J. DIFFERENTIAL GEOMETRY
5 (1971) 149-157

INTERIOR GRADIENT ESTIMATES FOR SURFACES $z = f(x, y)$ WITH PRESCRIBED MEAN CURVATURE

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Introduction

One of the main steps in solving the Dirichlet problem for a quasilinear elliptic equation with continuous boundary data (see [13] for a complete discussion) consists in deriving interior estimates for the gradient of the solutions in terms of their maximum bound. While such estimates are available for most uniformly elliptic equations (see [10] for further details), there are only limited classes of nonuniformly elliptic equations for which such conclusions are known. The first significant results in this field are due to Finn ([2] and [3]) who established such estimates for equations of minimal surface type (see [13, § 24] for further work on this subject). Very recently Serrin [14] succeeded in proving interior gradient estimates for the constant mean curvature equation in two variables:

$$(1) \qquad (1 + q^2)r - 2pqs + (1 + p^2)t = 2H(1 + p^2 + q^2)^{3/2} .$$

Serrin's method, which is entirely nonparametric, depends on a delicate construction of certain comparison functions associated with equation (1).

The question then naturally arises as to whether similar results can be established in the case where the mean curvature of the surface $z = f(x, y)$ is a given function $H = H(x, y)$ in a domain $D \subset \boldsymbol{R}^2$. In the present paper we shall answer that question in the affirmative, provided that $H(x, y)$ satisfies a Lipschitz condition (Theorems 1 and 2). In contrast to Serrin's method our approach is entirely parametric. Our methods are related to those of an earlier paper [7] in which we derived estimates for the jacobian of certain mappings $w \to z(w)$ relevant for the discussion of the Weyl embedding problem, and also gave references to previous work on this subject. So far as the nature of our estimates is concerned, we have presented them here in an indirect form thus rendering our arguments more transparent to the reader. Using the results of [6] and [8], it is even possible to obtain explicit bounds for the gradient of the surface $z = f(x, y)$ at a point $P \in D$ in terms of the area of the surface, the Lipschitz constant of H, and the distance of P from the boundary of D; the author intends to take up these matters elsewhere.

Received February 2, 1970.

1. Preliminary lemmas

Throughout this paper we frequently use the complex notation $w = u + iv$, $z = x + iy$, thus designating functions of (u, v), (x, y) by $f(w)$ and $g(z)$, respectively, without implying holomorphy. We set

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) .$$

Moreover, let $J_z(w)$ denote the jacobian of the mapping $w \to z(w)$. In this section we are primarily concerned with estimating the quantity $|J_z(w)|$ for a class of mappings satisfying certain nonlinear elliptic systems, and our main result is Lemma 3 which forms the basis of the subsequent proof of the interior gradient estimates for equation (1) to be carried out in § 2. We begin with the following lemma.

Lemma 1. *Let D be a domain in $\boldsymbol{R}^2$, and $g(w) = (g_1(w), \cdots, g_m(w)) \in \boldsymbol{R}^m$ a vector function of class $C^2(D)$ satisfying in D the differential inequality*

$$|g_{w\bar{w}}| \leq M(|g_w| + |g|) , \tag{1.1}$$

where M is a fixed positive constant. Furthermore, let

$$g(w_0) = g_w(w_0) = 0 , \tag{1.2}$$

where w_0 is a point in D, and let

$$g^{(p)}(w) = (g_1^{(p)}(w), \cdots, g_m^{(p)}(w)) \in \boldsymbol{R}^m \qquad (p = 1, 2, \cdots)$$

be a sequence of vector functions of class $C^1(D)$ such that the relations

$$g^{(p)}(w) \longrightarrow g(w) \qquad (p \to \infty) , \tag{1.3}$$

$$g_w^{(p)}(w) \longrightarrow g_w(w) \qquad (p \to \infty) \tag{1.3'}$$

hold uniformly in a neighborhood of w_0. Then either $g(w)$ vanishes identically or there exist a point $w_1 \in D$ and positive integers k, p with $1 \leq k \leq m$ such that $g_{kw}^{(p)}(w_1) = 0$.

Proof. Assume that g does not vanish identically. Then according to Hartman-Wintner ([4], especially pp. 455–458) we have an asymptotic expansion of the form

$$g_w = a(w - w_0)^l + o(|w - w_0|^l) \qquad (w \to w_0) , \tag{1.4}$$

where $a = (a_1, \cdots, a_m) \neq 0$, and l is a positive integer. Let $a_k \neq 0$. Then (1.4) implies that the index of the mapping $w \to g_{kw}$ is positive at w_0. Consequently, for sufficiently large p, the function $g_{kw}^{(p)}$ has a zero in a neighborhood of w_0, which proves the lemma.

We shall now specify the class of mappings $w \to z(w)$ to be studied in this paper.

Definition 1. Let M_0, M_1 be positive constants, and D a domain in $\boldsymbol{R}^2$. Then $\Gamma(M_0, M_1, D)$ is the class of mappings $w \to z(w) = x(u, v) + iy(u, v)$ with the following properties:

(a) The functions $x(u, v)$ and $y(u, v)$ belong to $C^2(D)$ and satisfy in D the quasilinear elliptic system

$$x_{w\bar{w}} = ia(x, y)(\bar{\varphi} y_w - \varphi y_{\bar{w}}) , \quad y_{u\bar{w}} = ib(x, y)(\bar{\varphi} x_w - \varphi x_{\bar{w}}) , \tag{1.5}$$

where $\varphi = \varphi(u, v)$ belongs to $C^0(D)$ and is subject to the restriction

$$\varphi^2 + x_w^2 + y_w^2 = 0 . \tag{1.6}$$

(b) The coefficients $a(x, y)$ and $b(x, y)$ occurring in (1.5) are real-valued for $|z| < +\infty$, and we have the estimates

$$|a(z)| + |b(z)| \leq M_0 \qquad (|z| < +\infty) , \tag{1.7}$$

$$|a(z') - a(z'')| + |b(z') - b(z'')| \leq M_1 |z' - z''| \quad (|z'|, |z''| < +\infty) . \tag{1.8}$$

Lemma 2. *Let $z(w) \in \Gamma(M_0, M_1, D)$, and let $w \to z_p(w) = x_p(w) + iy_p(w)$ $(p = 1, 2, \cdots)$ be a sequence of mappings of class $C^1(D)$ with nonvanishing jacobians such that the limit relations*

$$z_p(w) \to z(w) \qquad (p \to \infty) , \tag{1.9}$$

$$z_{pu} \to z_u , \quad z_{pv} \to z_v \qquad (p \to \infty) , \tag{1.9'}$$

hold uniformly in every compact subset of D. Then either $J_z(w)$ vanishes identically, or we have $J_z(w) \neq 0$ for $w \in D$.

Proof. Suppose that we have $J(w_0) = 0$ at a point $w_0 \in D$. Without loss of generality we may assume that $z(w_0) = 0$. If both x_w and y_w vanish for $w = w_0$, then on account of (1.5) the vector function $g(w) = (x(w), y(w))$ satisfies the hypotheses of Lemma 1. Consequently, either $g(w)$ vanishes identically or there exists an integer p such that one of the functions $x_p(w), y_p(w)$ has a vanishing gradient at some point $w_1 \in D$. Since by assumption $J_{z_p}(w_1)$ does not vanish, the second case cannot occur, and we thus have $g(w) \equiv 0$; hence $J(w) \equiv 0$. It therefore remains to consider the case where $x_w(w_0)$ and $y_w(w_0)$ do not vanish simultaneously. Obviously we may assume that $y_w(w_0) \neq 0$. Furthermore, on replacing φ, a, b by $-\varphi$, $-a$, $-b$ if necessary, we can achieve that

$$\text{(1.10)} \qquad \operatorname{Im} \frac{\varphi(w_0)}{y_w(w_0)} > 0 \,.$$

Now observe that $\dfrac{x_w(w_0)}{y_w(w_0)}$ is real, and let $f(y)$ be a real-valued function of class C^2 in the interval $|y| < \delta$ satisfying the conditions

$$\text{(1.11)} \qquad f(0) = 0 \,, \qquad f'(0) = \frac{x_w(w_0)}{y_w(w_0)} \,.$$

Choose $\rho > 0$ such that for $|w - w_0| < \rho$ the inequality $|z(w)| < \frac{1}{2}\delta$ holds, and consider the function

$$\text{(1.12)} \qquad \xi(w) = x(w) - f(y(w)) \qquad (|w - w_0| < \rho) \,.$$

In virture of (1.11) both ξ and ξ_w vanish for $w = w_0$. We proceed to show that for a suitable choice of f the function ξ satisfies a differential inequality of the form

$$\text{(1.13)} \qquad |\xi_{w\bar{w}}| \leq M(|\xi_w| + |\xi|) \qquad (|w - w_0| < \rho_1 < \rho) \,,$$

where M is a positive constant (in this regard see also [11], especially p. 691). To this end we first observe that on account of (1.6) and (1.10) we have

$$\text{(1.14)} \qquad |\varphi(w) - i(1 + f'(y)^2)^{1/2} y_w| \leq \kappa_0 |\xi_w|$$

for $|w - w_0| < \rho_1 < \rho$, where κ_0 is a positive constant. Furthermore, on differentiating (1.12) and using (1.5) we obtain the equation

$$\text{(1.15)} \qquad \begin{aligned} \xi_{w\bar{w}} = {} & i(a(x, y) - b(x, y) f'(y)^2)(\bar{\varphi} y_w - \varphi y_{\bar{w}}) \\ & - f''(y) |y_w|^2 - i f'(y) b(x, y)(\bar{\varphi} \xi_w - \varphi \xi_{\bar{w}}) \,. \end{aligned}$$

Now (1.14) and (1.15) imply

$$\text{(1.16)} \qquad \begin{aligned} & |\xi_{w\bar{w}}| \leq \kappa_1 |\xi_w| \\ & \quad + |2(a(x, y) - b(x, y) f'(y)^2)(1 + f'(y)^2)^{1/2} - f''(y)| \, |y_w|^2 \end{aligned}$$

for $|w - w_0| < \rho_1$, where κ_1 is a positive constant. If we choose $f(y)$ to be a solution of the ordinary differential equation

$$\text{(1.17)} \qquad f''(y) = 2(a(f(y), y) - b(f(y), y) f'(y)^2)(1 + f'(y)^2)^{1/2}$$

subject to the initial conditions (1.11), and take (1.8) into account, then the desired inequality (1.13) follows directly. To complete the proof, let

$$(1.18)\qquad \xi_p(w) = x_p(w) - f(y_p(w)) \qquad (|w - w_0| < \rho_1, p \geq p_0) ,$$

where p_0 is taken sufficiently large. Since by hypothesis the jacobians $J_{z_p}(w)$ do not vanish in D, we have $\xi_{pw} \neq 0$ for $|w - w_0| < \rho_1$ and $p \geq p_0$. Moreover, the relations

$$(1.19)\qquad \xi_p(w) \to \xi(w) \qquad (p \to \infty) ,$$

$$(1.19')\qquad \xi_{pw} \to \xi_w \qquad (p \to \infty)$$

hold uniformly in the disk $|w - w_0| < \rho_1$. Applying Lemma 1, we immediately infer $J(w) = 0$ for $|w - w_0| < \rho_1$. Hence $J(w) \equiv 0$ proving the lemma.

Proceeding now as in [7, § 2] we can derive global estimates for the jacobians by imposing further normalization conditions on the mappings. In what follows B denotes the unit disk $|w| < 1$.

Definition 2. Let M_0, M_1, N be positive parameters. Then $\Omega(M_0, M_1, N)$ is the subset of functions $z(w) \in \Gamma(M_0, M_1, B)$ with the following properties:

a) $z = z(w)$ maps $\bar{B}$ topologically onto itself such that $z(0) = 0$.

b) We have $J_z(w) \neq 0$ in B and

$$(1.20)\qquad D(z) = \iint_B (|z_u|^2 + |z_v|^2)\, dudv \leq N .$$

Lemma 3. *Let $z(w) \in \Omega(M_0, M_1, N)$. Then for $|w| \leq r < 1$ we have estimates of the form*

$$(1.21)\qquad |z_u|^2 + |z_v|^2 \leq \lambda(M_0, N, r) < +\infty ,$$

$$(1.22)\qquad |J_z(w)| \geq \tau(M_0, M_1, N, r) > 0 .$$

Proof. (I) First of all, on account of (1.5)–(1.7) the function $z(w)$ satisfies the differential inequality

$$(1.23)\qquad |\Delta z| \leq 2M_0(|z_u|^2 + |z_v|^2) .$$

Moreover, according to classical results (see for instance [6, Lemma 16]) the conditions (a) and (b) of Definition 2 imply that the set $\Omega(M_0, M_1, N)$ is equicontinuous in $\bar{B}$. Hence inequality (1.21) follows directly by applying Lemma 5 of [7]. An elementary argument then shows that the functions z_u, z_v, and φ ($z \in \Omega(M_0, M_1, N)$) satisfy a uniform Hölder condition with any exponent less than 1/2 in every closed disk $|w| \leq r < 1$. Consequently the set $\Omega(M_0, M_1, N)$ is precompact in $C^2(T)$ for any compact subset $T \subset B$.

(II) In order to estimate the jacobian we observe that on account of the equicontinuity of $\Omega(M_0, M_1, N)$ we can determine a fixed positive quantity $R < 1$ such that

$$(1.24) \qquad \iint_{|w|\leq R} |J_z(w)|\, dudv \geq \frac{\pi^2}{4}$$

holds for all mappings $w \to z(w)$ belonging to $\Omega(M_0, M_1, N)$. Now assume that (1.22) is false. Then from the facts hitherto established it follows that there exists a sequence of mappings $w \to z_k(w)$ ($k = 1, 2, \cdots$) of class $\Omega(M_0, M_1, N)$ such that the relations

$$(1.25) \qquad z_k(w) \to z(w) \qquad (k \to \infty) ,$$

$$(1.26) \qquad \begin{matrix} z_{ku} \to z_u , \\ z_{kv} \to z_v \end{matrix} \qquad (k \to \infty) ,$$

hold uniformly on every compact subset of B, where the limit mapping $w \to z(w)$ belongs to $\Gamma(M_0, M_1, B)$ and has a vanishing jacobian at some point $w^* \in B$. According to Lemma 2 this entails $J_z(w) \equiv 0$, hence $J_{z_k}(w) \to 0$ $(k \to \infty)$ uniformly in every closed disk $|w| \leq r < 1$, contradicting (1.24). This completes the proof of our lemma.

2. The main results

In this section we study the partial differential equation (1), where $H = H(x, y)$ is a given function in the disk $|z - z_0| < R < +\infty$ satisfying a Lipschitz condition

$$(2.1) \qquad |H(z') - H(z'')| \leq M\, |z' - z''| .$$

If we note that equation (1) can also be written in the divergence form

$$(2.2) \quad \mathscr{L}(f) = \frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1 + p^2 + q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1 + p^2 + q^2}}\right) = 2H(x, y) ,$$

then some elementary conclusions can be drawn at once. First of all, on integrating (2.2) over the disk $|z - z_0| < R' < R$ and passing to the limit $(R' \to R)$ we obtain (see [5, p. 452])

$$(2.3) \qquad \left| \iint_{|z - z_0| < R} H(x, y) dxdy \right| \leq \pi R ,$$

which implies, in virtue of (2.1),

$$(2.4) \qquad |H(x, y)| \leq 1/R + 2RM \qquad (|z - z_0| < R) .$$

Suppose now that in addition we also have an upper bound for the modulus of f, say,

$$(2.5) \qquad |f(x, y)| \leq \gamma \qquad (|z - z_0| < R) .$$

Then on multiplying (2.2) by f and integrating by parts we deduce an inequality of the form (see [2, p. 198] for the case $H \equiv 0$)

$$(2.6) \qquad \iint_{|z-z_0|<R} \sqrt{1 + |\nabla f|^2}\, dxdy \leq c(R, M, \gamma) < +\infty .$$

Obviously we can extend $H(x, y)$ to the whole $\boldsymbol{R}^2$ such that (2.1) and (2.4) hold everywhere in $\boldsymbol{R}^2$.

We are now prepared to establish the principal results of this paper.

Theorem 1. *Let $f = f(x, y)$ be a solution of the equation $\mathscr{L}(f) = 2H$ in the disk $|z - z_0| < R$, which belongs to C^2, where $H = H(x, y)$ satisfies the Lipschitz condition* (2.1). *Furthermore, let*

$$(2.7) \qquad \iint_{|z-z_0|<R} \sqrt{1 + |\nabla f|^2}\, dxdy \leq A < +\infty .$$

Then we have an estimate of the form

$$(2.8) \qquad |\nabla f(x_0, y_0)| \leq \Theta(M, A, R) < +\infty .$$

Proof. It evidently suffices to prove our assertion for the case $z_0 = 0$ and $R = 2$. Then according to a well-known differentiability theorem for elliptic equations due to E. Hopf [9] (see also [1], especially p. 344) the function f belongs to $C^{2+\alpha}(0 < \alpha < 1)$ for $|z| \leq R' < 2$. Consequently, in virtue of the uniformization theorem, there exists a homeomorphic map $w \to z(w) = x(w) + iy(w)$ of $\bar{B}$ onto itself, which belongs to $C^2(B)$ and has a positive jacobian, such that the equation

$$(2.9) \qquad ds^2 = dx^2 + dy^2 + df^2 = \mu |dw|^2$$

holds in B. The mapping can be normalized such that $z(0) = 0$.

Now consider the vector function

$$(2.10) \qquad X(w) = (x(w), y(w), f(x(w), y(w))) ,$$

where $w \in \bar{B}$. An elementary computation yields the differential equations

$$(2.11) \qquad X_{w\bar{w}} = -iH(z)(X_w \wedge X_{\bar{w}}) ,$$

$$(2.12) \qquad X_w^2 = 0$$

for $w \in B$, where $\wedge$ denotes the cross product of two vectors in $\boldsymbol{R}^3$. From (2.12) we infer

$$\{\sqrt{1+|\nabla f|^2}\}_{z=z(w)} = \frac{X_u^2 + X_v^2}{2J_z(w)}, \tag{2.13}$$

$$|z_u|^2 + |z_v|^2 \leq X_u^2 + X_v^2 \leq 2(|z_u|^2 + |z_v|^2) . \tag{2.14}$$

On integrating (2.13) over B and using (2.7) and (2.14), we obtain

$$D(z) \leq 2A . \tag{2.15}$$

Putting now $a(x, y) = -H(x, y), b(x, y) = H(x, y)$ and $\varphi = f_w$ it is obvious from (2.11)–(2.12) that $z(w)$ is a solution of the system (1.5)–(1.6). The estimates (2.1), (2.4) and (2.15) then show that the mapping $w \to z(w)$ belongs to $\Omega(1 + 8M, 2M, 2A)$. Now combining (2.13) and (2.14) and using Lemma 3, we obtain

$$\begin{aligned} |\nabla f(0,0)| &\leq \left\{\frac{|z_u|^2 + |z_v|^2}{J_z(w)}\right\}_{w=0} \\ &\leq \frac{\lambda(1 + 8M, 2A, 0)}{\tau(1 + 8M, 2M, 2A, 0)} = \Theta_1(M, A) < +\infty , \end{aligned} \tag{2.16}$$

which implies the conclusion of our theorem.

Applying a result of Nirenberg [12, Theorem IV]) and the Schauder estimates, we easily deduce from Theorem 1 the following proposition:

Theorem 2. *Let $\mathscr{F} = \mathscr{F}(M, \gamma, D)$ be the set of solutions of the differential equation $\mathscr{L}(f) = 2H$ in a domain $D \subset \boldsymbol{R}^2$, where H and f satisfy the inequalities* (2.1) *and* (2.5) *in D, respectively. Then $\mathscr{F}$ is precompact in $C^2(T)$ for any compact subset $T \subset D$.*

Added in proof. After this paper had been submitted to this Journal for publication, there appeared the joint paper by O.A. Ladyzhenskaya & N.N. Ural'tseva, *Local estimates for the gradients of solutions of nonuniformly elliptic and parabolic equations*, Comm. Pure Appl. Math. **23** (1970) 677-703, in which the authors used completely different methods to establish interior gradient estimates for a hypersurface $z = f(x_1, \cdots, x_n)$, $n \geq 2$, in terms of the gradient bound of its mean curvature and the maximum norm of f.

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