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INTERIOR ESTIMATES AND LONGTIME SOLUTIONS FOR MEAN CURVATURE FLOW OF NONCOMPACT SPACELIKE HYPERSURFACES IN MINKOWSKI SPACE

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Introduction

Spacelike hypersurfaces with prescribed mean curvature have played a major role in the study of Lorentzian manifolds. Maximal (mean curvature zero) hypersurfaces were used in the first proof of the positive mass theorem ([17]). Constant mean curvature hypersurfaces provide convenient time gauges for the Einstein equations ([7]). For a survey of results we refer to [3].

In [5] and [6], it was shown that entire solutions of the maximal surface equation

$$H(u) = \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0$$

for spacelike hypersurfaces in Minkowski space are linear. The proof of this remarkable result is based on an interior a priori estimate for the gradient function

$$v = \frac{1}{\sqrt{1-|Du|^2}}.$$

In fact, estimates for this quantity form the basis of existence proofs for spacelike hypersurfaces with prescribed mean curvature functions in a variety of contexts. These surfaces are described by nonlinear elliptic

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partial differential equations of the same type as the maximal surface equation. The a priori gradient estimate implies that the equation is uniformly elliptic so that topological fixed point arguments can be employed in order to prove the existence of a solution (see [4], [1]-[3], [14]). These arguments are indirect in nature.

In [9] and [8], a direct approach to the existence problem was taken. Solutions of mean curvature equations were constructed as stationary limits of a geometric heat flow which evolves the spacelike hypersurfaces in the direction of their future directed unit normal with speed given by the difference of the actual and the desired mean curvature. This so-called mean curvature flow has been extensively studied in Euclidean space (see [12]). In [9], the case of cosmological spacetimes was treated where one is dealing with the flow of compact hypersurfaces. In [8], noncompact hypersurfaces in asymptotically flat spacetimes were considered but with the strong restriction that the initial surface be asymptotic to a time slice of the spacetime. This essentially amounts to assuming a global bound for the gradient function.

In this paper, we study mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space without any restrictions on their behaviour at infinity. It turns out that some of the most interesting solutions of this flow have exponentially growing mean curvature at infinity and therefore cannot be dealt with in the framework of any standard theory for parabolic differential equations. In Euclidean space, mean curvature flow of noncompact hypersurfaces was studied in [10],[11].

Minkowski space $\mathbf{L}^{n+1}$ is $\mathbf{R}^{n+1}$ endowed with the metric $\langle\cdot,\cdot\rangle$ defined by $\langle X, Y\rangle = x\cdot y - x_{n+1}y_{n+1}$ for vectors $X = (x, x_{n+1}), Y = (y, y_{n+1})$. Spacelike hypersurfaces $M \subset \mathbf{L}^{n+1}$ have an everywhere timelike normal field which we assume to be future directed and to satisfy the condition $\langle\nu,\nu\rangle = -1$. Such surfaces can be expressed as graphs of functions $u : \mathbf{R}^n \to \mathbf{R}$ satisfying $|Du(x)| < 1$ for all $x \in \mathbf{R}^n$.

We consider a family of spacelike embeddings

$$X_t = X(\cdot, t) : \mathbf{R}^n \to \mathbf{L}^{n+1}$$

with corresponding hypersurfaces $M_t = X_t(\mathbf{R}^n)$ satisfying the evolution equation

$$\frac{\partial X}{\partial t} = H\nu \tag{1}$$

on some time interval. Here, $H = \operatorname{div}_{M_t}\nu$ denotes the mean curvature of the hypersurface M_t. Each M_t is the graph of a function $u(\cdot, t)$ satisfying

$|Du(\cdot,t)| < 1$. Equation (1) is equivalent up to diffeomorphisms in $\mathbf{R}^n$ to the equation

$$\frac{\partial u}{\partial t} = \sqrt{1-|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right), \tag{2}$$

which is the parabolic analogue of the maximal surface equation.

Examples of solutions of (1) are the spacelike hyperboloids of constant mean curvature $\sqrt{\frac{n}{2t}}$ given by the functions

$$\delta(x,t) = \sqrt{|x|^2 + 2nt}.$$

These are homothetic solutions with initial data given by the upper light cone at 0 which remain asymptotic to the same light cone for all $t > 0$, i.e., do not move at spatial infinity.

More interestingly, there are also solutions of (1) (or equivalently (2)) which move by vertical translation: Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be an initial spacelike hypersurface with mean curvature satisfying the equation

$$H(u_0) = \frac{1}{\sqrt{1-|Du_0|^2}} \tag{3}$$

or equivalently

$$\sqrt{1-|Du_0|^2}\,\mathrm{div}\left(\frac{Du_0}{\sqrt{1-|Du_0|^2}}\right) = 1.$$

Then a solution u of (2) is given by

$$u(x,t) = u_0(x) + t.$$

In the case n=1, $u_0(x) = \log\cosh x$ is a particular solution of (3). Note that the graph of this solution is geodesically incomplete. The mean curvature grows exponentially at infinity. In particular, the maximum principle does not apply in this case. The translating solution given by $u(x,t) = \log\cosh x + t$ lies initially underneath the homothetic solution given by $\sqrt{x^2+2t}$ but crosses it at infinity at time $t = \log 2$.

In Section 2, we establish the existence of solutions of (3) for general n. The construction of translating solutions of mean curvature flow in more general asymptotically flat spacetimes and their possible applications in general relativity are the subjects of further investigation. Translating solutions can be regarded as a natural way of foliating spacetimes by almost null like hypersurfaces. Particular examples may give insight into the structure of certain spacetimes at null infinity.

In Section 3, we prove an interior estimate inside

$$K_R(0) = \{X \in \mathbf{L}^{n+1},\ \langle X, X\rangle \leq R^2\}$$

for the gradient function and the mean curvature of M_t which has the form

$$\sup_{M_t \cap K_R(0)} (v + |H|) \leq c_1$$

for $t \leq c(n)R^2$ where c_1 depends on n, R and $\sup_{M_0 \cap K_{2R}(0)}(v + |H|)$. We furthermore derive similar bounds for the second fundamental form of M_t and its covariant derivatives.

In Section 4, we prove that the initial-boundary value problem corresponding to (2) on bounded domains of $\mathbf{R}^n$ has a smooth solution for all time which converges for $t \to \infty$ to the unique maximal hypersurface with the given boundary values. This result applied on increasing domains in combination with the interior estimates is then used to establish the following main result of this paper.

Theorem. *For arbitrary spacelike initial data $u_0 : \mathbf{R}^n \to \mathbf{R}$, equation (2) admits a smooth spacelike solution u for all $t > 0$ which satisfies $u(\cdot, 0) = u_0$.*

Note that in contrast to even the standard linear heat equation, no assumption about the behaviour of the initial data at infinity has to be imposed. A corresponding result for mean curvature flow in Euclidean space was established in [11].

1. Maximum principles and local height bounds

We list without proof the particular versions of the standard maximum and comparison principles used in this paper.

1.1. Proposition. *Let u_1 and u_2 be solutions of (2) on a bounded domain $\Omega \subset \mathbf{R}^n$. Suppose that $u_1(x,0) \leq u_2(x,0)$ for all $x \in \Omega$ and $u_1(x,t) \leq u_2(x,t)$ for all $x \in \partial\Omega$ and $t \geq 0$. Then $u_1(x,t) \leq u_2(x,t)$ for all $x \in \Omega$ and $t \geq 0$.*

1.2. Proposition. *Let $M_t = \operatorname{graph} u(\cdot, t)$ where $u : \Omega \times [0,T) \to \mathbf{R}$ solves (2). Suppose the function $f : \Omega \times [0,T) \to \mathbf{R}$ satisfies $f \geq 0$ and the inequality*

$$\left(\frac{d}{dt} - \Delta\right) f \leq 0,$$

where Δ denotes the Laplace-Beltrami operator on M_t. If $\mathrm{spt} f(\cdot,t)$ *is compact for every $t \in [0,T)$, then*

$$\sup_{\bar\Omega} f(\cdot,t) \leq \sup_{\bar\Omega} f(\cdot,0).$$

If Ω is bounded, then

$$\sup_{\bar\Omega\times[0,T)} f \leq \max\{\sup_{\bar\Omega} f(\cdot,0), \sup_{\partial\Omega\times[0,T)} f\}.$$

1.3. Proposition. *Let Ω be a bounded domain in $\mathbf{R}^n$ and $u_0 : \overline{\Omega} \to \mathbf{R}$ be spacelike. Let u be a solution of (2) in $\Omega \times (0,T)$ which satisfies $u(\cdot,0) = u_0$ in Ω and $u(\cdot,t) = u_0$ on $\partial\Omega$ for $t \geq 0$. Then for all $x \in \overline{\Omega}$ and $t \in [0,T]$ the inequality*

$$|u(x,t) - u_0(x)| \leq \sqrt{2nt}$$

holds.

Proof. Since u_0 is spacelike the inequality

$$u_0(y) - |x-y| < u_0(x) < u_0(y) + |x-y|$$

holds for all $x, y \in \overline{\Omega}$. For every $y \in \overline{\Omega}$ we use Proposition 1.1 to compare the solution u with the homothetic solutions corresponding to the initial data $u_0(y) \pm |x-y|$ given by

$$u_0(y) \pm \sqrt{|x-y|^2 + 2nt}.$$

This yields the inequality

$$u_0(y) - \sqrt{|x-y|^2+2nt} \leq u(x,t) \leq u_0(y) + \sqrt{|x-y|^2+2nt}$$

for every $x \in \overline{\Omega}$. Setting $x = y$ implies the estimate.

1.4. Proposition. *There exists a spacelike solution $u : \mathbf{R}^n \to \mathbf{R}$ of the equation*

$$H(u) = \frac{1}{\sqrt{1-|Du|^2}}. \tag{3}$$

Proof. Note that for radially symmetric solutions (3) reduces to the equation

$$\frac{u''}{1-(u')^2} + \frac{n-1}{r}u' = 1$$

on the real line which can be solved using ODE methods.

We outline an alternative approach, which seems to be overkill in the case of radially symmetric solutions but has the potential to generalize to asymptotically flat spacetimes. It is based on the observation that constant mean curvature spacelike hypersurfaces can be used as barriers for solutions of (3) on bounded domains.

For $k \in \mathbf{N}$ and fixed $a > 0$ we solve the Dirichlet problems

$$\begin{aligned} H(u_k) &= \frac{1}{\sqrt{1-|Du_k|^2}} && \text{in} \quad B_k(0), \\ u_k &= k - a && \text{on} \quad \partial B_k(0) \end{aligned}$$

on balls in $\mathbf{R}^n$. By symmetry, the solutions u_k are of course radially symmetric having constant boundary values. In general, solvability follows from [1] since the right-hand side of the equation satisfies the mean curvature structure conditions required there. Interior estimates for

$$v(u_k) = \frac{1}{\sqrt{1-|Du_k|^2}}$$

and higher derivatives of u_k independent of k hold on any fixed ball $B_R(0) \subset \mathbf{R}^n$. This uses again the symmetry of our solution which guarantees constant height on $\partial B_R(0)$. In a more general construction, this is the step which requires additional work. The crucial observation is the inequality

$$u_k(x) \leq \sqrt{|x|^2 + n^2} - a,$$

which holds for all $x \in B_k(0)$. This follows by comparing u_k with the constant mean curvature hypersurfaces given by

$$\delta_k^+(x) = \sqrt{|x|^2 + n^2} - \sqrt{k^2 + n^2} + k - a.$$

Their mean curvatures satisfy the inequality $H(\delta_k^+) = 1 \leq H(u_k)$ in $B_k(0)$ while $\delta_k^+ = u_k$ on $\partial B_k(0)$. The comparison principle of [4] then yields that $u_k \leq \delta_k^+$ in $B_k(0)$.

This argument provides a height bound for u_k over fixed balls $B_R(0)$ independent of k. In view of the uniform derivative estimates we can therefore let $k \to \infty$ to obtain the result.

2. Evolution equations

Most of the evolution equations in this section were derived in [9]. They will form the basis for the a priori estimates of the next section.

We first recall that in view of the geometric identity $\Delta X = H\nu$ for the position vector of the hypersurface M_t, equation (1) is equivalent to the nonlinear heat equation

$$\frac{\partial X}{\partial t} = \Delta X. \tag{4}$$

Here, Δ denotes the Laplace-Beltrami operator on M_t.

2.1. Proposition. *The Lorentz distance function $z = \langle X, X\rangle$ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)(z + 2nt) = 0.$$

Proof. Using (1), we calculate

$$\frac{d}{dt}z = 2H\langle X, \nu\rangle,$$

while (see [4])

$$\Delta z = 2(n + H\langle X, \nu\rangle).$$

The length of the tangential projections of vectors onto spacelike hypersurfaces $M = \operatorname{graph} u$ with normal ν is controlled by the gradient function

$$v = -\langle \nu, e_{n+1}\rangle = \frac{1}{\sqrt{1 - |Du|^2}}.$$

We will frequently use the inequalities $|\nabla x_{n+1}| \le v$ and $|\nabla r| \le v$ (see [4]), where ∇ denotes the tangential gradient on M_t, and $r = |x|$ is Euclidean distance on $\mathbf{R}^n$.

2.2. Proposition. *The Euclidean distance function $r = |x|$ on $\mathbf{R}^n$ satisfies*

$$\left|\left(\frac{d}{dt} - \Delta\right) r^2\right| \le c(n)v.$$

Proof. Since $r^2 = z + x_{n+1}^2$ for $X = (x, x_{n+1})$, from (4) and Proposition 2.1 we obtain that

$$\left(\frac{d}{dt} - \Delta\right) r^2 = -2n - 2|\nabla x_{n+1}|^2.$$

In view of the inequality $|\nabla x_{n+1}| \leq v$ this implies the result.

2.3. Proposition. *The gradient function satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v.$$

Here, $|A|^2$ denotes the square of the norm of the second fundamental form $A = (h_{ij})$ on M_t, which is defined by

$$h_{ij} = \langle \tau_i, D_{\tau_j} \nu \rangle$$

for a local orthonormal frame $\tau_1, \ldots, \tau_n$ of M_t.

For general spacelike hypersurfaces we recall the following inequality which is the key to the basic gradient type estimates (see [1], [2], [6], [14]).

2.4. Lemma. *On any spacelike hypersurface M in $\mathbf{L}^{n+1}$ we have the inequality*

$$|A|^2 v^2 \geq (1 + \frac{1}{n})|\nabla v|^2 - H^2 v^2.$$

In combination with Proposition 2.3 this implies

2.5. Corollary. *The gradient function satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right) v^2 \leq -4(1 + \frac{1}{2n})|\nabla v|^2 + 2H^2 v^2.$$

2.6. Proposition. *The second fundamental form and its derivatives satisfy*

$$(i) \qquad \left(\frac{d}{dt} - \Delta\right) H = -H|A|^2,$$

$$(ii) \qquad \left(\frac{d}{dt} - \Delta\right) |A|^2 \leq -2|\nabla A|^2 - |A|^4 + c(n),$$

$$(iii) \qquad \left(\frac{d}{dt} - \Delta\right) |\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + c_m(1 + |\nabla^m A|^2),$$

where $c_m = c_m(m, n, v, \sum_{j=1}^m |\nabla^{j-1} A|)$.

For later use we note that by combining Proposition 2.6 (i) with the inequality $|A|^2 \geq \frac{1}{n} H^2$ one obtains the following.

2.7. Corollary. *The square of the mean curvature satisfies*

$$\left(\frac{d}{dt}-\Delta\right)H^2 \leq -\frac{2}{n}H^4 - 2|\nabla H|^2.$$

3. Interior estimates

The proof of the existence theorem in the last section is based on the following interior estimate which simultaneously controls the gradient function and the mean curvature of M_t inside the set

$$K_R(0) = \left\{X \in \mathbf{L}^{n+1},\, z = \langle X, X\rangle \leq R^2\right\}.$$

The estimates will also be applied to solutions of (1) with boundary. However, we will always assume that the hypersurfaces M_t have no boundary inside the sets in which we are estimating.

For related estimates in the elliptic case we refer to [2], [3], [6] and [16].

3.1. Theorem. *Suppose that $M_t \cap K_R(0)$ is compact in $\mathbf{L}^{n+1}$ for $t \in [0, \frac{R^2}{2n}]$. Let $\Lambda > \sup_{M_0 \cap K_R(0)} H^2$. There are constants $p, q > 0$ which only depend on n such that for all $t \in [0, \frac{R^2}{2n}]$*

$$\begin{aligned}\sup_{M_t}&\left(v^2\frac{1}{(\Lambda - H^2)^{1/q}}(R^2 - z - 2nt)^p\right)\\ &\leq e^{c(n)q\Lambda t}\sup_{M_0}\left(v^2\frac{1}{(\Lambda - H^2)^{1/q}}(R^2 - z)^p\right).\end{aligned}$$

We present the proof in two steps contained in the following lemmata.

3.2. Lemma. *For sufficiently large $q = q(n)$, the quantity $g = v^2\frac{1}{(\Lambda - H^2)^{1/q}}$ satisfies the inequality*

$$\left(\frac{d}{dt}-\Delta\right)g \leq -(1+\delta)\frac{|\nabla g|^2}{g} + cg,$$

where $\delta = \delta(n) > 0$ and $c = c(n, q, \Lambda)$.

Proof. Let $g = v^2 h(H^2)$, where we will substitute $h(y) = (\Lambda - y)^{-1/q}$ later. Denoting derivatives of h by $'$ and using Corollaries 2.5 and 2.7 as well as $h \geq 0$ and $h' \geq 0$, we calculate

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g \leq &- 4(1 + \frac{1}{2n})|\nabla v|^2 h + 2H^2 v^2 h - \frac{2}{n} H^4 h' v^2 \\ &- h''|\nabla H^2|^2 v^2 - 2h'\langle \nabla v^2, \nabla H^2\rangle.\end{aligned} \tag{6}$$

By Young's inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$ with $\epsilon = \frac{1}{n}$, we estimate

$$|2h'\langle \nabla v^2, \nabla H^2\rangle| = |8h' v H \langle \nabla v, \nabla H\rangle| \leq \frac{1}{n}|\nabla v|^2 h + 16n \frac{(h')^2}{h} v^2 H^2 |\nabla H|^2$$

which yields

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g \leq &- 4(1 + \frac{1}{4n})|\nabla v|^2 h + 2H^2 v^2 h - \frac{2}{n} H^4 h' v^2 \\ &+ 4\left(4n\frac{(h')^2}{h} - h''\right) v^2 H^2 |\nabla H|^2.\end{aligned} \tag{7}$$

¿From the inequality $|a + b|^2 \leq (1 + \delta)|a|^2 + (1 + 1/\delta)|b|^2$ for $\delta > 0$ we derive

$$\begin{aligned}|\nabla g|^2 &= |2hv\nabla v + 2v^2 h' H \nabla H|^2 \\ &\leq 4(1 + \delta) v^2 |\nabla v|^2 h^2 + 4\left(1 + \frac{1}{\delta}\right) v^4 (h')^2 H^2 |\nabla H|^2.\end{aligned}$$

This implies

$$(1+\delta)\frac{|\nabla g|^2}{g} \leq 4(1+\delta)^2 |\nabla v|^2 h + 4(1+\delta)\left(1 + \frac{1}{\delta}\right) v^2 \frac{(h')^2}{h} H^2 |\nabla H|^2. \tag{8}$$

Choosing $\delta = \delta(n) > 0$ such that $(1 + \delta)^2 = 1 + \frac{1}{4n}$ and substituting (8) into (7) we arrive at

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g \leq &- (1 + \delta)\frac{|\nabla g|^2}{g} + 2H^2 v^2 h - \frac{2}{n} H^4 h' v^2 \\ &+ 4\left(c(n)\frac{(h')^2}{h} - h''\right) v^2 H^2 |\nabla H|^2.\end{aligned}$$

Again using Young's inequality, we estimate

$$2H^2 v^2 h \leq \frac{2}{n} H^4 h' v^2 + \frac{n}{2}\frac{h^2}{h'} v^2,$$

which implies

$$\left(\frac{d}{dt}-\Delta\right)g\leq-(1+\delta)\frac{|\nabla g|^2}{g}+\frac{n}{2}\frac{h}{h'}g+4\left(c(n)\frac{(h')^2}{h}-h''\right)v^2H^2|\nabla H|^2.$$

The function given by $h(y)=(\Lambda-y)^{-1/q}$ satisfies

$$c(n)\frac{(h'(y))^2}{h(y)}-h''(y)=\frac{1}{q^2}\left(c(n)-(q+1)\right)(\Lambda-y)^{-\frac{1}{q}-2}\leq 0$$

for sufficiently large q depending on n. Moreover, since $0\leq\frac{h}{h'}\leq q\Lambda$ we conclude

$$\left(\frac{d}{dt}-\Delta\right)g\leq-(1+\delta)\frac{|\nabla g|^2}{g}+(\frac{n}{2}q\Lambda)g.$$

3.3. Lemma. *Let $\eta=(R^2-z-2nt)^p$. Then for sufficiently large p depending only on $\delta=\delta(n)$ the inequality*

$$\left(\frac{d}{dt}-\Delta\right)g\eta\leq cg\eta$$

holds where c depends on n and Λ.

Proof. Abbreviating $\eta=\eta(r)$, denoting derivatives of η by $'$ and using the previous lemma we calculate that

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g\eta\leq&-(1+\delta)\frac{|\nabla g|^2}{g}\eta+cg\eta+g\eta'\left(\frac{d}{dt}-\Delta\right)(z+2nt)\\&-g\eta''|\nabla z|^2-2\langle\nabla g,\nabla\eta\rangle.\end{aligned}$$

Here c depends on n,q and Λ. We estimate

$$|2\langle\nabla g,\nabla\eta\rangle|\leq(1+\delta)\frac{|\nabla g|^2}{g}\eta+\frac{1}{1+\delta}\frac{|\nabla\eta|^2}{\eta}g$$

and use the identities $|\nabla\eta|^2=(\eta')^2|\nabla z|^2$ and $\left(\frac{d}{dt}-\Delta\right)(z+2nt)=0$ (see Proposition 2.1) to obtain

$$\left(\frac{d}{dt}-\Delta\right)g\eta\leq cg\eta+\left(\frac{1}{1+\delta}\frac{(\eta')^2}{\eta}-\eta''\right)|\nabla z|^2.$$

For $\eta(r)=(R^2-r)^p$ we have

$$\frac{1}{1+\delta}\frac{(\eta')^2}{\eta}-\eta''=\left(\frac{p^2}{1+\delta}-p(p-1)\right)(R^2-z-2nt)^{p-2}\leq 0$$

for sufficiently large p depending on δ.

Proof of Theorem 3.1. Since by assumption the hypersurfaces M_t are compact inside the support of η, we can apply the maximum principle of Proposition 1.2 to $f = e^{-ct} g\eta$ to conclude

$$\sup_{M_t} g\eta \le e^{ct} \sup_{M_0} g\eta$$

and therefore the desired estimate.

In the next section, the main estimate of Theorem 3.1 will be applied in balls in $\mathbf{R}^n$. Let C_R denote the cylinder $B_R \times \mathbf{R}$ where $B_R = \{x \in \mathbf{R}^n, |x| \le R\}$. Since $C_R \subset K_R$, the theorem applied with $2R$ instead of R immediately yields

3.4. Corollary. *For $t \in [0, c(n)R^2]$, the estimate*

$$\sup_{M_t \cap C_R} (v + |H|) \le c_1$$

holds where c_1 depends on n, R and $\sup_{M_0 \cap K_{2R}}(v + |H|)$.

3.5. Remark. Having obtained estimates for v we are now in a position to use the Euclidean distance function $r = |x|$ for further localization arguments. In view of Proposition 2.2 and the inequality $|\nabla r| \le v$, Corollary 3.4 implies that for $t \le c(n)R^2$

$$\left| \left(\frac{d}{dt} - \Delta \right) r^2 \right| \le c(n, c_1)$$

and $|\nabla r| \le c_1$ in C_R.

3.6. Proposition. *Suppose that $\sup_{M_t \cap C_R} v \le c_1$ for $t \in [0, T]$. Then the curvatures of M_t for $t \in [0, T]$ can be estimated by*

$$\sup_{M_t} |A|^2 (R^2 - r^2)^2 \le c_2,$$

where $c_2 = c_2(n, R, c_1, \sup_{M_0 \cap C_R} |A|^2)$.

Proof. Abbreviating $\eta(r^2) = (R^2 - r^2)^2$, we calculate, in view of Proposition 2.6 (ii),

$$\begin{aligned} \left(\frac{d}{dt} - \Delta \right) |A|^2 \eta \le & -2|\nabla A|^2 \eta - |A|^4 \eta + c(n)\eta \\ & + |A|^2 |\eta'| \left| \left(\frac{d}{dt} - \Delta \right) r^2 \right| \\ & - |A|^2 \eta'' |\nabla r^2|^2 + 2\eta' \langle \nabla |A|^2, \nabla r^2 \rangle. \end{aligned}$$

Estimating

$$|2\eta'\langle\nabla|A|^2,\nabla r^2\rangle| \le 2|\nabla A|^2\eta + 8\frac{(\eta')^2}{\eta}|A|^2|\nabla r^2|^2$$

we obtain

$$\left(\frac{d}{dt}-\Delta\right)|A|^2\eta \le -|A|^4\eta + c(n)\eta + |A|^2|\eta'|\left|\left(\frac{d}{dt}-\Delta\right)r^2\right| + \left(8\frac{(\eta')^2}{\eta}-\eta''\right)|A|^2|\nabla r^2|^2.$$

We now substitute η into the last term, estimate $|\eta'| \le R^2$ and use Remark 3.5 to arrive at

$$\left(\frac{d}{dt}-\Delta\right)|A|^2\eta \le -|A|^4\eta + c(n)\eta + c(n,c_1)R^2|A|^2.$$

At a point where $|A|^2\eta$ first reaches a maximum larger than $\sup_{M_0}|A|^2\eta$, in view of the inequality $\eta \le R^4$ we therefore obtain that

$$|A|^4\eta \le c(n)R^4 + c(n,c_1)R^2|A|^2.$$

Multiplying by η yields

$$(|A|^2\eta)^2 \le c(n)R^8 + c(n,c_1)R^2|A|^2\eta,$$

and hence we conclude from Young's inequality that

$$|A|^2\eta \le c(n,R,c_1,\sup_{M_0\cap C_R}|A|^2)$$

at the maximum point. This implies for $t\in[0,T]$

$$\sup_{M_t}|A|^2\eta \le c(n,R,c_1,\sup_{M_0\cap C_R}|A|^2).$$

3.7. Proposition. *Suppose* $\sup_{M_t\cap C_R} v \le c_1$ *and* $\sup_{M_t\cap C_R}|A|^2 \le c_2$ *for* $t\in[0,T]$. *Then for every* $m\ge 1$ *and* $t\in[0,T]$ *we have the estimate*

$$\sup_{M_t\cap C_{\frac{R}{2}}}|\nabla^m A|^2 \le c_m,$$

where $c_m = c_m(n,m,R,c_1,c_2,\max_{1\le j\le m}\sup_{M_0\cap C_R}|\nabla^j A|^2)$.

Proof. We proceed exactly as in the corresponding case of mean curvature flow in Euclidean space (see [11, Ch.3], [8]). Utilizing the bounds on v and $|A|^2$ we show that for sufficiently large $K > 0$ the function $f = |\nabla A|^2(K + |A|^2)$ satisfies an inequality of the type

$$\left(\frac{d}{dt} - \Delta\right) f \leq -\delta f^2 + C,$$

where $\delta > 0$. By Remark 3.5, the quantities $|\nabla r|$ and $|\left(\frac{d}{dt} - \Delta\right) r^2|$ are controlled in C_R. We can therefore estimate $f(R^2 - r^2)^2$ as in [11]. Bounds for the higher derivatives are established inductively.

4. Longtime existence theorems

In this section, we will first consider the initial-boundary value problem associated with equation (2) for bounded domains $\Omega \subset \mathbf{R}^n$. We will then construct a global solution of (2) for arbitrary spacelike initial data $u_0 : \mathbf{R}^n \to \mathbf{R}$. This is achieved by solving initial -boundary value problems on increasing domains, and then using the interior estimates of the previous section to extract a subsequence of solutions which converges smoothly on compact subsets.

4.1. Theorem. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Let $u_0 : \bar{\Omega} \to \mathbf{R}$ be smooth and strictly spacelike in the sense that* $\sup_{\bar{\Omega}} |Du_0| < 1$. *Then the equation*

$$\frac{\partial u}{\partial t} = \sqrt{1 - |Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) \tag{2}$$

has a smooth solution in Ω for all $t > 0$, which satisfies $u(\cdot, 0) = u_0$ in Ω and $u(\cdot, t) = u_0$ on $\partial\Omega$. Moreover, as $t \to \infty$, $u(\cdot, t)$ converges smoothly to the solution of the maximal surface equation with boundary data u_0.

Proof. Proposition 1.2 implies that any solution of (2) on a time interval $[0, T]$ satisfies

$$\sup_{\bar{\Omega} \times [0,T]} |u| = \sup_{\bar{\Omega}} |u_0|.$$

In view of Proposition 2.3 and Proposition 1.2 we also have

$$\sup_{\bar{\Omega} \times [0,T]} v = \max\{\sup_{\bar{\Omega}} v(0), \sup_{\partial\Omega \times [0,T]} v\}.$$

Estimates on $\sup_{\partial\Omega\times[0,T]} v$ are derived as in [4]. Note that again in view of the maximum principle the radially symmetric barriers used in [4, Proposition 3.1] will work in the parabolic setting as well. Being maximal hypersurfaces these barriers are stationary solutions of (2), and therefore by Proposition 1.1 the solution u will remain between them if it does so initially. We therefore obtain for arbitrary $T > 0$

$$\sup_{\bar{\Omega}\times[0,T]} (|u| + v) \leq c_0,$$

where c_0 depends only on the initial data. This implies that equation (2) is uniformly parabolic. Estimates for higher derivatives then follow from standard theory for uniformly parabolic equations. The estimates ensure the existence of a unique smooth solution for all $t > 0$ (see [15]). To prove the convergence to the unique (see [4]) solution of the maximal surface equation we proceed similarly as in [13]. From (2) we calculate for $v^{-1} = \sqrt{1 - |Du|^2}$ that

$$\frac{\partial}{\partial t}\sqrt{1 - |Du|^2} = -vD_i uD_i(v^{-1}H)$$

and therefore

$$\begin{aligned}\frac{d}{dt}\int_\Omega \sqrt{1 - |Du|^2} &= -\int_\Omega vD_i uD_i(v^{-1}H)\\ &= \int_\Omega D_i(vD_i u)Hv^{-1} = \int_\Omega H^2 v^{-1},\end{aligned}$$

where we have integrated by parts in the second last step using that $v^{-1}H = \frac{\partial u}{\partial t}$ vanishes on $\partial\Omega$ for $t > 0$. Therefore,

$$\int_0^\infty \int_\Omega H^2 v^{-1} \leq \int_\Omega \sqrt{1 - |Du_0|^2} \leq |\Omega|.$$

Since $v \leq c_0$ on $\bar{\Omega} \times [0, \infty)$ we obtain

$$\int_0^\infty \int_\Omega H^2 \leq c(c_0, |\Omega|).$$

Moreover, by the global estimates for higher devivatives and Proposition 2.6 (i) one verifies that

$$\sup_{[0,\infty)} |\frac{d}{dt}\int_\Omega H^2| \leq C.$$

This gives

$$\lim_{t\to\infty}\int_\Omega H^2 = 0.$$

An interpolation argument using the global estimates then implies that $\sup_\Omega |H| \to 0$ uniformly as $t \to \infty$.

4.2. Theorem. *Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be spacelike and smooth. Then the equation*

$$\frac{\partial u}{\partial t} = \sqrt{1-|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) \tag{2}$$

has a smooth solution for all $t > 0$ with initial data u_0.

Proof. Suppose without loss of generality that $u_0(0) = 0$. For $k \in \mathbf{N}$, we let u_k be the smooth solution of the initial-boundary value problem

$$\begin{aligned}
\frac{\partial u_k}{\partial t} = \sqrt{1-|Du_k|^2}\,\operatorname{div}\left(\frac{Du_k}{\sqrt{1-|Du_k|^2}}\right) \qquad &\text{in} \qquad B_k(0)\times(0,\infty),\\
u_k(\cdot,0) = u_0 \qquad \text{in} \qquad & B_k(0),\\
u_k(\cdot,t) = u_0 \qquad \text{on} \qquad & \partial B_k(0)\times(0,\infty).
\end{aligned}$$

Fix $R > 0$. Since u_0 is spacelike and $u_0(0) = 0$, an easy argument using the mean value theorem (see [6] or [16]) shows that

$$|x|^2 - u_0^2(x) \to \infty$$

as $|x| \to \infty$. Hence for sufficiently large k depending on R we have that $|x|^2 - u_0^2(x) > 16R^2$ for $|x| = k$. For $M_t^k = \operatorname{graph} u_k(\cdot,t)$, this implies that $\partial M_t^k \cap K_{4R}(0) = \emptyset$ for all $t \geq 0$. Also, $M_t^k \cap K_{4R}(0)$ is compact for $t \geq 0$ as these sets are contained in the cylinders $B_k(0)\times\mathbf{R}$. We can therefore apply the interior estimates of Theorem 3.1 or Corollary 3.4 to the solution (M_t^k) inside $K_{4R}(0)$ to obtain for $t \leq c(n)R^2$

$$\sup_{M_t^k\cap C_{2R}(0)} (v + |H|) \leq c_1.$$

Propositions 3.6 and 3.7 with $T = c(n)R^2$ imply that

$$\sup_{M_t^k\cap C_R(0)} |\nabla^m A|^2 \leq c_m$$

for $t \in [0, c(n)R^2]$ and for all $m \geq 0$. These estimates translate into uniform bounds (independent of k) on $B_R(0) \times [0, c(n)R^2]$ for $v(u_k)$ and derivatives of all orders of u_k . The height estimate of Proposition 1.3 furthermore yields that

$$\sup_{B_R(0)\times[0,c(n)R^2]} |u_k| \leq c(n, R, \sup_{B_R(0)} |u_0|).$$

Since $R > 0$ is arbitrary, we can select a subsequence of (u_k) which converges smoothly on compact subsets of $\mathbf{R}^n \times [0, \infty)$ to a solution of (2) with initial data u_0.

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