2.6.) is carried out. The next step is to calculate the following program:

$$(2.11.) \left\{ \begin{array}{l} \text{Min } v = \sum_{i=1}^m (z_i^+ + z_i^-) \\ \text{subject to} \\ y_i - z_i^+ + z_i^- = \bar{y}_i \quad , \forall i \\ y_i = \sum_{j=1}^n c_j^i \cdot x_j \quad , \forall i \\ x_j \in R \quad , \forall j \end{array} \right.$$

The next step is to solve a rather specific parametric linear program, which constitutes the basis for a game procedure by means of which the weights associated with each objective function can be determined.

Fichet's method is a rather simple and straightforward procedure which looks rather operational. Some problems inherent in this method are the rather mechanical way of determining the trade-offs w_i . Furthermore, it is not quite clear why the procedure described in steps (2.6.) - (2.9.) of the STEM procedure cannot be directly applied to (2.11.).

Another application of interactive goal programming techniques is contained in Monarchi et al [1975]. In the latter study the unweighted goal programming model is first solved in order to find a provisional feasible solution. The next step is to propose this initial solution to the decision-maker. When the decision-maker judges certain outcomes as unsatisfactory, the weights corresponding to the objectives concerned are increased in order to find a set of compromise solutions which are satisfactory for the decision-maker. The procedure is repeated until a final most satisfactory solution is found.

The latter procedure is rather practical, but yields the problem of a possibly large number of iterations. To a certain extent the whole spectrum of values related to unsatisfactory objectives might even be presented to the decision-maker. This leads to the question whether it is possible to structure in a more systematic way the successive phases of the interactive procedure. This question will be touched upon later.

A similar approach is contained in Price [1976] in which the author assumes that goals can be ranked in a hierarchical fashion. During the successive stages of the analysis the results of this hierarchical procedure are displayed to the decision-maker, so that the decision-maker may vary the ranking of the goals and achieve a more adequate compromise solution.

2.4. Other Sequential Optimization Methods.

Besides goal programming, with its possibility to include lexicographic orderings, there are several other methods treating the goal variables sequentially in decreasing order of importance. Two different approaches will be discussed briefly here.

Van Delft and Nijkamp [1977] discuss the use of the hierarchical optimization method. This approach is meant for problems in which the goal variables can be ranked in an ordinal way as 'most important', 'next most important', etc. The method consists of a series of constrained programming problems. First, the most important goal variable $g_1(\underline{x})$ is optimized (subject to a set of constraints), yielding the optimal value $g_1^{\underline{x}}$. For this optimal value, a tolerance limit is defined as an inequality constraint, which is added to the already existing set of constraints. Then the next most important goal variable $g_2(\underline{x})$ is optimized subject to the new set of constraints. For the optimal value $g_2^{\underline{x}}$ again a tolerance limit is defined and added to the set of constraints. This is followed by the optimization of $g_3(\underline{x})$ and so on, until all goal variables have been treated successively. Van Delft and Nijkamp also present some variants of this optimization procedure, which differ in the information need concerning relative preferences with respect to the hierarchically ordered set of goal variables and in the quantity of information from previous stages in the model used in following stages. The choice in favour of one of this variants may depend on the quantity and quality of information available.

Another approach is followed by Holmes [1971]. This author presents an ordinal method of evaluating a finite number of alternatives in terms of a set of possibly unquantifiable criteria. These criteria may be called goal variables - as in the preceding (sub)sections- except for the fact they may have no numerical description. Consequently the contribution of the alternatives to each of the criteria cannot be measured on a cardinal scale. However, it is assumed that these contributions can be measured on an ordinal scale. This means that the decision maker must decide which attribute contributes 'best' to a given criterion, which 'second best' and so on. Furthermore it is assumed he is able to define an ordinal ranking between the criteria. In these rankings criteria may be judged to be 'more important than', 'less important than' or 'neither more nor less important than' other criteria. This means that some criteria may obtain the same rank (The same holds true for the contributions to the criteria). Any alternative can be described in terms of its contributions to the various criteria. For example, the contributions of an alternative A

may be given by:

$$(2.12) \quad \underline{C}'' = (3, 1, 1, 4, 2)$$

which means that this alternative is third best for the first criterion, best for the second and third criteria, fourth best for the fourth criterion and second best for the fifth. Let us further assume that the first two criteria are of equal importance, but more important than the third, which on its turn is more important than the fourth and fifth criteria, while the last ones are again of equal importance. The rank of each criterion is then added to its respective score for alternative A. We then get:

$$(2.13) \quad \underline{C}' = (4, 2, 3, 7, 5)$$

When an alternative assumes the lowest possible value with respect to one of the criteria (which is 2 for an alternative contributing best to one of the class of most important criteria), the alternative is said to take the first position with respect to that criterion. The next higher value is associated with the second position and so on. The positions for alternative A are thus given by:

$$(2.14) \quad \underline{C} = (3, 1, 2, 6, 4)$$

For each alternative such a position vector can be calculated. The alternative to be chosen contains the largest number of first positions. If two or more alternatives have an equal number of first positions, then the one with the largest number of second corresponding positions will be chosen. As emphasized by Holmes (p.191) the method is not a calculation in the accepted sense of the word, but rather a way of presenting a complex argument systematically. In order to leave decision-makers free to deviate from the results obtained by the method described, Holmes suggests some variants of that method, which can also be found in the original article.

The conclusion can be drawn from this section that several sequential optimization techniques in the field of multi-criteria analysis may be distinguished. Now the question arises whether some of the features of such sequential optimization techniques can be used in order to construct an interactive goal programming procedure. This will be the subject of the next sections.

3 INTERACTIVE MULTIPLE GOAL PROGRAMMING (I.M.G.P.)

In the following sections we discuss a new, interactive version of multiple goal programming. Subsection 3.1. lists the prerequisites of the method in its most general form. The method itself is presented in 3.2. and illustrated with a simple example in 3.3. In section 4 we discuss the technical elaboration of the method.

3.1. Introduction to I.M.G.P.

In the method described here we assume that the decision-maker has defined a number of goal variables $g_1(\underline{x}), \dots, g_m(\underline{x})$, being functions of the instrumental variables x_1, \dots, x_n (\underline{x} in vector notation). We assume that the decision-maker's preferences with respect to the possible configurations of goal variables can be represented, at least in principle, by a preference function f . The value of the preference function is thus determined by the values of the goal variables which are in turn determined by the values of the instrumental variables \underline{x} . This means that the maximal value of the preference function $f(\underline{x})$ must be found by choosing appropriate values for \underline{x} . In doing so, the choice of \underline{x} is subject to a set of constraints describing the feasible region R . Consequently, we have to find those values of $\underline{x} \in R$, which determine values of the goal variables $g(\underline{x})$, for which $f(\underline{x})$ is maximized. However, the description of the problem is not complete without explicating the assumptions about the functional form of the functions, variables and restrictions. Another important factor is the nature of the information provided by the decision-maker concerning his preferences.

To start with the assumptions regarding the functional form of the goal variables, the preference function f and the restrictions, we assume the feasible region R to be convex. Interactive multiple goal programming aims at maximizing $f(\underline{x})$ within the feasible region R . During successive iterations, also the goal variables $g_i(\underline{x})$ must be maximized within R . We therefore presuppose that both $f(\underline{x})$ and $g_i(\underline{x})$ ($i=1, \dots, m$) are concave functions of \underline{x} . Besides, we assume f to be a concave function in the $g_i(\underline{x})$ (The latter is a weaker condition than the frequently adopted assumption⁴⁾ that the first (partial) derivative of f with respect to each of the $g_i(\underline{x})$'s is either positive or negative. So, there are three concavity conditions from which no

⁴⁾ For instance, all methods aimed at deriving efficient solutions.

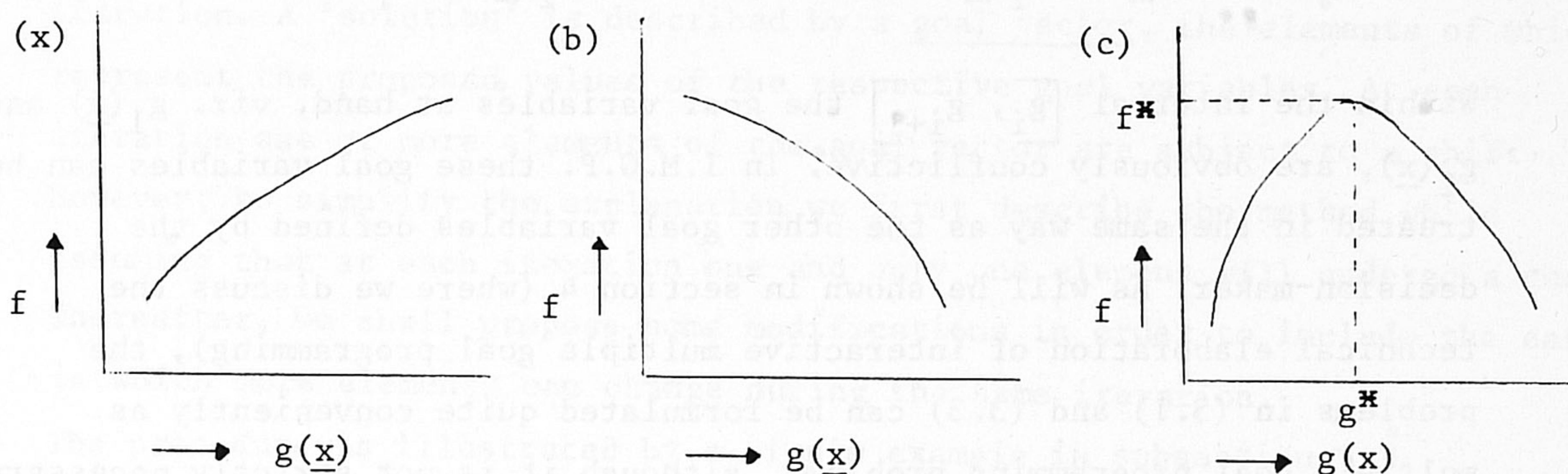
one can be eliminated. For instance, f being concave in the $g_i(\underline{x})$'s and the $g_i(\underline{x})$'s being concave in \underline{x} not necessarily⁵⁾ implies f being concave in \underline{x} .

As mentioned earlier in this subsection the kind of information required from the decision-maker offers another important characterization of the problem at hand. To start with, we assume the representation of the feasible region is known to the analyst, together with the functional form of all relevant goal variables. Although accurate a priori information about the decision-maker's preference function is difficult to obtain, there is at least some information in many decision problems. It would be a pity to let this information unused. On the other hand it must be recognized that the a priori information is not always waterproof. Furthermore, the decision-maker may change his mind while dealing with the problem. Interactive multiple goal programming tries to use the a priori information in a fruitful manner, by offering the decision-maker the opportunity during the interactive process to reconsider his a priori information. The a priori information used in this method consists of aspiration levels and of information about existing priorities. Aspiration levels may be of a psychological-institutional or of a technical nature. The first are expressed as a matter of habit, as a matter of keeping up (or beating up) with the Joneses or for some other reason. The second kind of aspiration levels occur for example when it is known that exceeding a production level of 100,000 units per year will sharply increase production costs. Such a production level may then be formulated as an aspiration level, because it is very likely that the trade-off between 'production level' and other goal variables changes at that point. Sometimes, the attainment of one aspiration has preemptive priority above the attainment of another aspiration. Also relative priority factors may be known. In both cases, the information may be used by interactive multiple goal programming.

Although f was assumed to be concave in the $g_i(\underline{x})$, it does not need to be a monotone non-decreasing or monotone non-increasing function of the $g_i(\underline{x})$, as shown for the simple example in figure 3.1., where f has been given as a function of one goal variable only (hence we omit the subscript of $g_i(\underline{x})$ in this case).

⁵⁾ See Nijkamp and Spronk [1977], subsection 4.1. for an example.

Figure 3.1. The preference function f as a function of one goal variable only.



Because f is not always a known function of $g(\underline{x})$, it is very helpful when we know that f is monotone non-decreasing or monotone non-increasing in $g(\underline{x})$, for then we can accomplish the maximization of f by means of the maximization (or minimization, respectively) of the goal variable $g(\underline{x})$. If f has a shape like in figure 3.1.c, the procedure is less straightforward. Let us first assume, the decision-maker knows that his preference function is maximized for $g(\underline{x})=g^*$. The problem can then be formulated as:

$$(3.1) \begin{cases} \text{Max } \{ g(\underline{x}) \} & \text{subject to } \underline{x} \in R \text{ and } g(\underline{x}) < g^* \\ \text{and} \\ \text{Min } \{ g(\underline{x}) \} & \text{subject to } \underline{x} \in R \text{ and } g(\underline{x}) > g^*. \end{cases}$$

In fact we have split up the goal variable $g(\underline{x})$ in two other goal variables, one to be maximized (which will be referred to as $g_1(\underline{x})$) and one to be minimized (which will be referred to as $g_2(\underline{x})$). Unfortunately, the optimal value g^* is often unknown. For that case, we assume the decision-maker can specify a number of aspiration levels g_1, g_2, \dots, g_k , such that:

$$(3.2) \quad f(g_1) < f(g_2) < \dots < f(g^*) > \dots > f(g_{k-1}) > f(g_k)$$

This means that the decision-maker can at least specify an interval, in which g^* can be found. Assuming $g_i < g^* < g_{i+1}$, we define the problem as

$$(3.3) \left\{ \begin{array}{l} \text{Max } \{ g(\underline{x}) = g_1(\underline{x}) \} \quad \text{for } \underline{x} \in R \quad \text{and } g_1(\underline{x}) \leq g_{i+1} \\ \text{and} \\ \text{Min } \{ g(\underline{x}) = g_2(\underline{x}) \} \quad \text{for } \underline{x} \in R \quad \text{and } g_2(\underline{x}) \geq g_i \end{array} \right.$$

Within the interval $[g_i, g_{i+1}]$ the goal variables at hand, viz. $g_1(\underline{x})$ and $g_2(\underline{x})$, are obviously conflictive. In I.M.G.P. these goal variables can be treated in the same way as the other goal variables defined by the decision-maker. As will be shown in section 4 (where we discuss the technical elaboration of interactive multiple goal programming), the problems in (3.1) and (3.3) can be formulated quite conveniently as multiple goal programming problems, although it is not strictly necessary to restrict these problems to the multiple goal programming format. This will also be shown in the fourth section.

3.2. Description of the Solution Procedure.

In this interactive procedure a new solution must be calculated at each iteration. A 'solution' is described by a goal vector, the elements of which represent the proposed values of the respective goal variables. At each iteration one or more elements of the goal vector are subject to a shift. However, to simplify the explanation we first describe the method while assuming that at each iteration one and only one element will undergo a change. Thereafter, we shall propose some modifications in order to include the case in which more elements can change during the same iteration. The procedure is illustrated by a simple example in subsection 3.3.

Step 0 - First identify the goal variables $g_i(\underline{x})$, $i = 1, \dots, m$, as linear or piecewise linear functions of \underline{x} , the vector of instrumental variables x_1, x_2, \dots, x_n . Then specify the set of feasible solutions R , within which the preference function f (not made explicit so far) must be maximized. Notice that some $g_i(\underline{x})$ may not be defined for all $\underline{x} \in R$, due to its division in two goal variables (as in (3.1) and (3.3)), which occurs when f is not a monotone function of $g_i(\underline{x})$.

Step 1 - Now successively maximize (or minimize when f is decreasing in $g_i(\underline{x})$)⁶⁾ each of the goal variables $g_i(\underline{x})$, $i = 1, \dots, m$, separately. Thus

$$(3.4) \quad \text{Max } \left\{ g_i(\underline{x}) \right\} \quad \text{subject to} \\ \underline{x} \in R \quad \text{for } i = 1, 2, \dots, m.$$

Denote the maxima by g_i^* , $i = 1, 2, \dots, m$ and the corresponding combinations of the instrumental variables by \underline{x}_i^* , $i = 1, 2, \dots, m$. The solution, resulting from the maximization of $g_i(\underline{x})$ can then be given by the goal vector:

$$(3.5) \quad \left[g_1(\underline{x}_1^*), g_2(\underline{x}_2^*), \dots, g_i(\underline{x}_i^*), \dots, g_m(\underline{x}_m^*) \right] \\ \text{where of course } g_i(\underline{x}_i^*) \equiv g_i^*.$$

An 'ideal' solution \underline{I} , although generally infeasible, is provided by the goal vector of the calculated 'absolute' maxima of the goal variables:

$$(3.6) \quad \underline{I} = \left[g_1^*, g_2^*, \dots, g_m^* \right]$$

6) To ease the exposition we assume f to be monotone-increasing in all $g_i(\underline{x})$. Whenever f is monotone decreasing in some $g_i(\underline{x})$, this goal variable must be handled analogously except for some sign reversals.

In contrast with this mostly infeasible 'ideal' solution, a 'most pessimistic' solution Q can be obtained with the help of the same information (3.4) and (3.5). Moreover such a pessimistic solution is frequently feasible (see however section 4.4). To find the most pessimistic solution let us first write the values of the goal variable $g_i(\underline{x})$ resulting from the successive maximizations (3.4) in the following vector (which is not a goal vector !):

$$(3.7) \quad \left[g_1(\underline{x}_1^*), g_1(\underline{x}_2^*), \dots, g_1^*, \dots, g_1(\underline{x}_m^*) \right]$$

Let us denote the smallest element of this vector by g_i^{\min} :

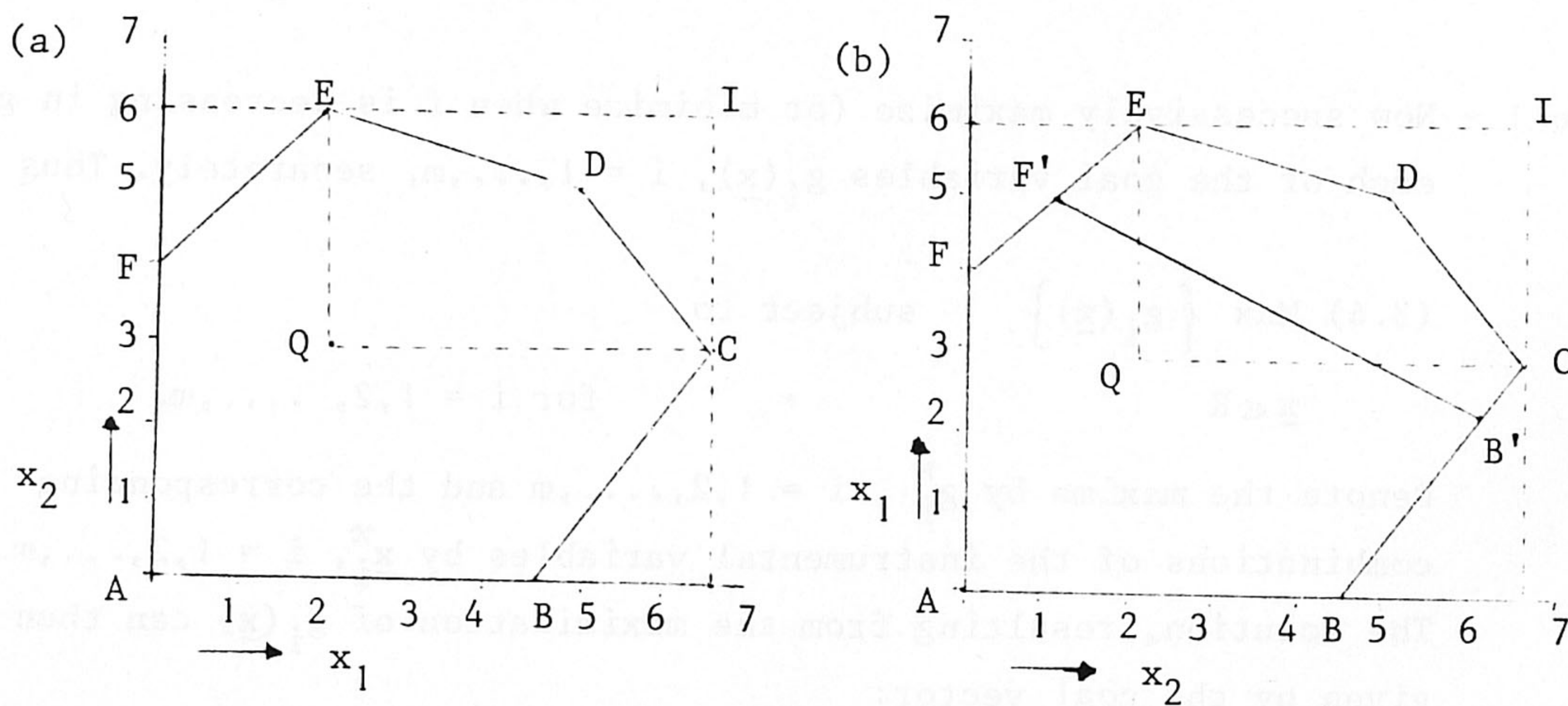
$$(3.8) \quad g_i^{\min} = \text{Min}_{j=1}^m \left\{ g_i(\underline{x}_j^*) \right\}$$

The most pessimistic solution Q is then given by the goal vector:

$$(3.9) \quad \underline{Q} = \left[g_1^{\min}, g_2^{\min}, \dots, g_m^{\min} \right]$$

To illustrate both (3.6) and (3.7) let us consider the simple examples in figure 3.2.

Figure 3.2. - An illustration of 'ideal' and 'pessimistic' solutions.



(a) feasible region: ABCDEF

(b) feasible region: B'CDEF'

In both cases the final solution must be found in the intersection of the feasible region and the rectangle QCIE.

In figure 3.2.a the feasible region R is given by ABCDEF, in which two goal variables, $g_1(\underline{x}) = x_1$ and $g_2(\underline{x}) = x_2$ should be maximized. As can be seen easily, $g_1^* = 6.5$ and $g_2^* = 6$. The ideal solution thus becomes:

$$(3.10) \underline{I} = [6.5, 6]$$

which can be found in figure 3.2a. Note that I exists but is not feasible. Clearly, the pessimistic solution Q can be identified as:

$$(3.11) \underline{Q} = [2, 3]$$

which solution, also shown in figure 3.2.a, is both existent and feasible. In figure 3.2.b. the same goal variables should be maximized. However, the feasible region has been reduced to B'CDEF'. The 'ideal' point I and the pessimistic point Q are exactly equal to those in figure 3.2.a. In this case, however, both I and Q are infeasible.

Thus the first step of the procedure shows that a final solution vector \underline{S}^* is bounded both by the ideal solution \underline{I} and the pessimistic solution Q (for different reasons, however). We have now:

$$(3.12) \underline{S}^* \leq \underline{I}$$

because, by definition none of the elements of \underline{S}^* can exceed the analogous element of \underline{I} and because $\underline{S}^* = \underline{I}$ only in the case where all goal variables coincide (which case may be generally left out of consideration). On the other hand, we have

$$(3.13) \underline{Q} \leq \underline{S}^*$$

Although other solutions do exist in this case it would be unwise to choose them. Assume there was a solution \underline{S}^{**} which was judged to be optimal and nevertheless did not satisfy (3.13), meaning there would be at least one i for which $g_i(\underline{x})$ is lower than g_i^{\min} . Clearly, \underline{S}^{**} can not be optimal, because one or more other solutions exist for which $g_i(\underline{x})$ is at least equal to g_i^{\min} , while the other values in \underline{S}^{**} remain the same. Because we assumed the preference function for all $g_i(\underline{x})$ to be monotone increasing, such a solution is preferred to \underline{S}^{**} . In section 4 we further discuss the character of the solutions, including their existence and feasibility.

In I.M.G.P. a solution \underline{S} can be considered as a set of minimum values imposed on the respective goal variables. The method starts with a solution having very low minimum values. This solution can be improved by raising one or more of the minimum values. For each solution \underline{S} we can calculate the potential shifts in the minimum values, by which we mean the maximal improvement of the value of a single goal variable subject to the condition that the other goal variables equal or exceed their respective minimum values. When for instance in figure 3.2. the values $g_1^{\min} = 2$ and $g_2^{\min} = 3$ are considered as minimum values for $g_1(\underline{x})$ and $g_2(\underline{x})$ respectively, the potential shift of $g_1(\underline{x})$ may continue until the point where it equals $g_1^* = 6.5$ (the potential shift of $g_2(\underline{x})$ may continue until the point where it equals $g_2^* = 6$). It is clear, that these potential shifts cannot be realized simultaneously. Furthermore, when one of the minimum values is augmented during the interactive process, this implies that the potential shift of one or more other goal variables decreases. Such a decline can be considered as the 'sacrifice' (or 'cost') needed to realize the higher minimum values for the first mentioned goal variables. To give a comprehensive idea of the potential shifts, we introduce for each solution \underline{S} a $(2 \times m)$ potence matrix P , the columns of which can be associated with the respective goal variables. The upper row shows the respective maxima of the goal variables $g_i(\underline{x})$, $i = 1, \dots, m$ when they are maximized subject to the minimum conditions which are listed in the lower row of P . Thus:

$$(3.14) \quad P = \begin{bmatrix} g_1^*(\underline{x})_{\underline{S}} & \cdot & \cdot & \cdot & g_m^*(\underline{x})_{\underline{S}} \\ g_1(\underline{x})_{\underline{S}} & \cdot & \cdot & \cdot & g_m(\underline{x})_{\underline{S}} \end{bmatrix},$$

where $g_i(\underline{x})_{\underline{S}}$ stands for the value of $g_i(\underline{x})$ in the solution \underline{S} and $g_i^*(\underline{x})_{\underline{S}}$ for the maximum of $g_i(\underline{x})$ given the solution \underline{S} . As suggested in the above example, when a solution \underline{S}_1 is given by the pessimistic values g_i^{\min} , the potence matrix P_1 is given by:

$$(3.15) \quad P_1 = \begin{bmatrix} g_1^* & \cdot & \cdot & \cdot & g_m^* \\ \min g_1 & \cdot & \cdot & \cdot & \min g_m \end{bmatrix}$$

Step 2 - Next compile the information available about the decision-maker's aspiration levels. To start with aspiration levels of $g_i(\underline{x})$, let us define the 'artificial' aspiration levels g_{i1} and g_{ik_i} as

$$(3.16) \begin{cases} g_{i1} = \min g_i \\ g_{ik_i} = g_i^* \end{cases}$$

assuming there are $k_i - 2$ aspiration levels of $g_i(\underline{x})$ provided by the decision-maker which have the property

$$(3.17) \quad g_{i1} < g_{i2} < g_{i3} < \dots < g_{ik_i-1} < g_{ik_i}$$

As suggested before (see section 2) the decision-maker may have assigned preemptive priorities between aspiration levels of different goal variables. If available, this information should be stored because it can be taken into account in the fifth step of the procedure.

In the following steps the decision-maker is first confronted with an initial solution (step 3). Then he has to indicate whether this solution should be improved or not (step 4). If not, the procedure terminates. Otherwise, the decision-maker has to point out which goal variable should be augmented (step 5). The procedure then determines a new (increased) value for this goal variable (step 6), which is next presented to the decision-maker, together with some information regarding the shifts in the potency matrix (in case this new solution would be accepted). In step 8 the decision-maker has to judge whether these 'sacrifices' counterbalance the proposed improvement of the solution. If so, the procedure returns to step 4, where the decision-maker has to indicate whether the solution should be further improved. If the sacrifices are judged to be too heavy, the proposed increase of the goal level is obviously too large. Then the procedure calculates a lower trial value for the goal variable (step 9), which in its turn has to be evaluated by the decision-maker (step 8).

In order to determine how much a selected goal variable must be augmented, we use a m -dimensional auxiliary vector $\underline{\delta}$ with elements δ_j ($j = 1, \dots, m$) corresponding to the goal variables $g_j(\underline{x})$, $j=1, \dots, m$.

We define δ_j as the difference of the lowest level of $g_j(\underline{x})$ being rejected by the decision-maker and the level of $g_j(\underline{x})$ in the accepted (current) solution \underline{S}_i (for example, compared with a solution $\underline{S}_i = (3, 5, 2)$ a proposed solution $\hat{\underline{S}}_{i+1} = (3, 9, 2)$ could have been rejected; in this case $\delta_2 = 4$). During step 2 of the procedure no proposals have been made yet to the decision-maker and consequently no goal level can have been rejected (accordingly $\delta_j = 0$ for $j = 1, \dots, m$). In step 8 proposed goal levels may be rejected, so that some of the δ_j 's can become positive.

Step 3 - Define the initial solution, given by the goal vector \underline{S}_1 , as

$$(3.18) \quad \underline{S}_1 = [g_{11}, g_{21}, \dots, g_{m1}],$$

which is thus equal to the pessimistic solution (3.9). Present this solution together with the potency matrix P_1 to the decision-maker.

Step 4 - If the proposed solution is satisfactory, accept it; if not, proceed to step 5. Denote the subset of the feasible region R , defined by the goal levels in \underline{S}_i , by R_i .

Step 5 - The decision-maker then has to answer the following question: "Given the solution presented to you (denoted by \underline{S}_i), which goal variable should be augmented first?" Let us assume the decision-maker wishes to raise the value of the j 'th goal variable.

Step 6 - In the fifth step we assumed that the decision-maker wants to improve the solution \underline{S}_i , by augmenting the j 'th goal variable. In this step a proposal solution $\hat{\underline{S}}_{i+1}$ is calculated, differing from the solution \underline{S}_i only with respect to the value of the j 'th goal variable. This new value $g_j(\underline{x})_{\hat{\underline{S}}_{i+1}}$ can be calculated in two ways. First, the model may use the list of aspiration levels for $g_j(\underline{x})$, as compiled in step 2. In this case the model simply chooses the first aspiration level which is preferred to the level of $g_j(\underline{x})$ in solution \underline{S}_i (denoted by $g_j(\underline{x})_{\underline{S}_i}$). For example, when $g_j(\underline{x})_{\underline{S}_i}$ was equal to g_{j1} , being one of the aspiration levels listed in step 2, the proposal solution $\hat{\underline{S}}_{i+1}$ can be constructed by putting

$g_j(\underline{x})_{\hat{S}_{i+1}}$ equal to g_{j1+1} , being the next higher aspiration level listed in step 2. To investigate the sacrifices needed to reach this new solution, the constraint

$$(3.19) \quad g_j(\underline{x}) \geq g_{j1+1}$$

is introduced. The method then proceeds to step 7. The second way of calculating $g_j(\underline{x})_{\hat{S}_{i+1}}$ is opportune when it is known that the decision-maker does not accept the next higher aspiration level listed in step 2 (such information may have come available during earlier iterations of the procedure). Let us assume that we know that the decision-maker has evaluated a solution $g_j(\underline{x}) = g_j(\underline{x})_{S_i} + \delta_j$ and that he judged this rise of $g_j(\underline{x})$ not to compensate the sacrifices needed to reach it. We know already that $g_j(\underline{x})_{S_i}$ was too low. Therefore we choose a value of $g_j(\underline{x})_{\hat{S}_{i+1}}$ exactly between these values:

$$(3.20) \quad g_j(\underline{x})_{\hat{S}_{i+1}} = g_j(\underline{x})_{S_i} + \frac{1}{2} \cdot \delta_j$$

As before, the sacrifices needed to reach this new solution must be investigated. Therefore, we introduce the constraint:

$$(3.21) \quad g_j(\underline{x}) \geq g_j(\underline{x})_{S_i} + \delta_j$$

and proceed to step 7.

In order to know whether $g_j(\underline{x})$ must be augmented in the first or in the second way described above, again δ_j can be used. If $\delta_j = 0$, the first way should be used, which means that we can choose for $g_j(\underline{x})_{\hat{S}_{i+1}}$ the first value in the list of aspiration levels in step 2, which exceeds $g_j(\underline{x})_{S_i}$. If $\delta_j > 0$, we know that a value $g_j(\underline{x}) = g_j(\underline{x})_{S_i} + \delta$ is judged by the decision-maker as being too expensive in terms of

the sacrifices needed to reach this value. Therefore we choose a proposal value $g_j(\underline{x})_{\hat{S}_{i+1}} = g_j(\underline{x})_{S_i} + \frac{1}{2} \cdot \delta_j$.

Step 7 - Add the restriction formulated in step 6 or in step 9 (see later) to the set of restrictions describing the feasible region R_i . Denote the part of the feasible region that remains feasible after adding the additional restriction by \hat{R}_{i+1} (Notice that \hat{R}_{i+1} may be empty, which occurs when the augmented aspiration level cannot be attained within the feasible region R_i).

Calculate next a new potency matrix (see step 4), but now subject to the new set of restrictions. Label this potency matrix \hat{P}_{i+1} (the hat is added because we still have a proposed and not an accepted solution). Proceed to step 8.

Step 8 - Confront the decision-maker with S_i and \hat{S}_{i+1} on the one hand and with P_i and \hat{P}_{i+1} on the other hand. The shifts in the potency matrix can be viewed as a 'sacrifice' for reaching the proposed solution. If the decision-maker judges this sacrifice to be justified, accept the proposed solution by putting $S_{i+1} = \hat{S}_{i+1}$ and $P_{i+1} = \hat{P}_{i+1}$. By accepting $S_{i+1} = \hat{S}_{i+1}$ we may have accepted a higher level of a goal variable $g_j(\underline{x})$, for which $\delta_j > 0$. Because δ_j was defined as the difference between the lowest level of $g_j(\underline{x})$ being rejected by the decision-maker and the level of $g_j(\underline{x})$ in the accepted (current) solution, we now have to adapt δ_j for the change in the current solution. When $\delta_j > 0$, the successive proposal solutions for $g_j(\underline{x})$ are found by adding $\frac{1}{2} \delta_j$ to a current accepted solution (step 6) or subtracting $\frac{1}{2} \cdot \delta_j$ from a rejected proposal solution (step 9). This means that by accepting a proposal solution, the value of δ_j is exactly halved. Therefore put $\delta_j = \frac{1}{2} \delta_j$ when $g_j(\underline{x})$ has been increased and this increase has been accepted by the decision-maker. (Note that when $\delta_j = 0$, meaning that no augmented value of $g_j(\underline{x})$ has been rejected thus far, the value of δ_j does not change). Then return to step 4⁷⁾ (where the value of $g_j(\underline{x})$ can be augmented again, if desired). If the decision-maker considers the sacrifice unjustified (or when \hat{R}_{i+1} is empty), the proposed value of $g_j(\underline{x})$ is obviously too high. Therefore, drop then the constraint added in step 7. Then proceed to step 9.

7) Note however, that the acceptance of this solution may make some of the aspiration levels listed in step 2 unattainable. These aspiration levels must be eliminated from the list of stated aspiration levels.

Step 9 - We now know that the decision-maker in the given situation wants a value of $g_j(\underline{x})$ which exceeds its value in \underline{S}_i , but which is smaller than the value in \underline{S}_{i+1} (By definition, $\delta_j = g_j(\underline{x})_{\hat{\underline{S}}_{i+1}} - g_j(\underline{x})_{\underline{S}_i}$). Because this is the only information available, it is reasonable to choose the new value of $g_j(\underline{x})$ exactly between the values of $g_j(\underline{x})$ in \underline{S}_i and $\hat{\underline{S}}_{i+1}$ respectively. Label this new proposal value by $\hat{\underline{S}}_{i+1}$, add the restriction that $g_j(\underline{x})$ must equal or exceed the new proposal value and return to step 7 in order to calculate a new potency matrix \hat{P}_{i+1} .

Besides the a priori information in step 2, the decision-maker has to give his judgments in step 4, 5 and 8. In step 4 he has to indicate whether a given solution should be improved or not, while in step 5 he has to point out which goal variable should be augmented. In step 8 he has to evaluate whether the augmentation (as proposed by the model) counterbalances the loss of potency induced by it.

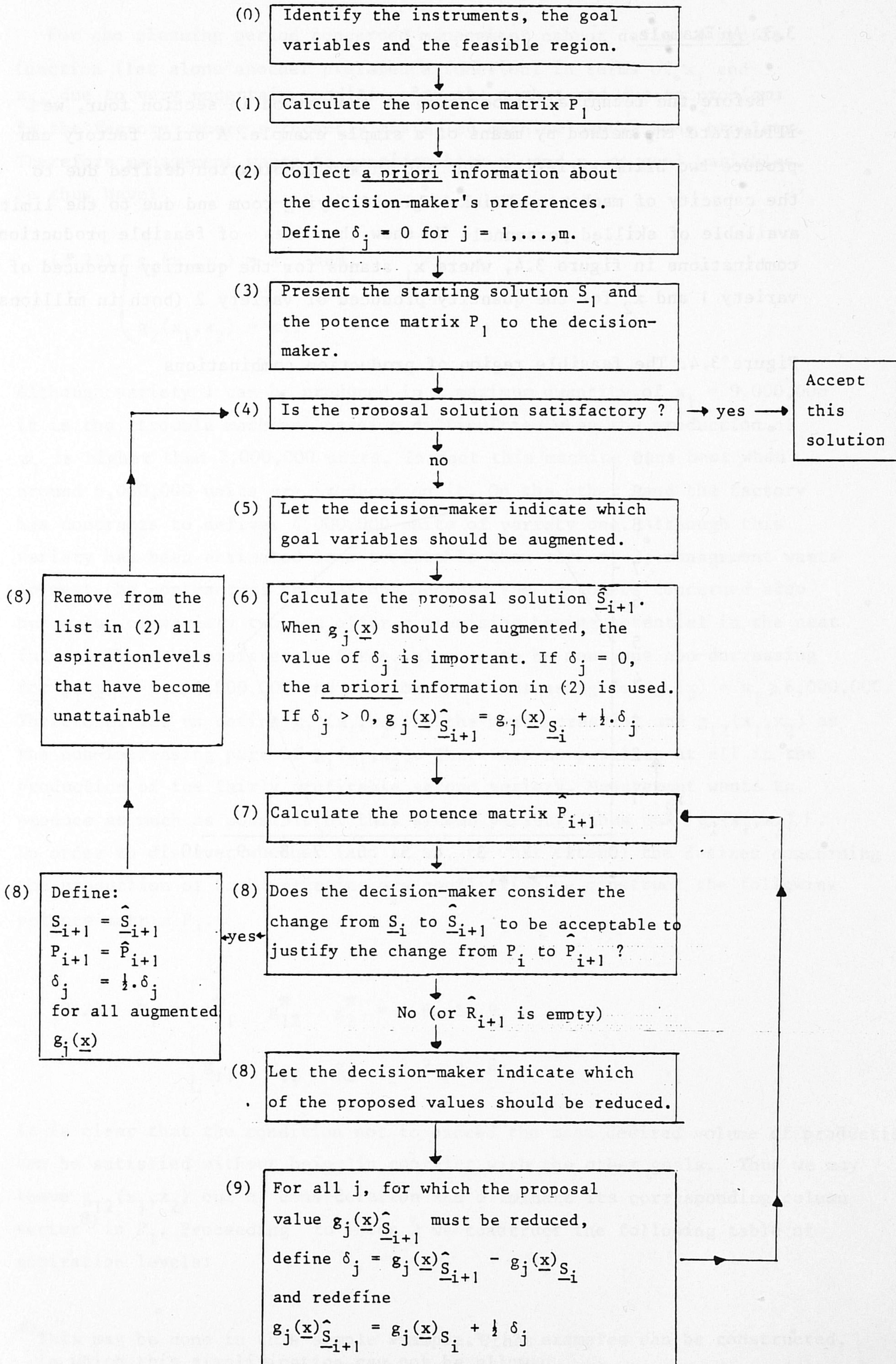
When the decision-maker is not able to indicate which single goal variable should be improved in value, we assume he is at least capable to define a set of goal variables which are to be augmented in value. Then, the procedure must be modified slightly. We only give the modifications:

- Step 5^{*} - Instead of one goal variable, more than one goal variable to be augmented is chosen.
- Step 6^{*} - Find proposal values for all goal variables selected in 5^{*} in the way a new value was calculated for the single goal variable in 6.
- Step 7^{*} - Before calculating the potency matrix \hat{P}_{i+1} the set of restrictions is extended with the restrictions formulated in 6^{*}.
- Step 8^{*} - When the decision-maker judges the sacrifices too heavy to approve the solution, he should indicate which of the goal variables having a higher value in $\hat{\underline{S}}_{i+1}$ than in \underline{S}_i , must be reduced in 9^{*}.

Step 9* - Calculate a new proposal solution by reducing all goal variables indicated in 8* in the same way in which the single goal variable was reduced in 9.

To illustrate this procedure we have given a flow chart in figure 3.3.

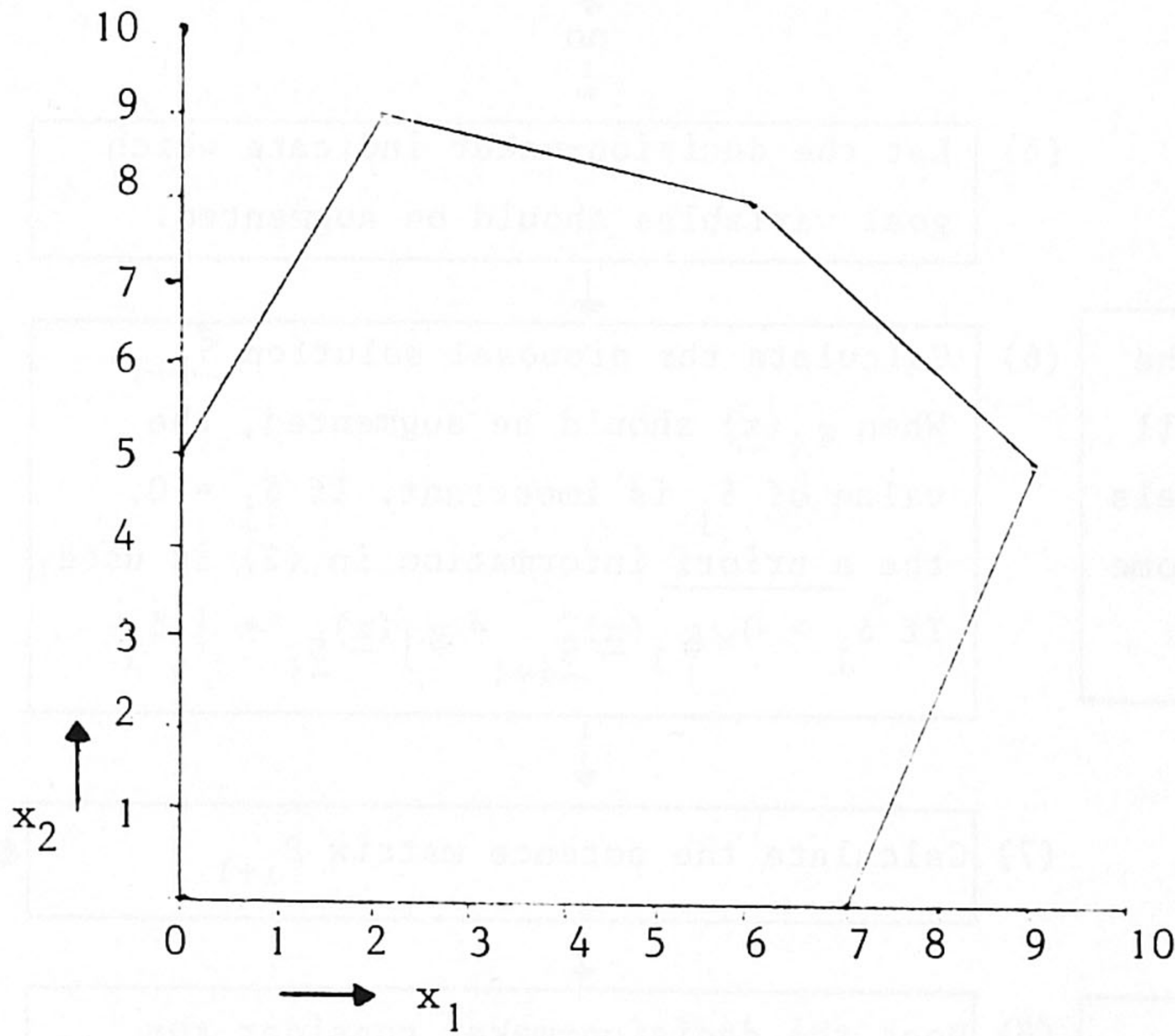
Figure 3.3. A flow chart of the extended interactive goal programming procedure



3.3. An Example.

Before the technical elaboration of the method in section four, we illustrate the method by means of a simple example. A brick factory can produce two brick varieties, but not in any combination desired due to the capacity of machines, brick-kiln and drying-room and due to the limited available of skilled personnel. We show the area of feasible production combinations in figure 3.4, where x_1 stands for the quantity produced of variety 1 and x_2 for the quantity produced of variety 2 (both in millions).

Figure 3.4. The feasible region of production combinations



For the planning period concerned management cannot define a profit function (let alone another preference function) in terms of x_1 and x_2 , due to very uncertain conditions of the market and due to problems in the factory where a recently installed machine causes many problems. Therefore management wants to consider both x_1 and x_2 as goal variables. We thus have:

$$(3.22) \begin{cases} g_1(x_1, x_2) = x_1 & \text{and} \\ g_2(x_1, x_2) = x_2. \end{cases}$$

Although variety 1 can be produced in a maximum quantity of $x_1 = 9,000,000$ it is the 'trouble machine' causing difficulties when the production of x_1 is higher than 7,000,000 units. In fact this machine runs best when around 6,000,000 units are produced on it. On the other hand the factory has contracts to deliver 4,000,000 units of variety one. Although this variety has been estimated less profitable than variety 2, management wants to meet the contractual obligations because the customers concerned also buy a lot of variety two and offer a promising buying potential in the near future. Thus the preferences for $g_1(\underline{x})$ seem to be monotone non-decreasing for $g_1(\underline{x}) = x_1 \leq 6,000,000$ and monotone non-increasing for $g_1(\underline{x}) = x_1 \geq 6,000,000$. Therefore, let us define $g_{11}(x_1, x_2)$ as the non-decreasing and $g_{12}(x_1, x_2)$ as the non-increasing part of $g_1(x_1, x_2)$. There are no problems at all in the production of the fairly profitable second variety. Management wants to produce as much as possible of this second variety (thus $\max\{g_2(x_1, x_2)\}$). In order to discover whether (and if so, to what extent) the desires concerning the production of both varieties are conflictive we construct the following potency matrix P_1 .

$$(3.23) \quad P_1 = \begin{bmatrix} g_{11}^* & g_{12}^* & g_2^* \\ g_{11} & g_{12} & g_2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 9 \\ 2 & 6 & 8 \end{bmatrix}$$

It is clear that the condition not to exceed the most desired volume of production can be satisfied without being in conflict with the other goals. Thus we may leave $g_{12}(x_1, x_2)$ out of consideration and eliminate its corresponding column vector⁸⁾ in P_1 . Proceeding to step 2 we construct the following table of aspiration levels:

⁸⁾This may be done in this simple example. Other examples can be constructed, in which this simplification can not be allowed.

$$(3.24) \begin{cases} g_{113} = 6 & g_{22} = 9 \\ g_{112} = 4 & g_{21} = 8 \\ g_{111} = 2 \end{cases}$$

The table was constructed as follows. The most optimistic and pessimistic values of $g_{11}(x_1, x_2)$ were 6 and 2 millions, as can be read from (3.16). The only aspiration level defined by management is 4 mln, the production of variety one, needed to meet the contracts. In this way there are three aspiration-levels for $g_{11}(x_1, x_2)$ of which the most preferred is labeled g_{113} , the next preferred g_{112} and the less preferred g_{111} . Because management did not define any aspiration level for $g_2(x_1, x_2)$, its most and less preferred values can be labeled simply as g_{22} and g_{21} respectively.

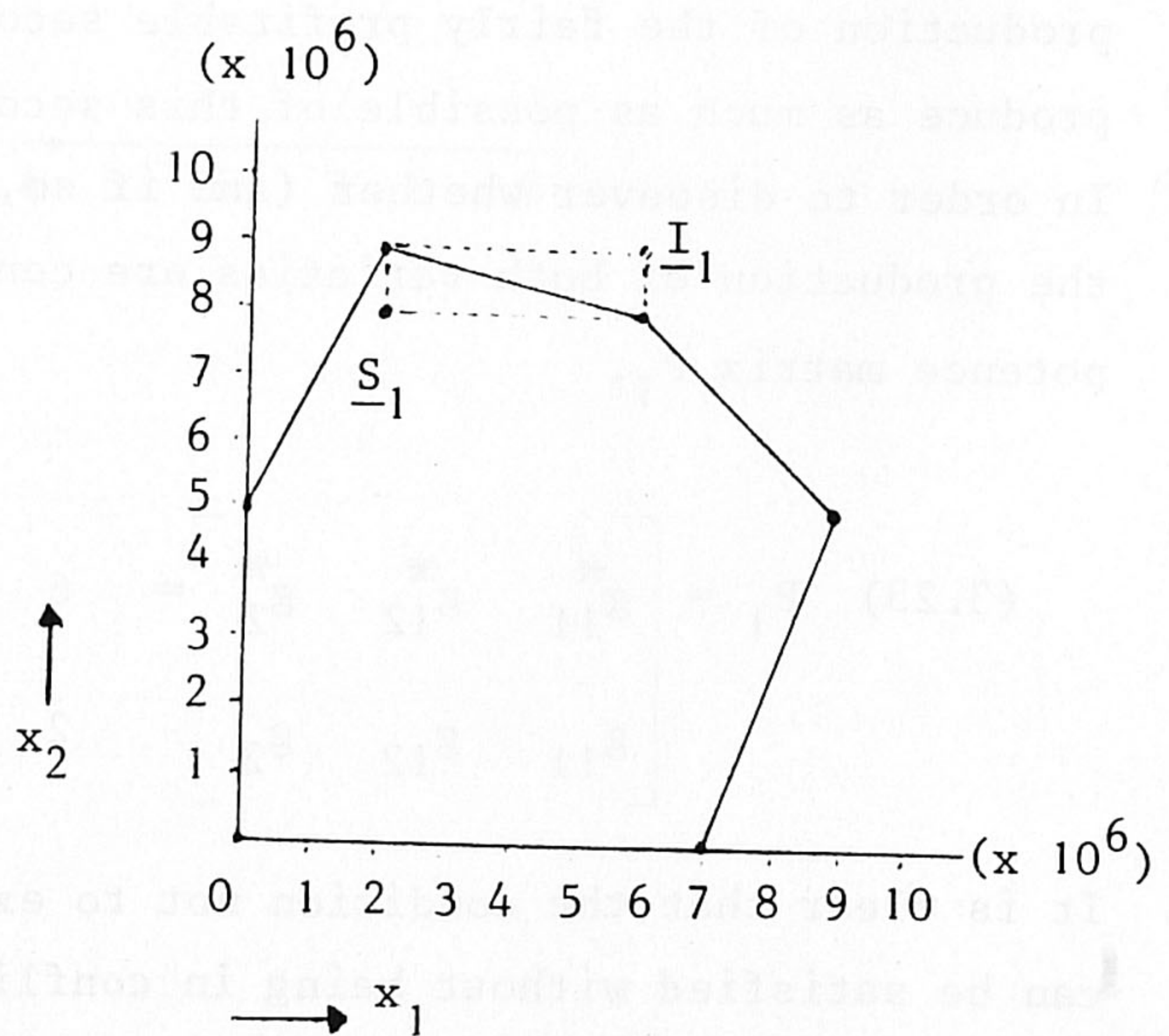
In the third step the most pessimistic solution, defined as

$$(3.25) \underline{S}_1 = [g_{111}, g_{21}] = [2, 8]$$

is presented to the decision-maker, together with the potency matrix P_1 . We illustrated this pessimistic starting solution \underline{S}_1 together with the 'ideal' solution \underline{I}_1 in figure 3.5.

Figure 3.5.

The feasible region R of the production combinations (the pessimistic and ideal starting solutions are indicated by \underline{S}_1 and \underline{I}_1)



We assume the decision-maker (in step 4) is not satisfied with \underline{S}_1 and that in step 5 he proposes to augment the production of the first variety. In step 6 we then can construct a new proposal solution with the help of table (3.22). We get

$$(3.26) \quad \hat{\underline{S}}_2 = [g_{112}, g_{21}] = [4, 8]$$

Having added the proposed values of the goal variables as restrictions to the already existing set of restrictions, we can in step 7 calculate the potency matrix belonging to $\hat{\underline{S}}_2$ as

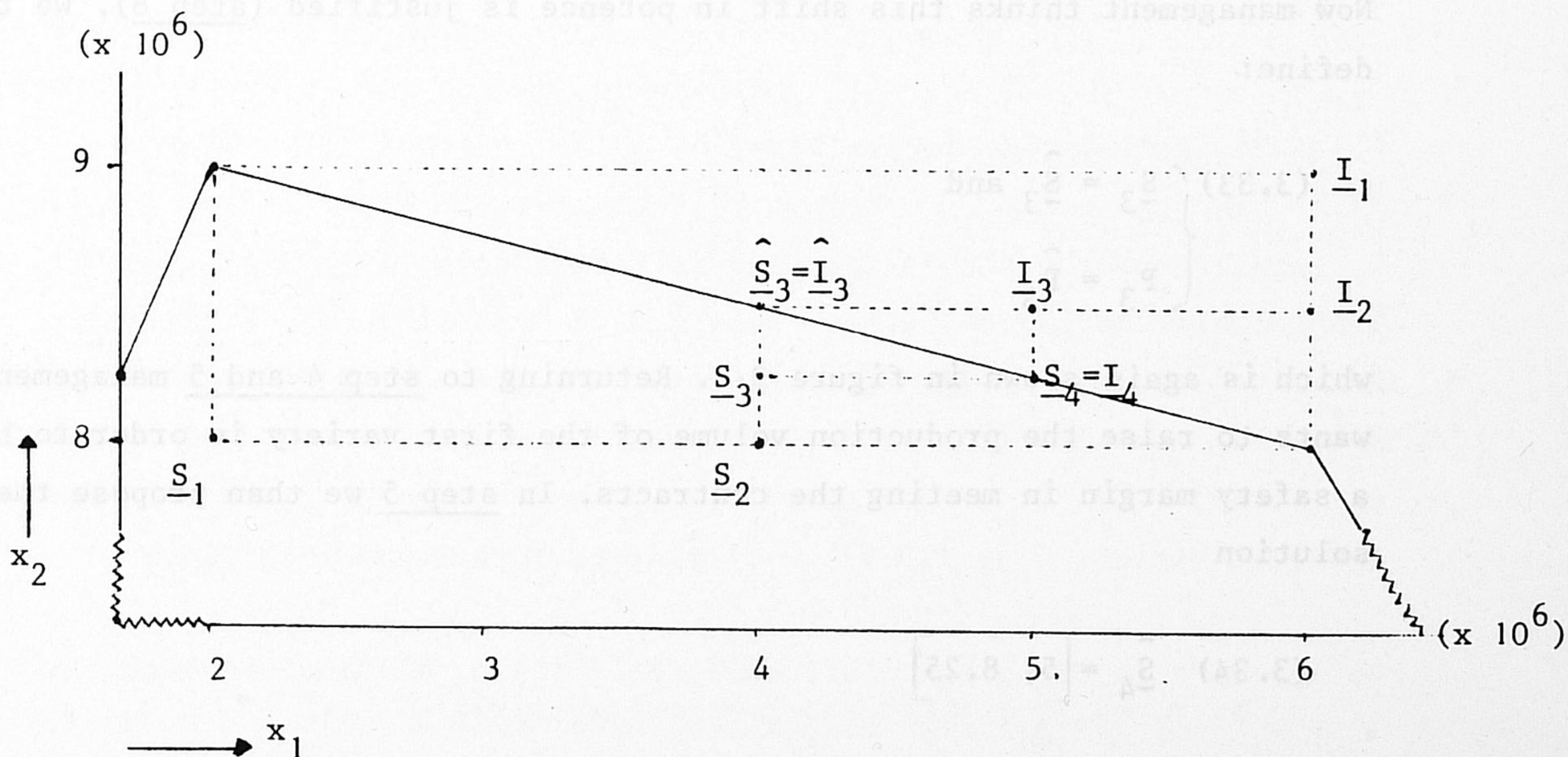
$$(3.27) \quad \hat{\underline{P}}_2 = \begin{bmatrix} 6 & 8,5 \\ 4 & 8 \end{bmatrix}$$

When the decision-maker thinks the sacrifice needed to reach the new solution is justified, we can define

$$(3.28) \quad \begin{cases} \underline{S}_2 = \hat{\underline{S}}_2 & \text{and} \\ \underline{P}_2 = \hat{\underline{P}}_2 \end{cases}$$

which solution can be found in figure 3.6 where we only show the relevant part of the feasible region R.

Figure 3.6. All solutions from the starting solution up to and including the final solution.



Returning to step 4 we assume the decision-maker wants to raise, given \underline{S}_2 , the production of the second variety (step 5). In step 6 we propose the only aspiration level for $g_2(x_1, x_2)$ left, which is now 8.5. We thus have the proposal solution

$$(3.29) \quad \hat{\underline{S}}_3 = [4, 8.5]$$

for which we can calculate (step 7) the following potence matrix:

$$(3.30) \quad \hat{P}_3 = \begin{bmatrix} 4 & 8.5 \\ 4 & 8.5 \end{bmatrix}$$

This solution too is shown in figure 3.6. In the 8th step management argues that this sacrifice ($P_2 \rightarrow \hat{P}_3$) does not justify the improvement of the solution ($\underline{S}_2 \rightarrow \hat{\underline{S}}_3$). We then have to go to step 9 in which we have to define a new, lower proposal value for $g_2(x_1, x_2)$. By choosing this new proposal value exactly between its value in \underline{S}_2 and the old $\hat{\underline{S}}_3$ we get the new proposal:

$$(3.31) \quad \hat{\underline{S}}_3 = [4, 8.25]$$

for which we can calculate (by returning to step 7) the new potence matrix

$$(3.32) \quad \hat{P}_3 = \begin{bmatrix} 5 & 8.50 \\ 4 & 8.25 \end{bmatrix}$$

Now management thinks this shift in potence is justified (step 8). We thus can define:

$$(3.33) \quad \begin{cases} \underline{S}_3 = \hat{\underline{S}}_3 \text{ and} \\ P_3 = \hat{P}_3 \end{cases}$$

which is again shown in figure 3.6. Returning to step 4 and 5 management wants to raise the production volume of the first variety in order to have a safety margin in meeting the contracts. In step 5 we than propose the solution

$$(3.34) \quad \hat{\underline{S}}_4 = [5, 8.25]$$

for which the potence (calculated in step 7) again reduces to zero. To terminate the example, we assume management accepts this solution as the final one by which we can write:

$$(3.35) \quad \underline{S}_4 = \hat{\underline{S}}_4,$$

which solution is given again in figure 3.6.

In this example we have omitted the definition and redefinition of the δ_j , because they were not needed in the calculations. To be complete we will now show how the δ_j 's should have been defined during the successive iterations. At the beginning, in step 2, we have $\delta_{11} = \delta_2 = 0$. In (3.28) the proposal solution is accepted without modifying the proposed goal levels. Consequently, δ_{11} and δ_2 remain unchanged. The proposal solution (3.29) is judged to have a strongly positive influence on the second goal variable. Here δ_2 set equal to 0.5. The new proposal solution in (3.31) is subsequently accepted and δ_2 is halved to 0.25. Finally, the proposal solution in (3.24) is accepted without any problem, by which δ_{11} remains zero. When this last proposal solution would not have been accepted directly, δ_{11} would have become positive too.

4 TECHNICAL ASPECTS OF I.M.G.P.

In the description of I.M.G.P. in the preceding section we formulated some fairly general requirements. The feasible region R had to be convex, the preference function f had to be concave (both in $g_i(\underline{x})$ and in \underline{x}) and the goal variables $g_i(\underline{x})$ had to be concave in \underline{x} . Furthermore, the decision-maker had to provide only limited information on his preferences. He had to evaluate only concrete solution alternatives (whether these were satisfactory or not) and to indicate in which direction they had to be improved. No explicit trade-offs or even weights were asked from the decision-maker. Although we have represented the decision-maker's preferences as if they could be described by some preference function f , we did not try to describe such a function⁹⁾. Our only aim was to generate a solution to the problem at hand which meets the decision-maker's preferences in an optimal way. This can be done when at least the decision-maker's preferences are not in conflict with the concavity conditions for f .

Given these not very restrictive requirements, many methods are available for use within the I.M.G.P. procedure. Besides many mathematical programming techniques also other methods may be useful. As an illustration, we shall describe (subsection 4.1) I.M.G.P. in linear terms (with respect to the instruments \underline{x}), by which it becomes accessible for linear programming and multiple goal programming routines. The advantages of such a linear format of I.M.G.P. are discussed in subsection 4.2. We conclude this section with a discussion on the convergence properties of I.M.G.P. and the character of the generated solutions (e.g. existence, uniqueness and feasibility).

4.1. The Problem in Linear Terms.

Formulating the problem at hand in linear terms with respect to the instrumental variables is less restrictive than it seems. First there is no extra restriction needed with respect to f . Thus it is sufficient that f meets the two concavity conditions. We further assume that the feasible region R can be described in linear terms, notwithstanding the convexity condition. The goal variables $g_i(\underline{x}), i=1, \dots, m$ must be formulated in a linear

9) Attempts to explicate preference functions by means of implicit information can be found in Nijkamp and Somermeijer [1971].

way too. This does not mean that the $g_i(\underline{x})$ cannot be non-linear. Among others, Ijiri ([1965], pp. 15-22)¹⁰⁾ has shown that piecewise linear functions can be formulated within the linear programming context. In fact, he argues quite convincingly (Ibid, pp. 23) that many non-linear functions in economics are in fact substitutes for piecewise linear functions. On the other hand piecewise linear functions may be used as approximations of continuous non-linear functions (see for instance Goodman [1974] and Laurent [1976]).

The calculatory steps in I.M.G.P. consist of the computation of the potency matrices $P_i, i=1,2,\dots$. The first potency matrix, P_1 , is calculated in step 1. Whenever the lower bounds on the values of one or more of the $g_i(\underline{x})$ are augmented (in step 6) or when some of these bounds are lowered (in step 9), the accompanying potency matrix is calculated in step 7. In all three cases the structure of the problem is exactly the same. Each of the goal variables must successively be maximized (or minimized) within the feasible region R and conditioned by a set of lower bounds (or upper bounds) on the values of the goal variables. The problem can thus be written as:

$$(4.1) \left\{ \begin{array}{l} \text{Max (c.q. min)} \{g_i(\underline{x})\}, \text{ subject to} \\ \underline{x} \in R \text{ and} \\ g_j(\underline{x}) \geq (<) \hat{g}_j \quad \text{for } j=1,\dots,m \end{array} \right\} \text{ for } i=1,\dots,m.$$

where \hat{g}_j denotes the proposed value of $g_j(\underline{x})$ for the problem at hand.

It is possible to formulate the set of problems in (4.1) in a uniform way. All problems are then viewed as minimization problems only differing in the coefficients of the objective function. Even the set of restrictions is the same for all problems to be solved for the same proposed goal vector. In order to demonstrate this, we formulate for each of the goal variables two restrictions:

$$(4.2) \left\{ \begin{array}{l} g_j(\underline{x}) - \hat{y}_j^+ + \hat{y}_j^- = \hat{g}_j \quad \text{for } j=1,\dots,m; \text{ and} \\ g_j(\underline{x}) - \hat{y}_j^* + \hat{y}_j^* = g_j^* \quad \text{for } j=1,\dots,m; \end{array} \right.$$

10)

See for an overview on this topic Nijkamp and Spronk [1977], section 4.

where \hat{g}_j denotes the proposed value of $g_j(\underline{x})$ in the problem at hand, and g_j^* its maximum value (or minimum value) in the first solution S_1 (thereby only constrained by $\underline{x} \in R$). The y^+ and y^- values measure the overattainment and underattainment with respect to the aspired levels \hat{g} and g^* . The problem can then be formulated as a multiple goal programming problem. Let us assume $g_i(\underline{x})$ should be maximized, given proposed values for the goal variables. We then get:

$$(4.3) \left\{ \begin{array}{l} \text{Min } \{ M_1 \cdot \sum_{j=1}^m (\alpha_j^+ \cdot \hat{y}_j^+ + \alpha_j^- \cdot \hat{y}_j^-) + M_2 \cdot \hat{y}_i^- \} \text{ subject to} \\ x \in R \text{ and subject to (4.2)} \\ \alpha_j^+ = 1 \text{ and } \alpha_j^- = 0 \text{ when } f \text{ is a decreasing function of } g_j(\underline{x}) \\ \alpha_j^+ = 0 \text{ and } \alpha_j^- = 1 \text{ when } f \text{ is an increasing function of } g_j(\underline{x}). \end{array} \right.$$

The non-Archimedean (see Charnes and Cooper [1977]) weighting factors M_1 and M_2 have the property $M_1 \gg M_2$ by which preemptive priority is given to attain the proposal values $\hat{g}_j, j=1, \dots, m$, before $g_i(\underline{x})$ can be maximized by means of the minimization of \hat{y}_j^- . We assumed that the variables $g_j(\underline{x})$ could be formulated in such a way that f was monotone non-decreasing or monotone non-increasing in $g_j(\underline{x})$. In the first case, the proposal value \hat{g}_j must be considered as a lower bound (which means \hat{y}_j^- must be zero) and in the second case, \hat{g}_j must be considered as an upper bound (by which \hat{y}_j^+ must be zero). In (4.3.) we assumed $g_i(\underline{x})$ was to be maximized. When f would have been a monotone non-increasing function of $g_i(\underline{x})$, the latter should have been minimized. This can easily be achieved by replacing \hat{y}_i^- in (4.3.) by \hat{y}_i^+ .

According to our assumption, f needs to be a monotone non-decreasing or monotone non-increasing function of all the $g_j(\underline{x})$, as was shown for the simple example in figure 3.1, where f was given as a function of one goal variable only. In section 3.1 we proposed to solve this problem (see (3.1) and (3.3)) by dividing such a $g_i(\underline{x})$ in two other goal variables: one to be maximized and one to be minimized. Again, a multiple goal programming formulation may be useful. Assume the decision-maker has formulated for the goal variable $g(\underline{x})$ the aspiration levels $g_j, j=1, \dots, k$. First assume that $g^* = g_h$ is the most preferred aspiration level, that the

preferences for $g(\underline{x})$ are monotone non-decreasing for $g(\underline{x}) \leq g_h$ and monotone non-increasing for $g(\underline{x}) \geq g_h$. Given a lower bound (g_{h-1}) and an upper bound level (g_{h+1}) for $g(\underline{x})$, the potential shifts of $g(\underline{x})$ from below and from above can be calculated at the same time by the following goal programming formulation:

$$(4.4) \left\{ \begin{array}{l} \text{Min } \{ M_{h-1}^- \cdot y_{h-1}^- + M_h^- \cdot y_h^- + M_h^+ \cdot y_h^+ + M_{h+1}^+ \cdot y_{h+1}^+ \} \text{ subject to} \\ g(\underline{x}) - y_{h-1}^+ + y_{h-1}^- = g_{h-1} \\ g(\underline{x}) - y_h^+ + y_h^- = g_h \\ g(\underline{x}) - y_{h+1}^+ + y_{h+1}^- = g_{h+1} \\ \text{and } \underline{x} \in R \\ \text{with } M_{h-1}^- \gg M_h^- \\ \text{and } M_h^+ \ll M_{h+1}^+ \\ \text{and } y_i^+ \cdot y_i^- = 0 \end{array} \right.$$

In this formulation we assumed g^* to be exactly known. Let us now assume that the only information available is that g^* must be found in the interval $[g_h, g_{h+1}]$, and that the preferences for $g(\underline{x})$ are monotone non-decreasing for $g(\underline{x}) \leq g_h$ and monotone non-increasing for $g(\underline{x}) \geq g_{h+1}$. There are again two possible shifts, one from g_h to the right and one from g_{h+1} to the left. The problem can be written in a goal programming format as:

$$(4.5) \left\{ \begin{array}{l} \text{Min } \{ M_h^- \cdot y_h^- + M_{\hat{g}}^- \cdot y_{\hat{g}}^- + M_{\hat{g}}^+ \cdot y_{\hat{g}}^+ + M_{h+1}^+ \cdot y_{h+1}^+ \} \text{ subject to} \\ g(\underline{x}) - y_h^+ + y_h^- = g_h \\ g(\underline{x}) - y_{\hat{g}}^+ + y_{\hat{g}}^- = \hat{g} \\ g(\underline{x}) - y_{h+1}^+ + y_{h+1}^- = g_{h+1} \\ \underline{x} \in R \\ \text{with } M_h^- \gg M_{\hat{g}}^- \\ \text{and } M_{\hat{g}}^+ \ll M_{h+1}^+ \\ \text{and } y_i^+ \cdot y_i^- = 0 \end{array} \right.$$

Unfortunately, \hat{g} is unknown in this case. However, given a solution in which $g_h \leq g(\underline{x}) \leq g_{h+1}$, we may either put $\hat{g} = g_h$ or $\hat{g} = g_{h+1}$. When such a proposal value is not accepted, a new proposal value \hat{g} is calculated by the δ procedure described in subsection 3.2.

Thus for each solution \underline{S}_i a potence matrix P_i can be calculated by means of goal programming problems as shown in (4.3), (4.4) and (4.5). For each new solution a new set of problems must be solved. These differ from the preceding set of problems in one or more of the aspiration levels, which are in fact right hand constants in the goal programming formulation. To illustrate this we use the goal programming formulation of the example in subsection 3.3. In this example there are two goal variables given in (3.22):

$$(3.22) \begin{cases} g_1(\underline{x}) = x_1 \\ g_2(\underline{x}) = x_2 \end{cases}$$

for which is given that $g_1^* = 6$ is the most preferred value of $g_1(\underline{x})$ and that f is monotone non-decreasing in $g_2(\underline{x})$. From the construction of the potence matrix P_1 (in (3.23)) we learn that the final solution must satisfy the conditions

$$(4.6) \begin{cases} 2 \leq g_1(\underline{x}) \leq 6 \\ 8 \leq g_2(\underline{x}) \leq 9 \end{cases}$$

Besides these goal levels the decision maker himself has formulated the aspiration level $g_1(\underline{x}) = 4$. Starting with $\underline{S}_1 = [2, 8]$, the decision-maker wants to raise the value of $g_1(\underline{x})$, by which he is confronted with the proposal $\hat{\underline{S}}_2 = [4, 8]$ and the potence matrix \hat{P}_2 which can be calculated successively by means of:

$$(4.7a) \text{ Min } \{ M_1 \cdot (\hat{y}_1^- + \hat{y}_2^-) + M_2 \cdot \frac{x^-}{y_1} \} \text{ and}$$

$$(4.7b) \text{ Min } \{ M_1 \cdot (\hat{y}_1^- + \hat{y}_2^-) + M_2 \cdot \frac{x^-}{y_2} \} \text{ both subject to}$$

$$(4.7c) \begin{cases} g_1(\underline{x}) - \hat{y}_1^+ + \hat{y}_1^- = \hat{g}_1 = 4, \\ g_1(\underline{x}) - \frac{x^+}{y_1} + \frac{x^-}{y_1} = \frac{x}{g_1} = 6, \\ g_2(\underline{x}) - \hat{y}_2^+ + \hat{y}_2^- = \hat{g}_2 = 8, \\ g_2(\underline{x}) - \frac{x^+}{y_2} + \frac{x^-}{y_2} = \frac{x}{g_2} = 9, \end{cases}$$

$$\left\{ \begin{array}{l} \underline{x} \in R \text{ and} \\ y_i^+ \cdot y_i^- = 0 \text{ for all } i. \\ \text{with } M_1 \gg M_2 \text{ being non-Archimedean weights.} \end{array} \right.$$

The solution to (4.7a) will be $\hat{y}_1^- = \hat{y}_2^- = \hat{y}_1^* = 0$, which means that the proposed solution \hat{S}_2 is feasible and that this solution does not affect the potential maximum of $g_1(\underline{x})$. The solution to (4.7b) will also give $\hat{y}_1^- = \hat{y}_2^- = 0$, but here $\hat{y}_2^* = \frac{1}{2}$ which means that the potential maximum of $g_2(\underline{x})$ reduces to $g_2(\underline{x}) = 8.5$. In the same way the potence matrices of the succeeding proposals can be calculated. For the third proposal solution the right-hand side \hat{g}_2 is first augmented to $\hat{g}_2 = 8.5$ and then to $\hat{g}_2 = 8.25$ which is accepted by the decision-maker. For the fourth proposal solution the right-hand side \hat{g}_1 is augmented to 5, which solution is accepted as a final solution by the decision-maker. Thus, in this final solution, $g_1(\underline{x}) = 5$, $g_2(\underline{x}) = 8.25$, $\hat{y}_1^* = 1$ and $\hat{y}_2^* = 0.75$.

4.2. Advantages Related to the Linear Format of I.M.G.P.

As shown above, I.M.G.P. can be used together with any (optimization) method which meets the fairly unrestrictive requirements¹¹⁾ imposed by I.M.G.P. (see section 3.1.). Nevertheless, it may be advantageous to formulate the problem in linear terms.

As suggested in (4.2.) and (4.3.) of the preceding section, I.M.G.P. can make a straightforward use of goal programming routines. That is, for each proposal solution a set of goal programs can be formulated. These differ mutually only with respect to one element in the objective function, being the \bar{y}_i , $i = 1, \dots, m$ to be minimized. By means of these goal programs a potency matrix based on the proposal solution can be constructed. For each new proposal solution a new potency matrix must be calculated in this way. However, the goal programs belonging to different solutions only differ with respect to some of the right hand side constants, being the goal levels which have been changed. Clearly, this formulation gives access to specific goal programming routines as for instance proposed by Lee [1972]. However, standard linear programming packages can also be used. In that case, the Extended Control Language proposed by Benichou et al. [1977] is well suited to implement the modifications of the right-hand side values and the successive reoptimizations of the model (Ibid, p.317).

A main advantage of the linear format of the problem is that each solution of a goal program contains useful information about the effects of a shift of the right-hand side constants (see Nijkamp and Spronk [1977], pp.23-28). In an extensive overview Isermann [1977] argues that duality in multiple objective linear programming is even more relevant than in standard linear programming. Besides the economic implications of duality he illustrates its decision-oriented relevance. He shows how information from the dual may be used in the decision-maker's search for a compromise solution. In the same sense Kornbluth [1977] proposes a method in which information from the (fuzzy) dual is systematically used in an interactive way.

Furthermore, the linear format of I.M.G.P. has all advantages of multiple goal programming which were discussed in section 2.1. However, this linear format has some additional advantages. Especially, the preemptive priority

¹¹⁾ In some cases, I.M.G.P. (or rather a slightly modified version of it) can even be used when not all these requirements are fulfilled. In an earlier study (Nijkamp and Spronk [1978]) we illustrated the use of I.M.G.P. in a decision situation with a finite number of alternatives (which is in conflict with the requirement that the feasible region must be convex).

factors mentioned before are most easily handled in the linear format of the goal program (see Nijkamp and Spronk [1977]).

Finally, the linear format of I.M.G.P. may benefit by the widespread attention paid to the linear programming problem in theory and practice. Special procedures developed for linear programming may also be useful in linear I.M.G.P. As an example, procedures to identify redundant constraints in a linear programming problem (see Telgen [1977]) may be used to identify 'redundant goal constraints' within the successive linear goal problems.

4.3. Existence, Feasibility, Uniqueness and Convergence

In subsection 3.2. we identified a 'solution' by means of a goal vector, of which the elements represent the proposed values of the respective goal variables. These proposed values are in fact minimum (or maximum) values imposed on the respective goal variables. As shown in the same subsection, the solution which corresponds to the minimum values themselves, is not necessarily feasible. For example, the ideal solution \underline{I} is generally infeasible. But also the pessimistic solution \underline{Q} is not always feasible, as was shown in figure 3.2.b. One can even show that there does not always exist a unique (either feasible or infeasible) combination of the instrumental variables, which can be associated with a given solution \underline{S} . This can be seen with the help of the following example:

$$(4.8) \begin{cases} \text{Max } g_1(x) = x \text{ and} \\ \text{Min } g_2(x) = x, \text{ subject to} \\ 0 \leq x \leq 1 \end{cases}$$

In this case, the optimistic value for $g_1(x)$ is at the same time the pessimistic solution for $g_2(x)$ and vice versa. The ideal solution is calculated as $\underline{I} = \begin{bmatrix} g_1^* \\ g_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The pessimistic solution is in this case $\underline{Q} = \begin{bmatrix} g_1(x_2^*) \\ g_2(x_1^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Clearly, there does not exist a combination of the instrumental variables which corresponds either with \underline{I} or \underline{Q} . However, as mentioned above, a solution is to be considered as a set of minimum (or maximum) values imposed on the goal variables. Thus in this example, \underline{I} represents the conditions $g_1(x) \geq 1$ and $g_2(x) \leq 0$. In the same way, \underline{Q} represents the conditions $g_1(x) \geq 0$ and $g_2(x) \leq 1$. Although in the latter case, there does not exist a value of the instrumental variable which corresponds to the pessimistic solution itself, there exists at least one value (which moreover is feasible) which meets the conditions represented by the pessimistic solution. In this case obviously any solution $\underline{S} = \begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix}$, $0 \leq \bar{x} \leq 1$, exists and is feasible¹²⁾. For any goal program which meets the condition as formulated in subsection 3.1., it is easily seen that, given a feasible region R, there is always at least one solution \underline{S} , which is bounded by both the ideal and the pessimistic solution, for which a feasible combination of the instrumental variables exists. For instance, when the maximum of the i'th goal variable

12) Because of the completely opposite goal variables in this example, the whole feasible region remains to be evaluated by the decision-maker.

$g_i(\underline{x})$ is attained for $\underline{x}_i^* \in R$, the resulting values of the m goal variables can be grouped in one goal vector, yielding the following solution:

$$(4.9) \quad S \left| g_i^* = \left[g_1(\underline{x}_1^*), g_2(\underline{x}_2^*), \dots, g_i(\underline{x}_i^*), \dots, g_m(\underline{x}_m^*) \right] \right.$$

By definition, this solution is bounded both by the ideal and the pessimistic solution. Furthermore, given the convexity of R and the concavity of $g_i(\underline{x})$ ($i=1, \dots, m$) in \underline{x} , the solution \underline{x}_i^* is known to exist and to be feasible (cf. Kuhn-Tucker [1951]). By the convexity of R , also the weighted combinations of the \underline{x}_i^* exist and are feasible.

During the successive iterations of I.M.G.P. the goal values in the successive solutions are repeatedly shifted upwards by the decision-maker. As noticed before, one can imagine this process as adding new constraints to the already existing set of constraints. Because R was assumed to be convex in \underline{x} and because the newly added constraints are linear in \underline{x} , the part of the feasible region R which remains feasible after adding the constraints (denoted by R_i , $i=1, 2, \dots$; see subsection 3.2) remains convex in \underline{x} . This means that at each iteration of I.M.G.P. there exists a feasible solution, which is bounded by the ideal and the pessimistic solution of the reduced feasible region (provided R_i is non-empty, of course). We will discuss the convergence properties of the procedure in more detail below.

From above it will be clear that, at each iteration of I.M.G.P., there exist more than one feasible solution. One may wonder whether I.M.G.P. produces a unique final solution. These are in fact two questions. One is whether there exists, given the feasible region R , a unique solution \underline{S}^* which is preferred by the decision-maker to any other solution which can be achieved within the feasible region. Obviously, the answer to this question depends on the decision-maker's preferences. In subsection 3.1., we assumed that these preferences could, at least in principle, be described by means of a concave function of the goal variables. By this assumption, the variety of preference patterns which can be incorporated is not very restricted. For instance, even satisficing behaviour can be represented by the decision-maker. This means, that the concavity condition is not a sufficient condition to guarantee a unique final solution¹³⁾.

13) Note furthermore, that a unique solution in the space of solutions does not guarantee a unique solution in the space of the instrumental variables.

The second question is, whether I.M.G.P. converges to an optimal solution whenever it exists (either unique or not) and whether it produces satisfactory results in the case the decision-maker requires the achievement of some aspiration levels.

The convergence properties of I.M.G.P. depend, of course, on the ability of the decision-maker to answer the questions which are posed to him during the interactive process, and on the ability to do so in a consistent manner. We therefore first assume that the decision-maker is able to answer the questions posed by I.M.G.P., that his answers are consistent and finally that his preferences (which must meet the concavity conditions formulated in subsection 3.1.) do not change during the interactive process of I.M.G.P. Given these assumptions it can be shown that I.M.G.P. terminates in a finite number of iterations within an ϵ -neighbourhood from the final optimum. First, starting from an accepted solution, a next solution will be accepted after a finite number of steps. To show this, let us assume that the decision-maker accepted the solution \underline{S}_i , representing a set of minimally required values of a number of goal variables which are to be maximized. The decision-maker next indicates that the k'th goal variable should be augmented first. We thus may infer:

$$(4.10) \quad \left. \frac{\delta U}{\delta g_k} \right|_{\underline{S}_i} > \left. \frac{\delta U}{\delta g_j} \right|_{\underline{S}_i} \quad \text{for } j=1, \dots, m \text{ and } j \neq k.$$

From this it follows that there must be a solution \underline{S}_{i+1} which differs from \underline{S}_i only with respect to the value of $g_k(\underline{x})$, for which:

$$(4.11) \quad U(\underline{S}_{i+1}) > U(\underline{S}')$$

for all \underline{S}' which exist for $\underline{x} \in \{R_i - R_{i+1}\}$, the part of the feasible region which becomes infeasible when \underline{S}_{i+1} is accepted¹⁴⁾. Let us consider such a

14) This assertion can be proven indirectly, by assuming that for any $\lambda, \lambda > 0$ there does not exist a solution \underline{S}_{i+1} , of which the $g_k(\underline{x})$ value exceeds the corresponding value in \underline{S}_i by an amount λ (all other minimal goal values remaining equal), for which (4.11) holds. In other words, for any λ with its corresponding \underline{S}_{i+1} , there exists some $\underline{x}' \in \{R_i - R_{i-1}\}$ for which the goal vector \underline{S}' is preferred to \underline{S}_{i+1} . In this case, \underline{S}' must be preferred to \underline{S}_{i+1} because it has one or more goal variables different from $g_k(\underline{x})$ which exceed their corresponding values in \underline{S}_i . (If \underline{S}' would have a value of $g_k(\underline{x})$ exceeding its value in \underline{S}_i , a solution \underline{S}_{i+1} could be constructed for which (4.11) holds. This could be achieved by substituting this higher value of $g_k(\underline{x})$ in \underline{S}_i). Since this, by assumption, must hold for any λ , however small, the value of $g_k(\underline{x})$ cannot exceed its value in \underline{S}_i . This is in contradiction with our earlier assumption that the decision-maker, given \underline{S}_i , preferred a shift of $g_k(\underline{x})$ to any other shift in \underline{S}_i . This completes the proof. A similar proof can be given for the case in which the decision-maker, given \underline{S}_i , prefers to have a simultaneous change of two or more goal variables at the same time.

solution \underline{S}_{i+1} , and let us further assume that its $g_k(\underline{x})$ value exceeds the corresponding value \underline{S}_i by an amount λ , $\lambda > 0$. How is such a solution found for the decision-maker? As described in subsection 3.2., a first proposal solution $\hat{\underline{S}}_{i+1}$ is generated by augmenting the value of $g_k(\underline{x})$ in \underline{S}_i by a given amount, which shall be labeled here as d . When $d < \lambda$, the proposal solution will be accepted. On the other hand, when $d > \lambda$ the proposal is not necessarily accepted¹⁵⁾. Then a new proposal solution is calculated by halving the value of d . If $\frac{d}{2} < \lambda$ the proposal solution is accepted. If not, the value of d is divided by 2^2 and so on. Clearly, the proposal solution is accepted as soon as $\frac{d}{2^n} < \lambda$ which for $\lambda > 0$ will happen for a finite value n . We thus have shown that each new solution \underline{S}_{i+1} is reached in a finite number of steps and furthermore that all possible solutions which have become infeasible by accepting \underline{S}_{i+1} , are less preferred than \underline{S}_{i+1} .

Next we have to show, that only a finite number of solutions must be calculated in order to reach a final solution. To be more precise, we want to show that only a finite number of solutions must be calculated before a final solution \underline{S}^f is obtained in which the values of the respective goal variables differ less than some predetermined ϵ - value from the respective goal values in the optimal solution \underline{S}^o . At each iteration of I.M.G.P. at least one goal variable's value is raised. Because there is a finite number (m) of goal variables, it is sufficient to show that an arbitrary goal variable $g_k(\underline{x})$ reaches its 'optimal' value g_k^o , apart from a small distance of at most ϵ_k , within a finite number of iterations. The k 'th goal variable is chosen to be augmented whenever condition (4.10) holds. Without further requirements with respect to the decision-maker's preference structure, it is not sure whether condition (4.10) will ever hold for each goal variable, at least within the feasible region R . For example, one goal variable may be judged to be far more important than each of the other goal variables. When the decision-maker has to meet many and/or very strong restrictions it may be that this important goal variable 'consumes' the whole feasible region. In that case, the accompanying values of the other goal variables must be considered as being optimal. However, let us assume, that $g_k(\underline{x})$ reaches its most preferred value within the feasible region R for $g_k^o > g_k^{\min}$, the pessimistic value of $g_k(\underline{x})$ and

15) We thus assume that the decision-maker evaluates a shift of one of the goal variables in a way which is consistent with (4.11). For this evaluation he uses the information presented in the potency matrices.

that this value is not automatically attained, as in the above example. Let us further assume, that the decision-maker has not defined any aspiration level for $g_k(\underline{x})$. In this case we only know that $g_k^{\min} < g_k^o < g_k^*$, where g_k^* again is the maximum value of $g_k(\underline{x})$ for $\underline{x} \in R$. As described in subsection 3.2., a proposal solution is calculated as $\hat{g}_k = (g_k^* - g_k^{\min})/2$. From the answer of the decision-maker we can infer whether $g_k^o > \hat{g}_k$ or $g_k^o < \hat{g}_k$. We then know either that $g_k^{\min} < g_k^o < \hat{g}_k$ or that $\hat{g}_k < g_k^o < g_k^*$. At the next iteration a new proposal solution is chosen exactly in the middle of the chosen region. Thus, the range in which g_k^o must be found is exactly halved at each time the decision-maker is consulted. This means that the ϵ -neighbourhood of g_k^o is reached when,

$$(4.12) \quad \frac{(g_k^* - g_k^{\min})}{2^n} < \epsilon_k,$$

where n is the number of times the decision-maker gives his opinion on \hat{g}_k . In general, this ϵ_k -neighbourhood will be attained in less steps. This is because the aspiration levels which have been defined a priori, may be of great help during the search procedure. Furthermore g_k^* is influenced by the values which are required for the other goal variables.

Some final remarks need to be made on the assumed consistency and the preference structure which has been supposed not to change during the interactive process. Obviously, the decision-maker may make errors while expressing his preferences. Furthermore, as mentioned in section 2, the decision-maker may learn from the interactive process. For example, the decision-maker may recognize that a proposed shift in a single goal variable is outweighed by a simultaneous shift in two other goal variables. He may then wish to return to the preceding solution to ask for such a simultaneous shift. In our opinion, two devices are needed to circumvent this kind of difficulties. First, the decision-maker must have the possibility to stop the interactive procedure and to return to an earlier solution. Second, it is wise to repeat the whole interactive procedure when a final solution has been found.

5. Evaluation

In this final section we mention some of the main features of I.M.G.P. point by point.

- * I.M.G.P. is a continuous multi-dimensional optimization method (or multi-objective-programming model), because it is based on an infinite number of possible values for the decision arguments and hence for the objective function¹⁶⁾.
- * I.M.G.P. is interactive, because it is based on a mutual and successive interplay between a decision-maker and an expert (or analyst).
- * I.M.G.P. needs no more a priori information than other interactive programming models. However, all available a priori information can be incorporated within the procedure. Notably, aspiration levels and preemptive priorities which have been defined by the decision-maker can be incorporated in the interactive process quite easily. Besides, I.M.G.P. offers the decision-maker the opportunity to reconsider this a priori information during the interactive process.
- * In I.M.G.P. the goal variables are assumed to be known and concave in the instrumental variables. The preference function of the decision-maker is not assumed to be known. However, it is assumed to be concave, both in the goal variables and in the instrumental variables. Clearly, these assumptions are not very restrictive. For instance, both optimizing and satisficing behaviour can be incorporated.
- * The decision-maker only has to give information on his local preferences. This is done on basis of a solution and a potence matrix presented to him. A solution is a vector of minimum values for the respective goal variables. The potence matrix shows for each of these goal variables separately the maximum value, given the solution concerned. The decision-maker only has to indicate whether a solution is satisfactory or not, and if not, which of the minimum goal values should be raised. He does not have to specify how much these goal values should be raised. Nor there is any need to specify weighing factors. A new solution is presented to him together with a new potence matrix. He then has to indicate whether the shifts in the solution outweigh the shifts in the potence matrix. If not, a new solution is calculated and so on.

¹⁶⁾ However, with some minor modifications I.M.G.P. can also be applied to discrete decision models (see Nijkamp and Spronk [1978]).

* Given a consistent decision-maker, I.M.G.P. converges within a finite number of iterations to a final solution, which exists and is feasible. Apart from an ϵ - neighbourhood, this solution is optimal. Whether this optimal solution is unique or not, depends on the decision-maker's preferences.

* Given a new (proposal) solution, the maxima of the goal variables must be (re)calculated at each iteration of I.M.G.P. This can be done with the help of any optimization method which meets the not very restrictive requirements imposed by I.M.G.P. (i.e. convexity of the feasible region R and concavity of the preference function and the goal variables). Nevertheless, it may be advantageous to formulate the problem in linear terms. Then, I.M.G.P. can make a straightforward use of goal programming routines. The advantages of this approach are sketched in subsections 2.1. and 4.2.

* It must be stressed that the described advantages only hold, when the decision-maker meets the requirements of I.M.G.P. He must be able to answer the questions posed by I.M.G.P. His answers must be consistent, although he is allowed to make some errors during the interactive process. Finally, because of the possible learning effects, the procedure must be repeated several times to be sure that a final solution is found which is as close as possible to the optimum.

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