Interest rate rules and macroeconomic stability under heterogeneous expectations

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Interest Rate Rules and Macroeconomic Stability under Heterogeneous Expectations

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Abstract

The recent macroeconomic literature stresses the importance of managing heterogeneous expectations in the formulation of monetary policy. We use a stylized macro model of Howitt (1992) to investigate inflation dynamics under alternative interest rate rules when agents have heterogeneous expectations and update their beliefs based on past performance as in Brock and Hommes (1997). The stabilizing effect of different monetary policies depends on the ecology of forecasting rules, on agents' sensitivity to differences in forecasting performance and on how aggressively the monetary authority sets the nominal interest rate in response to inflation. In particular, if the monetary authority only responds weakly to inflation, a cumulative process with rising inflation is likely. On the other hand, a Taylor interest rate rule

that sets the interest rate more than point for point in response to inflation stabilizes inflation dynamics, but does not always lead the system to converge to the rational expectations equilibrium as multiple equilibria may persist, even when a fully rational, but costly, expectations rule is part of the ecology of forecasting strategies.

JEL codes: E52, D83, D84, C62.

Keywords: Heterogeneous Expectations, Monetary Policy, Cumulative Process, Taylor Rule.

1 Introduction

The rational representative agent approach is still the core assumption in macroeconomics. In contrast, in behavioral finance models with bounded rationality and heterogeneous expectations have been developed as a concrete alternative to the standard rational representative agent approach. These heterogeneous agent models mimic important observed stylized facts in asset returns, such as fat tails, clustered volatility and long memory, as discussed e.g. in the extensive surveys of LeBaron (2006) and Hommes (2006). Although bounded rationality and adaptive learning have become increasingly important in macroeconomics, most models still assume a representative agent who is learning about the economy (see e.g. Evans and Honkapohja (2001) and Sargent (1999) for extensive overviews) and thus ignore the possibility of heterogeneity in expectations and its consequences for monetary policy and macroeconomic stability. Some recent examples of macro models with heterogeneous expectations include Brock and de Fontnouvelle (2000), Evans and Honkapohja (2003, 2006), Branch and Evans (2006), Honkapohja and Mitra (2006), Branch and McGough (2006, 2009), Berardi (2007), Tuinstra and Wagener (2007) and Brazier, Harrison, King, and Yates (2008). Branch (2004), Santoro and Pfajfar (2006) and Pfajfar (2008) recently provided empirical evidence in support of heterogeneous expectations using survey data on inflation expectations, while Hommes, Sonnemans, Tuinstra, and van de Velden (2005) and Adam (2007) find evidence for heterogeneity in learning-to-forecast laboratory experiments with human subjects.

The importance of managing expectations for conducting monetary policy has been recognized and stressed e.g. in Woodford (2003) (p. 15). However, the question how to manage expectations when forecasting rules are heterogeneous has hardly been addressed. The aim of our paper is to investigate whether the Central Bank can enhance macroeconomic stability, in the presence of *heterogeneous* expectations about future inflation, by implementing simple interest rate rules. In particular, we investigate how the ecology of potential forecasting rules affects the
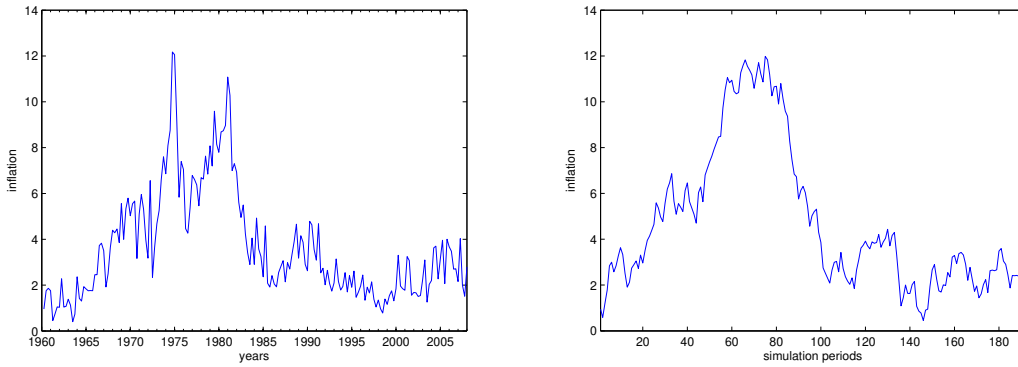
Figure 1: Inflation time series. **Left:** US annualized inflation, quarterly data for a period 1960-2007. **Right:** Simulated data.

stabilizing properties of a simple Taylor rule. Moreover we study how, in a world where expectations are heterogenous, the aggressiveness of the monetary authority in responding to fluctuations of the inflation rate affects these stabilizing properties. See also De Grauwe (2008) for a recent discussion on how heterogeneous expectations may affect monetary policy.

In order to study the potential (de-)stabilizing role of heterogeneous expectations we take the simple, but influential model of Howitt (1992, 2006). In this macro framework we investigate the dynamics generated by two different monetary policy scenarios: an interest rate pegging and a Taylor-type interest rate rule. In our stylized model agents form expectations about the future rate of inflation using different forecasting rules. We employ the heterogeneous expectations framework of Brock and Hommes (1997), where the ecology of forecasting rules is disciplined by endogenous, evolutionary selection of strategies with agents switching between forecasting rules on the basis of their past performance.

Albeit being very stylized, our model is able to reproduce some qualitative features of US inflation time series. In Fig. 1 we confront the simulated dynamics of a stochastic version of our model buffeted with shocks to economic fundamentals (right panel) with actual time series of US inflation (left panel). The model is simulated for 192 periods corresponding to quarters. The monetary policy in the model exhibits a structural break in period 80, when the Central Bank changes the

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coefficient (measuring its aggressiveness in responding to actual inflation) of the interest rate rule from 0.8 to 2. These values are of the same order of magnitude as the estimated coefficients of a Taylor rule in the period 1960-1979 (the pre-Volcker era) and the period 1979-1997 (the Volcker-Greenspan era), see e.g. Taylor (1999) and Clarida, Gali, and Gertler (2000). Before the structural break, in setting the interest rate the Central Bank responds relatively weakly to inflation. In our nonlinear model with heterogeneous expectations, when the Central Bank only responds weakly to inflation multiple steady states arise and, as a consequence, self-fulfilling expectations contribute to and reinforce a strong rise in inflation initially triggered by shocks to fundamentals, consistent with US data. In period $T = 80$, after the structural break, the Central Bank modifies the monetary policy rule to respond more aggressively, i.e., adapts the nominal interest rate more than point for point in response to inflation. Because of this policy change some of the high level steady states disappear and inflation stabilizes to low levels, consistent with US data[1]. Our model, thus, explains the strong rise in US inflation between 1960 and 1980 as being triggered by shocks to economic fundamentals (such as the Oil shocks in 1973 and 1979), reinforced by evolutionary selection among heterogeneous forecasting rules under a too weakly responding Taylor rule in the pre-Volcker period, and the subsequent strong decline in US inflation data between 1980 and 2007 (the Great Moderation) enforced by a more aggressive interest rate rule.

Our paper also contributes to a long lasting debate about the feasibility of a policy of interest rate pegging. Friedman (1968) pointed out that controlling interest rates tightly is not a feasible monetary policy. He argues that if the real interest rate in the economy is pegged to a value different from the "natural" level (corresponding to full employment), due to an expectations-augmented Phillips curve and the Fisher effect, inflation will follow a cumulative process of accelerating inflation or deflation. The cumulative process argument disappeared from

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[1]In our model under the Taylor rule inflation does not necessarily converge to the RE level, as different co-existing equilibria may persist. This result differs from the standard representative agent adaptive learning literature, where the interest rate rules that satisfy the Taylor principle lead to a unique, E-stable RE equilibrium, see, e.g., Bullard and Mitra (2002) and Preston (2005).

the literature after the rational expectations revolution. Howitt (1992) pointed out that in an economy in which people try to acquire rational expectations through *adaptive learning*, a monetary policy aimed at controlling tightly the interest rate will lead inevitably to the cumulative process. Indeed, in a world in which any departure of expected inflation from its equilibrium level causes an overreaction of actual inflation and generates a misleading signal for the agents, a forecasting rule that tries to learn from past mistakes will lead the economy away from equilibrium causing inflation/deflation to accelerate until the interest rate pegging policy is abandoned. Howitt (1992) shows that the cumulative process arises for any plausible adaptive learning rule in a homogeneous expectations setting. Moreover he shows that by reacting aggressively to inflation when setting the interest rate, the monetary authority can avoid the cumulative process.

The present paper investigates the potentially destabilizing effect of interest rate pegging and the potentially stabilizing effect of a Taylor rule in a world with endogenously evolving heterogeneous expectations. As we will see, the answer whether a Taylor rule can stabilize the cumulative process depends in interesting ways on the ecology of forecasting rules and on how aggressively the monetary authority adjusts the interest rate in response to inflation.

The paper is organized as follows. Section 2 briefly recalls the ideas behind the cumulative process and presents the benchmark model as in Howitt (1992, 2006). The model with heterogeneous expectations is introduced in Section 3, where an example of a perfectly rational, but costly, expectations rule versus freely available naive expectations is analyzed under an interest rate pegging as well as a Taylor rule. Sections 4 and 5 consider the model with an ecology of constant forecasting rules in the case of interest rate pegging as well as a Taylor rule. We study both the case when the number of forecasting rules is small (e.g. 3 or 5) and the case of an arbitrarily large number of rules, applying the notion of Large Type Limit (Brock, Hommes, and Wagener (2005)). We also study the effect of adding a costly rational expectations rule to the ecology of forecasting strategies. In Section 5.4 we discuss

a calibration of our model to US inflation data. Finally, Section 6 concludes.

2   Interest Rate Rules and Cumulative Process

In this section we recall the formalization developed by Howitt (2006) of the instability problem implied by the Wicksellian cumulative process. Consider the following system of equations

$$y_t = -\sigma\left(i_t - \pi_t^e - r^*\right), \tag{2.1}$$

$$\pi_t = \pi_t^e + \varphi y_t, \tag{2.2}$$

where y_t is the output gap, i_t is the nominal interest rate, π_t and π_t^e are respectively the actual and expected inflation rates, r^* is the natural interest rate, and σ and φ are positive coefficients. Equation (2.1) is the usual IS curve in which the real interest rate $i_t - \pi_t^e$ must equal the natural rate in order for output to equal its "full employment" capacity, here normalized to zero. Equation (2.2) is the expectations-augmented Phillips curve expressed in terms of inflation and output.

Let us assume that the monetary authority decides to peg the nominal interest rate at level $\bar{\imath}$. Under rational expectations the expected inflation rate coincides with the actual inflation, and, according to (2.2), the economy is in the state of full employment, $y^* = 0$. From (2.1) the inflation rate in the RE equilibrium depends positively on the pegged nominal interest rate:

$$\pi^* = \bar{\imath} - r^*.$$

Thus, assuming rational expectations interest rate pegging is a feasible monetary policy: accelerating or decelerating inflation will not arise because the system will immediately reach the equilibrium level.

However, the policy implications change dramatically when the rational expectations assumption is relaxed, and expectations are revised in an adaptive, bound-

edly rational way. To illustrate the failure of the interest rate pegging policy, let us assume that the nominal interest rate is pegged too low, so that the real interest rate $\bar{\imath} - \pi_t^e$ is below its natural level r^*. In this case, inflation expectations will be higher than the equilibrium inflation π^*. Actual inflation will be even higher than expected inflation because of the expectations augmented Phillips curve:

$$\pi_t = \pi_t^e + \varphi\sigma(\pi_t^e - \pi^*)\,.$$

This means that the signal that agents receive from the market is misleading. Even though inflation was overestimated with respect to the equilibrium level, i.e., $\pi_t^e > \pi^*$, realized inflation suggests that agents underestimated it, i.e., $\pi_t > \pi_t^e$. Any reasonable rule that tries to learn from past mistakes will then lead agents to expect even higher inflation, causing a cumulative process of accelerating inflation. Similarly, pegging the interest rate too high will lead to a cumulative process of accelerating deflation. Hence, interest pegging is not a feasible monetary policy.

The actual dynamics depends, of course, on the forecasting rule that agents use to form their expectations. As an illustrative example, consider the case of naive expectations, when agents expect that past inflation will persist in the current period, $\pi_t^e = \pi_{t-1}$. In deviations from the RE steady state, the model (2.1)–(2.2) becomes

$$\begin{aligned} y_t &= \sigma x_t^e\,, \\ x_t &= x_t^e + \varphi y_t\,, \end{aligned}$$

where $x_t^e = \pi_t^e - \pi^*$ and $x_t = \pi_t - \pi^*$ are respectively the *deviations* of the expected and actual inflation from the RE steady state. The dynamics under naive expectations is described by the following linear equation

$$x_t = (1 + \varphi\sigma)\, x_{t-1}\,, \tag{2.3}$$

whose unique steady-state corresponds to the RE equilibrium, $x^* = 0$. This steady-state is, however, unstable. Thus pegging the interest rate at a non-equilibrium level, will lead to a cumulative process.

So far we have discussed a model considering a monetary institution that follows a nominal interest rate pegging monetary policy rule. Howitt (1992) proposed an alternative strategy to model monetary policy in order to stabilize inflation under adaptive learning, i.e. under the assumption that agents are not rational. He showed that the cumulative process can be avoided when the Central Bank adopts a monetary policy rule that makes the nominal interest rate respond to the rate of inflation more than point for point. This monetary policy rule has become known as the "Taylor principle", after Taylor (1993).

Consider the example above and assume that the Central Bank responds to the inflation rate according to the following relation:

$$i_t = \phi_\pi \pi_t \,. \tag{2.4}$$

The coefficient ϕ_π measures the response of the nominal interest rate to changes in the inflation rate π_t. When the Taylor rule (2.4) is implemented and agents hold naive expectations, the dynamics is described by

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} x_{t-1} \,,$$

which differs from (2.3) only in the slope coefficient. It is immediately clear that for a Taylor rule (2.4) with $\phi_\pi > 1$, the RE equilibrium is globally stable and the cumulative process will not arise.[2]

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[2]The RE equilibrium in this model depends on the Taylor rule itself. The inflation at the REE is given by $\pi^* = r^*/(\phi_\pi - 1)$. In particular, even if the aggressive (more than point for point) response of the interest rate to the inflation rate is always sufficient to stabilize dynamics, the response should be *very* aggressive to lead dynamics to the REE with low inflation.

3 Rational versus Naive

Will the cumulative process arise in an economy where agents have *heterogeneous expectations* about the future level of the inflation rate? Will a Taylor type interest rate rule succeed in stabilizing inflation? To address these questions we employ the framework of Adaptive Belief Systems proposed in Brock and Hommes (1997) to model heterogeneous expectations. Assume that agents can form expectations choosing from H different forecasting rules. We denote by $x_{h,t}^e$ the forecast of the deviation of inflation from its RE equilibrium level by rule h. The fraction of agents using forecasting rule h at time t is denoted by $n_{h,t}$. Assuming linear aggregation of individual expectations[3], actual inflation (in deviations from the steady-state) in the model (2.1)–(2.2) is given by

$$x_t = k \sum_{h=1}^{H} n_{h,t} x_{h,t}^e \,, \tag{3.1}$$

where k depends on the structural parameters of the model as well as on the monetary policy type. Under interest rate pegging we have $k = 1 + \varphi\sigma$, as in (2.3). Under a Taylor rule (2.4) k is given by

$$k = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \,. \tag{3.2}$$

The evolutionary part of the model describes the updating of beliefs over time. Fractions are updated according to an *evolutionary fitness* measure. The fitness measures of all strategies are publicly available, but subject to noise. Fitness is derived from a random utility model and given by

$$\widetilde{U}_{h,t} = U_{h,t} + \varepsilon_{h,i,t} \,,$$

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[3]Averaging of individual forecasts represents a first-order approximation to general nonlinear aggregation of heterogeneous expectations. Recent papers following the same approach include Adam (2007), Arifovic, Bullard, and Kostyshyna (2007), Brazier, Harrison, King, and Yates (2008) and De Grauwe (2008).

where $U_{h,t}$ is the deterministic part of the fitness measure and $\varepsilon_{h,i,t}$ represent IID noise at date t, across types $h = 1, \ldots, H$ and agents i. Assuming that the noise $\varepsilon_{h,i,t}$ is drawn from a double exponential distribution, in the limit as the number of agents goes to infinity, the probability that an agent chooses strategy h is given by the well known discrete choice fractions (see Manski and McFadden (1981)):

$$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \,. \tag{3.3}$$

Note that the higher the fitness of a forecasting rule h, the higher the probability that an agent will select strategy h. The parameter β is called the *intensity of choice* and reflects the sensitivity of the mass of agents to selecting the optimal prediction strategy. The intensity of choice β is inversely related to the variance of the noise term. The case $\beta = 0$ corresponds to the situation of infinite variance in which differences in fitness can not be observed, so agents do not switch between strategies and all fractions are constant and equal to $1/H$. The case $\beta = \infty$ corresponds to the situation without noise in which the deterministic part of the fitness can be observed perfectly and in every period all agents choose the best predictor. A natural performance measure is past squared forecast errors

$$U_{h,t-1} = -(x_{t-1} - x_{h,t-1}^e)^2 - C_h \,, \tag{3.4}$$

where C_h is the per period information gathering cost of predictor h.

Consider the case where agents can choose between a sophisticated but costly forecasting rule and a simple, freely available one. As a typical, stylized example we consider the case with rational expectations (perfect foresight), i.e. $x_{1,t}^e = x_t$, versus naive expectations, i.e. $x_{2,t}^e = x_{t-1}$. In a world with heterogeneous expectations perfect foresight requires knowledge about the predictions of all other agents in the population. Therefore, we assume that in order to obtain the perfect foresight forecast agents have to pay information gathering costs $C \geq 0$ per period, whereas the naive forecast is available for free. We investigate and compare two possible

monetary policy rules, interest rate pegging and the Taylor rule.

3.1 Interest Rate Pegging

Substituting the predictions of rational and naive agents into (3.1), with $k = 1 + \varphi\sigma$, and solving for x_t leads to

$$x_t = \frac{(1 + \varphi\sigma)(1 - n_{1,t})}{1 - n_{1,t}(1 + \varphi\sigma)} x_{t-1} \, , \tag{3.5}$$

where the fraction of agents with perfect foresight evolves according to

$$n_{1,t} = \frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(x_{t-1} - x_{t-2})^2}} \, . \tag{3.6}$$

The next result describes steady state properties of this two-dimensional system:

Proposition 3.1. *The dynamics given by* (3.5) *and* (3.6) *has a unique steady-state with* $x^* = 0$ *and* $n_1^* = \dfrac{e^{-\beta C}}{1 + e^{-\beta C}} \leq \dfrac{1}{2}$. *This "Rational Expectations" steady state is unstable for all costs* $C \geq 0$.

Proof. See Appendix A. □

The RE equilibrium with full employment is the only steady-state of the model. In this steady-state both types of agents give the same (correct) forecast. The population, however, is split between the two types, with naive agents constituting at least half of the population. The RE equilibrium is a locally unstable steady-state, which suggests that interest rate pegging is not a feasible policy, not even when the information gathering costs for rationality $C = 0$.

In order to get some intuition for the dynamics of the model, in Fig. 2 we plot the graph of the slope of the right-hand side of (3.5) as a function of the fraction $n_{1,t}$ of rational agents. In this way one can interpret the behavior of rational agents, given their knowledge about the distribution of agents over the two types. Recall that if all agents are naive, i.e. $n_{1,t} = 0$ (labeled N in Fig. 2), the cumulative process
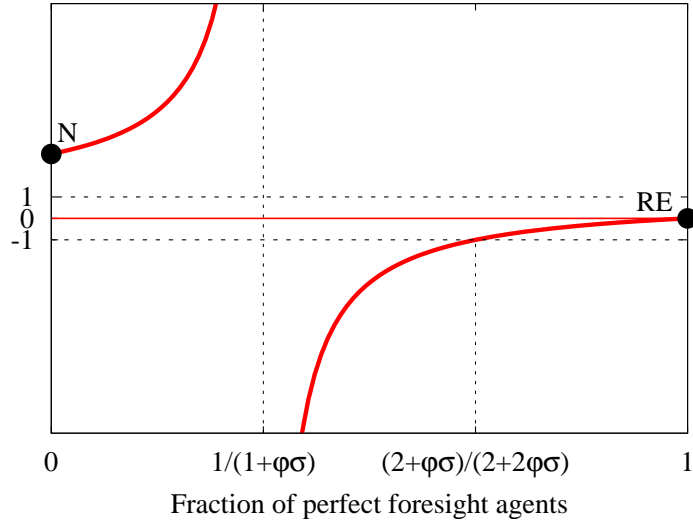
Figure 2: Graph of the slope, $\dfrac{(1 + \varphi\sigma)(1 - n_{1,t})}{1 - n_{1,t}(1 + \varphi\sigma)}$, of the map in Eq. (3.5) as a function of the fraction of rational agents $n_{1,t}$. In this figure parameters are such that $\varphi\sigma = 2$.

in (2.3) arises. At the other extreme, when all agents have perfect foresight, $n_{1,t} = 1$ (labeled RE in Fig. 2), the system immediately jumps to the RE steady-state. Fig. 2 shows that in the intermediate case, when agents have heterogeneous expectations, the perfect foresight agents can either reinforce (the left branch of the curve) or counterbalance (the right branch of the curve) the cumulative process, depending on the relative weight of rational agents in the population. When the fraction of rational agents is relatively low, i.e. $n_{1,t} < 1/(1 + \varphi\sigma)$, the cumulative process is reinforced, accelerating inflation or deflation even stronger than under naive expectations. When the fraction of rational agents is relatively high, i.e. $n_{1,t} > 1/(1 + \varphi\sigma)$, rational agents counterbalance and reverse the cumulative process. But only when the fraction of rational agents is sufficiently large, i.e. when $n_{1,t} > (2+\varphi\sigma)/(2+2\varphi\sigma)$, the counterbalancing of rational agents leads to a stable process. Notice that at the steady state $n_1^* \leq 1/2 < (2+\varphi\sigma)/(2+2\varphi\sigma)$, so that at the steady state the counterbalancing effort of rational agents can not prevent an unstable inflation process.

Thus, only when naive agents dominate the population, the cumulative process
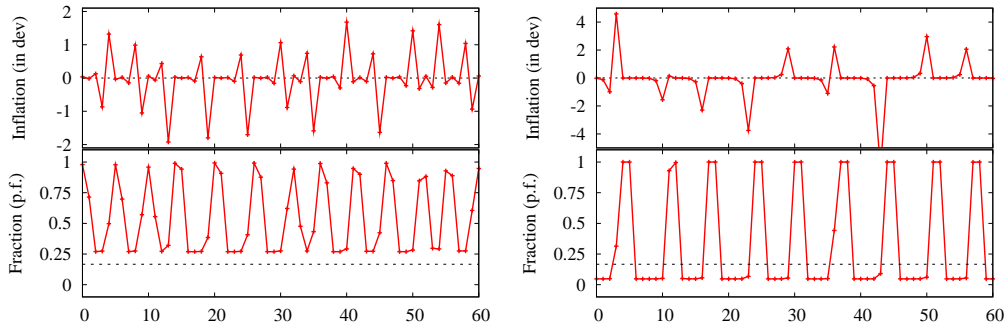
Figure 3: Dynamics of the evolutionary model with rational vs. naive agents under interest rate pegging. For two values of the intensity of choice, $\beta = 1$ (**Left**) and $\beta = 3$ (**Right**) the deviations of inflation from the RE level (*upper parts*) are shown together with evolution of the fraction of the perfect foresight agents (*lower parts*). When the fraction of rational traders falls below the threshold value $1/(1 + \varphi\sigma)$ (shown by the dotted line), a temporary cumulative process with accelerating inflation or deflation starts.

will start. However, this process will never be permanent. Indeed, along such a process, the prediction errors of naive agents accumulate. At some point, the information gathering costs for rational expectations outweigh the errors of naive forecast, and the majority of agents will switch to the rational predictor. This will lead the system back close to the RE equilibrium. However, as Proposition 3.1 shows, this equilibrium is unstable. Close to the equilibrium both forecasting rules give approximately the same predictions and, because of the information gathering costs for rationality, the majority of agents will switch back to the naive forecasting rule thus destabilizing the inflationary process.

This mechanism is illustrated in Fig. 3, where the dynamics of the actual deviation of inflation from the RE steady state and the evolution of the fraction of perfect foresight agents are shown for two levels of the intensity of choice β. In both cases we observe phases in which actual deviations of inflation from the RE steady state is relatively small and phases in which the deviations are relatively high. As explained above, during the phases with small deviations from the steady state, the economy is dominated by naive agents, while the phases with high deviations always end up by massive switching to the perfect foresight predictor. There is an
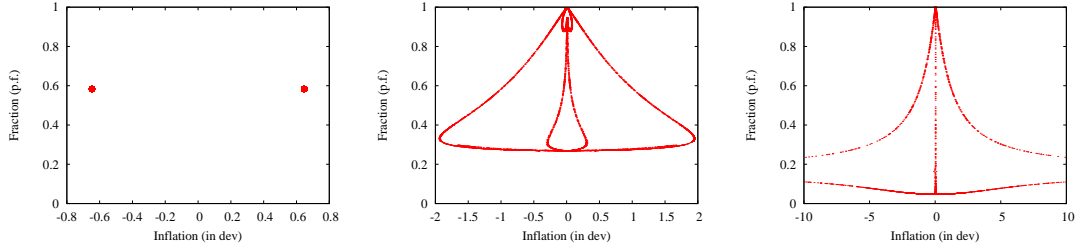
Figure 4: Phase diagram, $(x_t, n_{1,t})$, in the evolutionary model with perfect foresight and naive agents. **Left panel:** $\beta = 0.5$. **Middle panel:** $\beta = 1$. **Right panel:** $\beta = 3$.

Figure 5: Delay plot, (x_t, x_{t-1}), in the evolutionary model with perfect foresight and naive agents. **Left panel:** $\beta = 0.5$. **Middle panel:** $\beta = 1$. **Right panel:** $\beta = 3$.
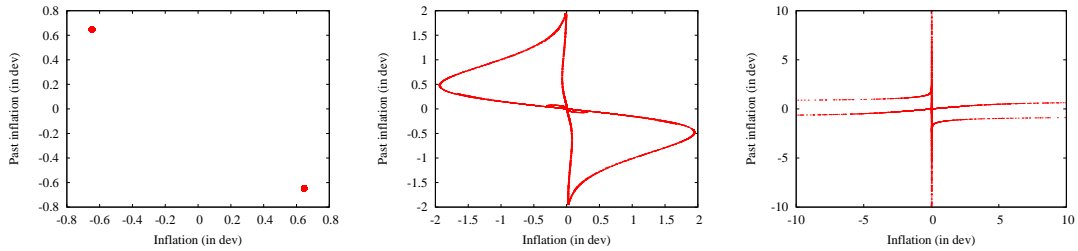
important difference between the two cases. When the intensity of choice is low (the left panel), the fraction of rational agents never falls below the threshold value $1/(1 + \varphi\sigma)$. The cumulative process never arises in this case because the fraction of rational agents is always sufficiently large to counterbalance the cumulative process. When the intensity of choice is high (the right panel), the cumulative process occurs when the fraction of the rational agents falls below the threshold value.

Figs. 4 and 5 compare the phase diagrams and delay plots for three different values of the intensity of choice. We observe that the system converges to a two-cycle for small values of β, but as soon as the intensity of choice increases strange attractors and chaotic behavior occur. Indeed a *rational route to randomness* in inflation rates, that is, a bifurcation route to complicated dynamics, arises when the intensity of choice becomes large[4].

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[4]A difference with the rational route to randomness in the cobweb model of Brock and Hommes (1997) is that in our macro model it starts off from a stable 2-cycle (the steady state is always unstable), while in the cobweb model it starts off from a stable steady state.

It is worthwhile pointing out an important difference with Howitt (1992). In his model with a representative agent, the cumulative process always arises for any reasonable adaptive learning rule. In our heterogeneous expectations model with rational versus naive agents, a cumulative process only arises temporarily. Similar results hold under heterogeneous expectations when we replace the naive forecasting rule by a (freely available) adaptive learning rule. Along a cumulative process of inflation or deflation the forecasting errors from the naive or adaptive learning rule will accumulate and at some point the benefits of rationality will outweigh its costs and the majority of agents will switch to the rational expectations rule. Rational agents thus prevent an everlasting process of inflation or deflation. However, in a heterogeneous expectations world a group of perfectly rational agents can not fully stabilize the cumulative process under interest rate pegging, but rather the economy switches irregularly between phases of high and phases of low inflation.

3.2   Taylor Rule

In this section we consider a Central Bank that responds to the inflation rate by means of a simple Taylor rule as defined in equation (2.4). The dynamics of the model is described by

$$x_t = \frac{k(1 - n_{1,t})}{1 - kn_{1,t}} x_{t-1} \,, \tag{3.7}$$

where the constant $k \equiv \dfrac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi}$ and, as before, the fraction of agents with perfect foresight evolves according to (3.6). Under a Taylor rule with $\phi_\pi > 1$, the coefficient k belongs to the interval $(0, 1)$, and it becomes smaller as the fraction of rational agents increases, as illustrated in Fig. 6. It is then obvious that for any $n_{1,t}$ the map (3.7) is a contraction. It leads to

Proposition 3.2. *The dynamics* (3.7) *and* (3.6) *under a Taylor rule with* $\phi_\pi > 1$ *has a unique, globally stable RE steady-state with* $x^* = 0$ *and* $n_1^* = \dfrac{e^{-\beta C}}{1 + e^{-\beta C}}.$
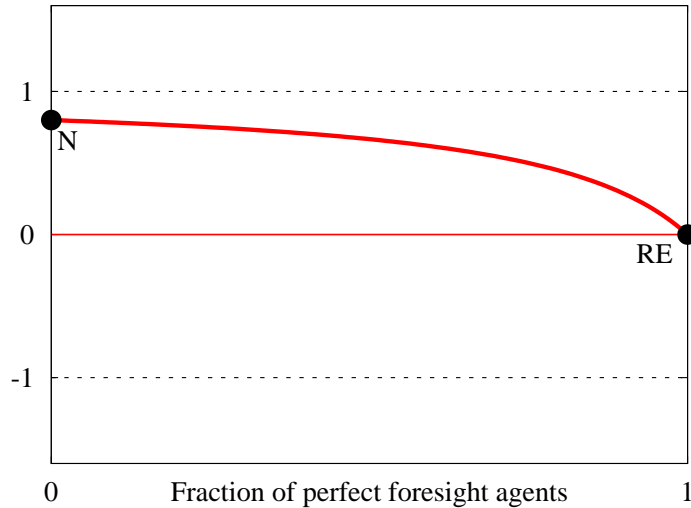
Figure 6: Graph of the slope, $\dfrac{k(1-n_{1,t})}{1-kn_{1,t}}$, with $k = \dfrac{1+\varphi\sigma}{1+\varphi\sigma\phi_\pi}$, of the map in Eq. (3.7) as a function of the fraction of rational agents $n_{1,t}$ in the case of a Taylor rule. Parameters are such that $\varphi\sigma = 2$.

Hence, in an economy with rational versus naive agents, for any costs of the rational forecast, the Taylor rule stabilizes the cumulative inflationary process. In this simple 2-type ecology of forecasting rules, the Central Bank can thus manage heterogeneous expectations by using a Taylor rule that adjusts the nominal interest rate to inflation more than point for point.

4 Interest Rate Pegging with Fundamentalists and Biased Beliefs

In this section we consider an environment in which agents can choose between different constant "steady state" predictors to forecast future inflation, under the assumption that the monetary authority pegs the interest rate. This represents a situation in which agents roughly know the fundamental steady state of the economy, but agents are boundedly rational and disagree about the correct value of the fundamental inflation rate. Forecasting the RE equilibrium value of inflation, $x^* = 0$, requires some cognitive efforts and information gathering costs, which will

be incorporated in the cost $C \geq 0$.[5] Realized inflation and expectations will co-evolve over time and evolutionary selection based on reinforcement learning will decide which forecasting rule performs better and will survive in the evolutionary environment. The class of constant forecasts is extremely simple, but it should be emphasized that it may include all possible point-predictions of next period's inflation level. For this simple ecology of rules it will be possible to obtain analytical results under heterogeneous expectations. We will consider simple examples with only a few rules as well as examples with a large number, even a continuum of rules representing an ecology including all possible steady state predictions.

4.1 Evolutionary Dynamics with Few Constant Belief Types

As a first step we consider the simplest scenario in which agents can choose between three different forecasting rules:

$$x_{1,t}^e = 0,$$
$$x_{2,t}^e = b,$$
$$x_{3,t}^e = -b,$$

with bias parameter $b > 0$. Type 1 agents believe that the inflation rate will always be at its RE level and so the expected deviation will be zero. Type 2 agents have a positive bias, expecting that inflation will be above its fundamental level, while type 3 agents have a negative bias, expecting an inflation level below the fundamental value[6]. Assuming that the equilibrium predictor is available at cost

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[5]In our model formulation the fundamental inflation rate is constant, but the model in deviations can easily be reformulated with a time-varying fundamental. The costs $C \geq 0$ thus represent information gathering costs of a time-varying fundamental inflation rate.

[6]Notice that this example is symmetric in the sense that the positive and negative bias are exactly balanced around the REE. The main reason why we assume symmetry of the belief types in this and other examples below is that under such an assumption the REE is among the steady states of the dynamical system. Thus, with symmetric belief types we can address the important question of stability of the REE. We stress, however, that symmetry of beliefs is not essential for our qualitative results on bifurcations towards multiple steady states. The insight of the model can therefore be used to study the consequences of policy changes (after which the symmetry would be lost since the belief types would not respond to the policy shift immediately), as we do in Section 5.4.

$C \geq 0$ and substituting the forecasting rules of the three types into (3.1) we get

$$x_t = (1 + \varphi\sigma)(n_{2,t}b - n_{3,t}b) = f_\beta(x_{t-1}), \tag{4.1}$$

where fractions are updated according to the discrete choice model (3.3), that is,

$$n_{2,t} = \frac{e^{-\beta(x_{t-1}-b)^2}}{Z_{t-1}}, \qquad n_{3,t} = \frac{e^{-\beta(x_{t-1}+b)^2}}{Z_{t-1}},$$

and

$$Z_{t-1} = e^{-\beta(x_{t-1}^2+C)} + e^{-\beta(x_{t-1}-b)^2} + e^{-\beta(x_{t-1}+b)^2}.$$

In what follows we will fix the parameters φ, σ, and b and consider the intensity of choice β as bifurcation parameter[7]. In Appendix B we derive conditions for the local stability of the rational expectations steady state when the intensity of choice varies; here we will simply illustrate some of these results. The bifurcation scenarios are different in the two cases, when $C \geq b^2$, i.e., the equilibrium predictor is available at a relatively high cost, and in the opposite case when $C < b^2$, i.e., the equilibrium predictor is available at a relatively low cost or for free. Notice that the dynamics in (4.1) is described by a 1-dimensional map f_β, and a straightforward computation shows that f_β is increasing.

Let us start with the case in which the fundamental predictor has relatively high cost. Fig. 7 shows the maps f_β for small, medium and high values of the intensity of choice β. When the intensity of choice is relatively low, there exists only one steady state, the RE steady state, which is globally stable. For low intensity of choice agents are more or less evenly distributed over the different forecasting rules, thus realized inflation will remain relatively close to the fundamental steady state. As the intensity of choice increases, the RE steady state loses stability in a

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[7]Changes in the product of the IS slope σ and the Phillips curve's slope φ, as well as changes in the bias parameter b will only affect the numerical values of the non-RE steady states and the critical values of the bifurcation parameter at which multiple steady states appear, but they will not alter the *qualitative* bifurcation scenario discussed below.
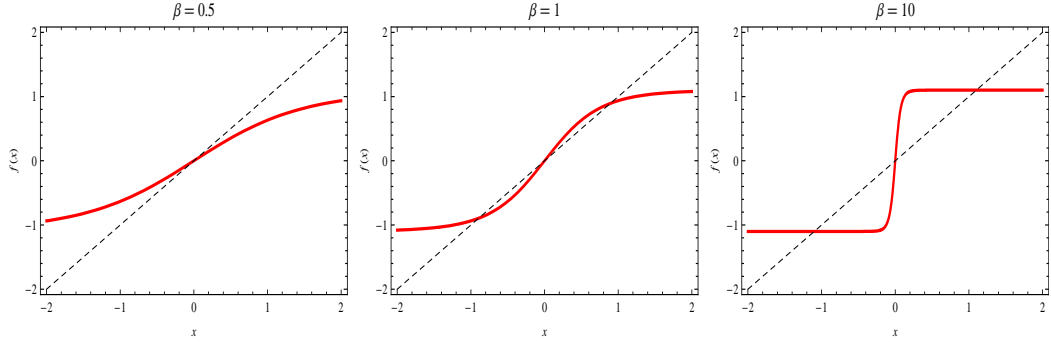
Figure 7: The map f_β in (4.1) with 3 belief types, 0, b and $-b$, high information costs C and different values of β. Other parameter values are $\varphi\sigma = 0.1$, $b = 1$ and $C = 1$.
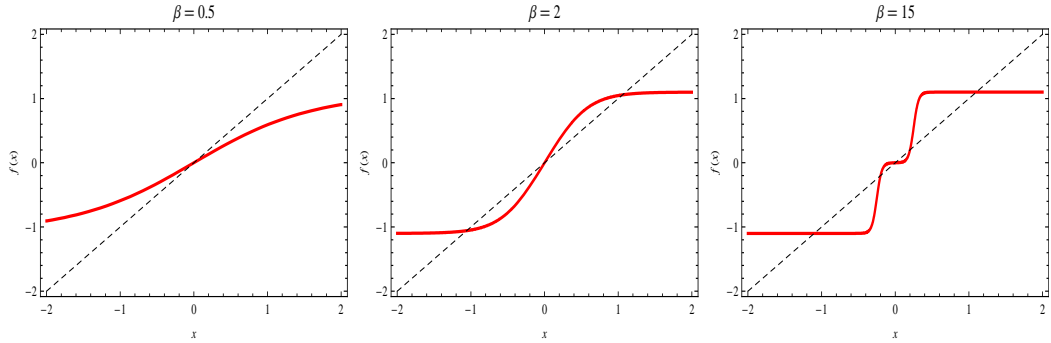

Figure 8: The map f_β in (4.1) with 3 belief types, 0, $+b$ and $-b$, and low information costs C. Parameter values are $\varphi\sigma = 0.1$, $b = 1$ and $C = 0.5$.

(supercritical) pitchfork bifurcation[8] and two new stable non-fundamental steady states are created. The economic intuition behind the fact that non-fundamental steady states exist for high intensity of choice is simple. Suppose that the intensity of choice is high and that, at time t, the deviation x_t is close to the optimistic belief, that is, $x_t \approx b$. The positive bias forecast will perform better than the negative bias and the fundamental belief. Therefore, when the intensity of choice is high, almost all agents will forecast inflation with the positive bias, i.e., $n_{2,t+1} \approx 1$, implying that $x_{t+1} \approx b(1 + \varphi\sigma)$. The same intuition explains existence of a negative non-fundamental steady state for high intensity of choice.[9]

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[8]See Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory.

[9]In fact, it is easily seen that for $\beta = \infty$ the map f_β has two symmetric non-fundamental steady states $x^+ = b(1 + \varphi\sigma)$, with only positive bias forecasters ($n_2^+ = 1$) and $x^- = -b(1 + \varphi\sigma)$, with only negative bias forecasters ($n_3^+ = 1$).
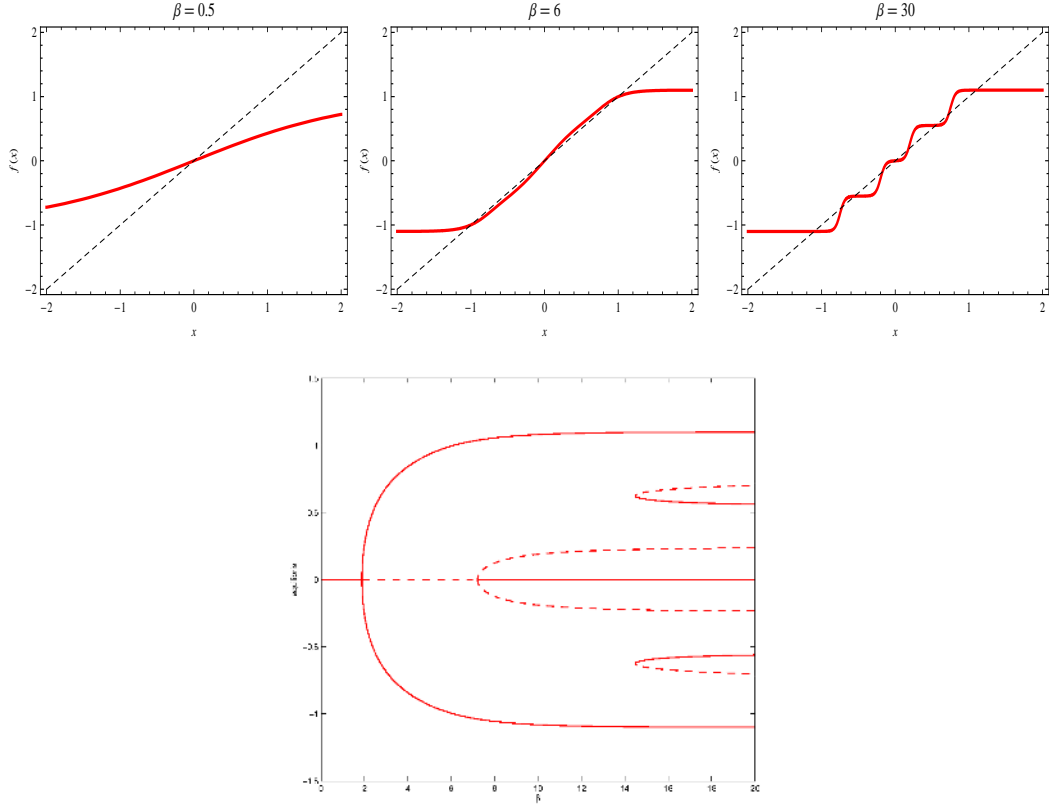
Figure 9: **Top panels:** Maps with 5 types of beliefs, $b_h \in \{-1, -1/2, 0, 1/2, 1\}$, and low cost C for different values of β. **Lower panel:** Bifurcation diagram for 5 belief types (cost $C = 0$) with respect to the intensity of choice. Solid lines indicate stable equilibria and dashed lines unstable equilibria. For high values of β 9 different steady states co-exist, 5 stable separated by 4 unstable steady states.

Consider now the case in which the equilibrium predictor has zero (or relatively low) costs. In this scenario all agents have free access to the relevant information, but they make some computational mistakes or they just think that in a heterogeneous world not every agent will behave the same and try to anticipate deviations from RE equilibrium. Fig. 8 shows graphs of the map f_β for small, medium and high values of the intensity of choice β.

As before, when the intensity of choice β is relatively low we have a unique globally stable fundamental steady state $x^* = 0$. As β increases the fundamental steady state loses stability in a (supercritical) pitchfork bifurcation in which two additional stable non-fundamental steady states are created. However, as β increases further, we have a secondary pitchfork bifurcation (this time a subcritical

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pitchfork bifurcation) in which the RE steady state becomes stable again and two additional unstable steady states are created. In the case of low costs for fundamentalists, we thus have *three stable* steady states, $x^+ > 0$, $x^- < 0$ and also $x^* = 0$, for high values of the intensity of choice β. The economic intuition that, if the costs for the fundamental rule are low, the fundamental steady state will be stable for high intensity of choice is simple: when the system is close to the fundamental steady state, a cheap fundamental rule is the best predictor, causing more agents to switch to the fundamental rule.

A similar analysis can be made for other examples with larger number of constant beliefs. Fig. 9 illustrates graphs of the 1-D map when there are five strategy types $b_h \in \{-1, -1/2, 0, 1/2, 1\}$ and the costs C of the fundamental predictor are low. We also show the creation of five multiple steady state equilibria as the intensity of choice increases by means of the bifurcation diagram. For small and medium values of β the bifurcation scenario is similar to the three types case. However for high values of the intensity of choice, four additional steady states, two stable and two unstable, are created via saddle-node bifurcations. The intuition for the appearance of the new stable steady states is similar as before. Any available predictor would give the most precise forecast if the past inflation rate is sufficiently close to it. A high intensity of choice causes a large group of agents to choose this successful predictor, locking the inflation dynamics into a self-fulfilling stable equilibrium steady state close to that predictor.

4.2   Many Belief Types

The previous analysis shows that in an economy with an ecology of 3 or 5 fundamentalists and biased beliefs, a cumulative process leading to accelerating inflation or deflation does *not* arise. Rather, for high intensity of choice, the system will lock in into one of multiple steady state equilibria, with a majority of agents using the forecasting rule with the smallest error at that equilibrium steady state. A natural question addressed in this section is: *what happens when the number of constant*

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forecasting rules increases and approaches infinity? As we will see, if agents select beliefs from a *continuum* of forecasting rules, representing an ecology containing *all* possible steady state predictions, the cumulative process will reappear.

Suppose there are H belief types b_h, all available at zero costs. Under interest rate pegging, the evolutionary dynamics with H belief types is given by

$$x_t = (1 + \varphi\sigma)\frac{\sum_{h=1}^{H} b_h e^{-\beta(x_{t-1}-b_h)^2}}{\sum_{h=1}^{H} e^{-\beta(x_{t-1}-b_h)^2}} =: f_\beta^H(x_{t-1}). \qquad (4.2)$$

The dynamics of the system with H belief types b_h is described by a 1-D map f_β^H. What can be said about the dynamical behavior when H is large? In general, it is difficult to obtain analytical results for systems with many belief types. We apply the concept of *Large Type Limit* (LTL henceforth) introduced in Brock, Hommes, and Wagener (2005) to approximate the evolutionary system with many belief types in (4.2). Suppose that at the beginning of the economy, i.e. at period $t = 0$, all H belief types $b = b_h \in \mathbb{R}$ are drawn from a common initial distribution with density $\psi(b)$. We then can derive the LTL of the system as follows. Divide both numerator and denominator of (4.2) by H and rewrite the "H-type system" as

$$x_t = (1 + \varphi\sigma)\frac{\frac{1}{H}\sum_{h=1}^{H} b_h e^{-\beta(x_{t-1}-b_h)^2}}{\frac{1}{H}\sum_{h=1}^{H} e^{-\beta(x_{t-1}-b_h)^2}}.$$

The LTL is then obtained by replacing the sample mean with the population mean in both the numerator and the denominator, yielding

$$x_t = (1 + \varphi\sigma)\frac{\displaystyle\int b e^{-\beta(x_{t-1}-b)^2}\psi(b)db}{\displaystyle\int e^{-\beta(x_{t-1}-b)^2}\psi(b)db} =: F_\beta(x_{t-1}). \qquad (4.3)$$

As shown in Brock, Hommes, and Wagener (2005), when the number of strategies H is sufficiently large, the LTL dynamical system (4.3) is a good approximation of the dynamical system with H belief types given by (4.2). In particular, if H is large then with high probability the steady-states and their local stability conditions as
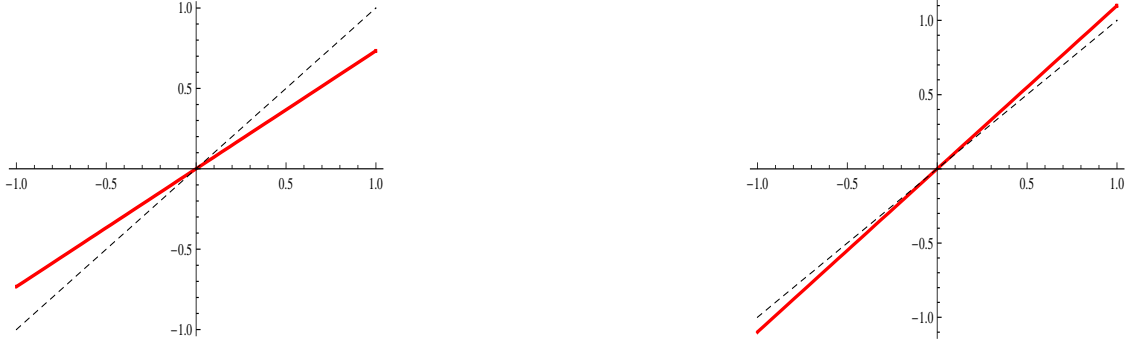
Figure 10: Graphs of the LTL map F_β in (4.4) under interest rate pegging for a normal distribution $\psi(b) \simeq N(0, s^2)$ of initial beliefs. **Left panel:** $\beta = 1$, **Right panel:** $\beta = 1000$.

functions of β coincide for both the LTL map F_β and the H-belief system map f_β^H. In other words, properties of the evolutionary dynamical system with many types of agents can be studied using the LTL system.

For suitable distributions $\psi(b)$ of initial beliefs, the LTL (4.3) can be computed explicitly. As an illustrative example consider the case when $\psi(b)$ is a normal distribution, $\psi(b) \simeq N(m, s^2)$. Plugging the normal density in (4.3), a straightforward computation shows that the LTL map F_β is linear in this case, given by

$$F_\beta(x) = (1 + \varphi\sigma)\frac{m + 2\beta s^2 x}{1 + 2\beta s^2} . \tag{4.4}$$

In particular, when the initial beliefs distribution is centered around $m = 0$, the unique steady-state of the LTL map is the RE equilibrium, $x^* = 0$. This case is illustrated in Fig. 10, where we show the LTL map for different values of the intensity of choice. For $\beta = \beta^* = \dfrac{1}{2s^2\varphi\sigma}$ the slope of the linear map is exactly 1. Hence, the RE equilibrium is globally stable for $\beta < \beta^*$ and unstable otherwise. When $m \neq 0$, i.e., when the initial belief distribution is not symmetric with respect to the fundamental equilibrium, the unique steady state of the LTL system is not the REE, but the stability result and critical value β^* do not change (cf. footnote 6).

We can conclude, therefore, that when initial beliefs are drawn from a normal distribution centered around the REE and the number of belief types is sufficiently

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high, an increase in the intensity of choice, beyond the bifurcation value β^*, leads to instability of the system. Indeed, when β is low, agents are more or less equally distributed among predictors. This means that the average expected deviation of inflation from the steady state will be close to zero. Hence realized inflation will be close to the steady state value, more agents will adopt the steady state predictor and inflation will converge. However, when the intensity of choice increases and agents can switch faster to better predictors, the system becomes unstable. This is so because, for example, when the inflation rate is above its steady state value most agents will switch to an even more positive bias belief, leading to an even higher realized inflation rate. A cumulative process of ever increasing inflation arises again.

Note that increasing the variance s^2 of the normal distribution of initial beliefs has exactly the same effect on the LTL dynamics (4.4) as increasing the intensity of choice. For $s^2 < \dfrac{1}{2\beta\varphi\sigma}$ the LTL map is globally stable, it is unstable otherwise. Hence, when many initial beliefs are drawn from a normal distribution with small variance, the system will be stable, otherwise it will be unstable and a cumulative process will arise. The *spread* of initial beliefs is therefore an important element for the stability of the economy.

In the previous example we have assumed a normal distribution $\psi(b)$ of initial beliefs. Applying the results derived in Hommes and Wagener (2008), similar conclusions can be obtained for general distribution functions of initial beliefs. In fact, for systems with many belief types b_h and initial beliefs drawn from a fixed strictly positive distribution function, when the intensity of choice becomes sufficiently large, a cumulative process arises with high probability[10].

To get some intuition for this result, it will be instructive to look at the limiting case $\beta = \infty$. When there is a continuum of beliefs, the best predictor in every period, according to past forecast error, will be the predictor that exactly coincides

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[10]This result follows by applying Lemma 1, p. 10 of Hommes and Wagener (2008), stating that for any strictly positive distribution function ψ describing initial beliefs, as the intensity of choice goes to infinity, the corresponding LTL map converges to a linear map with slope $1 + \varphi\sigma$.

with last period's inflation realization, $b_h = x_{t-1}$. For $\beta = \infty$, all agents will switch to the optimal predictor. Hence, for $\beta = \infty$, the economy with heterogenous agents updating their beliefs through reinforcement learning behaves exactly the same as an economy with a representative naive agent, for which we have shown that a cumulative process will arise (see Section 2, eq. (2.3)).

5 Taylor Rule with Fundamentalists and Biased Beliefs

Now we turn to the analysis of the inflation dynamics under an alternative monetary policy, namely a Taylor rule, $i_t = \phi_\pi \pi_t$, introduced in (2.4). Plugging the Taylor rule into the system (2.1)-(2.2) and rewriting the model in deviations from the RE equilibrium yields the dynamics (3.1) with constant k given by (3.2), i.e.,

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \sum_{h=1}^{H} n_{h,t} x_{h,t}^e . \tag{5.1}$$

5.1 Many types

When the central bank implements a Taylor interest rate rule (2.4), the LTL of the system can be derived as

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \cdot \frac{\int b e^{-\beta(x_{t-1}-b)^2} \psi(b) db}{\int e^{-\beta(x_{t-1}-b)^2} \psi(b) db} = F_\beta(x_{t-1}) . \tag{5.2}$$

Under the assumption that the distribution of initial beliefs is normal, $\psi(b) \equiv N(0, s^2)$, the LTL map (5.2) is a linear map with slope increasing in β, similar to the illustration in Fig. 10. However, the "unstable" situation shown in the right panel of Fig. 10 cannot occur if the Central Bank follows a Taylor rule with $\phi_\pi > 1$

(i.e., if the Taylor principle holds). Indeed, in this case we will have that

$$\lim_{\beta \to \infty} F_\beta(x) = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi} \, x \,. \tag{5.3}$$

Hence an interest rate rule that responds aggressively to actual inflation, i.e. $\phi_\pi > 1$, will fully stabilize the system, for all values of the intensity of choice β. In contrast, if the policy rule of the Central Bank is not sufficiently aggressive, i.e. $\phi_\pi < 1$, then inflation dynamics will only be stable for small values of the intensity of choice, but the cumulative process will reappear when the intensity of choice is large. The same result holds for a normal initial distribution of beliefs centered around $m \neq 0$, even though the steady state of the dynamics will differ from the RE equilibrium in this case.

5.2   Few types

Now consider the simple case in which the Central Bank implements a Taylor-type interest rate rule and there are only three steady state predictors, $\{-b, 0, +b\}$, available in the economy. The map describing the dynamics of the system is given by

$$x_t = \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi}(n_{2,t}b - n_{3,t}b) = f_\beta(x_{t-1}) \,. \tag{5.4}$$

As before, we consider the two different cases in which the equilibrium predictor is available at a relatively high cost C and freely available respectively. Fig. 11 depicts the dynamics of the system using the same parameterization as in Section 4.1, for the coefficient $\phi_\pi = 1.5$ of the Taylor rule and with a relatively high cost C. When the intensity of choice is relatively low the RE equilibrium is unique and globally stable. However, as β increases the fundamental steady becomes unstable after a supercritical pitchfork bifurcation. We thus observe that when agents switch faster between different predictors, inflation dynamics locks in into
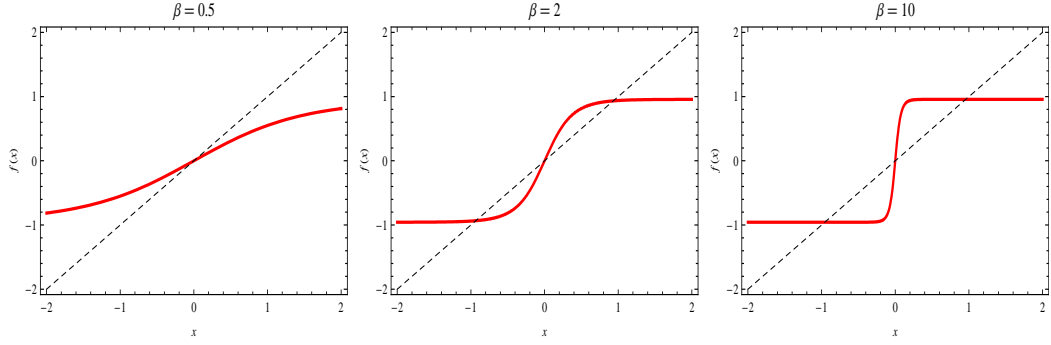
Figure 11: Graphs of the map f_β in (5.4) with 3 belief types, 0, $+b$ and $-b$, and high information costs. Parameter values are $\varphi\sigma = 0.1$, $b = 1$, $C = 0.5$ and $\phi_\pi = 1.5$.
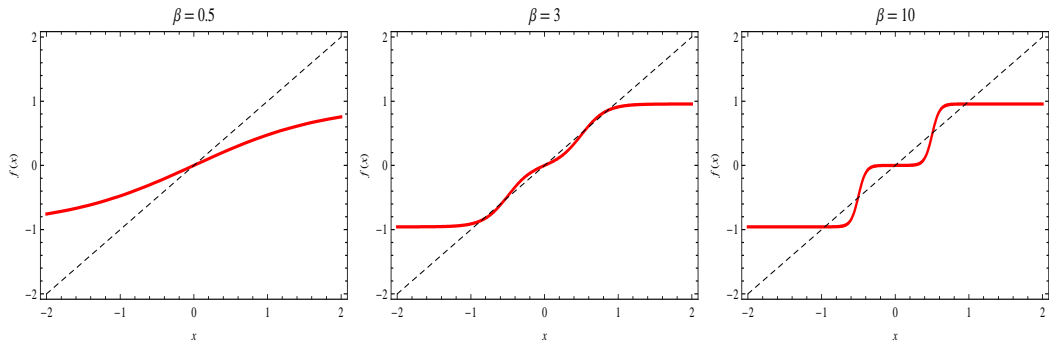

Figure 12: Graphs of the map f_β in (5.4) with 3 belief types, 0, $+b$ and $-b$, at zero information costs. Parameter values are $\varphi\sigma = 0.1$, $b = 1$, $C = 0$ and $\phi_\pi = 1.5$.

non-fundamental equilibria because of the relatively high costs of the fundamental predictor.

Fig. 12 illustrates the dynamics of the model when the equilibrium predictor is freely available. In this case we observe that the RE equilibrium remains locally stable when the intensity of choice increases and four additional steady states, two stable and two unstable, are created via saddle node bifurcation.

The previous analysis shows that even if the interest rate rule followed by the Central Bank obeys the Taylor principle and responds more than point for point to the rate of inflation, multiple equilibria can arise when only a few prediction strategies are available in the economy. One can construct similar examples for any finite (odd) number, $H = 2K + 1$, of forecasting strategies generating H multiple stable equilibria. A finite class of forecasting rules seems reasonable as

boundedly rational agents may exhibit "digit preference" and restrict their inflation predictions to values in integer numbers, e.g., 2%, 3%, or to half percentages, e.g., 2.5% or 3.5%, within the range of historically observed values from say -5% to $+15\%$[11].

5.3 Adding rational agents

The previous subsection has shown that in a world of heterogeneous expectations and an ecology of finitely many constant, biased forecasting rules, a Taylor rule that adjusts the interest rate more than point for point to the inflation rate does not stabilize inflationary dynamics and multiple steady state equilibria may arise. While the ecology of constant forecasting rules may be a reasonable first order description of individual forecasting behavior the question arises whether multiple steady state equilibria will persist when introducing more sophisticated forecasting strategies. The purpose of this subsection is to add a fully rational forecasting strategy to the ecology of predictors. We thus extend the model by including an optimal, perfect foresight forecasting rule. Perfect foresight implies strong cognitive knowledge and perfect information about the economy, including the beliefs of all other trader types, and therefore we follow Brock and Hommes (1997) and assume that information gathering costs $C \geq 0$ must be incurred to obtain the rational forecast.

Interest Rate Pegging

Under interest rate pegging the extended model with three constant belief types 0, $+b$ and $-b$ (as in Section 4.1) and an additional rational type, whose forecast is

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[11]Digit preference has, e.g., been observed in the learning-to-forecast experiments in Hommes, Sonnemans, Tuinstra, and van de Velden (2007). 118 subjects had to forecast prices for 50 periods in an interval $[0, 10]$ in two digits. The distribution of 5400 forecasts was clearly peaked at integer values (27%) and .5-forecasts (12.5%).
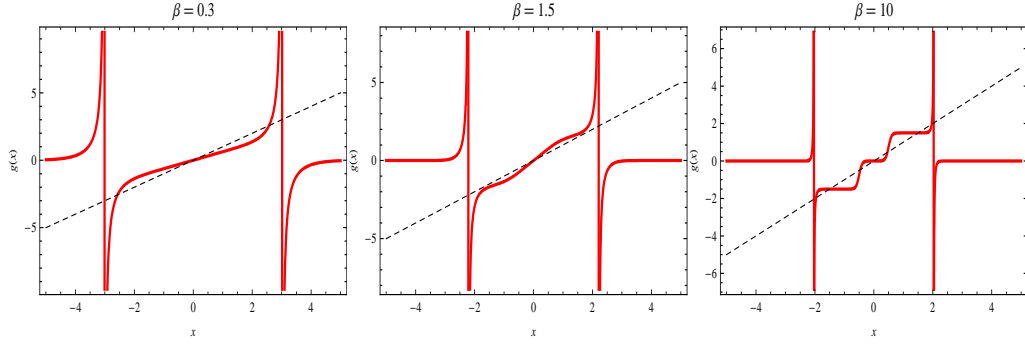
Figure 13: Graphs of the maps g_β in (5.5) with a rational agent and 3 types of constant beliefs for high information cost $C = 1$.

$x_{0,t}^e = x_t$ and fraction $n_{0,t}$, reads:

$$x_t = \frac{(1 + \varphi\sigma)b(n_{2,t} - n_{3,t})}{1 - (1 + \phi\sigma)n_{0,t}} = g_\beta(x_{t-1}). \tag{5.5}$$

We assume that the perfect foresight predictor is available at cost $C \geq 0$, so that the fractions are given by

$$n_{0,t} = \frac{e^{-\beta C}}{Z_{t-1}}, \quad n_{1,t} = \frac{e^{-\beta x_{t-1}^2}}{Z_{t-1}}, \quad n_{2,t} = \frac{e^{-\beta(x_{t-1}-b)^2}}{Z_{t-1}}, \quad n_{3,t} = \frac{e^{-\beta(x_{t-1}+b)^2}}{Z_{t-1}}, \tag{5.6}$$

with normalization factor

$$Z_{t-1} = e^{-\beta C} + e^{-\beta x_{t-1}^2} + e^{-\beta(x_{t-1}-b)^2} + e^{-\beta(x_{t-1}+b)^2}.$$

Fig. 13 shows some typical features of the system's dynamical behavior for different values of the intensity of choice β. When the intensity of choice is relatively low agents are evenly distributed among the predictors and the RE steady state is globally stable (Fig. 13, left panel)[12]. In the second scenario depicted in the middle panel of Fig. 13, the fundamental steady state is unstable and a temporary cumulative process arises, until the point when the forecast errors of the cheap bi-

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[12]The vertical asymptotes in Fig. 13 are due to the presence of rational agents in the market. Indeed when the fraction of rational agents $n_{0,t}$ is equal to the threshold level $1/(1 + \varphi\sigma)$ the denominator in (5.5) becomes zero. When the deviation x_{t-1} approaches an asymptote, the errors of the steady state predictor become so large that almost all agents switch to the rational forecast, pushing inflation back close to its fundamental equilibrium.

ased predictors become so large that it becomes worthwhile to switch to the costly sophisticated perfect foresight predictor. The rightmost graph of Fig. 13 illustrates that for high values of the intensity of choice parameter β the RE steady state becomes locally stable (after a subcritical pitchfork bifurcation) and four additional steady states are created (via saddle-node bifurcations). We thus notice that even in the presence of a rational agent multiple equilibria may still arise.

For the sake of completeness we include some intuition about the behavior of the system when the perfect foresight strategy is available at zero or relatively low cost. For small and intermediate values of the intensity of choice β the first two scenarios described in Fig. 13 occur. However, for a high intensity of choice agents switch fast to the best predictor available in the economy, i.e. to the perfect foresight predictor, and the RE steady state becomes globally stable. Indeed, when there are no information gathering costs for rationality, it is optimal for agents to adopt the perfect foresight predictor and inflation will stabilize.

Taylor Rule

When the Central Bank implements a Taylor type interest rate rule as in (2.4) the inflation dynamics of the model with three constant belief types 0, $+b$ and $-b$ and a rational type is given by

$$x_t = \frac{kb(n_{2,t} - n_{3,t})}{1 - kn_{0,t}} = h_\beta(x_{t-1}), \qquad k \equiv \frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi}, \tag{5.7}$$

with the fractions of different types as in (5.6). Fig 14 illustrates the two typical features of the model when the cost for the perfect foresight predictor is relatively high.

When the intensity of choice is low, the zero steady state is unique and globally stable, but when the intensity of choice β increases multiple equilibria arise. Multiple equilibria arise when the perfect foresight predictor is available at a relatively high cost compared to the forecasting errors of the biased steady state forecasts.
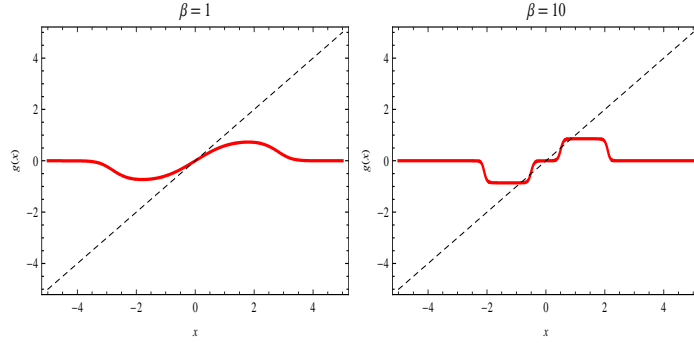
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Figure 14: Maps $h_\beta(x_{t-1})$ in (5.7) with a rational agent and 3 types of constant beliefs for high cost $C = 1$ and a Taylor monetary policy rule with $\phi_\pi = 1.5$.

Indeed at the non-fundamental equilibria the biased predictors make a forecast error that is smaller than the information gathering costs for rationality. In the case when the rational predictor is available at zero or relatively low cost, the RE fundamental steady state is unique and globally stable as the majority of agents will switch to the best predictor in the economy.

Hence, in a world with an ecology of finitely many heterogeneous forecasting rules including constant rules as well as a costly fully rational rule and where agents are sensitive to differences in strategy performance, an aggressive monetary policy, that adjusts the interest rate in response to inflation more than point for point does not fully stabilize inflation dynamics, but non-fundamental equilibria may exist.

5.4 Stochastic Simulations

In this section we discuss stochastic simulations of our nonlinear model with heterogeneous expectations in order to match some characteristics of US inflation quarterly data over the period 1960-2007. We consider an ecology of $H = 12$ forecasting rules, $b_h \in \{0, 1, \ldots, 11\}$, so that the dynamics of inflation is given by

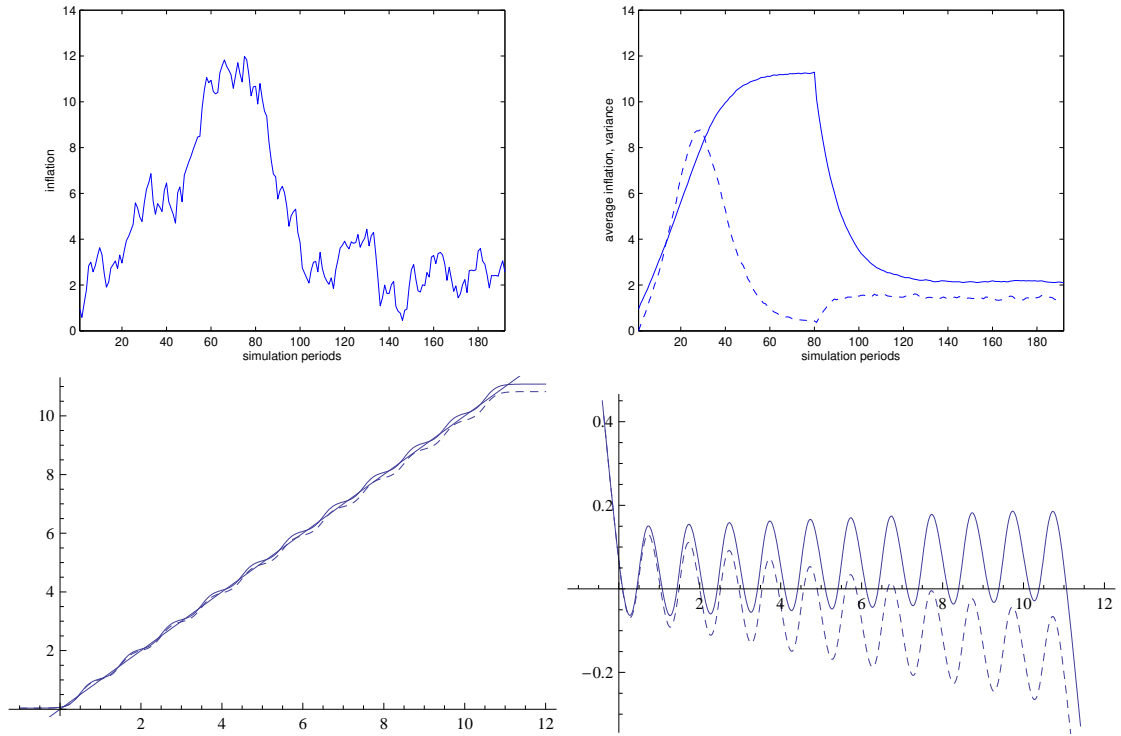$$\pi_t = f_{\phi_\pi}^H(\pi_{t-1}) + \varepsilon_t \,,$$

Figure 15: **Top Panels:** Simulated inflation time series. **Left:** Simulated inflation for a particular realization of the stochastic shocks. **Right:** Average inflation (solid line) and its variance (dashed line) over 1000 simulations. **Bottom Panels:** Steady-states of the dynamics before and after the structural break. **Left:** The steady-states of the dynamics as intersection points of the 45-degree line with the map $f_{0.8}^H(\pi)$ before the structural break (solid) and the map $f_2^H(\pi)$ after the structural break (dashed). **Right:** Plot of the maps $f_{\phi_\pi}^H(\pi) - \pi$. The same steady-states are now clearly visible as intersections with the horizontal axis. The stable (unstable) steady-states correspond to the intersections with negative (positive) slope. The basin of attraction of a stable steady-state is the interval between two adjacent unstable steady-states.

where the map $f_{\phi_\pi}^H$ is defined as

$$f_{\phi_\pi}^H(\pi_{t-1}) = \left(\frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi}\right) \frac{\sum_{h=1}^H b_h e^{-\beta(\pi_{t-1} - b_h)^2}}{\sum_{h=1}^H e^{-\beta(\pi_{t-1} - b_h)^2}} + \frac{\varphi\sigma r^*}{1 + \varphi\sigma\phi_\pi}, \tag{5.8}$$

and the exogenous random shocks ε_t are drawn from a normal distribution with mean 0 and standard deviation $\sigma_\varepsilon = 0.5$. Recall that r^* denotes the natural interest rate, which we fix at 2 percent in our simulations. The notation $f_{\phi_\pi}^H(\pi_{t-1})$ stresses the fact that the nonlinear map depends on the monetary policy parameter ϕ_π, the coefficient in the Taylor rule. In all stochastic simulations there is a structural

break in period $T = 80$, when the Central Bank changes the coefficient of the Taylor rule from $\phi_\pi = 0.8$ to $\phi_\pi = 2$. These values are of the same order of magnitude as the estimated coefficients of a Taylor rule in the period 1960-1979 and the period 1979-1997, see e.g., Taylor (1999) and Clarida, Gali, and Gertler (2000). Other parameters are fixed at $\varphi\sigma = 0.1$, and $\beta = 3.5$. This choice affects the functional form of the map $f_{\phi_\pi}^H$, and therefore the number of steady-states and their levels. The qualitative features of our simulations are, however, fairly robust to the changes in these parameter values.

The stochastic time series in Figure 15 replicates the observed pattern of a strong rise in US inflation until 1980 and a sharp decline and stabilization of inflation thereafter (see also Fig. 1). Of course the particular realization shown in the top left panel is affected by stochastic shocks, but this pattern is quite common and reproduced by the time series of average inflation, averaged over 1000 stochastic simulations, in Figure 15 (top right panel). The plot of the corresponding variance of the stochastic simulations shows that the variance is low after the structural break, implying that the strong decline in inflation after the structural break is a robust feature of the nonlinear model with heterogeneous beliefs and a monetary authority that responds aggressively to high inflation (i.e. uses a Taylor rule with a high value of the coefficient ϕ_π). On the other hand, before the structural break the variance of the stochastic simulations is large, showing that the rise in inflation can be either slow or fast depending upon the realizations of the exogenous stochastic shocks. In particular, a few large positive shocks to inflation, such as large oil shocks, may trigger an increase in inflation which then becomes amplified by evolutionary pressure of self-fulfilling forecasting rules predicting high inflation.

The bottom panel in Figure 15 illustrates how the number of steady states in the nonlinear model with heterogeneous expectations changes when the monetary policy coefficient ϕ_π in the Taylor rule increases from 0.8 to 2. Before the structural break ($\phi_\pi = 0.8$) there are 23 steady states, 12 stable ones separated by 11 unstable steady states, ranging from a low level of 0 to a high level of 11. A careful look

at Figure 15 (bottom right panel) reveals an important asymmetry in the basins of attraction of each stable steady state: the basin of attraction (whose endpoints consist of the two neighboring unstable steady states) is relatively large to the left of the stable steady state and relatively small to the right. In the presence of (symmetric) stochastic shocks to inflation, jumps to the basin of attraction of a higher stable steady states are therefore more likely than jumps to a lower level. This explains why for $\phi_\pi = 0.8$ on average inflation will rise from low levels to high levels as shown by the average inflation of the stochastic simulations.

After the structural break, when the Central Bank switches to a more aggressive Taylor rule ($\phi_\pi = 2$), the number of steady states has decreased from 23 to 15, with 8 stable ones separated by 7 unstable steady states, ranging from approximately 0 to 7. Hence, an increase of the monetary policy parameter ϕ_π causes a number of high level steady states to disappear[13], implying more stable inflation dynamics in the stochastic nonlinear system as illustrated in the stochastic simulations after the structural break.

It is interesting to note that similar results occur when we allow for (infinitely) many constant prediction rules. Indeed our results concerning the LTL system in (5.2) in Section 5.1 show that, when agents are sensitive to difference in forecasting performance (i.e. for high values of the intensity of choice β), the inflation dynamics with an ecology of *many* steady state predictors drawn from a normal distribution of initial beliefs is well approximated by the linear map in (5.3), with slope $\frac{1 + \varphi\sigma}{1 + \varphi\sigma\phi_\pi}$. This implies globally stable inflation dynamics approaching the RE equilibrium rate of inflation when $\phi_\pi > 1$, but exploding inflation dynamics when $\phi_\pi < 1$. Hence, in an ecology with many steady state predictors when the Central Bank uses a Taylor rule with $\phi_\pi < 1$ a cumulative process of rising inflation is very likely, while the monetary authority can manage heterogeneous expectations and achieve global macro economic stability by using a more aggressive Taylor rule with $\phi_\pi > 1$.

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[13]As ϕ_π increases from 0.8 to 2 the high level steady states disappear in pairs of two (one stable and one unstable) through a number of subsequent saddle-node bifurcations.

6   Concluding Remarks

We have used the stylized model of Howitt (1992) to study the role of heterogeneous expectations about future inflation and the potential (de-)stabilizing effect of different interest rate rules. We use the heterogeneous expectations framework of Brock and Hommes (1997), where the ecology of forecasting rules is disciplined by endogenous, evolutionary selection of strategies with agents switching towards more successful rules.

Macroeconomic stability and inflation dynamics depend in interesting ways on the ecology of forecasting strategies and the coefficient of an interest rate rule à la Taylor. When the monetary authority responds weakly to inflation, a cumulative process of rising inflation occurs, triggered by exogenous shocks to economic fundamentals and reinforced by self-fulfilling expectations of high inflation. In contrast, when the nominal interest rate is adjusted more than point for point in response to inflation, the monetary authority can manage heterogeneous expectations and stabilize inflation, although multiple low-level steady state equilibria may persist. Our nonlinear model with heterogeneous expectations and a structural break in monetary policy matches US inflation quarterly data from 1960-2007 fairly well.

This paper has used the simple, stylized model of Howitt (1992) to study the effect of heterogeneous expectations on the dynamics of inflation. Future work should further investigate the effect of heterogeneous expectations on the dynamics of aggregate output and inflation in more realistic micro-founded models, such as the New Keynesian framework, and the conditions under which monetary policy rules may stabilize or may fail to stabilize aggregate macroeconomic variables.

APPENDIX

A Proof of Proposition 3.1

Substituting (3.6) into (3.5), the dynamical system becomes a difference equation of second order:

$$x_t = \frac{(1 + \varphi\sigma)\left(1 - \frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(x_{t-1} - x_{t-2})^2}}\right)}{1 - (1 + \varphi\sigma)\left(\frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(x_{t-1} - x_{t-2})^2}}\right)} x_{t-1} .$$

We can rewrite the latter equation as a two-dimensional system by introducing $z_t = x_t$ and $w_t = x_{t-1}$

$$z_t = \frac{(1 + \varphi\sigma)\left(1 - \frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(z_{t-1} - w_{t-1})^2}}\right)}{1 - (1 + \varphi\sigma)\left(\frac{e^{-\beta C}}{e^{-\beta C} + e^{-\beta(z_{t-1} - w_{t-1})^2}}\right)} z_{t-1}$$

$$w_t = z_{t-1} .$$

The Jacobian of the system computed in the RE steady-state $(0, 0)$ is given by

$$J(0,0) = \begin{bmatrix} \frac{(1+\varphi\sigma)(1-n_1^*)}{1-(1+\varphi\sigma)n_1^*} & 0 \\ 1 & 0 \end{bmatrix} ,$$

where $n_1^* = \frac{e^{-\beta C}}{1 + e^{-\beta C}}$. The eigenvalues are

$$\lambda_1 = 0 ,$$
$$\lambda_2 = \frac{(1 + \varphi\sigma)(1 - n_1^*)}{1 - (1 + \varphi\sigma)n_1^*} = \frac{1 + \varphi\sigma - n_1^* - \varphi\sigma n_1^*}{1 - n_1^* - \varphi\sigma n_1^*} .$$

The numerator in expression of λ_2 is always positive since $0 < n_1^* < 1$, while the denominator is positive if $\varphi\sigma < 1/e^{-\beta C}$. In this case we have that $\lambda_2 > 1$. When $\varphi\sigma > 1/e^{-\beta C}$, the stability condition implies

$$\frac{(1 + \varphi\sigma)(1 - n_1^*)}{1 - (1 + \varphi\sigma)n_1^*} > -1 \Rightarrow (1+\varphi\sigma)(1-n_1^*) < (1+\varphi\sigma)n_1^* - 1 \Rightarrow \varphi\sigma < \frac{2(n_1^* - 1)}{1 - 2n_1^*} \Rightarrow \varphi\sigma < \frac{-2}{1 - \exp(-\beta C)}$$

Since both φ and σ are positive coefficients, the stability condition is never satisfied and thus we conclude that $|\lambda_2| > 1$, and the steady state is always unstable ∎

B   Stability of RE steady state in 3 types system

This appendix investigates the (local) stability of the RE steady state in the 3-types systems with an interest rate pegging rule (4.1) and with a Taylor rule (5.4), given by

$$x_t = kb \frac{e^{-\beta(x_{t-1}-b)^2} - e^{-\beta(x_{t-1}+b)^2}}{e^{-\beta(x_{t-1}^2+C)} + e^{-\beta(x_{t-1}-b)^2} + e^{-\beta(x_{t-1}+b)^2}} = f(x_{t-1}),$$

where $k = 1 + \varphi\sigma$ resp. $k = \frac{1+\varphi\sigma}{1+\varphi\sigma\phi_\pi}$. The derivative of the map f in the RE steady state is

$$f'(0) = k4\beta b^2 \frac{e^{-\beta b^2}}{2e^{-\beta b^2} + e^{-\beta C}} = k4\beta b^2 \frac{1}{2 + e^{-\beta(C-b^2)}}\,.$$

The stability condition is thus given by

$$f'(0) < 1 \Rightarrow \frac{4b^2\beta}{2 + e^{-\beta(C-b^2)}} < \frac{1}{k}\,.$$

Now define

$$h(\beta) = \frac{4b^2\beta}{2 + e^{-\beta(C-b^2)}}$$

and consider the following two cases.

If $C \geq b^2$ we have that $h(\beta)$ is monotonically increasing in β. Thus, when β is higher than the bifurcation value β^* defined as

$$\beta^* : h(\beta^*) = \frac{1}{k}$$

the zero steady state looses stability, as shown in the left panel of Fig. 16.

If $C < b^2$ we have that the function $h(\beta)$ is initially increasing in β and then decreas-

Figure 16: Stability/Instability of the REE. **Left panel:** The case of high cost, $C > b^2$. **Right panel:** The case of small cost, $C < b^2$.

ing. We indeed have that

$$h'(\beta) = \frac{4b^2}{\left[2 + e^{-\beta(C-b^2)}\right]^2} \left[2 + e^{-\beta(C-b^2)} + \beta e^{-\beta(C-b^2)}(C - b^2)\right] .$$

We have that $h'(\beta) = 0$ when

$$2 + e^{-\beta(C-b^2)} + \beta e^{-\beta(C-b^2)}(C - b^2) = 0 \,.$$

Now define $z \equiv (C - b^2)\beta$, so that the previous equation becomes

$$2 + e^{-z} + ze^{-z} = 0 \,,$$

which can be rewritten as

$$2e^z = -z - 1 \,. \tag{B.1}$$

Now, when $C < b^2$, we have that z is a variable defined over $(-\infty, 0)$ since β is increasing from 0 to ∞. This means that there is only one solution $z^* < 0$ to the previous equation, i.e. $h(\beta)$ has only one optimum as shown in the right panel of Fig. 16.

We can find an approximate numerical solution to (B.1) which is given by $z^* \approx -1.46306$. We then have that the maximum point β^* is defined through $(C - b^2)\beta^* = z^*$.

Plugging β^* in $h(\beta)$ we find the maximum value of the function, which is given by

$$h(\beta^*) \approx \frac{4b^2\beta^*}{2 + e^{1.46306}} \approx 0.926111 \frac{b^2}{b^2 - C} \ .$$

The condition for the RE steady state to remain stable when $C < b^2$ is given by

$$0.926111 \frac{b^2}{b^2 - C} < \frac{1}{k} = \frac{1 + \varphi\sigma\phi_\pi}{1 + \varphi\sigma} \ , \tag{B.2}$$

where the latter equality applies in the case of a Taylor rule. This implies that given parameters b and C, the Central Bank can always implement an interest rate rule, by choosing ϕ_π large enough, to satisfy (B.2) and keeps the RE steady state locally stable.

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