Internal Definability And Completeness In Modal Logic

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Dedicated to my parents, Susanne Kracht and Alfred Kracht,
to my sister Ina and my brother Martin

Three pillars of wisdom support the edifice of Modal Logic. There is the ubiquitous *Completeness Theory*, the present Correspondency—or, more generally, *Definability Theory* – and finally, the *Duality Theory* between Kripke-frames and 'modal algebras' has become an area of its own. [...]

The relation between correspondency and completeness [...] turns out to be rather complex—and indeed only partially understood.

J. van Benthem

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Introduction

This dissertation intends to bridge the gap between completeness and correspondency theory. It was initiated by the insight that almost all completeness proofs can be reinterpreted as definability results in certain classes of general frames and that also correspondency theory is nothing but a special kind of definability theory. For example, in his essay on extensions of $\mathsf{K4}$, Fine shows that if a transitive logic Λ is of finite width then the weak reduced canonical frame is a frame for Λ iff its underlying Kripke-frame is a frame for Λ. Logics of finite width are therefore "weakly r-canonical", that is, persistent with respect to weak reduced canonical frames; and in the second part he shows that if the Kripke-frame underlying a weak reduced canonical frame subreduces to a finite rooted frame f then the weak canonical frame fails the subframe axiom associated with f. Such results always require a partial answer to the following fundamental problem: Given a generalized frame $\mathcal{G} = \langle g, A \rangle$ and a formula P, decide whether or not $\mathcal{G} \models P$. The intuitive strategy for solving this problem is this. First tackle the question whether $g \models P$. If γ and s can be found such that $\langle g, \gamma, s \rangle \models \neg P$, try to establish that the sets $\gamma(p)$ are *internal*, that is, that they are members of A. Although the two problems are of separate nature, they are nevertheless not independent. For suppose that the sets $\gamma(p)$ have been described in some formal or informal language, for example first order predicate logic with equality and a binary relation $\lhd$. Then if there is a way to tell whether such descriptions do define internal sets, an answer to the first problem might immediately answer the second problem. The whole dissertation is centered around this question. The historical aspect of this endeavour is that by bringing together completeness and correspondency we reunite two fields which separated in the mid-seventies, when the first results on the incompleteness of Kripke-semantics appeared.

The prevailing intuition in modal logic when used in argumentations and proofs is that of alternatives to a given situation. These alternatives were called *possible worlds* by Leibniz. They have given rise to what is now known as Kripke-semantics but is in fact due to a number of people, most prominently by [Hintikka, 1962] and [Kripke, 1963]. Tarski, in collaboration with McKinsey and Jónsson, developed an *algebraic* semantics for modal logic (see [McKinsey and Tarski, 1944], [Jónsson and Tarski, 1951]). This algebraization helped to fit modal logic into the large program of algebraization of logic and into well established diciplines such as model theory and boolean algebra. Despite their success, the algebraic models lacked the intuitive appeal of the relational models and this made them less attractive from a practical point of view. Consequently, the question arouse whether it was always possible to work with relational rather than algebraic models and for some time there was hope that this might turn out to be the case. Such hopes were thwarted, however, by the incompleteness results of [Fine, 1974b] and [Thomason, 1974]. Thus, relational models had to be handled with care in semantic analysis. With the balance between pros and cons of relational semantics being thus equalized, effort was put into providing a semantics combining the subtlety of algebraic models with the manageability of relational models. This led to the development of generalized relational models ([Thomason, 1972]) and duality theory ([Sambin and Vaccaro, 1988], [Goldblatt, 1989]).

At the same time another discipline emerged as a flash-back from the research into completeness: correspondency theory. It was observed that a fair number of logics hitherto studied and proven complete for relational semantics determined an elementary class of frames and that furthermore the elementary condition on the frames can be derived from the modal axiom via a syntactic translation. Much research was therefore directed into the connection between elementary and modal logic ([Fine, 1975], [Goldblatt, 1975], [Goldblatt, 1976]), culminating in [Sahlqvist, 1975] and [Benthem, 1983]. It became clear that while the connection is not completely arbitrary, it is in most cases unpredictable and the problem of specifying which elementary properties of frames determine a modally definable class of frames is in the general case probably undecidable. The research into correspondency theory has for the last years been conducted separately from the study of completeness and it seemed that beyond the early success of Sahlqvist, correspondency theory would not be able to contribute to completeness theory, witness, for example, the results in [Fine, 1974c; Fine, 1985]. At present, both theories have acquired a status of their own and neither seems to be in great need of the other. But this is true only superficially. At the level of *generalized correspondency*, which is nothing but correspondency theory of generalized frames, both fall together into one. For the sin of omission of correspondency theory was not so much the emphasis on first-order properties but the neglect of generalized frames because a correspondency result for Kripke-frames does not yield any completeness proof for the logic in question whereas such a proof would be immediate if the same result would hold in a suitably large class of generalized frames. In fact, Sahlqvist was aware of this problem when he wrote his famous [Sahlqvist, 1975] and gave a double layered correspondency proof; first he established the elementarity of what we now call Sahlqvist formulas in Kripke-frames and later he showed the same for descriptive frames. He actually notes that the proofs run strictly similar but he does not draw any conclusion from this. It was [Sambin and Vaccaro, 1988] who observed that a simultaneous proof of this result for descriptive frames and for Kripke-frames can be given.

This essay contains two major parts. Part I is of more theoretical nature. It spans the chapters § 1 – 4 and deals with generalized correspondency and internal definability. Part II contains § 5 – 7 and shows how internal definability bears on completeness questions. Special focus is placed on extensions of $\mathsf{K4}$. In contrast to the first part which covers mostly known material—although from a new perspective—the second part contains a number of new results and techniques. In § 1 we start with an overview of the technical apparatus and prove some elementary theorems on completeness and persistence. § 2 introduces the notion of a *concept* and of *internal definability* of a concept in a class of generalized frames and we show that the class of internally definable concepts is closed with respect to a number of logical operations. Concepts are abstract properties of general frames which may depend on a finite number of propositions or worlds. To make matters simple, a concept C is *internally described* in $\mathbf{X}$ by an n-tuple $\overline{P}$ of modal formulae if C holds of an n-tuple $\overline{w}$ of worlds in $\mathcal{G} \in \mathbf{X}$ iff for some valuation γ into $\mathcal{G}$ $w_i \in \gamma(P_i)$ for all $i \in n$. A concept is *internally definable* in $\mathbf{X}$ iff its complement or negation is internally describable in $\mathbf{X}$. Concepts are independent of the language in which they are expressed; although one could actually fix a formal language of concepts with connectives and quantifiers, we see no theoretical gain in doing so. Given the fact that the distinction between first- and second-order logic does not bear at all on completeness and that every now and then it would have been necessary to enrich the power of the language of concepts to deal with, say, infinite operations and nonstandard quantifiers, we have chosen to deemphasize the role of formal languages. However, there are occasions when we want to connect our results with classical model theory and this is why we have singled out a fragment of two-sorted (= second-order) predicate logic and called it *the* external language well knowing that we face a certain terminological confusion since there are plenty of concepts which are external even to our so-called external language. In § 3 we study internal definability from the viewpoint of general correspondency. The link between the two is the following. A concept C is *internally described* in the class $\mathbf{X}$ by the n-tuple $\overline{P}$ if C corresponds to the concept $\bigwedge\langle w_i \in P_i \,:\, i \in n\rangle$ in $\mathbf{X}$. Some results of [Goldblatt, 1989] and [Benthem, 1983] will be applied to see how closure properties of a class narrow down the class of concepts internally definable in this class. In contrast to this chapter which tells us what concepts are internally definable at best, § 4 establishes, which concepts are in actual fact definable in the union of the class of Kripke-frames with the class of descriptive frames. As a result, Sahlqvist's theorem will be proved. This proof differs from its predecessors in two ways. First, we establish a correspondency between sequences of modal formulae and elementary n-ary relations rather than between a single modal formula and a unary relation. Second, we do not ask which concepts are internally *definable* but which concepts are internally *describable*. This seems to be a marginal step since a concept is internally describable iff its negation is internally definable. But both together result in substantial simplification of notation and of the proof itself. Any reader of [Sahlqvist, 1975] will appreciate this. Also, unlike [Sambin and Vaccaro, 1988] we presuppose no knowledge of duality theory.

In § 5 we define a very abstract class of logics which we call *sketch-omission logics*. Sketch-omission logics include subframe logics and splitting logics and are an attempt to characterize logics which are defined by a set of geometric conditions. The canonical axioms of [Zakharyaschev, 1987] are also sketch-omission axioms. Sketch-omission logics are logics which omit a set of *sketches*, where a sketch is just a tree of worlds with a first-order condition associated with it. Each world in the tree actually represents an internal set of worlds, but it is best not to think this way, because it mostly turns out that it can indeed be treated as a world. $\mathsf{K4}$ is an example of a logic which omits a sketch, namely the three point chain with the condition that the first point does not see the last. We believe that all Sahlqvist logics can be interpreted as sketch-omission logics, but we have found no way of proving it. In § 6 we investigate the structure of finitely generated $\mathsf{K4}$-frames and prove some results which will be needed in the final chapter. The structure theory for finitely generated $\mathsf{K4}$-frames is particularly attractive in the case of logics of finite width where we prove that each point in such frames can be assigned depth, which in general is an ordinal number. Also we develop a language of generating concepts which is both close to natural language and allows to read off easily whether sets constructed in this language are internal. The ideas for this language result from a close investigation of the techniques used in [Fine, 1985]. In § 7 it will be shown that all subframe logics have the finite model property ([Fine, 1985]). Using a well-known theorem of [Kruskal, 1960] we can show for a large class of subframe logics that they are decidable. The general question whether all subframe logics are decidable is unsolved since the proof in [Fine, 1985] that there are uncountably many subframe logics is flawed. We believe to the contrary that all subframe logics are decidable but considering the strength of Kruskals theorem (it is true but not provable in **PA**), the answer to this question will probably depend on the additional assumptions on the arithmetical universe. In particular, we believe that if true it is nevertheless not constructively provable in **PA**. We will also extend the well-known theorem by [Bull, 1966] and [Fine, 1971] that all extensions of $\mathsf{S4.3}$ are decidable, finitely axiomatizable and have the finite model property by proving that this applies to all logics containing $\mathsf{S4}$ which are of tightness 2. In the end we will extend results shown in [Kracht, 1990a] on the conservation property of splitting logics. These results deserve special attention for they are both of a new type and of a new spirit. In contrast to previous work we do not establish completeness or fmp for a logic simpliciter; rather we prove that certain logics Λ have the property that whichever logic Ξ containing Θ has fmp or is complete—where Θ is mostly either $\mathsf{K4}$ or $\mathsf{S4}$—, the join $\Lambda \sqcup \Xi$ has the same property as Ξ. In other words, Λ preserves completeness and fmp beyond Θ. Moreover, in all cases considered it turns out that if Λ is decidable or the size of a minimal model for a formula can be estimated in advance, the same holds for $\Lambda \sqcup \Xi$. Finally, the methods are totally constructive and so if a constructive proof can be given that Λ is complete then a constructive proof can be given for the completeness of $\Lambda \sqcup \Xi$. Thus no canonical models are needed and finite model property does not have to be proved with the help of infinite models.

Throughout this paper the following notation will be used. An ordinary arrow between two structures $\mathcal{A} \to \mathcal{B}$ will always denote a homomorphism. If a map between structures is not necessarily a homomorphism we write $\mathcal{A} \rightharpoondown \mathcal{B}$. $\rightarrowtail$ denotes an injective morphism, $\twoheadrightarrow$ a surjective morphism. The categorial product will be denoted by $\otimes$, the coproduct by $\oplus$. We write $\leftrightharpoons$ for definitional equivalence. When we define a term it is printed in boldface letters when it is defined. Natural numbers are defined such that $n = \{0, 1, \ldots, n-1\}$. $\sharp N$ is the cardinality of the set N. If $g = \langle g, \lhd \rangle$ is a frame with g a set and $\lhd$ the accessibility relation, we draw g in such a way that $\bullet$ always represents a reflexive point, that is a point s satisfying $s \lhd s$, and x represents an irreflexive point. We let $\square$ stand for a point which is either reflexive or irreflexive. A completed proof is indicated by $\dashv$. Short and easy proofs are mostly omitted, which is signalled by $\dashv$ occurring right after the statement of the theorem. If a theorem is cited without a proof this is marked by ($\dashv$).

We have tried to keep our symbols coherent so that if a symbol denotes an entity of a particular type once, it will always denote entities of the same type. For example, $\mathcal{G}$ and $\mathcal{H}$ always denote general frames.

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Part I

Internal Definability

Chapter 1

Basic Definitions

1.1 The Internal Language of Modal Logic

The language $\mathcal{L}^i$ of modal logic consists of a denumerable set $\mathbb{P}_\omega = \{p_i : i \in \omega\}$ of proposition variables denoted by lower case Roman letters p, q, r, ..., the constants $\top, \bot$ and the connectives $\neg, \wedge$, $\to$, $\Box$, $\Diamond$, out of which formulas are built in the usual way. We will take $\neg, \wedge, \top, \Box$ as primitive symbols, the others being defined from them. Formulas of $\mathcal{L}^i$ will be denoted by upper case Roman letters P, Q, ... A **normal modal logic** is a set Λ of well-formed formulas which contains the axioms of classical logic, BD : $\Box(p \to q) \to .\Box p \to \Box q$, which is closed under Modus Ponens, Substitution and $MN : p/\Box p$. In the sequel we will omit the words 'normal' and 'modal' and simply talk about logics when we mean normal modal logics. We write $\mathbb{P}$ for an arbitrary subset of $\mathbb{P}_\omega$ and $\mathcal{L}^i(\mathbb{P})$ for the sublanguage of all propositions with variables from $\mathbb{P}$. Also we set $\mathbb{P}_k = \{p_i : i \in k\}$. The **modal degree** $\mathrm{dg}(P)$ of a formula is defined as usual by

$$
\begin{array}{lrcl}
(g0) & \mathrm{dg}(p) & = & 0 \\
(g\neg) & \mathrm{dg}(\neg P) & = & \mathrm{dg}(P) \\
(g\wedge) & \mathrm{dg}(P \wedge Q) & = & \max\{\mathrm{dg}(P), \mathrm{dg}(Q)\} \\
(g\Box) & \mathrm{dg}(\Box P) & = & \mathrm{dg}(P) + 1
\end{array}
$$

1.2 Modal Algebras

The simplest kind of model for $\mathcal{L}^i$ is an algebraic model, that is, a model based on an algebra of the appropriate signature. We therefore define a **(normal) modal algebra** to be an algebra $\mathcal{A} = \langle A, -, \cap, 1, \blacksquare\rangle$ where $\langle A, \backslash, \cap, 1\rangle$ is a boolean algebra and $\blacksquare : A \longrightarrow A$ a **hemimorphism** , that is, a map such that for all $a, b \in A$

$$\begin{array}{llcl} (h\cap) & \blacksquare(a \cap b) & = & \blacksquare a \cap \blacksquare b \\ (h1) & \blacksquare 1 & = & 1 \end{array}$$

A modal algebra can also be written as a pair $\langle \mathbf{A}, \blacksquare\rangle$ with $\mathbf{A} = \langle A, -, \cap, 1\rangle$ a boolean algebra and $\blacksquare$ a hemimorphism. For any set $\mathbb{P} \subseteq \mathbb{P}_\omega$ of sentence letters there is a natural modal algebra $\mathcal{A}(\mathbb{P}) = \langle A(\mathbb{P}), -, \cap, 1, \blacksquare\rangle$ where the elements of $A(\mathbb{P})$ are the equivalence classes of formulas of the relation $P \equiv Q \leftrightharpoons \vdash P \leftrightarrow Q$. It is easy to see that $\equiv$ is a congruence relation and therefore $-, \cap, 1$ and $\blacksquare a := \{s : (\forall t)(s \lhd t \Rightarrow t \in a)\}$ are well defined if we put $-(P/\equiv) \leftrightharpoons (\neg P)/\equiv$, $P/\equiv \ \cap Q/\equiv \leftrightharpoons (P \wedge Q)/\equiv$ and $(\blacksquare a)/\equiv \leftrightharpoons \blacksquare(a/\equiv)$. $\mathcal{A}(\mathbb{P})$ is the free modal algebra on $\sharp\mathbb{P}$ generators, the so-called *Lindenbaum algebra*. A map $\gamma : \mathbb{P} \longrightarrow A$ with $\mathbb{P} \subseteq \mathbb{P}_\omega$ is called a **valuation** into $\mathcal{A}$; such a valuation defines a unique extension $\overline{\gamma} : \mathcal{L}^i(\mathbb{P}) \longrightarrow A$ via

$$\begin{array}{llcl} (ev) & \overline{\gamma}(p) & = & \gamma(p) \\ (e\neg) & \overline{\gamma}(\neg P) & = & -\overline{\gamma}(P) \\ (e\wedge) & \overline{\gamma}(P \wedge Q) & = & \overline{\gamma}(P) \cap \overline{\gamma}(Q) \\ (e\Box) & \overline{\gamma}(\Box P) & = & \blacksquare\overline{\gamma}(P) \end{array}$$

We will mostly write $\gamma(P)$ instead of $\overline{\gamma}(P)$. If γ, δ are two valuations into $\mathcal{A}$, then $\gamma \cap \delta$ is defined by $\gamma \cap \delta(p) = \gamma(p) \cap \delta(p)$.

A **point** is defined in $\mathcal{A}$ as an ultrafilter of $\mathbf{A}$. If U is a point of $\mathcal{A}$, then the triple $\langle \mathcal{A}, \gamma, U\rangle$ is called an (**algebraic**) **model**. We write $\langle \mathcal{A}, \gamma, U\rangle \models P$ for $\overline{\gamma}(P) \in U$ and say that $\langle \mathcal{A}, \gamma, U\rangle$ is a **model for** P. Abstracting away from points we write $\langle \mathcal{A}, \gamma\rangle \models P$ iff $\langle \mathcal{A}, \gamma, U\rangle \models P$ for all points U. Equivalently, $\langle \mathcal{A}, \gamma\rangle \models P$ iff $\overline{\gamma}(P) = 1$. Similarly we write $\langle \mathcal{A}, U\rangle \models P$ iff $\langle \mathcal{A}, \gamma, U\rangle \models P$ for all $\gamma : \mathrm{var}(P) \longrightarrow A$ and $\mathcal{A} \models P$ iff $\langle \mathcal{A}, \gamma\rangle \models P$ for all $\gamma : \mathrm{var}(P) \longrightarrow A$ and say that P **is in the logic of** $\mathcal{A}$ or $P \in \mathrm{Th}\,\mathcal{A}$. The concepts of subalgebras, products and homomorphisms are as in universal algebra. If $S \subseteq A$ then $[S]$ denotes the subalgebra (as well as its carrier set) generated by S. $\mathcal{A}$ is called n-**generated** if there is an S with $\sharp S = n$ such that $[S] = A$. This is equivalent to saying that if $\gamma : p_i \longrightarrow s_i, i \in n$ for $S = \{s_0, \ldots, s_{n-1}\}$ then $\overline{\gamma}$ is surjective. If we define

$[\gamma] = \{\overline{\gamma}(P) : \mathrm{var}(P) \subseteq \mathrm{dom}(\gamma)\}$ we can write this as $[\gamma] = A$. In that case we say that γ **generates** $\mathcal{A}$. We will frequently use the notation $[\gamma]$ to denote the subalgebra generated by γ. Modal algebras form an equational class and there is a well known one-to-one correspondence between logics and varieties of modal algebras, that is, between logics and classes of algebras closed under the formation of products (P), homomorphic images (H) and subalgebras (S). The variety corresponding to a logic Λ is denoted by $\mathcal{V}(\Lambda)$ and the logic corresponding to a class $\mathcal{K}$ (not necessarily a variety) by $\mathrm{Th}\,\mathcal{K}$.

1.3 Frames as Models

More intuitive than algebras are the so-called *frames*. A **frame** is a pair $\langle f, \lhd\rangle$ where f is a set—the set of **worlds**—and $\lhd$ an arbitrary relation on f—the **accessibility relation** on f. A **valuation** on f is a map $\gamma : \mathbb{P} \longrightarrow 2^f$, $\mathbb{P}$ a subset of $\mathbb{P}_\omega$. For $s \in f$ and $p \in \mathbb{P}$ we define as usual

(fv)	$\langle f, \gamma, s\rangle \models p$	iff	$s \in \gamma(p)$
$(f\neg)$	$\langle f, \gamma, s\rangle \models \neg P$	iff	$\langle f, \gamma, s\rangle \not\models P$
$(f\wedge)$	$\langle f, \gamma, s\rangle \models P \wedge Q$	iff	$\langle f, \gamma, s\rangle \models P, Q$
$(f\Box)$	$\langle f, \gamma, s\rangle \models \Box P$	iff	$\forall t \in f : s \lhd t \Rightarrow \langle f, \gamma, t\rangle \models P$

for P, Q formulas based on variables in $\mathbb{P}$. As above, the abbreviations $\langle f, \gamma\rangle \models P$, $\langle f, s\rangle \models P$ etc. are defined by universal abstraction over the nonoccurring symbols.

Morphisms between frames are defined as follows: $p : f \to g$ is called a **p-morphism** if for all $s, t \in f$:

1. $s \lhd_f t \Rightarrow p(s) \lhd_g p(t)$,
2. if for some u $p(s) \lhd_g u$ then $u = p(t')$ for some $t' \in f$ and
3. for every s' such that $p(s) = p(s')$ given $p(s) \lhd p(t)$ there is a t' such that $s' \lhd_f t'$ and $p(t') = p(t)$.

If $p : f \rightharpoondown g$ is a map satisfying only the first condition namely that $s \lhd_f t$ implies $p(s) \lhd_g p(t)$ then p is called a $\lhd$-**homomorphism** or a **filtration map**. A p-morphism $p : f \to g$ is **admissible** for a valuation $\gamma : \mathbb{P} \longrightarrow 2^f$ if for all $s \in g$ there is a Q_s such that $p^{-1}(s) = \gamma(Q_s)$. γ then uniquely defines a valuation on g via $\delta(q) = p[\gamma(q)]$ and we will write γ instead of δ to indicate that this valuation is induced by γ. Similarly, we write γ for the unique valuation induced by γ on a generated subframe $g \rightarrowtail f$. Every p-morphism p can be factored into a surjection $c : f \twoheadrightarrow h$ and an injection $i : h \rightarrowtail g$. We call c a **contraction** and h a **generated subframe** of g. If $s \in f$, then $\mathrm{Tr}_f(s)$ denotes the subframe of f generated by s. We call $\mathrm{Tr}_f(s)$ the **transit** of s in f.

For a logic Λ there is a unique class $\mathrm{Fr}(\Lambda)$ of frames with $f \in \mathrm{Fr}(\Lambda) \Leftrightarrow \mathrm{Th}(f) \supseteq \Lambda$ and that class is closed under the formation of p-morphic images (C), generated subframes (W), ultrafilter extensions (UE) and coproducts (CP) . With the notation introduced below the **ultrafilter extension** of g is the frame $({g^{\sharp}}_{+})^{+}{}_{\sharp}$. A well-known theorem by [Benthem, 1983] states that a class F of Kripke-frames is the class of frames of a logic iff it is closed under the aforementioned operators while also its complement is closed under UE. Such classes are called **modally definable**. We prefer to call them *internally definable*. For a modally definable class F of frames there usually is a spectrum of logics Λ such that $\mathrm{Fr}(\Lambda) = F$.

1.4 General Frames—both Algebras and Frames

The two kinds of models can be integrated in the concept of a *general frame*. A **general frame** is a pair $\mathcal{G} = \langle g, A\rangle$ such that g is a frame and A a subset of 2^g closed under $-, \cap$ and $\blacksquare$, which is defined by $\blacksquare A = \{s : (\forall t) s \lhd t \Rightarrow t \in A\}$. We say that the elements of A are the **internal sets** of $\mathcal{G}$. General frames work just like frames except that valuations may draw values only from A; or, equivalently, the value of a proposition must always be an internal set. General frames provide the ideal synthesis of frames and algebras, because they are as subtle as algebras in that every logic is complete with respect to its general frames and they allow us to use our geometric intuitions by always having a frame at hand.

A morphism between general frames is a pair $\langle p, q\rangle : \langle g, A\rangle \to \langle h, B\rangle$ with $p : g \to h, q : B \to A$ such that for $b \in B$ $q(b) = p^{-1}[b] := \{x : p(x) \in b\}$. Thus, in practice a morphism between general frames is determined by the p-morphism between the underlying frame and so we simply write $p : \langle g, A\rangle \to \langle h, B\rangle$. A general frame $\mathcal{G}$ not only defines a frame $\mathcal{G}_\sharp$ via the forgetful functor $(-)_\sharp : \langle g, A\rangle \mapsto g$ in a unique way but also the algebra $\langle A, -, \cap, 1, \blacksquare\rangle = \mathcal{G}_+$ with the operations defined as in § 1.2. $(-)_\sharp$ is a covariant functor from the category of general frames Gfr into the category Frm of frames and $(-)_+$ a contravariant functor from Gfr into the category Mal of modal algebras. A frame g defines the general frame $g^\sharp = \langle g, 2^g\rangle$ and we have $\mathrm{Th}(g) = \mathrm{Th}(g^\sharp)$. This defines a covariant functor from Frm into Gfr but we will henceforth not distinguish between the frame g and the general frame $g^\sharp$. Important, however, is the contravariant functor $(-)^+$ from Mal to Gfr which for each algebra $\langle \mathbf{A}, \blacksquare\rangle$ returns the general frame $\mathcal{A}^+ = \langle \mathrm{pt}(\mathcal{A}), \lhd_{\mathcal{A}}, S_{\mathcal{A}}\rangle$ where $\mathrm{pt}(\mathcal{A})$ is the set of points or ultrafilters of $\mathcal{A}$ and $U \lhd_{\mathcal{A}} T$ iff $\forall \blacksquare a \in U : a \in T$ and $S_{\mathcal{A}} = \{\widehat{a} : a \in A\}$ with $\widehat{a} = \{U \in pt(\mathcal{A}) : a \in U\}$. It is easy to see that for every algebra $\mathcal{A} \cong (\mathcal{A}^+)_+$ and that for every frame $g \cong (g^\sharp)_\sharp$. General frames can now also be viewed as injective maps $\mathcal{A} \rightarrowtail g^\sharp$ i.e. as objects in the arrow category $\mathsf{Mal}^\rightarrow$. From that one can easily deduce that if $\mathcal{G}_i = \langle g_i, A_i\rangle$ are frames, then the coproduct $\mathcal{G}_1 \oplus \mathcal{G}_2$ is isomorphic to $\langle g_1 \oplus g_2, A_1 \otimes A_2\rangle$ and that a subframe $\mathcal{G}_1 \rightarrowtail \mathcal{G}_2$ corresponds to a subframe $g_1 \rightarrowtail g_2$ where $A_1 = \{a \cap g_2 : a \in A_2\}$. More on the categorical approach to modal logic and duality can be found in [Sambin and Vaccaro, 1988].

The class of general frames such that $\mathcal{G} \models \Lambda$ for a logic Λ is denoted by $\mathrm{Md}(\Lambda)$, the logic of a class of frames $\mathcal{K}$ by $\mathrm{Th}\,\mathcal{K}$. This class is closed under generated subframes (W) $i : \mathcal{G} \rightarrowtail \mathcal{H}$ where $i^\sharp \upharpoonright B$ is onto, p-morphic images (C), coproducts (Cp), biduals (B) where B: $\mathcal{G} \mapsto (\mathcal{G}_\sharp)^\sharp$, subalgebras (S) and *equivalence* (Eq): a map $e : \langle g, A\rangle \rightharpoonup \langle h, B\rangle$ is a **subalgebra** if e is iso and an **equivalence** if $e^\sharp : B \to A$ is iso as a map between algebras.

1.5 The External Language of Modal Logic

Now we introduce the *external language* $\mathcal{L}^e$ of modal logic which allows to talk about general frames. This language is a fragment of 2-sorted predicate language with the two sorts *proposition* and *world*. As before, $\mathbb{P}_\omega$ is the set of proposition variables; similarly, $\mathbb{W}_\omega$ is the set of world variables, an arbitrary subset of which is denoted by $\mathbb{W}$. On the side of the propositions it contains the symbols of $\mathcal{L}^i$ i.e. the proposition variables from $\mathbb{P}_\omega$ and the connectives $1, \wedge, \neg, \Box$; and on the side of the worlds it contains equality $=$ and the accessibility relation $\lhd$. Finally, there is binary relation ϵ, the *acceptance relation*. If w is a world and P a proposition, then $x \,\epsilon\, P$ is a formula which can be translated as "x accepts P" or "x is in P", if propositions are thought of as sets of worlds. To distinguish the logical connectives of $\mathcal{L}^e$ from the connectives of $\mathcal{L}^i$ we write them as t, f, $\sim$, &, $\supset$, $\equiv$. We will call proposition-variables simply p-variables and world-variables w-variables. By

$\mathcal{L}^f$ we denote the sublanguage of $\mathcal{L}^e$ consisting only of the world-variables , -quantifiers and -relations plus the formula connectives. $\mathcal{L}^f$ is the *frame part* of $\mathcal{L}^e$ and formulas from $\mathcal{L}^f$ are called **frame formulas** or simply **f-formulas** to distinguish them from the frame-formulas in the sense of [Fine, 1974a].

The appropriate structures for $\mathcal{L}^e$ are general frames $\mathcal{G} = \langle g, A\rangle$ and interpretations are pairs $\langle \gamma, \iota\rangle$ with $\gamma : \mathbb{P} \longrightarrow A, \iota : \mathbb{W} \longrightarrow g$, Propositions receive their value in A as defined in § 1.2. In $\mathcal{L}^e$, $\langle \mathcal{G}, \gamma, \iota\rangle \models P$ is clearly meaningless. However, we will allow ourselves to write $\langle \mathcal{G}, \gamma, s\rangle \models P$ instead of $\langle \mathcal{G}, \gamma, \iota\rangle \models x \epsilon P$ for $\iota : x \mapsto s$ when there is no risk of misunderstanding. The structure $\langle \mathcal{G}, \gamma, \iota\rangle$ is called a **triple** . There are the following base clauses for interpretation:

(gp)	$\langle \mathcal{G}, \gamma, \iota\rangle \models w \epsilon P$	iff	$\iota w \in \overline{\gamma}(P)$
$(g =)$	$\langle \mathcal{G}, \gamma, \iota\rangle \models w = x$	iff	$\iota w = \iota x$
$(g\lhd)$	$\langle \mathcal{G}, \gamma, \iota\rangle \models w \lhd x$	iff	$\iota w \lhd \iota x$

If $\langle \mathcal{G}, \gamma, \iota\rangle, \langle \mathcal{H}, \delta, \kappa\rangle$ are triples and $p : \mathcal{G} \rightharpoondown \mathcal{H}$ a map, then p can be lifted to a map between the two triples (denoted by the same letter p) if $dom(\gamma) = dom(\delta), dom(\iota) = dom(\kappa)$ and $p \circ \iota = \kappa, \gamma = p^+ \circ \delta$. The triple map is a homomorphism iff p is a p-morphism.

For an arbitrary e-formula α we denote the set of w-variables contained in α by $wvar(\alpha)$ and the set of free w-variables in α by $fwvar(\alpha)$. Likewise, the set of p-variables in α is denoted by $pvar(\alpha)$ and the set of free p-variables by $fpvar(\alpha)$.

Quite often we will use 'restricted' world-quantifiers instead of the global ones. The definitions run as follows

$(r\exists)$	$(\exists w \rhd v)\alpha$	$\leftrightharpoons$	$(\exists w)(v \lhd w \,\&\, \alpha)$
$(r\forall)$	$(\forall w \rhd v)\alpha$	$\leftrightharpoons$	$(\forall w)(v \lhd w \supset \alpha)$

The restricted quantifiers range over successors of points called **restrictors** rather than all points of the frame. In the above definitions, v is a restrictor. Furthermore, we will define quantifers ranging over points up to a certain depth.

$(d \lhd 0)$	$v \lhd^0 w$	$\leftrightharpoons$	$v = w$
$(d \lhd +)$	$v \lhd^{n+1} w$	$\leftrightharpoons$	$(\exists w')(v \lhd^n w' \,\&\, w' \lhd w)$
$(d\lhd^{(n)})$	$v \lhd^{(n)} w$	$\leftrightharpoons$	$\bigvee\langle v \lhd^i w : i \in n\rangle$
$(d\forall^n)$	$(\forall w \rhd^n v)\alpha$	$\leftrightharpoons$	$(\forall w)(v \lhd^n w. \supset .\alpha)$
$(d\forall^{(n)})$	$(\forall w \rhd^{(n)} v)\alpha$	$\leftrightharpoons$	$(\forall w)(v \lhd^{(n)} w. \supset .\alpha)$

and likewise for $\exists$. Transitivity can then be written as $(\forall w \rhd^2 v)(v \lhd w)$ and reflexivity as $(\forall w \rhd^0 v)(v \lhd w)$. It is useful to note that $\alpha[v/w]$ is equivalent to $(\forall w \rhd^0 v)\alpha$ and also to $(\exists w \rhd^0 v)\alpha$.

If we denote by $L_{\mathbf{G}}$ the logic of all frames, that is, if $L_{\mathbf{G}} = \{\alpha \in \mathcal{L}^e : \forall \mathcal{G} \in \mathbf{G} : \mathcal{G} \models \alpha\}$, then, as a consequence of (gp), we have the following axioms in addition to the usual axioms of 2-sorted predicate logic with equality:

$(d1)$ $\vdash (\forall x)(x \epsilon \top)$
$(d\wedge)$ $\vdash (\forall p \forall q \forall x)(x \epsilon (p \wedge q) \equiv x \epsilon p \,\&\, x \epsilon q)$
$(d\neg)$ $\vdash (\forall p \forall x)(x \epsilon \neg p \equiv \sim x \epsilon p)$
$(d\Box)$ $\vdash (\forall p \forall x)(x \epsilon \Box p. \equiv .(\forall y \rhd x)(y \epsilon p))$
$(d\lhd)$ $\vdash (\forall x)(\forall y \rhd x)(\forall p)(y \epsilon p \supset x \epsilon \Diamond p))$

1.6 Some Classes of General Frames

From now on we will speak of *Kripke-frames* rather than frames and call the general frames simply *frames* instead. With the help of $\mathcal{L}^e$ we can now define some properties of frames which we will need later on:

Definition 1.6.1 *Let $\mathcal{G} = \langle g, A\rangle$ be a general frame. We call $\mathcal{G}$ 1-**refined** or **differentiated** if*

(df) $\mathcal{G} \models (\forall x \forall y)(x = y. \equiv .(\forall p)(x \epsilon p \supset y \epsilon p))$

*and **2-refined** or **tight** if*

(ti) $\mathcal{G} \models (\forall x \forall y)(x \lhd y. \equiv .(\forall p)(y \epsilon p \supset x \epsilon \Diamond p))$.

*We call $\mathcal{G}$ **atomic** if*

(at) $\mathcal{G} \models (\forall x \exists p \forall y)(y \epsilon p. \equiv .y = x)$.

*Finally, $\mathcal{G}$ is said to be **refined** or **natural** if $\mathcal{G}$ is both 1- and 2-refined.*

The term 'refined' is due to [Thomason, 1972], 1- and 2-refinedness is taken from [Bull and Segerberg, 1984]. [Fine, 1975] uses the words differentiated, tight and natural. Refinedness will play a great role later on, because it states that world relations are recoverable from the set of worlds and the set of internal sets alone. Thus, refined frames allow to define equality and accessibility of worlds, a property which is of importance for defining internal sets from properties of certain worlds. Some properties of frames are not definable within $\mathcal{L}^e$.

Definition 1.6.2 *A general frame $\mathcal{G}$ is called **descriptive** if $\mathcal{G}$ is isomorphic to its own bidual. $\mathcal{G}$ is **full** if $\mathcal{G} \cong (\mathcal{G}_\sharp)^\sharp$. A class $\boldsymbol{X}$ of frames is called **closed** if $\mathcal{G} \in \boldsymbol{X} \Rightarrow \mathrm{F}(\mathcal{G}) \in \boldsymbol{X}$, where $\mathrm{F} : \mathcal{G} \mapsto (\mathcal{G}_\sharp)^\sharp$. For an arbitrary class $\boldsymbol{X}$ we denote the closure $\boldsymbol{X} \cup \mathrm{F}(X)$ by $\boldsymbol{X}^\bullet$.*

[Goldblatt, 1976] has coined the word 'descriptive' and proved that a frame is descriptive iff it is refined and for every point $U : \bigcap U \neq \emptyset$ (thus every point in $\mathcal{G}_+$ exactly corresponds to a point in $\mathcal{G}_\sharp$). In [Sambin and Vaccaro, 1988] it also shown that $\mathcal{G}$ is descriptive iff $\mathcal{G}$ is refined and compact as a topological space. For a logic Λ the **co-free frame** $\mathcal{F}_\Lambda(k)$ generated by k elements is defined by $\mathcal{F}_\Lambda(k) = \mathcal{A}_\Lambda(k)^+$, where $\mathcal{A}_\Lambda(k)$ is the free Λ-algebra on k generators. If these generators are denoted by $\{a_i : i \in k\}$ then $\kappa : p_i \mapsto a_i, i \in k$ is the **canonical valuation** on $\mathcal{F}_\Lambda(k)$. The **canonical frame** for Λ is the co-free frame generated by ω elements.

For future nomenclature we will abbreviate the properties and respective classes of frames as follows: G stands for *general*, **Ti** for *tight*, **Df** for *differentiated*, **R** for *refined*, **D** for *descriptive*, **C** for *canonical*, **K** for *Kripke* i.e. full frames and **F** for *finite Kripke-frames*. **X** will stand for any property or class of frames. For example, a logic Λ is called **K**-complete if it is complete with respect to Kripke-frames, i.e. if it is complete simpliciter; Λ is called **X**-*persistent* if it is persistent for the class **X**, which is an arbitrary class. If only frames are taken from a class **X** which satisfy an axiom this is denoted by adding the name of the axiom. Thus the class of refined $\mathsf{K4}$-frames is denoted by **R4**. Each of the classes defines its own logic via $L_{\mathbf{X}} = \{\alpha \in \mathcal{L}^e : \forall \mathcal{G} \in \mathbf{X} : \mathcal{G} \models \alpha\}$. We have for example $L_{\mathbf{Df}} = L_{\mathbf{G}} + (df)$ with $df := (\forall xy)(x = y. \equiv .(\forall p)(x \epsilon p \supset y \epsilon p))$ (see Def. **??**). In some cases, when the class is actually defined by a $\mathcal{L}^e$-formula, it is possible to study the properties of the frames in that class abstractly, that is, through investigating the logic of that class. This opens up the possibility of recasting proofs about those frames into pure symbol crunching within the logic.

We will briefly remind the reader of a way to turn an arbitrary frame $\mathcal{G} = \langle g, A\rangle$ into a refined frame. The construction is taken from [Thomason, 1972]. Define an equivalence relation $\equiv$ on g as follows.

(cg1) $(\forall xy)(x \equiv y \Rightarrow (\forall a \in A)(x \in a \Leftrightarrow y \in a))$
(cg2) $(\forall xyx'y')(x \equiv y$ and $x' \equiv y' \Rightarrow (x \lhd y \Leftrightarrow x' \lhd y'))$

Let $s/\equiv$ denote the equivalence class of s and set $g/\equiv = \{s/\equiv : s \in g\}$. Now define $s/\equiv \lhd t/\equiv$ by $s' \lhd t'$ for all $s' \equiv s, t' \equiv t$ or simply by $s \lhd t$ (by (cg2)). Finally, $A/\equiv = \{a/\equiv : a \in A\}$ with $a/\equiv = \{s/\equiv : s \in a\}$. Then $\mathcal{G}/\equiv = \langle\langle g/\equiv, \lhd\rangle, A/\equiv\rangle$ is the reduced frame corresponding to $\mathcal{G}$. The map $s \mapsto s/\equiv$ defines a p-morphism $\mathcal{G} \longrightarrow \mathcal{G}/\equiv$ which we will call the **refinement map**.

1.7 Completeness and Persistence

An important area of modal logic is the study of *completeness*. Initially, completeness was taken to be only **K**-completeness, but the concept can be studied with respect to other classes of frames as well.

Definition 1.7.1 *A logic Λ is called **X-complete** if Λ is the logic of its **X**-frames, that is, if* $\Lambda = \mathrm{Th}(\mathrm{Md}(\Lambda) \cap \boldsymbol{X})$.

The importance of completeness results from the fact that if a logic is complete with respect to a class of frames **X** then a model may be drawn from that class without loss of generality. Moreover, in the special case of **F**-completeness —otherwise known as the finite model property (fmp)—it is easily demonstrated that if Λ is finitely axiomatizable and **F**-complete then Λ is decidable. This cannot be strengthened to logics which are recursively axiomatizable as noted in [Urquhart, 1981] For even if the theorems of Λ can still be recursively enumerated, the models cannot; for there is no finite procedure for testing whether a frame is a frame for the logic or not.

Completeness is at the same time a very desirable property – especially if the class with respect to which the logic is complete is very small—and hard to prove in individual cases; for logics are usually given via an axiomatization $\mathbf{K}(X)$, and even if the logics $\mathbf{K}(P)$ of the individual axioms are complete, this need not be the case for $\mathbf{K}(X)$. In other words, completeness is not stable under union. But completeness is stable under intersection, that is, if $\Lambda = \bigsqcap\langle\Lambda_i : i \in I\rangle$ and all Λ_i are **X**-complete, so is Λ. For if $\Lambda_i = \mathrm{Th}(k_i)$ for a class of **X**-frames then $K = \bigcup\langle K_i : i \in I\rangle$ is a class of **X**-frames and it is easily checked that $\Lambda = \mathrm{Th}(K)$.

Definition 1.7.2 *A logic Λ is called* ***X-persistent*** *if $\mathcal{G} \models \Lambda$ implies $\mathcal{G}_\sharp \models \Lambda$ for all* ***X****-frames $\mathcal{G}$. Λ is called* ***locally X-persistent*** *if $\langle \mathcal{G}, v\rangle \models \Lambda$ implies $\langle \mathcal{G}_\sharp, v\rangle \models \Lambda$ for all* ***X****-frames $\mathcal{G}$ and $v \in \mathcal{G}_\sharp$.*

Equivalently, Λ is **X**-persistent if for all $\mathcal{G} \in \mathbf{X}$ $\mathcal{G} \models \Lambda$ implies $(\mathcal{G}_\sharp)^\sharp \models \Lambda$. Persistence does not necessarily imply **K**-completeness; however, call **X total** if every logic is **X**-complete. Then we have

Proposition 1.7.3 *If Λ is* ***X****-persistent and* ***X*** *total, then Λ is* ***K****-complete.*

Proof. A consistent proposition P has an **X**-model $\langle \mathcal{G}, \gamma, w\rangle \models P$. Then $\langle \mathcal{G}_\sharp, \gamma, w\rangle \models P$. ⊣

In contrast to completeness, persistence is stable under union:

Proposition 1.7.4 *Let $\Lambda = \bigsqcup\langle \Lambda_i : i \in I\rangle$. If all Λ_i are* ***X****-persistent, so is Λ.*

Proof. Let $\mathcal{G}$ be an **X**-frame such that $\mathcal{G} \models \Lambda$. Then $\mathcal{G} \models \Lambda_i$ for all $i \in I$ and so $\mathcal{G}_\sharp \models \Lambda_i$ for all $i \in I$; whence $\mathcal{G}_\sharp \models \Lambda$. ⊣

1.8 Some Small Theorems on Persistence

The above results can be refined somewhat by introducing the notion of a persistent axiom or property of frames:

Definition 1.8.1 *Let P be a proposition. Λ is called* ***X-persistent with respect to P****, if for any* ***X****-frame such that $\mathcal{G} \models \Lambda$: $\mathcal{G} \models P$ implies $(\mathcal{G}_\sharp)^\sharp \models P$. Λ is called* ***X-complete with respect to P*** *if $P \notin \Lambda$ implies $P \notin \mathrm{Th}(\mathrm{Md}(\Lambda) \cap \boldsymbol{X})$.*

We then have the following results:

Proposition 1.8.2 *Let* ***X*** *be total. If Λ is* ***X****-persistent with respect to P then Λ is* ***K****-complete with respect to P.*

Proof. Let P be consistent with Λ. Then there is an **X**-frame in $\mathrm{Md}(\Lambda)$ such that $\langle \mathcal{G}, \gamma, w\rangle \models \neg P$ from which $\langle \mathcal{G}_\sharp, \gamma, w\rangle \models \neg P$. ⊣

Proposition 1.8.3 *If $\Lambda_2 \supseteq \Lambda_1$ and Λ_1 is **X**-persistent with respect to P, so is Λ_2.*

Proof. If $\mathcal{G}$ is a Λ_2-frame, it is also a Λ_1-frame. So for any **X**-frame $\mathcal{G}$ such that $\mathcal{G} \models \Lambda_2$ we have $\mathcal{G} \models \Lambda_1$ and thus $\mathcal{G}_\sharp \models P$. ⊣

Proposition 1.8.4 *Let Λ_1 be **X**-persistent. Then $\Lambda_2 \supseteq \Lambda_1$ is **X**-persistent with respect to every $P \in \Lambda_1$.* ⊣

Theorem 1.8.5 *If* $\mathrm{var}(P) = \emptyset$ *then every logic is **G**-persistent with respect to P.*

Proof. From the above proposition it follows that it is enough to see that K is **G**-persistent for P. Since $\mathcal{G} \models K$ for every frame it is enough to show that $\mathcal{G} \models P$ implies $\mathcal{G}_\sharp \models P$. But $\mathcal{G} \models P$ iff for all $\gamma : \emptyset \longrightarrow A : \langle \mathcal{G}, \gamma\rangle \models P$ iff for all $\gamma : \emptyset \longrightarrow 2^{\mathcal{G}_\sharp} : \langle \mathcal{G}_\sharp, \gamma\rangle \models P$ iff $\mathcal{G}_\sharp \models P$. ⊣

[Bellissima, 1990] calls a logic k-**axiomatizable** if it can be axiomatized by a set of axioms based on no more than k sentence letters. The previous theorem shows that 0-axiomatizable logics are **G**-persistent and complete.

Chapter 2

Internal Describability

2.1 Concepts

Fix a finite set $\mathbb{P}$ of p-variables and a finite set $\mathbb{W}$ of w-variables. Suppose $C_{\mathbb{P},\mathbb{W}}$ is a property of triples $\langle \mathcal{G}, \gamma, \iota \rangle$ with $\mathrm{dom}(\gamma) \supseteq \mathbb{P}$ and $\mathrm{dom}(\iota) \supseteq \mathbb{W}$ which depends only on the values of $\gamma \upharpoonright \mathbb{P}$ and $\iota \upharpoonright \mathbb{W}$. Such a property is called a *concept*.

Definition 2.1.1 *Let $\mathbb{P}, \mathbb{W}$ be finite sets of proposition- and world-variables and $C_{\mathbb{P},\mathbb{W}}$ a property of triples. $C_{\mathbb{P},\mathbb{W}}$ is called a* ***concept*** *if $C(\langle \mathcal{G}, \gamma, \iota \rangle) \Leftrightarrow C(\langle \mathcal{G}, \gamma \upharpoonright \mathbb{P}, \iota \upharpoonright \mathbb{W} \rangle)$ whenever $\mathbb{P} \subseteq \mathrm{dom}(\gamma), \mathbb{W} \subseteq \mathrm{dom}(\iota)$. A concept is called an* ***e-concept****, if there is an $\alpha \in \mathcal{L}^e$ with $C(\langle \mathcal{G}, \gamma, \iota \rangle) \Leftrightarrow \langle \mathcal{G}, \gamma, \iota \rangle \models \alpha$. In that case we write $[\![\alpha]\!]$ instead of $C_{\mathbb{P},\mathbb{W}}$ where it is understood that $\mathbb{P} = \mathrm{fpvar}(\alpha)$ and $\mathbb{W} = \mathrm{fwvar}(\alpha)$. $C_{\mathbb{P},\mathbb{W}}$ is called an* ***f-concept*** *if there is an $\alpha \in \mathcal{L}^f$ with $C_{\mathbb{P},\mathbb{W}} = [\![\alpha]\!]$.*

We shall stress here that the definition of a concept is totally independent of the language $\mathcal{L}^e$. We will mostly suppress the reference to the sets $\mathbb{P}$ and $\mathbb{W}$ and simply write C for $C_{\mathbb{P},\mathbb{W}}$. Examples of concepts are "$\mathcal{G}$ is atomic", "$\mathcal{G}$ is transitive", "$\mathcal{G}/\equiv$ is transitive", "w satisfies p iff w has no successor" all of which are e-concepts. They are defined by the formulas $(\forall x \exists p \forall y)(y \in p. \equiv .x = y)$, $(\forall x \forall z \rhd^2 x)(x \lhd z)$, $(\forall x \forall p)(x \in \Diamond^2 p \to \Diamond p)$ and $(w \in p. \equiv .(\forall x)(w \not\lhd x))$. Concepts which are not e-concepts are "$\mathcal{G}$ is descriptive", "$\mathcal{G}$ is full". The property "w satisfies exactly one propositional variable" is not a concept at all. f-concepts are usually called **elementary** , since it is customary to use a language in which propositions are predicates over worlds instead of $\mathcal{L}^e$, so that quantifiers for propositions are second-order quantifiers. In $\mathcal{L}^e$ there is no asymmetry between worlds and propositions and so this terminology is obsolete, but we will nevertheless adhere to it. A concept can be characterized in various ways, for example with $\alpha = (\exists y)(x \lhd y)$ and $\beta = x \epsilon \Diamond\top$ we have $[\![\alpha]\!] = [\![\beta]\!]$ because $L_{\mathbf{G}} \vdash \alpha \equiv \beta$.

A concept $C_{\mathbb{P},\mathbb{W}}$ can naturally be understood as a concept $C_{\mathbb{P}',\mathbb{W}'}$ for $\mathbb{P} \subseteq \mathbb{P}', \mathbb{W} \subseteq \mathbb{W}'$. Given two concepts $C^i = C^i_{\mathbb{P}^i,\mathbb{W}^i}$, $i = 1, 2$ we can form the conjunction $C^1 \& C^2 = (C^1 \& C^2)_{\mathbb{P}^1 \cup \mathbb{P}^2, \mathbb{W}^1 \cup \mathbb{W}^2}$ and the negation $\sim C^1 = \sim C^1_{\mathbb{P}^1,\mathbb{W}^1}$ in the obvious way. For fixed $\mathbb{P}, \mathbb{W}$ the concepts defined on $\mathbb{P}, \mathbb{W}$ form a boolean algebra. For e-definable concepts we have $[\![\alpha]\!] \& [\![\beta]\!] = [\![\alpha \& \beta]\!]$ and $\sim [\![\alpha]\!] = [\![\sim \alpha]\!]$. Furthermore, we can abstract variables in concepts by

$$\begin{array}{rcl} (\forall p)C(\langle \mathcal{G}, \gamma, \iota \rangle) & \leftrightharpoons & \text{for all } \gamma' \text{ such that } \gamma'_p = \gamma_p : C(\langle \mathcal{G}, \gamma', \iota \rangle) \\ (\forall w)C(\langle \mathcal{G}, \gamma, \iota \rangle) & \leftrightharpoons & \text{for all } \iota' \text{ such that } \iota'_w = \iota_w : C(\langle \mathcal{G}, \gamma, \iota' \rangle). \end{array}$$

where γ_p denotes the restriction of γ to the set of all variables of the domain of γ different from p. We then get $(\forall p)[\![\alpha]\!] = [\![(\forall p)\alpha]\!]$ and $(\forall w)[\![\alpha]\!] = [\![(\forall w)\alpha]\!]$. If $C = [\![\alpha]\!]$ we will henceforth speak of "the concept α" rather than "the concept $[\![\alpha]\!]$" whenever we do not fear a risk of confusion.

Some important classes of concepts are the *world-* and the *frame-concepts*.

Definition 2.1.2 *If* card$(\mathbb{W}) = 1$, $C_{\mathbb{P},\mathbb{W}}$ *is called a* ***world-concept****.* $C_{\mathbb{P}}$ *is called a* ***frame-concept*** *if it is of the form* $(\forall w)D$ *with* $D = D_{\mathbb{P},\{w\}}$ *a world-concept.*

An e-concept $[\![\alpha]\!]$ is a world-concept precisely if $\sharp\,\mathrm{fwvar}(\alpha) = 1$ and a frame-concept if $\mathrm{fwvar}(\alpha) = \varnothing$ and the outermost quantifier is '$\forall$'. World-concepts can be seen as properties of worlds within a model. There is an intimate relation between world- and frame-concepts. Frame-concepts are the global versions of world-concepts; namely, they express that all worlds in the frame have a certain property. In practice it almost works out the same whether one deals with a frame-concept or its associated world-concept. However, in [Benthem, 1976] an example is given of a definable frame-concept whose associated world-concept is not definable. Nevertheless, in contrast to [Benthem, 1984] who prefers the global approach, we will use the local one, i.e. we prefer world-concepts over frame-concepts.

2.2 Definability—an Example

We motivate the definitions to come by working through a specific example. It is known that in Kripke-frames the property of being transitive is expressible by a modal formula, namely for all Kripke-frames f

(tf) $f \models (\forall v)(\forall w \rhd^2 v)(v \lhd w) \Leftrightarrow f \models \Diamond^2 p \to \Diamond p$

The formula $(\forall v)(\forall w \rhd^2 v)(v \lhd w)$ is said to *correspond globally* to the modal formula $\Diamond^2 p \to \Diamond p$. Such correspondency results are in the focus of correspondency theory as laid out, for example, in [Benthem, 1983] and [Benthem, 1984]. However, if we want to know more about the logic determined by the frames of this property, $\mathsf{K4}$, correspondency theory has not very much to offer. If we want to know whether $\mathsf{K4}$ is complete or has the finite model property, correspondence to a first-order formula is not a specifically helpful property. But this is only so because this correspondence has been established only *within the class of Kripke-frames*—a class for which not all logics are complete, or, to use our terminology here, a class which is not *total*. If we can prove this correspondency to hold in a closed total class of frames we can easily deduce that $\mathsf{K4}$ is complete. For suppose we can show

(tg) $\mathcal{G} \models (\forall v)(\forall w \rhd^2 v)(v \lhd w) \Leftrightarrow \mathcal{G} \models \Diamond^2 p \to \Diamond p$

for all $\mathcal{G} \in \mathbf{X}$ for a closed class $\mathbf{X}$, then $\mathsf{K4}$ is $\mathbf{X}$-persistent, for we have $\mathcal{G} \models \Diamond^2 p \to \Diamond p \Rightarrow \mathcal{G} \models (\forall v)(\forall w \rhd^2 v)(v \lhd w) \Rightarrow (\mathcal{G}_\sharp)^\sharp \models (\forall v)(\forall w \rhd^2 v)(v \lhd w) \Rightarrow (\mathcal{G}_\sharp)^\sharp \models \Diamond^2 p \to \Diamond p$. Since $\mathbf{X}$ is total, $\mathbf{X}$-persistence also implies completeness by Proposition 1.7.3.

It is easy to see that (tg) cannot hold in general. A counterexample is $\langle g, A\rangle$ with g being the frame $\boxed{\begin{array}{ccc} 0 & 1 & 2 \\ \times \longrightarrow & \times \longrightarrow & \bullet \end{array}}$ and $A = \{\varnothing, \{0,1,2\}\}$. For we have $\mathcal{G}/\equiv \; = \mathcal{H} := \langle h, B\rangle$ with $h = \boxed{\bullet}$ and $B = 2^h$ so that $\mathcal{G} \models \Diamond^2 p \to \Diamond p$ since $\mathcal{H} \models \Diamond^2 p \to \Diamond p$. But evidently $\mathcal{G} \not\models (\forall v)(\forall w \rhd^2 v)(v \lhd w)$. However, one can show that *tightness* alone suffices for (tg) to be true. This can be proved in two ways. The first is method is brute force using the logical laws of L_{Ti}:

$$\begin{array}{rl} & (\forall v)(\forall w \rhd^2 v)(v \lhd w) \\ \Leftrightarrow & (\forall v)(\forall w \rhd^2 v)(\forall p)(w \in p \supset v \in \Diamond p)) \\ \Leftrightarrow & (\forall v)(\forall w \rhd^2 v)(\forall p)(v \in \Diamond^2 p. \supset .v \in \Diamond p) \\ \Leftrightarrow & (\forall v)(\forall p)(v \in \Diamond^2 p \to \Diamond p) \end{array}$$

The other method is less formal but closer to intuitive reasoning. Instead of showing (tg) directly, we will prove that the negations of both sides are equivalent. This has the advantage that we can reason by counterexample. For suppose the left hand side fails. Then there are $r, s, t \in g$ such that $r \lhd s \lhd t$ but $r \not\lhd t$. By (ti), the latter implies that for some $\gamma : \langle \mathcal{G}, \gamma, r\rangle \models \Box\neg p$ and $\langle \mathcal{G}, \gamma, t\rangle \models p$. But since $r \lhd^2 t$, $r \models \Diamond^2 p$ and hence $\langle \mathcal{G}, \gamma, r\rangle \models \Diamond^2 p \wedge \Box\neg p$. The other direction is true in general.

2.3 Internal Describability

Let us say that C is *internally definable* if $\sim C$ is internally describable. Then how can internal describability be defined? To illustrate our way of thinking, let us take the example of $[\![v \lhd w]\!]$. We want to say that $[\![v \lhd w]\!]$ is internally definable in tight frames and hence that $[\![v \not\lhd w]\!]$ is internally describable in this class. The intuition is that $[\![v \not\lhd w]\!]$ is describable precisely if, whenever we are given a triple $\langle \mathcal{G}, \gamma, \iota\rangle$ such that $v, w \in \mathrm{dom}(\iota)$ and $v \lhd w$, we can name two formulas P, Q such that both $\langle \mathcal{G}, \gamma, \iota\rangle \models v \epsilon P$ and $\langle \mathcal{G}, \gamma, \iota\rangle \models w \epsilon Q$. (In our case, we let $P = \Box\neg Q$.) Or, we want to have a set $\mathcal{P}$ of pairs $\{\langle P^i, Q^i\rangle : i \in I\}$ such that $\langle \mathcal{G}, \gamma, \iota\rangle \models v \not\lhd w$ exactly if $\langle \mathcal{G}, \gamma, \iota\rangle \models v \epsilon P^i$ and $\langle \mathcal{G}, \gamma, \iota\rangle \models w \epsilon Q^i$ for some $i \in I$. There are, however, some adjustments to be made. First, we will require that the set is not just *any* set of pairs, but a set which derives from a *single* pair by a set of substitutions, namely, $\mathcal{P} = \{\langle \sigma(\Box\neg p), \sigma(p)\rangle : \sigma \text{ a substitution}\}$. Secondly, the describing pair contains the variable p which does not function in the concept (we will call such variables *parameters* of the description) and we want no free occurrence of p in the actual definition. This is accomplished in

Definition 2.3.1 *Assume $\mathbb{W} = \{w_0, \dots, w_{n-1}\}, n > 0$. Let $C = C_{\mathbb{P},\mathbb{W}}$ be a concept and $\boldsymbol{X}$ be a class of frames. C is said to be* ***internally describable*** *or simply* ***describable*** *in $\boldsymbol{X}$ if there is sequence $\langle Q_i : i \in n\rangle$ of modal formulae such that for all triples $\langle \mathcal{G}, \gamma, \iota\rangle$ with $\mathcal{G} \in \boldsymbol{X}$, $\mathbb{P} \subseteq \mathrm{dom}(\gamma), \mathbb{W} \subseteq \mathrm{dom}(\iota)$:*

(id) $C(\langle \mathcal{G}, \gamma, \iota\rangle) \Leftrightarrow \langle \mathcal{G}, \gamma, \iota\rangle \models (\exists \overline{q})(\&\langle w_i \epsilon Q_i : i \in n\rangle)$

where $\overline{q}$ contains all free variables of the Q_i which do not occur in $\mathbb{P}$. (Such variables are called ***parameters of the description*** *whereas the variables of $\mathbb{P}$ are called the* ***main variables****.) C is called* ***internally definable*** *or* ***definable*** *in $\mathbf{X}$, if $\sim C$ is internally describable in $\boldsymbol{X}$.*

In the case that $C = [\![\alpha]\!]$ (id) reduces to

$$\mathbf{X} \models (\forall\overline{p}\forall\overline{w})(\alpha(\overline{p},\overline{w}). \equiv .(\exists\overline{q})\&\langle w_i \epsilon\, Q_i : i \in n\rangle)$$

Our definition does not cover the case $\mathbb{W} = \varnothing$. We can extend the definition as follows. If $C = [\![\alpha]\!]$ is a frame-concept, say $\alpha = (\forall w)\beta$, then we say that $[\![\alpha]\!]$ is internally describable in $\mathbf{X}$ if there is a modal formula Q such that for all $\langle \mathcal{G}, \gamma\rangle$ with $\mathcal{G} \in \mathbf{X}$

$$\langle \mathcal{G}, \gamma\rangle \models \alpha \Leftrightarrow \langle \mathcal{G}, \gamma\rangle \models (\exists\overline{q})Q$$

Then $[\![\alpha]\!]$ is internally definable iff for some P

$$\langle \mathcal{G}, \gamma\rangle \models \alpha \Leftrightarrow \langle \mathcal{G}, \gamma\rangle \models (\forall\overline{q})P$$

The case where α is elementary, that is, no free p-variables occur, is also of importance. Here, the definability reduces to the condition $\mathcal{G} \models \alpha \Leftrightarrow \mathcal{G} \models Q$. This is the standard definition of definability; thus our definitions are in accord with standard terminology. A last remark: if C is *described* by $\langle Q_i : i \in n\rangle$ without any parameters then C is also *defined* by $\langle Q_i : i \in n\rangle$.

2.4 Definability and Completeness

If α is an f-sentence then obviously $\mathcal{G} \models \alpha$ exactly if $(\mathcal{G}_\sharp)^\sharp \models \alpha$; so, all frames are persistent with respect to α. Consequently, if α is internally defined in $\mathbf{X}$ by Q and $\mathbf{X}$ is closed, then for all $\mathcal{G} \in \mathbf{X}$: $\mathcal{G} \models Q \Leftrightarrow \mathcal{G} \models \alpha \Leftrightarrow (\mathcal{G}_\sharp)^\sharp \models \alpha \Leftrightarrow (\mathcal{G}_\sharp)^\sharp \models Q$. This proves

Proposition 2.4.1 *Let α be an f-sentence. If Q internally defines α in $\boldsymbol{X}^\bullet$, then $\mathsf{K}(Q)$ is $\boldsymbol{X}^\bullet$-persistent and therefore also $\boldsymbol{X}^\bullet \cap \boldsymbol{K}$-complete.* $\dashv$

Corollary 2.4.2 *Let $\boldsymbol{X}$ be total and α be a f-sentence. If Q internally defines α in $\boldsymbol{X}^\bullet$ then $\mathsf{K}(Q)$ is $\boldsymbol{X}$-complete and in particular ($\boldsymbol{K}$-)complete.* $\dashv$

For example, transitivity is **Ti**-definable and a fortiori **R**-definable and since **Ti** is closed and total, $\mathsf{K4} = \mathsf{K}(\Diamond^2 p \to \Diamond p)$ is complete. The condition that Q defines α in $\mathbf{X}^\bullet$ and not only in **X** alone is indeed a necessary one. It is for example not sufficient to have **D**-definability in order to conclude **D**-persistence. Rather, one needs definability in $\mathbf{D}^\bullet$. But it turns out that practically all definability results for **D** are valid in $\mathbf{D} \cup \mathbf{K}$ which is both closed and total.

2.5 Results on Describable Concepts

Here we will prove a number of easy theorems about describability. These results allow to generate large classes of describable properties and concepts. We will state the theorems using a special notation for finite sequences. Throughout, sequences are indicated by a bar, e.g. $\overline{P}$, $\overline{Q}$ denote sequences of formulae. The concatenation of $\overline{P}$ and $\overline{Q}$ is denoted by $\overline{P} \star \overline{Q}$. If $\overline{Q} = \langle Q_0 \rangle$ we write $\overline{P} \star Q$, dropping the outer brackets. If $\overline{P} = \langle P_i : i \in n\rangle$ and $\overline{Q} = \langle Q_i : i \in n\rangle$ are of equal length, we let $\overline{P} \wedge \overline{Q} \leftrightharpoons \langle P_i \wedge Q_i : i \in n\rangle$ and $\neg\overline{P} \leftrightharpoons \langle \neg P_i : i \in n\rangle$. Finally, we abbreviate $\&\langle w_i \,\epsilon\, P_i : i \in n\rangle$ by $\overline{w} \,\epsilon\, \overline{P}$.

Proposition 2.5.1 *$\langle \top \rangle$ describes $[\![t]\!]$ in **G** and $\langle \bot \rangle$ describes $[\![f]\!]$ in **G**.* ⊣

Proposition 2.5.2 *$\langle p, \neg p\rangle$ describes $[\![w_0 \neq w_1]\!]$ in **Df**. Moreover, $[\![w_0 \neq w_1]\!]$ is describable in **X** iff $\mathbf{X} \subseteq \mathbf{Df}$.*

Proof. Recall that $\mathcal{G} \in \mathbf{Df}$ iff $\mathcal{G} \models (\forall w_0 w_1)(w_0 = w_1. \equiv .(\forall p)(w_0 \,\epsilon\, p \supset w_1 \,\epsilon\, p))$ iff $\mathcal{G} \models (\forall w_0 w_1)(w_0 \neq w_1. \equiv .(\exists p)(w_0 \,\epsilon\, p \,\&\, w_1 \,\epsilon\, \neg p))$. Consequently, if $\mathbf{X} \subseteq \mathbf{Df}$ then $\langle p, \neg p\rangle$ describes $[\![w_0 \neq w_1]\!]$ in **Df**. If on the other hand $\mathbf{X} \not\subseteq \mathbf{Df}$, then **X** contains a non-differentiated frame and thus $[\![w_0 \neq w_1]\!]$ is not internally desribed by $\langle p, \neg p\rangle$. ⊣

Proposition 2.5.3 *$\langle \neg\Diamond p, p\rangle$ describes $[\![w_0 \ntriangleleft w_1]\!]$ in **Ti**. Moreover, $[\![w_0 \ntriangleleft w_1]\!]$ is describable in **X** iff $\mathbf{X} \subseteq \mathbf{Ti}$.*

Proof. Analoguous: $\mathcal{G} \in \mathbf{Ti}$ iff $\mathcal{G} \models (\forall w_0 w_1)(w_0 \triangleleft w_1. \equiv .(\forall p)(w_1 \,\epsilon\, p \supset w_0 \,\epsilon\, \Diamond p))$ iff $\mathcal{G} \models (\forall w_0 w_1)(w_0 \ntriangleleft w_1. \equiv .(\exists p)(w_1 \,\epsilon\, p \,\&\, w_0 \,\epsilon\, \neg\Diamond p))$. ⊣

Proposition 2.5.4 (Expansion) *Let $C_{\mathbb{P},\mathbb{W}}$ be a concept described by $\overline{Q}$ in $\boldsymbol{X}$. Then $\overline{Q} \star \top$ describes $C_{\mathbb{P},\mathbb{W}\cup\{w_n\}}$.* $\dashv$

Proposition 2.5.5 (Permutation) *Let C be described by $\langle Q_i : i \in n\rangle$ in $\boldsymbol{X}$. Then if $\pi : n \to n$ is a permutation and $\pi(C) := C[w_{\pi(i)}/w_i : i \in n]$ the concept derived from C by a permutation of variables, then $\pi(\overline{Q}) := \langle Q_{\pi(i)} : i \in n\rangle$ describes $\pi(C)$.* $\dashv$

Proposition 2.5.6 (Contraction) *Let $C_{\mathbb{P},\mathbb{W}}$ be a concept described by $\langle Q_i : i \in n+1\rangle$ in $\boldsymbol{X}$. Then $\langle\langle Q_i : i \in n-1\rangle, Q_{n-1} \wedge Q_n\rangle$ describes $C_{\mathbb{P},\mathbb{W}}[w_n/w_{n-1}]$, that is, the concept derived from C by identifying w_n with w_{n-1}.* $\dashv$

Proposition 2.5.7 (Swap) *If $\overline{P}$ describes C in $\boldsymbol{X}$ and $\overline{Q}$ results from substituting $\neg p$ for p, where p is a parameter variable, and p for $\neg p$ then $\overline{Q}$ describes C in $\boldsymbol{X}$. In particular, $\overline{Q}[\neg p_i/p_i]$ describes C in $\boldsymbol{X}$.* $\dashv$

Proposition 2.5.8 *If both $C^1 = C^1_{\mathbb{P},\mathbb{W}}$ and $C^2 = C^2_{\mathbb{P},\mathbb{W}}$ are describable in $\boldsymbol{X}$, so is $C^1\&C^2$.*

Proof. We can assume that if $\langle P_i : i \in n\rangle$ describes C^1 and $\langle Q_i : i \in n\rangle$ describes C^2 that the parametric variables the two descriptions are disjoint. Then $\langle P_i \wedge Q_i : i \in n\rangle$ describes $C^1\&C^2$ in $\mathbf{X}$:

$$\begin{array}{ll} & (C^1\&C^2)(\langle\mathcal{G},\gamma,\iota\rangle) \\ \Leftrightarrow & C^1(\langle\mathcal{G},\gamma,\iota\rangle) \text{ and } C^2(\langle\mathcal{G},\gamma,\iota\rangle) \\ \Leftrightarrow & \langle\mathcal{G},\gamma,\iota\rangle \models (\exists\overline{p})\overline{w} \mathrel{\epsilon} \overline{P} \text{ and } \langle\mathcal{G},\gamma,\iota\rangle \models (\exists\overline{q})\overline{w} \mathrel{\epsilon} \overline{Q} \\ \Leftrightarrow & \langle\mathcal{G},\gamma,\iota\rangle \models (\exists\overline{p}\exists\overline{q})\overline{w} \mathrel{\epsilon} \overline{P} \wedge \overline{Q}. \dashv \end{array}$$

Proposition 2.5.9 *If $\overline{Q} \star P^1$ describes $C^1 = C^1_{\mathbb{P},\mathbb{W}}$ in $\mathbf{X}$ and $\overline{Q} \star P^2$ describes $C^2 = C^2_{\mathbb{P},\mathbb{W}}$ in $\boldsymbol{X}$ then $\overline{Q} \star P^1 \vee P^2$ describes $C^1 \vee C^2$ in $\boldsymbol{X}$.*

Proof. $(C^1 \vee C^2)(\langle\mathcal{G},\gamma,\iota\rangle)$

$$\begin{array}{ll} \Leftrightarrow & C^1(\langle\mathcal{G},\gamma,\iota\rangle) \text{ or } C^2(\langle\mathcal{G},\gamma,\iota\rangle) \\ \Leftrightarrow & \langle\mathcal{G},\gamma,\iota\rangle \models (\exists\overline{p})(\overline{w} \mathrel{\epsilon} \overline{Q} \;\&\; (\exists\overline{p'})(w_n \mathrel{\epsilon} P^1)) \text{ or} \\ & \langle\mathcal{G},\gamma,\iota\rangle \models (\exists\overline{p})(\overline{w} \mathrel{\epsilon} \overline{Q} \;\&\; (\exists\overline{q'})(w_n \mathrel{\epsilon} P^2)) \\ \Leftrightarrow & \langle\mathcal{G},\gamma,\iota\rangle \models (\exists\overline{p})(\exists\overline{p'})(\exists\overline{q'})(\overline{w} \mathrel{\epsilon} \overline{Q} \;\&\; w_n \mathrel{\epsilon} P^1 \vee P^2). \dashv \end{array}$$

Corollary 2.5.10 *If $C^1 = C^1_{\mathbb{P},\{w\}}$ and $C^2 = C^2_{\mathbb{P},\{w\}}$ are world-concepts describable in* **X***, so is $C^1 \vee C^2$.* ⊣

Proposition 2.5.11 *If C is describable in* **X** *and $x \in \mathbb{W}$ then $(\exists y \rhd^n x)C$ is describable in* **X** *for all $n \in \omega$.*

Proof. We only need to consider the cases $n = 0, 1$. The rest is an easy induction. If $n = 1$ there are two cases: (i) $y \in \mathbb{W}$ (ii) $y \notin \mathbb{W}$. (ii) can be reduced to (i) by expanding the concept by y as in Proposition 2.5.4. So let us assume (i) and $\mathbb{W} = \{w_i : i \in n+1\}$ as well as $x = w_{n-1}, y = w_n$. Now let $\langle Q_i : i \in n+1\rangle$ describe C in $\mathbf{X}$. Then $\langle\langle Q_i : i \in n\rangle, Q_{n-1} \wedge \Diamond Q_n\rangle$ describes $(\exists w_n \rhd w_{n-1})C$:

$$
\begin{aligned}
& (\exists w_n \rhd w_{n-1})C(\langle \mathcal{G}, \gamma, \iota\rangle) \\
\Leftrightarrow\quad & \text{there is } \iota' \supset \iota : w_n \in \mathrm{dom}(\iota'), \iota(w_{n-1}) \lhd \iota'(w_n) \text{ and } C(\langle \mathcal{G}, \gamma, \iota'\rangle) \\
\Leftrightarrow\quad & \text{there is } \iota' \supset \iota : w_n \in \mathrm{dom}(\iota'), \iota(w_{n-1}) \lhd \iota'(w_n) \\
& \text{and } \langle \mathcal{G}, \gamma, \iota'\rangle \models (\exists \overline{q})\&\langle w_i \epsilon Q_i : i \in n+1\rangle \\
\Leftrightarrow\quad & \langle \mathcal{G}, \gamma, \iota\rangle \models (\exists \overline{q})(\&\langle w_i \epsilon Q_i : i \in n-1\rangle \;\&\; w_{n-1} \epsilon Q_{n-1} \wedge \Diamond Q_n).
\end{aligned}
$$

If $n = 0$ note that $(\exists y \rhd^0 x)C = C[y/x]$; consequently, by Proposition 2.5.6, $(\exists y \rhd^0 x)C$ is describable in **X** if C is. ⊣

A combination of Proposition 2.5.11 and Proposition 2.5.9 yields that if $i_0, \ldots, i_{k-1}$ is a sequence of numbers, then if C is describable in **X**, so is the disjunction $\bigvee\langle(\exists y \rhd^{i_\lambda} x)C : \lambda \in k\rangle$. As a special corollary we have

Corollary 2.5.12 *If C is describable in* **X***, so is $(\exists y \rhd^{(n)} x)C, n \in \omega$.*

2.6 Universal Logics

A major result of the preceding theorems is

Theorem 2.6.1 *If C is a world-concept derived from concepts describable in* **X** *using $\&, \vee$ and restricted existential quantification, then C is itself describable in* **X**.

Proof. Since the rules for & and restricted existential quantifier can be used without restriction, the theorem is clearly true for concepts not derived with $\vee$. However, if a world-concept C is derived with the help of $\vee$ it can be transformed into a disjunction of world-concepts built with & and restricted $\exists$, using standard logical laws. By Cor. 2.5.10, C is internally decsribable in **X**. $\dashv$

In **R**, $[\![w_0 \neq w_1]\!]$ and $[\![w_0 \ntriangleleft w_1]\!]$ are internally describable and therefore any elementary world-concept which is negative and contains only restricted existential quantifiers is internally describable. In the next chapter we will see that for elementary, existential α $[\![\alpha]\!]$ is internally describable in **R** iff it is of this form. As a consequence, if a logic is axiomatized by a set of positive and restricted universal frame-formula then it is **R**-persistent. We should warn the reader that there is a substantial difference between universal and restricted universal formulae. For the latter only completeness has been shown so far. For the former it can be shown that they have fmp. For if Λ is axiomatized by axioms expressing positive universal world-concepts then the class of Kripke-frames for Λ is closed under $\triangleleft$-homomorphisms and thus admits filtration ([Gabbay, 70]). We conjecture that all logics characterized by positive, restricted universal sentences have fmp. This has been shown so far only for extensions of $\mathsf{K4}$.

This theorem can also be used to strengthen a result by [Bellissima, 1988]. He showed that every extension of $\mathsf{K.Alt}_n$ is canonical, where

$$\mathsf{Alt}_n = \bigwedge \langle \Diamond p_i : i \in n+1 \rangle \to \bigvee \langle \Diamond (p_i \wedge p_j) : i \neq j \rangle \tag{2.1}$$

Close inspection of the proof reveals that since $\mathsf{K.Alt}_n$ is **Df**-persistent, every extension of $\mathsf{K.Alt}_n$ is **Df**-persistent. The reason is that if $\Lambda \supseteq \mathsf{K.Alt}_n$ then any differentiated Λ-frame $\mathcal{G}$ satisfies the corresponding first-order property

$$alt_n(v) = (\forall w_0 \rhd v) \ldots (\forall w_n \rhd v)(\bigvee \langle w_i = w_j : i \neq j \rangle) \tag{2.2}$$

As a consequence, for every point there are at most n^k successors of depth k, and thus $\sharp\, \mathrm{Tr}^k_{\mathcal{G}}(v) := \sharp\{w : v \triangleleft^{(k)} w\} \leq (n^{k+1} - 1)/(n-1)$ where $\mathrm{Tr}^k_f(s) := \{t : s \triangleleft^{(k)} t\}$. So, suppose $\langle \mathcal{G}, v \rangle \models P$ and $\mathrm{dg}(P) \leq k$ and $\mathcal{G}$ is differentiated. Then the algebra induced by $\mathcal{G}$ on $\mathrm{Tr}^k_{\mathcal{G}}(v)$ is the powerset-algebra and consequently $\langle \mathcal{G}_\sharp, v \rangle \models P$ showing that Λ is locally **Df**-persistent. By a result of [Fine, 1975] any **Df**-persistent logic is Δ-elementary. (This observation as well as the proof are due to F. Wolter.)

Theorem 2.6.2 (Wolter) *Every extension of $\mathsf{K.Alt}_n$ is locally **Df**-persistent and Δ-elementary.*
$\dashv$

This is of some importance since given a generalized frame it is easier to see whether it is differentiated than whether it is refined or descriptive.

Chapter 3

Some General Results on Internal Definability

3.1 General Correspondence

The notions of internal definability and internal describability can be viewed as specializations of *general correspondence*. If $C = C_{\mathbb{P},\mathbb{W}}$ and $D = D_{\mathbb{P},\mathbb{W}}$ are concepts and $\mathbf{X}$ a class of frames, we say that C **corresponds** to D in $\mathbf{X}$ if for all triples $\langle \mathcal{G}, \gamma, \iota \rangle$ such that $\mathcal{G} \in \mathbf{X}$, $\mathbb{P} \subseteq \mathrm{dom}(\gamma), \mathbb{W} \subseteq \mathrm{dom}(\iota)$ we have $C(\langle \mathcal{G}, \gamma, \iota \rangle) \Leftrightarrow D(\langle \mathcal{G}, \gamma, \iota \rangle)$. If C corresponds to D in $\mathbf{X}$ we write $C \underset{\mathrm{X}}{\leftrightsquigarrow} D$. Since C and D are both undefined whenever $\mathrm{dom}(\gamma) \not\supseteq \mathbb{P}$ or $\mathrm{dom}(\iota) \not\supseteq \mathbb{W}$ we can say that C and D correspond to each other in $\mathbf{X}$ iff they are identical as partial predicates over all triples in $\mathbf{X}$. Now if we call a concept **simple** if it is of the form $[\![(\exists \overline{q})\overline{w} \in \overline{Q}]\!]$ then a concept is internally describable in $\mathbf{X}$ iff it corresponds to a simple concept in $\mathbf{X}$. This way of reformulating describability has some advantages. For *correspondency in* $\mathbf{X}$ can be shown to be an equivalence relation and thus all closure properties of the class of simple concepts are inherited by the class of internally describable concepts. This makes the Propositions 2.5.4 – 2.5.11 straightforward. In addition, the class of simple concepts is closed under substitution.

In this section we will investigate the interplay between closure properties of classes of frames and definability. This will help to establish criteria for deciding which concepts are internally definable in a given class of frames. The results are not new and we will therefore not give any proofs. The interested reader is referred to [Benthem, 1983] and

[Goldblatt, 1989].

3.2 Preservation, Reflection and Invariance

The operators on classes of frames as defined in § 1.4 give rise to operators on classes of triples, denoted by the same name throughout. To give just one example, we take the operator W of generated subframes. Here, $\langle \mathcal{H}, \delta, \kappa \rangle \in \mathrm{W}(\mathbf{X})$ iff for some $\langle \mathcal{G}, \gamma, \iota \rangle \in \mathbf{X}$ there is a map $i : \langle \mathcal{H}, \delta, \kappa \rangle \longrightarrow \langle \mathcal{G}, \gamma, \iota \rangle$ such that $i_\sharp : \mathcal{H}_\sharp \rightarrowtail \mathcal{G}_\sharp$ is an injective p-morphism.

Definition 3.2.1 *Let C be a concept, $\mathbf{X}$ a class of triples and* O *an operator on classes of triples.* O ***preserves*** *C* ***in*** *$\mathbf{X}$ if for all $\langle \mathcal{G}, \gamma, \iota \rangle, \langle \mathcal{H}, \delta, \kappa \rangle$ such that $\mathcal{G}, \mathcal{H} \in \mathbf{X}$ and $\langle \mathcal{H}, \delta, \kappa \rangle \in$* O*$(\langle \mathcal{G}, \gamma, \iota \rangle)$, $C(\langle \mathcal{G}, \gamma, \iota \rangle)$ implies $C(\langle \mathcal{H}, \delta, \kappa \rangle)$.* O ***reflects*** *$C$* ***in*** *$\mathbf{X}$ if* O *preserves $\sim C$ in $\mathbf{X}$ and* O ***leaves*** *C* ***invariant in*** *$\mathbf{X}$ if it both preserves and reflects C in $\mathbf{X}$.*

Now we have the following results

Proposition 3.2.2 *Simple concepts are invariant under* W *and* Cp.

Proof. It suffices to prove this for the case O = W. Thus let $\langle \mathcal{H}, \delta, \kappa \rangle \overset{v}{\rightarrowtail} \langle \mathcal{G}, \gamma, \iota \rangle$, $\mathcal{G}, \mathcal{H} \in \mathbf{X}$. Then

$$\begin{array}{ll}
 & [\![(\exists \overline{q})\overline{w} \epsilon \overline{Q}]\!]\langle \mathcal{G}, \gamma, \iota \rangle \\
\Leftrightarrow & \langle \mathcal{G}, \gamma, \iota \rangle \models (\exists \overline{q})\overline{w} \epsilon \overline{Q} \\
\Leftrightarrow & \text{for some } \overline{\gamma} \supseteq \gamma : \langle \mathcal{G}, \overline{\gamma}, \iota \rangle \models \overline{w} \epsilon \overline{Q} \\
\Leftrightarrow & \text{for some } \overline{\gamma} \supseteq \gamma : \iota(w_i) \in \overline{\gamma}(Q_i) \text{ for all } i \in n \\
\Leftrightarrow & \text{for some } \overline{\delta} \supseteq \delta : \kappa(w_i) \in \overline{\delta}(Q_i) \text{ for all } i \in n \\
\Leftrightarrow & \text{for some } \overline{\delta} \supseteq \delta : \langle \mathcal{H}, \overline{\delta}, \kappa \rangle \models \overline{w} \epsilon \overline{Q} \\
\Leftrightarrow & \langle \mathcal{H}, \delta, \kappa \rangle \models (\exists \overline{q})\overline{w} \epsilon \overline{Q} \\
\Leftrightarrow & [\![(\exists \overline{q})\overline{w} \epsilon \overline{Q}]\!]\langle \mathcal{H}, \delta, \kappa \rangle. \dashv
\end{array}$$

The next two propositions are also easily proved.

Proposition 3.2.3 *Simple concepts are invariant under equivalence and under biduals.* ⊣

Simple concepts are not invariant under S and hence not under C. But

Proposition 3.2.4 *Simple concepts are reflected by* C *and* S. ⊣

Now that simple concepts are reflected by all four C, W, CP and B, the negations of simple concepts are preserved by all of them:

Theorem 3.2.5 *If D is internally definable in $\boldsymbol{X}$, D is preserved by* W, C, CP *and* B *and reflected by* B, CP *and* W. ⊣

3.3 Quasi-elementary Classes

In the remainder of this chapter we will exclusively deal with *quasi-elementary* concepts; a concept is **quasi-elementary** if it is of the form $[\![\alpha]\!]$ with α free of p-quantifiers. This paragraph is to establish criteria under which a concept corresponds to a quasi-elementary concept. We begin with two simple observations.

Observation 3.3.1 *Let $\boldsymbol{Y} = \mathrm{O}(\boldsymbol{X})$ and $C \underset{X}{\leftrightsquigarrow} D$. Then if both C and D are* O*-invariant, $C \underset{Y}{\leftrightsquigarrow} D$.* ⊣

Observation 3.3.2 *If $\boldsymbol{Y} = \mathrm{F}(\boldsymbol{Y})$ then C is invariant under* F *in $\boldsymbol{Y}$ iff it is invariant under* S *in $\boldsymbol{Y}$.* ⊣

To see this note that if $\mathcal{H} \in \mathrm{S}(\mathcal{G})$ then $(\mathcal{H}_\sharp)^\sharp \cong (\mathcal{G}_\sharp)^\sharp$.

Lemma 3.3.3 (van Benthem) *Let $\boldsymbol{Y} \subseteq \boldsymbol{K}$ be elementary. Then $C = C_{\mathbb{P},\mathbb{W}}$ corresponds to a quasi-elementary concept in $\boldsymbol{Y}$ iff C is preserved by ultraproducts (*UP*) in $\boldsymbol{Y}$.*

Proof. (⇒) Quasi-elementary concepts are preserved under UP. Thus this must hold for C as well. (⇐) Expand $\mathcal{L}^e$ to $\mathcal{L}^{e+}$ by adopting a new constant for each variable in $\mathbb{P}$ and $\mathbb{W}$, and let $\mathbf{Y}^+$ be the class of all models $\langle \mathcal{G}, \gamma, \iota \rangle$ such that $\mathcal{G} \in \mathbf{Y}$. Then $\mathbf{Y}^+$ is an elementary $\mathcal{L}^{e+}$-class. Therefore, since C is preserved under UP in $\mathbf{Y}$, it is also preserved under UP in $\mathbf{Y}^+$ and reflected by UP in $\mathbf{Y}^+$. Hence the class of $\langle \mathcal{G}, \gamma, \iota \rangle \in \mathbf{Y}^+$ satisfying C is $\mathcal{L}^{e+}$-elementary and hence C corresponds to a quasi-elementary concept. ⊣

If we call a class **X quasi-elementary** if it is closed under F and if $\mathbf{X} \cap \mathbf{K}$ is elementary, the lemma together with the observations give

Theorem 3.3.4 *Let* $\mathbf{X}$ *be quasi-elementary. Then* C *corresponds to a quasi-elementary concept in* $\mathbf{X}$ *iff* C *is invariant under* F *and preserved by* Up *in* $\mathbf{X}\cap\mathbf{K}$. ⊣

3.4 Closure Conditions and Syntactic Classes

We will briefly survey some results concerning the relationship between syntactic classes of quasi-elementary concepts and closure conditions on classes in which these concepts are definable. Let α be in prenex-normal form, with every variable being free everywhere or bound by exactly one quantifier. We say that α is **existentially restricted in** y (ER in y) if α is built from atoms and their negations using $\wedge$, $\vee$ and $\forall$ but unrestricted $\exists$ only for variables distinct from y. α is **existentially restricted** if α is existentially restricted in all occurring variables.

Theorem 3.4.1 *Let* $\mathbf{X}$ *be quasi-elementary and* W*-closed and* C *be quasi-elementary. Then* C *is preserved under* W *iff* $C \underset{X}{\leftrightsquigarrow} [\![\alpha]\!]$ *for an existentially restricted* α. (⊣)

Call α **two-way existentially restricted** or **two-way ER** if α is built from atoms and their negations with the help of $\wedge$, $\vee$ and $\forall$ but $(\exists y \rhd x)$ and $(\exists y \lhd x)$ for existential quantification only. Then, using analoguous methods one can prove a conjecture of [Benthem, 1983], namely that an elementary concept is reflected by coproducts iff it corresponds to a two-way ER formula.

Theorem 3.4.2 *Let* $\mathbf{X}$ *be quasi-elementary and closed under* Cp^{-1} *and let* C *be a quasi-elementary concept. Then* C *is reflected by* Cp *iff* $C \underset{X}{\leftrightsquigarrow} [\![\alpha]\!]$ *for a two-way ER formula* α.

There is a direct argument to derive this theorem out of Theorem 3.4.1. For if $\mathbf{X}$ is a quasi-elementary class of frames closed under inverse coproducts, $\mathbf{X}$ can be interpreted as a class $\mathbf{X}^t$ of tense-frames closed under W^t (generated tense-subframes) whose intersection with the class of all Kripke-tense-frames is elementary. Then the conditions of Theorem 3.4.1 are satisfied and we find that C is preserved by W^t iff it corresponds in $\mathbf{X}^t$ to $[\![\alpha]\!]$ for an α, which is existentially restricted as a tense-formula. The latter means nothing but that α is two-way ER and the former means that C is reflected by Cp.

Theorem 3.4.3 *Let* $\boldsymbol{X}$ *be quasi-elementary and closed under* C. *Let* C *be a quasi-elementary concept. Then* C *is preserved by* C *iff* $C \underset{X}{\leftrightsquigarrow} [\![\alpha]\!]$ *for a positive f-formula* α. $(\dashv)$

Here, α is **positive** iff it is built from atoms with the help of $\wedge, \vee$ and quantifiers, restricted or unrestricted. If we call α **restricted** if it contains only restricted quantifiers and **negative** if $\sim \alpha$ is equivalent to a positive formula, then this theorem combines with the preceding ones to

Theorem 3.4.4 *Let* $\boldsymbol{X}$ *be quasi-elementary and closed under* C, $\mathrm{C}\textsc{p}^{-1}$ *and* W. *Let* C *be a quasi-elementary concept. Then* C *is internally definable only if* $C \underset{X}{\leftrightsquigarrow} [\![\alpha]\!]$ *for a positive restricted* α *and internally describable iff* $C \underset{X}{\leftrightsquigarrow} [\![\alpha]\!]$ *for a negative restricted* α. $(\dashv)$

3.5 A Conjecture

Theorem 3.4.4 is a rough guide as to which concepts may turn out to be definable in a given class. However, the problem of characterizing *exactly* the concepts which are definable and which ones are not, is far from being solved. We believe that, depending on the class, this problem will receive different answers. As regards **C** and **D** we believe that the characterization problem is *provably undecidable*, whereas in **Ti**, **Df** and **R** it has a positive solution. This is due to the fact that **Df**, **Ti** and **R** are defined to be exactly the classes in which certain e-concepts are definable. Namely, **Df** is the class of frames in which equality is internally definable, **Ti** is the class in which accessibility is internally definable and **R** the class in which both equality and accessibility are internally definable.

Conjecture 3.5.1 *Let* S *be a set of* $\mathcal{L}^e$*-formulae and* $\boldsymbol{X} = \{\mathcal{G} : \mathcal{G} \models (\forall\overline{p})(\forall\overline{w})(\alpha(\overline{p},\overline{w}) \equiv (\exists\overline{q})(\overline{w} \in \overline{Q}^{\alpha}), \alpha \in S\}$*; in other words,* $\boldsymbol{X}$ *is the class of frames in which the concepts of* S *are internally describable. If* $\boldsymbol{X} \supseteq \boldsymbol{K}$*, then* $\overline{Q}$ *describes a concept in* $\boldsymbol{X}$ *iff* $\overline{Q}$ *can be derived from* $\{\overline{Q}^{\alpha} : \alpha \in S\}$ *with the following rules:*

(t)	$\langle\top\rangle$	(f)	$\langle\bot\rangle$
(exp)	$\overline{Q}\ /\ \overline{Q}\star\top$	(per)	$\overline{Q}\ /\ \pi(\overline{Q})$
(swa)	$\overline{Q},(\exists\tau)(\forall i)\vdash_K p_i\leftrightarrow\tau\sigma p_i\ /\ \sigma\overline{Q}$	(con)	$\overline{Q},\overline{R}\ /\ \overline{Q}\star\overline{R}$
(dis)	$\overline{Q}\star R^1,\overline{Q}\star R^2\ /\ \overline{Q}\star R^1\vee R^2$	(ex)	$\overline{Q}\star R\star S\ /\ \overline{Q}\star R\wedge\Diamond S$

A partial answer can be collected from [Lachlan, 1974] who proved that if **X** is characterized by a set of universal e-sentences then the set of **X**-persistent modal formulae is recursively enumerable. Obviously, the classes defined in the above conjecture are defined by universal e-sentences. If this conjecture is correct and if **X** is a class defined by a set S of p-quantifier free $\mathcal{L}^e$-formulae in the above way i.e. if **X** is the class of frames in which at least the concepts $[\![\alpha]\!]$ for $\alpha\in S$ are describable, then C is internally describable or definable only if it is F-invariant and therefore corresponds to a quasi-elementary concept. Specifically, a concept C is internally definable in **R** iff $C \underset{\mathrm{R}}{\leftrightsquigarrow} [\![\alpha]\!]$ for a universal and positive α.

Chapter 4

Sahlqvist's Theorem

4.1 The Theorem

In [Sahlqvist, 1975] the following theorem is proved

Theorem 4.1.1 (Sahlqvist) *Let T be a modal formula which is equivalent to a conjunction of formulae of the form $\Box^m(T_1 \to T_2)$ where $m \in \omega$, T_2 is positive and T_1 is obtained from propositional variables and constants in such a way that no positive occurrence of a variable is in a subformula of the form $U_1 \vee U_2$ or $\Diamond U_1$ within the scope of some $\Box$. Then T is effectively equivalent to a first-order formula and $K(T)$ is $\boldsymbol{D}$-persistent.*

We have stated the theorem roughly as in [Sambin and Vaccaro, 1988] whom we will follow in calling formulae of the described type **Sahlqvist** formulae. We will denote the set of Sahlqvist formulae by Σ_m and the set of first-order counterparts Σ_e. Here we will give a new proof of this theorem using the theory of internal descriptions and some ideas from [Sambin and Vaccaro, 1988]. The advantage of our proof over the previous ones is that it spells out in exact detail the inductive principles valid in the class of descriptive and Kripke-frames. Σ_e just happens to be the class of world-concepts generated from $w = x$ in $\mathbf{K} \cup \mathbf{D}$ via these principles. As it turns out, Σ_e is not the largest class of world-concepts describable in $\mathbf{K} \cup \mathbf{D}$. And so via the principles we can generate even more describable concepts.

4.2 Esakia's Lemma

Call variables, $\top, \bot$ **atoms**. Call a modal proposition P **positive** if it is built from atoms with the help of $\vee, \wedge, \Diamond$ and $\Box$ only; call Q **negative** if it is built from negated atoms with the help of $\vee, \wedge, \Diamond$ and $\Box$ only. Thus, substituting negated variables for variables turns a positive formula into a negative one and (modulo the equivalence $\neg\neg p \leftrightarrow p$) negative formulae into positive ones. Finally, say that S is **strongly positive** if it is built from atoms with the use of $\wedge$ and $\Box$. We will reserve the letter P for positive, S for strongly positive and Q for negative formulae. Notice that positive formulae are monotone, $p \rightarrow q \vdash_K P(\ldots, p, \ldots) \rightarrow P(\ldots, q, \ldots)$ and that negative formulae are antitone, $p \rightarrow q \vdash_K Q(\ldots, q, \ldots) \rightarrow Q(\ldots, p, \ldots)$. Strongly positive formulae satisfy $\vdash_K S(\ldots, p \wedge q, \ldots) \leftrightarrow S(\ldots, p, \ldots) \wedge S(\ldots, q, \ldots)$.

The following lemma is the key to Sahlqvists theorem as is observed in [Sambin and Vaccaro, 1988].

Lemma 4.2.1 (Esakia) *Let $\mathcal{G}$ be descriptive and $\mathcal{D} = (D_i)_{i \in I}$ an upward directed family of sets in A. Then*

$$\Box \lim \mathcal{D} = \lim \Box\mathcal{D}$$

where $\Box\mathcal{D} = (\Box D_i)_{i \in I}$.

Proof. It is equivalent to show that for a downward directed family C: $\Diamond \lim C = \lim \Diamond C$. Here, the inclusion "$\subseteq$" is generally valid; let us therefore show "$\supseteq$". Let $r \in \lim \Diamond C$. Then for all $i \in I : r \in \Diamond C_i$, that is, if $A = \{s : r \lhd s\}$, $A \cap C_i \neq \varnothing$ for every $i \in I$. The family $(A \cap C_i)_{i \in I}$ has the finite intersection property. By compactness, $A \cap \lim C \neq \varnothing$ from which $r \in \Diamond \lim C$. $\dashv$

This lemma can be exploited directly for proving an induction scheme for describability in $\mathbf{K} \cup \mathbf{D}$.

Proposition 4.2.2 *Let $\langle S, Q \rangle$ describe C in $\mathbf{K}\cup\mathbf{D}$. Then if S is strongly positive and Q negative, $\langle S \wedge \Box Q \rangle$ describes $(\forall w_1 \rhd w_0)C$ in $\mathbf{K}\cup\mathbf{D}$.*

Proof. We have to show that if $\mathcal{G} \in \mathbf{K}\cup\mathbf{D}$ then $(\forall w_1 \rhd w_0) C(\langle \mathcal{G}, \iota \rangle) \Leftrightarrow \langle \mathcal{G}, \iota \rangle \models (\exists \overline{p}) w_0 \epsilon S \wedge \Box Q$ on the hypothesis that for all $\iota' \supseteq \iota$ with $w_1 \in \mathrm{dom}(\iota')$ and $\iota'(w_0) \lhd \iota'(w_1)$: $C(\langle \mathcal{G}, \iota' \rangle) \Leftrightarrow$

$\langle\mathcal{G},\iota'\rangle \models (\exists\overline{p})(w_0 \epsilon S \,\&\, w_1 \epsilon Q)$. The implication from right to left is generally valid. Thus let us assume $(\forall w_1 \rhd w_0)C(\langle\mathcal{G},\iota\rangle)$. Then for all $\iota' \supseteq \iota$ with $w_1 \in \mathrm{dom}(\iota')$ and $\iota(w_0) \lhd \iota'(w_1)$: $C(\langle\mathcal{G},\iota'\rangle)$. Let $r := \iota'(w_1)$ and $A := \{s : \iota(w_0) \lhd s\}$. By assumption, there is a γ_r such that $\iota(w_0) \in \gamma_r(S)$ and $r \in \gamma_r(Q)$. If $\mathcal{G}$ is a Kripke-frame, we can let $\delta = \bigcap\langle\gamma_r : r \in A\rangle$. Then $\delta(S) = \bigcap\langle\gamma_r(S) : r \in A\rangle$, since S is strongly positive and therefore $\iota(w_0) \in \delta(S)$. Furthermore, $\delta(Q) \supseteq \gamma_r(Q)$ for all r, since Q is negative; consequently, $r \in \delta(Q)$. This concludes the case $\mathcal{G} \in \mathbf{K}$. If $\mathcal{G} \in \mathbf{D}$, that is, if $\mathcal{G}$ is descriptive, let I be set of finite subsets of A and $\delta_i = \bigcap\langle\gamma_r : r \in i\rangle$ for $i \in I$. As before, $\iota(w_0) \in \delta_i(S)$ and $r \in \delta_i(Q)$ for all $i \in I$. Now the family $(\delta_i(Q))_{i\in I}$ is an upward directed family and thus by Esakia's Lemma and the fact that $A \subseteq \lim(\delta_i(Q))_{i\in I}$ we have $\iota(w_0) \in \Box\lim(\delta_i(Q))_{i\in I} = \lim(\Box\delta_i(Q))_{i\in I} = \lim(\delta_i(\Box Q))_{i\in I}$; and thus there is a $j \in I$ with $\iota(w_0) \in \delta_j(\Box Q)$ which proves the theorem. $\dashv$

This proposition has a straightforward generalization to concepts which are described by sequences $\overline{S} \star \overline{Q}$ of strongly positive and negative formulae. It can be generalized still further; say that S is **strongly positive in** $\mathbb{P}$ (**positive in** $\mathbb{P}$) if S is built from (i) propositions R with $\mathrm{pvar}(R) \cap \mathbb{P} = \varnothing$ and (ii) sentence letters from $\mathbb{P}$ with the help of $\wedge$ and $\Box$ ($\wedge$, $\vee$, $\Diamond$, $\Box$). S is **negative in** $\mathbb{P}$ if $\neg S$ is positive in $\mathbb{P}$. It is clear that if S is strongly positive (positive) in $\mathbb{P}$ and $p \in \mathbb{P}$ then $\vdash_K S(\ldots,p,\ldots) \wedge S(\ldots,q,\ldots) \leftrightarrow S(\ldots,p\wedge q,\ldots)$ ($p \to q \vdash_K Q(\ldots,p,\ldots) \to Q(\ldots,q,\ldots)$). Likewise, **strongly positive in p**, **positive in p** and **negative in p** for a single variable is defined.

Let us in addition make the following convention: if $\overline{T} := \langle T_\nu : \nu \in n\rangle$ describes C then the variable corresponding to the i-th strongly positive T_ν is denoted by w_i and the variable corresponding to the j-th negative T_ν by x_j.

Theorem 4.2.3 *Let $\overline{S} \star \overline{Q} \star S_\lambda \star Q_\mu$ describe C in $\boldsymbol{K}\cup\boldsymbol{D}$ and let all $S_i, i \in \lambda+1$, be strongly positive in the set of parametric variables of C and all Q_j, $j \in \mu+1$, be negative in the set of parametric variables of C; then $\overline{S} \star \overline{Q} \star S_\lambda \wedge \Box Q_\mu$ describes $(\forall x_\mu \rhd w_\lambda)C$ in $\boldsymbol{K}\cup\boldsymbol{D}$.* $\dashv$

4.3 Proof of Sahlqvist's Theorem

For the proof we now only need a small

Lemma 4.3.1 *Let $\overline{S} \star \overline{Q^i}$ describe C^i in $\boldsymbol{X}$. Then $\overline{S} \star \overline{Q^1} \wedge \overline{Q^2}$ describes $C^1 \& C^2$ in $\boldsymbol{X}$.*

Proof. Suppose $C^1(\langle\mathcal{G},\gamma,\iota\rangle)$ and $C^2(\langle\mathcal{G},\gamma,\iota\rangle)$. Then there are extensions $\gamma^1,\gamma^2 \supseteq \gamma$ such that $\iota(w_i) \in \gamma^1(S_i), \gamma^2(S_i)$ and $\iota(x_j) \in \gamma^1(Q_j^1), \gamma^2(Q_j^2)$. Then, letting $\delta = \gamma^1 \cap \gamma^2$ we get $\iota(w_i) \in \gamma^1(S_i) \cap \gamma^2(S_i) = \delta(S_i)$ as well as $\iota(x_j) \in \gamma^1(Q_j^1) \cap \gamma^2(Q_j^2) \subseteq \delta(Q_j^1) \cap \delta(Q_j^2) = \delta(Q_j^1 \wedge Q_j^2)$. ⊣

Now suppose we are given a Sahlqvist formula $\Box^m(T_1 \to T_2)$. Then T_2 is positive and T_1 is built from atoms and their negations using $\wedge, \vee, \Diamond, \Box$ in such a way that no positive occurrence of a variable is in a subformula of the form $U_1 \vee U_2$ or $\Diamond U_1$ within the scope of some $\Box$. In order to prove that this formula defines a first-order world-concept in **K**∪**D**, we prove that $\langle \Diamond^m(T_1 \wedge \neg T_2)\rangle$ describes an elementary world-concept in **K**∪**D**. Using Proposition 2.5.11 we can reduce this to showing that $\langle \top, \ldots, \top, T_1 \wedge \neg T_2\rangle$ describes an elementary concept in **K**∪**D**. Now with Propositions 2.5.6, 2.5.9 and 2.5.11 we can eliminate the outer occurrences of $\wedge$, $\vee$ and $\Diamond$. In particular, we can decompose T_1 successively into strongly positive and negative formulae. Since $\neg T_2$ is negative as well, all that is left to show is

Proposition 4.3.2 *All sequences $\overline{S} \star \overline{Q}$ of strongly positive and negative formulae describe elementary concepts in* ***K***∪***D****.*

Proof. By induction on the complexity of the sequence. The base clauses are as follows:

$\langle p, \neg p\rangle$	describes	$w_0 \neq x_0$
$\langle \top \rangle$	describes	t
$\langle \bot \rangle$	describes	f
$\langle p \rangle$	describes	t

Successively applying Expansion and Permutation we can show with the help of Proposition 2.5.8 first that $\langle p : i \in \lambda\rangle \star \langle \neg p : j \in \mu\rangle$ describes $\&\langle w_i \neq x_j : i \in \lambda, j \in \mu\rangle$ and then that $\overline{S} \star \overline{Q}$ describes an elementary concept, where the S_i are atoms and the Q_j negated atoms. Further application of Proposition 2.5.9 shows this for the case where S_i is any variable or constant and Q_j is any negated atom. Now with Proposition 2.5.9 and 2.5.11 we can build any $\overline{Q}$ consisting of atoms, $\vee$ and $\Diamond$, in other words, negations of strongly positive formulae. With Swap we can now exchange the roles of $\overline{Q}$ and $\overline{S}$ and we have thus shown that $\overline{S} \star \overline{Q}$ is elementary if all S_i are strongly positive and all Q_j negated atoms. Now a straightforward induction on the complexity of the Q_j with the Propositions 4.3.1,2.5.9, 2.5.11 and 4.2.3 yields the conclusion. ⊣

4.4 A Worked Example

The proof of the theorem makes it clear that the corresponding elementary formula can be found in an easy and mechanical way. Let us demonstrate this with an example. Take $T = \Box p \to \Diamond\Box p$. Obviously, T is a Sahlqvist formula. The corresponding elementary concept $[\![\alpha]\!]$ is derived by first decomposing $\neg T$ according to the various schemata until we reach $\langle p, \neg p\rangle$ which we translate as $w_0 \neq x_0$ and then running all the steps backwards, this time assembling α:

	$\langle \Box p \wedge \Box\Diamond\neg p\rangle$
2.5.6	$\langle \Box p, \Box\Diamond\neg p\rangle$
4.2.3	$\langle \Box p, \top, \Diamond\neg p\rangle$
2.5.11	$\langle \Box p, \top, \top, \neg p\rangle$
2.5.7	$\langle \Box\neg p, \top, \top, p\rangle$
2.5.5	$\langle \top, \top, p, \Box\neg p\rangle$
4.2.3	$\langle \top, \top, p, \top, \neg p\rangle$
2.5.4	$\langle p, \neg p\rangle$
	$w_0 \neq x_0$
2.5.4	$w_2 \neq x_1$
4.2.3	$(\forall x_1 \rhd x_0) w_2 \neq x_1$
2.5.5	$(\forall x_1 \rhd x_0) w_2 \neq x_1$
2.5.7	$(\forall w_1 \rhd w_0) x_2 \neq w_1$
2.5.11	$(\exists x_2 \rhd x_1)(\forall w_1 \rhd w_0) x_2 \neq w_1$
4.2.3	$(\forall x_1 \rhd x_0)(\exists x_2 \rhd x_1)(\forall w_1 \rhd w_0) x_2 \neq w_1$
2.5.6	$(\forall x_1 \rhd w_0)(\exists x_2 \rhd x_1)(\forall w_1 \rhd w_0) x_2 \neq w_1$

The numbers to the left are the numbers of the theorems used in the derivation. The resulting formula can be simplified to $(\forall x_1 \rhd w_0)(\exists x_2 \rhd x_1) w_0 \not\lhd x_2$. Thus, $\Box p \to \Diamond\Box p$ corresponds to $(\forall w)(\exists x \rhd w)(\forall y \rhd x) w \lhd y$.

There also is the much more complicated problem of describing Σ_e, the class of elementary formulae corresponding to Sahlqvist formulae. We have not been able to give a full characterization of Σ_e. However, if we say that $\alpha(\overline{x}, \overline{w})$ is a **property** of $\overline{x}$ if α is composed from atomic formulae of type $x_i \lhd^k x_j$ and $x_i \lhd^k w_j$ by means of the logical connectives and quantifiers and if we call $\mathbb{W} \subseteq \mathrm{var}(\alpha)$ the **set of guards** for α (α in prenex normal form) if $\mathbb{W}$ is maximal such that every variable from $\mathbb{W}$ is either free in α or bound by a restricted universal quantifier with a restrictor from $\mathbb{W}$; then we have the

Theorem 4.4.1 *If $\alpha \in \Sigma_e$ then $\alpha \dashv\vdash \forall x.\beta$ where (i) β is positive and restricted and (ii) every subformula of β is a property of the set of guards of β.*

Proof. By induction it is shown that $\overline{Q} \star \overline{S}$ describes $\sim\beta$ for a β satisfying (i) and (ii) with respect to the set $\mathbb{W}$ of w variables, i.e. the variables corresponding to the $\overline{S}$ formulae. If we now apply Propositions 2.5.8,2.5.9, 2.5.6 and 2.5.11, (ii) remains valid with respect to $\mathbb{W}$. It turns out that $\mathbb{W}$ is exactly the set of guards of β. $\dashv$

4.5 Some Generalizations

We can at no extra costs generalize Sahlqvist's theorem in two ways. First, we allow any number of modal operators denoted as in dynamic logic by $[i]$, $i \in \alpha$. Secondly, we say that T is **regular in** $\mathbb{P}$ if $T = \overline{B}(T_1 \to T_2)$ where $\overline{B}$ is a prefix consisting of any number of boxes $[i]$ and T_2 is positive in $\mathbb{P}$ and T_1 is built from (i) propositions R with $\mathrm{pvar}(R) \cap \mathbb{P} = \varnothing$, (ii) atoms in such a way that no positive occurrence of a variable from $\mathbb{P}$ is in a subformula $U_1 \vee U_2$ or $\langle i \rangle U_1$ within the scope of some $[j]$.

Theorem 4.5.1 *Suppose that T is regular in* $\mathrm{var}(T) - \mathbb{P}$. *Then in* $\boldsymbol{K}\cup\boldsymbol{D}$ *T defines an e-concept $C_{\mathbb{P}}$ with $\mathbb{P}$ the set of main variables.* $\dashv$

For example take $T = p \wedge r \to \Diamond(p \wedge \neg r)$. T is regular in $\{p\}$ and thus corresponds to a concept $C_{\{r\}}$ with parameter p. The corresponding e-concept is determined as above:

	$\langle p \wedge r \wedge \Box(\neg p \vee r)\rangle$
2.5.6	$\langle p, r, \Box(\neg p \vee r)\rangle$
4.2.3	$\langle p, r, \top, \neg p \vee r\rangle$
2.5.9	$\langle p, r, \top, \neg p\rangle$ or $\langle p, r, \top, r\rangle$
2.5.8	$\langle p, \top, \top, \neg p\rangle$ and $\langle \top, r, \top, \top\rangle$ or $\langle p, \top, \top, \top\rangle$ and $\langle \top, r, \top, \top\rangle$ and $\langle \top, \top, \top, r\rangle$
	$w_0 \neq x_2$ and $x_0 \epsilon r$ or t and $x_0 \epsilon r$ and $x_2 \epsilon r$
2.5.8	$w_0 \neq x_2 \& x_0 \epsilon r$ or $x_0 \epsilon r \& x_2 \epsilon r$
2.5.9	$x_0 \epsilon r \& (w_0 \neq x_2 \vee x_2 \epsilon r)$
4.2.3	$(\forall x_2 \rhd x_1)(x_0 \epsilon r \& (w_0 \neq x_2 \vee x_2 \epsilon r))$
2.5.6	$(\forall x_2 \rhd w_0)(w_0 \epsilon r \& (w_0 \neq x_2 \vee x_2 \epsilon r))$
	$\equiv (\forall x \rhd w)(w \epsilon r \& (w \neq x \vee x \epsilon r))$

Hence, T defines $(\exists w \rhd x)(w \in \neg r \& w = x \vee x \in \neg r)$. Of course, if T is regular in $\mathbb{P}$ and $Y \subseteq \mathbb{P}$ then T is also regular in Y and thus corresponds to an e-concept with parameter set Y. In the present case the possibility $Y = \varnothing$ is open and thus there is an e-concept defined by $p \wedge r \to \Diamond(p \wedge \neg r)$ with empty parameter set. This concept is nothing but $w_0 \in p \wedge r \to \Diamond(p \wedge \neg r)$.

Theorem 4.5.1 has various applications. It has become necessary especially in dynamic logic to work with logics which have propositional constants such as **loop** and **fail**. For such logics we can note the

Corollary 4.5.2 *Let Θ be a modal logic with propositional constants. If Θ is axiomatized by a set of axioms regular in all occurring* ***variables*** *then Θ is elementary and locally* ***D****-persistent.* ⊣

A nontrivial application of this result is to show that the logic of knowledge representation systems designed in [Vakarelov, 1988] is **D**-persistent and therefore complete.

4.6 The Converse of Sahlqvist's Theorem does not Hold

Sahlqvist's theorem is not the best possible result. [Fine, 1975] shows that $\mathsf{S4.1} = \mathbf{K}(p \to \Diamond p, \Diamond^2 p \to \Diamond p, \Box\Diamond p \to \Diamond\Box p)$ is **D**-persistent and the class of Kripke-frames determined by these axioms is exactly the class determined by the first-order conditions

(r) $(\forall w)(w \lhd w)$

(t) $(\forall w)(\forall x \rhd^2 w)(w \lhd x)$

(m) $(\forall w)(\exists x \rhd w)(\forall y \rhd x)(x = y)$

But it is easily checked that McKinsey's axiom $\Box\Diamond p \to \Diamond\Box p$ is not a Sahlqvist formula. In fact, it is known that it does not correspond to an elementary frame-concept ([Goldblatt, 1975]) and recently it has been shown not to be canonical either in [Goldblatt, 1991]. Only in the presence of reflexivity and transitivity does it reduce to a first-order condition. [Fine, 1975] also shows that $\mathsf{S4.1}$ is not **R**-persistent. But, as we will see later, **S4.1** is **Rf**-persistent, if **Rf** denotes the class of finitely generated, refined frames. This follows from the discussions of chapter 7 where we will show that the world-concept $(\exists y \rhd x)(\forall z \rhd y)(z = y)$ corresponds to $\Box\Diamond p \to \Diamond\Box p$ in $\mathbf{Rf}^\bullet$.

In another sense the result by Sahlqvist is not optimal. It is not the case that the intersection of two logics axiomatized by Sahlqvist formulae is again a logic axiomatized by Sahlqvist formulae. But it can be shown that the intersection is again complete and **C**-persistent. For the class of definable world-concepts is closed under disjunctions. And it is known that a logic is globally Σ-elementary iff it is locally Σ-elementary and that a logic which is complete and Σ-elementary is also **C**-persistent ([Fine, 1975]). Now let Λ_1 and Λ_2 be two complete and Σ-elementary logics. Then their intersection $\Lambda_1 \cap \Lambda_2$ is complete; it is also Σ-elementary since both Λ_1 and Λ_2 are locally Σ-elementary and therefore $\Lambda_1 \cap \Lambda_2$ is locally Σ-elementary and so Σ-elementary. We have thus shown that the class of complete and Σ-elementary logics is closed with respect to arbitrary unions and finite intersections. Since Sahlqvist logics are both elementary and **C**-persistent we have the

Theorem 4.6.1 *Any finite intersection of Sahlqvist logics is **C**-persistent, Σ-elementary and complete.* $\dashv$

Part II

Completeness

Chapter 5

Logics from the Drawing-Board

5.1 Sketches

Reserve a world variable v_p for each propositional letter p. Now take a world variable v which is distinct from all variables v_p. Define a translation ${}^{\flat(v)}$ from modal formulae into first-order formulae as follows:

$$\begin{array}{lll}
(P \wedge Q)^{\flat(v)} & \leftrightharpoons & P^{\flat(v)} \,\&\, Q^{\flat(v)} \\
(\neg P)^{\flat(v)} & \leftrightharpoons & \sim P^{\flat(v)} \\
(\Diamond P)^{\flat(v)} & \leftrightharpoons & (\exists w \rhd v) P^{\flat(w)} \quad w \notin \mathrm{fwvar}(P^{\flat(v)}) \\
p^{\flat(v)} & \leftrightharpoons & v = v_p
\end{array}$$

This translation results in a type of first-order formula which we will call a *slim* formula. A precise definition will follow. If f is a Kripke-frame and ι a valuation on the v_p and if we let $\gamma(p) = \{\iota(v_p)\}$ for all p then

$$\langle f, \gamma, \iota(v) \rangle \models P \Leftrightarrow \langle f, \iota \rangle \models P^{\flat(v)}$$

However, from this it does not follow that

$$f \models P \Leftrightarrow f \models (\forall v) P^{\flat(v)}$$

This deplorable fact is a consequence of a result by [Goldblatt, 1975] that $\Box\Diamond p \to \Diamond\Box p$ does not define an elementary condition on Kripke-frames. Nevertheless, despite its dangers the illusion that propositions are really just worlds is quite useful when assessing the geometrical meaning of axioms. *Sketches* are designed to exploit the merits of this illusion.

A **slim** formula is an f-formula $\alpha(\overline{w})$ which is composed from atomic formulae and their negations using $\wedge, \vee$ and restricted quantifiers in such a way that (i) $w_i = w_j$, $w_i \lhd w_j$ with $w_i, w_j \in \mathrm{fwvar}(\alpha)$ are not bound by a quantifier and (ii) every subformula β is a property of the free variables of α (see the definition preceding Theorem 4.4.1) and $\sharp(\mathrm{fwvar}(\beta) - \mathrm{fwvar}(\alpha)) \leq 1$. $\phi^{\flat(v)}$ is slim. On the other hand, slim formulae can be translated into modal formulae. This is done by

$$
\begin{array}{lcll}
(\alpha \& \beta)^\sharp & \leftrightharpoons & \alpha^\sharp \wedge \beta^\sharp & \\
(\alpha \vee \beta)^\sharp & \leftrightharpoons & \alpha^\sharp \vee \beta^\sharp & \\
((\exists w \rhd v)\alpha)^\sharp & \leftrightharpoons & p_v \to \Diamond\alpha^\sharp & \text{if } v \in \mathrm{fwvar}(\alpha) \\
 & \leftrightharpoons & \Diamond\alpha^\sharp & \text{else} \\
((\forall w \rhd v)\alpha)^\sharp & \leftrightharpoons & p_v \to \Box\alpha^\sharp & \text{if } v \in \mathrm{fwvar}(\alpha) \\
 & \leftrightharpoons & \Box\alpha^\sharp & \text{else} \\
(v \lhd^k w)^\sharp & \leftrightharpoons & p_v \to \Diamond^k p_w & \text{if } v \in \mathrm{fwvar}(\alpha) \\
 & \leftrightharpoons & \Diamond^k p_w & \text{else} \\
(v \ntriangleleft^k w)^\sharp & \leftrightharpoons & p_v \to \neg\Diamond^k p_w & \text{if } v \in \mathrm{fwvar}(\alpha) \\
 & \leftrightharpoons & \neg\Diamond^k p_w & \text{else}
\end{array}
$$

where for each world variable w a distinct proposition variable p_w is reserved. $(\phi^{\flat(v)})^\sharp$ and ϕ are mostly not identical.

A **sketch** is a quadruple $\Sigma = \langle \mathbb{W}, \prec, r; \alpha\rangle$ where $\mathbb{W}$ is a finite set of w-variables, $\langle \mathbb{W}, \prec, r\rangle$ an irreflexive, intransitive tree with root r and α a slim formula with $\mathrm{fwvar}(\alpha) \subseteq \mathbb{W}$. A map $\sigma : \mathbb{W} \longrightarrow \mathcal{G}_+$ **realizes** Σ in $\mathcal{G}$ if for $\gamma_\sigma : p_v \mapsto \sigma(v)$ there is an $s \in \mathcal{G}_\sharp$ such that

$$
\begin{array}{lll}
(skr) & \langle \mathcal{G}, \gamma_\sigma, s\rangle \models & p_r; \{\Box^{(k)} \bigwedge \langle p_v \to \Diamond p_w : v \prec w\rangle : k \in \omega\}; \\
 & & \{\Box^{(k)}\alpha^\sharp : k \in \omega\}
\end{array}
$$

If we abbreviate $\bigwedge\langle p_v \to \Diamond p_w : v \prec w\rangle \wedge \alpha^\sharp$ by $\Sigma^\sharp$ then this can be stated more succinctly as

$$(skr) \quad \langle \mathcal{G}, \gamma_\sigma, s\rangle \models p_r; \{\Box^{(k)}\Sigma^\sharp : k \in \omega\}$$

If Σ cannot be realized in $\mathcal{G}$, $\mathcal{G}$ is said to **omit** Σ. $\mathcal{G}$ omits Σ iff whenever $\langle \mathcal{G}, \gamma, s\rangle \models \{\Box^{(k)}\Sigma^\sharp : k \in \omega\}$ for some γ, s then $\langle \mathcal{G}, \gamma, s\rangle \models \neg p_r$. Similarly, realization and omission of sketches are defined for algebras. We call a sketch Σ **elementary** in **X** if "$\mathcal{G}$ omits Σ" corresponds to an elementary concept in **X**. We emphasise here once more that although α is elementary, the sketch associated with it is not, because a world-variable is represented by a proposition variable.

5.2 Sketch–Omission Logics

Given a variety $\mathcal{V}$ of algebras and a sketch Σ there is a natural question as to whether the class $\mathcal{V}_\Sigma$ of all algebras of $\mathcal{V}$ omitting Σ is a variety or not. Here we have the interesting result that $\mathcal{V}_\Sigma$ is a variety iff it is closed under ultraproducts, generalizing a theorem first proved in [Kracht, 1990a].

Theorem 5.2.1 *The following are equivalent:*

(i) $\mathcal{V}_\Sigma$ *is a variety.*

(ii) $\mathcal{V}_\Sigma$ *is closed under ultraproducts.*

(iii) There is a $k \in \omega$ *such that for all* $\mathcal{A} \in \mathcal{V}$ *and all* $\ell \in \omega$*, if* $\langle \mathcal{A}, \gamma, U\rangle \models p_r \wedge \Box^{(k)}\Sigma^\sharp$ *for some* γ, U*, there are* γ', U' *such that* $\langle \mathcal{A}, \gamma', U'\rangle \models p_r \wedge \Box^{(\ell)}\Sigma^\sharp$.

In that case $\mathcal{V}_\Sigma$ *is the class of all algebras of* $\mathcal{V}$ *satisfying* $\Box^{(k)}\Sigma^\sharp \to \neg p_r$.

Proof. $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$ Suppose, no such k exists. Then for all k there is a $\mathcal{A}_k$ and a $\ell > k$ such that $\langle \mathcal{A}_k, \gamma_k, U_k\rangle \models p_r \wedge \Box^{(k)}\Sigma^\sharp$ for some γ_k, U_k but $\mathcal{A}_k \models \Box^{(\ell)}\Sigma^\sharp \to \neg p_r$. Then clearly $\mathcal{A}_k$ omits Σ. But if U is a nonprincipal ultrafilter over ω, then $\prod_U \langle \mathcal{A}_k, \gamma_k, U_k\rangle \models p_r; \{\Box^{(k)}\Sigma^\sharp : k \in \omega\}$ showing that $\prod_U \mathcal{A}_k$ does not omit Σ. This contradicts (ii).

$(iii) \Rightarrow (i)$ If such a k exists, then $\mathcal{A}$ omits Σ iff $\mathcal{A} \models \Box^{(k)}\Sigma^\sharp \to \neg p_r$ for that k. For under that condition we have for all $\mathcal{A} \in \mathcal{V}$: $\mathcal{A} \not\models \Box^{(k)}\Sigma^\sharp \to \neg p_r$ iff for all $\ell \in \omega$ $\mathcal{A} \not\models \Box^{(\ell)}\Sigma^\sharp \to \neg p_r$. Thus, $\mathcal{V}_\Sigma$ is a variety. ⊣

Corollary 5.2.2 *If $\mathcal{V} \models \Box^{(k)} p \to \Box^{(k+1)} p$ then for every Σ $\mathcal{V}_\Sigma$ is a variety.* $\dashv$

If Λ is a modal logic and $\mathcal{V} = \mathcal{V}(\Lambda)$ and if Σ is a sketch satisfying either of (i)—(iii) in Theorem 5.2.1, then by Λ_Σ we denote the logic $\mathrm{Th}(\mathcal{V}_\Sigma)$ of all Λ-frames omitting Σ. For some $k \in \omega$, $\Lambda_\Sigma = \Lambda(\Box^{(k)} \Sigma^\sharp \to \neg p_r)$. Of course, several sketches can be omitted in succession. An interesting case is provided when $\Sigma_i = \langle \mathbb{W}, \prec, r; \alpha_i \rangle$, $i \in n$, are all based on the same tree $\langle \mathbb{W}, \prec, r \rangle$. If all Σ_i can individually be omitted from $\mathcal{V}$, then for the disjunction of all sketches $\Sigma = \langle \mathbb{W}, \prec, r; \alpha \rangle$ with $\alpha = \bigvee \langle \alpha_i : i \in n \rangle$ we get

$$\begin{aligned} \mathcal{V}_\Sigma &= \bigcap \langle \mathcal{V}_{\Sigma_i} : i \in n \rangle \\ \Lambda_\Sigma &= \bigsqcup \langle \Lambda_{\Sigma_i} : i \in n \rangle \end{aligned}$$

We will call a logic Λ a **sketch-omission logic** if Λ can be obtained from K by iterated omission of sketches. Sketch-omission is an extremely finetuned tool for classifying logics as will be seen below. In addition to the examples that will follow we add that the standard modal logics $\mathsf{K4}$, KB and KT are sketch-omission logics using the following sketches

$$\begin{tikzcd} \overset{r}{\square} \arrow[r] & \square \arrow[r] & \overset{v}{\square} \end{tikzcd} \qquad r \ntriangleleft v$$

$$\begin{tikzcd} \overset{r}{\square} \arrow[r] & \overset{v}{\square} \end{tikzcd} \qquad v \ntriangleleft r$$

$$\overset{r}{\square} \qquad r \ntriangleleft r$$

In fact, all positive universal logics are sketch-omission logics. A more elaborate example is McKinsey's logic $\mathsf{KM} = \mathsf{K}(\Box\Diamond p \to \Diamond\Box p)$. It is obtained by omitting the sketch

$$\begin{tikzcd} & & \overset{w}{\square} \\ \overset{r}{\square} \arrow[r] & \overset{v}{\square} \arrow[ur] \arrow[dr] & \\ & & \underset{x}{\square} \end{tikzcd} \qquad x \neq w \& (\forall y \rhd r)(y = v)$$

For suppose this sketch is realized in $\mathcal{G}$ by σ. Then there is a point $r \in \gamma_\sigma(p_r)$ with $\langle \mathcal{G}, \gamma_\sigma, r \rangle \models \Box p_v$ and

$$\langle \mathcal{G}, \gamma_\sigma, r\rangle \models \Box(p_v \to \Diamond p_w), \Box(p_v \to \Diamond p_x), \Box^2(p_x \to \neg p_w)$$

whence $\langle \mathcal{G}, \gamma_\sigma, r\rangle \models \Box\Diamond p_w \wedge \Box\Diamond\neg p_w$. Conversely, if $\langle \mathcal{G}, \gamma, r\rangle \models \Box\Diamond p \wedge \Box\Diamond\neg p$ then $\sigma : r \mapsto \Box\Diamond p \wedge \Box\Diamond\neg p, v \mapsto \Diamond p \wedge \Diamond\neg p, w \mapsto p, x \mapsto \neg p$ clearly realizes the above sketch.

5.3 Subframe Logics as Sketch-Omission Logics

[Fine, 1985] has introduced the notion of a *subframe*. If $\langle g, \lhd\rangle$ is a Kripke-frame and $h \subseteq g$ a subset, then $\langle h, \lhd_h\rangle$ is a **subframe** of $\langle g, \lhd\rangle$ if $\lhd_h = \lhd \cap h^2$. In that case we write $h \overset{\subset}{\to} g$. If $\mathcal{G} = \langle g, A\rangle$ and $\mathcal{H} = \langle h, B\rangle$ are (general) frames and $h \overset{\subset}{\to} g$ then $\mathcal{H}$ is a subframe of $\mathcal{G}$ if h is an internal set in B, i.e. $h \in B$, and $B = A \cap h := \{a \cap h : a \in A\}$. Again we denote this by $\mathcal{H} \overset{\subset}{\to} \mathcal{G}$. The class of subframes of $\mathcal{G}$ is denoted by $\mathrm{W}^\circ(\mathcal{G})$. $\mathcal{G}$ **subreduces** to $\mathcal{H}$ if $\mathcal{H} \in \mathrm{CW}^\circ(\mathcal{G})$.

Consider a **rooted frame** f, that is a frame which is generated by a single point which will always be denoted by r. Then f can be ordered by an irreflexive, intransitive relation such that $\prec \subseteq \lhd$. If f is finite, introduce a w-variable $\underline{w}$ for each point $w \in f$ and define

$$\begin{aligned} \alpha_f = \quad & \bigwedge\langle \underline{v} \neq \underline{w} : v \neq w\rangle \\ & \wedge \bigwedge\langle \underline{v} \lhd \underline{w} : v \lhd w\rangle \\ & \wedge \bigwedge\langle \underline{v} \not\lhd \underline{w} : v \not\lhd w\rangle \end{aligned}$$

Then with $\mathbb{W}_f = \{\underline{v} : v \in f\}$ and $\underline{v} \prec \underline{w} \Leftrightarrow v \prec w$ we get a sketch $\mathrm{SF}(f) = \langle \mathbb{W}_f, \prec, r; \alpha_f\rangle$, the **subframe sketch** for f. If a subframe sketch $\mathrm{SF}(f)$ can be omitted from Λ, we denote the logic $\Lambda_{\mathrm{SF}(f)}$ simply by Λ_f and call Λ_f a **FINE-splitting** of Λ by f. A frame $\mathcal{G}$ realizes $\mathrm{SF}(f)$ iff there is a subframe $\mathcal{H} \overset{\subset}{\to} \mathcal{G}$ such that $\mathcal{H} \twoheadrightarrow f^\sharp$. For if $\mathcal{G}$ realizes $\mathrm{SF}(f)$ then for some γ, s

$$\langle \mathcal{G}, \gamma, s\rangle \models p_r; \{\Box^{(k)}\,\mathrm{SF}(f)^\sharp : k \in \omega\}$$

Let $\mathcal{H} \overset{\subset}{\to} \mathcal{G}$ be defined as the subframe of $\mathrm{Tr}_{\mathcal{G}}(s)$ of all points satisfying a formula $p_w, w \in f$. Then again

$$\langle \mathcal{H}, \gamma, s\rangle \models p_r; \{\Box^{(k)}\,\mathrm{SF}(f)^\sharp : k \in \omega\}$$

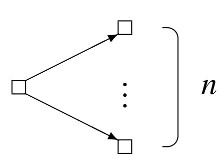
Now define $p : \mathcal{H} \twoheadrightarrow f^\sharp$ by $p_\sharp(t) = v$ iff $\langle \mathcal{H}, \gamma, t\rangle \models p_v$. It is readily checked that p is p-morphism. Conversely, if there is a subframe $i : \mathcal{H} \overset{\subset}{\to} \mathcal{G}$ and a p-morphism $p : \mathcal{H} \twoheadrightarrow f^\sharp$ then since $\sigma : \underline{v} \mapsto \{v\}$ realizes $\mathrm{SF}(f)$ in $f^\sharp$, $p^+ \circ \sigma : \underline{v} \mapsto p^{-1}(v)$ realizes $\mathrm{SF}(f)$ in $\mathcal{H}$ and $2^i \circ p^+ \circ \sigma : p \mapsto i[p^{-1}(v)]$ realizes $\mathrm{SF}(f)$ in $\mathcal{G}$. So, $\mathcal{G}$ omits $\mathrm{SF}(f)$ iff $f^\sharp \notin \mathrm{CW}^\circ(\mathcal{G})$ iff $\mathcal{G}$ does not subreduce to $f^\sharp$.

FINE-splittings can be iterated and if $\Lambda = \mathsf{K4}_g$, then $\Lambda_h = \mathsf{K4}_g \sqcup K4_h$. Thus, for a set N of finite rooted frames we write $\mathsf{K4}_N$ for $\bigsqcup\langle \mathsf{K4}_g : g \in N\rangle$. $\mathsf{K4}_N$ is also called a FINE-splitting of $\mathsf{K4}$.

Call a sketch $\Sigma = \langle \mathbb{W}, \prec, r; \alpha\rangle$ **quantifier-free** iff α is quantifier-free and **negative** if α is negative. By an earlier remark it follows that if Σ is a quantifier-free sketch, then, as α can be written as a disjunction $\alpha = \bigvee\langle \gamma_i : i \in n\rangle$ where γ_i is a conjunction of $\underline{v} = \underline{w}$, $\underline{v} \lhd \underline{w}$ and their negations in such a way that either $\underline{v} = \underline{w}$ or its negation and either $\underline{v} \lhd \underline{w}$ or its negation occurs in γ_i for every $v, w \in f$, Σ itself is the disjunction of the sketches $\Sigma_i = \langle \mathbb{W}_f, \prec, r; \gamma_i\rangle$. But each Σ_i is a subframe sketch $\mathrm{SF}(f_i)$ for some f_i, except of course if γ_i is inconsistent, in which case it can be dropped anyway. This proves a nice theorem on quantifier-free sketches, underlining the special importance of subframe sketches.

Theorem 5.3.1 *Suppose that* $\Box^{(k)}p \to \Box^{(k+1)}p \in \Lambda$. *Then for any quantifier-free sketch* Σ, Λ_Σ *is the union of FINE-splittings of* Λ. $\dashv$

For example, logics of width n are defined by $\Sigma = \langle\{w_i : i \in n+2\}, \prec, w_0; \alpha\rangle$ with $w_0 \prec w_i, 0 \in i \in n+2$, and $\alpha = \&\langle w_i \not\lhd w_j : 0 \in i \in j\rangle.\&.\&\langle w_i \neq w_j : i \neq j\rangle$. Here, α underspecifies the frame as regards $w_i \lhd w_i$ or not. Hence, $\mathsf{K4}.I_n = \mathsf{K4}_\Sigma$ is the union of FINE-splittings $\mathsf{K4}_f$ where f is of type



5.4 Splittings

Let f be a rooted finite frame. As before we can define a tree $\langle \mathbb{W}_f, \prec, \underline{r}\rangle$ based on f. But we now let

$$
\begin{array}{rll}
\beta_f := & & \bigwedge\langle \underline{v} \neq \underline{w} : v \neq w\rangle \\
& \wedge & \bigwedge\langle \underline{v} \lhd \underline{w} : v \lhd w\rangle \\
& \wedge & \bigwedge\langle \underline{v} \ntriangleleft \underline{w} : v \ntriangleleft w\rangle \\
& \wedge & \bigwedge\langle (\forall x \rhd \underline{v}) \bigvee\langle x = \underline{w} : v \lhd w\rangle : v \in f\rangle
\end{array}
$$

$\Sigma := \langle \mathbb{W}_f, \prec \underline{r}; \beta_f\rangle$ is called a **frame sketch**. If Σ meets the condition of Theorem 5.2.1, the frame f is said to **split** the variety $\mathcal{V}$ or the corresponding logic Λ. In that case we write Λ/f for Λ_Σ and $\mathrm{Sp}(f)$ for the splitting axiom $\Box^{(k)}\Sigma^\sharp \to \neg p_r$. It can be checked as in the case of the subframe logics that the concept "$\mathcal{G}$ omits Σ" corresponds to the concept "$f^\sharp \notin \mathrm{CW}(\mathcal{G})$". In contrast to the subframe conditions, splittings express a purely algebraic condition on the frame. In the study of the lattice of nomal extensions of $\mathsf{K4}$, splittings have provided an elegant tool for analysis (see [Blok, 1978], [Fine, 1974a], [Rautenberg, 1977] and [Rautenberg, 1980]). Since the algebraic theory of splittings has been developed already in [Kracht, 1990a], we will not expand on this theme any further.

5.5 Differentiation Sketches

Define a *differentiation sketch* on a finite, rooted $\mathsf{S4}$-frame f as follows. First, say that a pair $\langle A, B\rangle$ of subsets of f is a **disjunctive domain** if A and B each consists of pairwise incomparable elements and no $v \in A$ precedes a $w \in B$ and whenever $x \in f$ precedes all elements of A it precedes some element of B. Let Δ be a set of disjunctive domains over f. Δ is then finite and we can define the **differentiation sketch** $D(f, \Delta)$ with respect to f and Δ by

$$
\begin{aligned}
\gamma_{f,\Delta} \quad = \quad & \bigwedge\langle \underline{v} \neq \underline{w} : v \neq w\rangle \\
& \wedge \bigwedge\langle \underline{v} \lhd \underline{w} : v \lhd w\rangle \\
& \wedge \bigwedge\langle \underline{v} \ntriangleleft \underline{w} : v \ntriangleleft w\rangle \\
& \wedge(\forall x \rhd \underline{r}) \bigvee\langle x \lhd \underline{v} : v \in f\rangle \\
& \wedge \bigwedge\langle(\forall x \rhd \underline{r})(\&\langle x \lhd \underline{v} : v \in A\rangle. \supset . \bigvee\langle x \lhd \underline{w} : w \in B\rangle) : \langle A, B\rangle \in \Delta\rangle
\end{aligned}
$$

Then if $\langle \mathbb{W}_f, \prec, \underline{r}\rangle$ is a tree based on f, $D(f, \Delta) := \langle \mathbb{W}_f, \prec, \underline{r}; \gamma_{f,\Delta}\rangle$.

Theorem 5.5.1 (Zakhar'yashchev) *For every P there are finitely many differentiation sketches $D(i)$, $i \in n$, such that $\mathsf{S4}(P) = \bigsqcup\langle \mathsf{S4}_{D(i)} : i \in n\rangle$. Consequently, every extension of $\mathsf{S4}$ is a sketch-omission logic.* $(\dashv)$

This theorem illustrates the power of the sketch-omission method. It might be reasonable to expect that perhaps all modal logics or at least all transitive logics are sketch-omission logics.

Chapter 6

The Structure of Finitely Generated K4-Frames

6.1 Localization

If $j : h \rightarrowtail g$ is a generated subframe, there exists a natural homomorphism $j^\sharp : \langle h, 2^h \rangle \rightarrowtail \langle g, 2^g \rangle$ and by duality an epimorphism $(j^\sharp)_+ : 2^g \twoheadrightarrow 2^h$ defined by $a \mapsto a \cap h$. If, however, $\rho : h \overset{\subset}{\to} g$ is not a generated subset, the induced map $a \mapsto a \cap h$ still is a boolean homomorphism but no longer a homomorphism between the modal algebras. For it is easily seen that if $2^g = \langle 2^g, \Box \rangle$ and $2^h = \langle 2^h, \blacksquare \rangle$ then $\blacksquare\rho(a) = \rho(\Box(\rho(1) \to a))$ and $\blacklozenge\rho(a) = \rho(\Diamond(\rho(1) \wedge a))$. Now let $\gamma : \mathbb{P} \longrightarrow 2^g$ be a valuation and $\delta = \rho \circ \gamma : \mathbb{P} \longrightarrow 2^h$. Then if $\delta(Q)$ is also interpreted as the intersection of an element of 2^g with $\rho(1)$, it can be computed inductively via

$$
\begin{array}{llcl}
(\delta p) & \delta(p) & = & \gamma(p) \cap \rho(1) \\
(\delta\wedge) & \delta(P \wedge Q) & = & \delta(P) \cap \delta(Q) \\
(\delta\neg) & \delta(\neg P) & = & (\backslash \delta(P)) \cap \rho(1) \\
(\delta\Box) & \delta(\Box P) & = & \Box(\rho(1) \to \delta(P)) \cap \rho(1)
\end{array}
$$

Hence if we define $P \!\downarrow\! Q$ by

$$
\begin{array}{llcl}
(\downarrow p) & p\downarrow Q & \leftrightharpoons & p \wedge Q \\
(\downarrow \wedge) & (P_1 \wedge P_2)\downarrow Q & \leftrightharpoons & (P_1\downarrow Q) \wedge (P_2\downarrow Q) \\
(\downarrow \neg) & (\neg P)\downarrow Q & \leftrightharpoons & Q \wedge \neg(P\downarrow Q) \\
(\downarrow \Box) & (\Box P)\downarrow Q & \leftrightharpoons & Q \wedge \Box(Q \to P\downarrow Q)
\end{array}
$$

and if $h = \rho(1) = \gamma(Q)$ then one has $\delta(P) = \gamma(P \downarrow Q)$. Furthermore, $\mathrm{dg}(P \downarrow Q) = \mathrm{dg}(P) + \mathrm{dg}(Q)$. This leads us to a result first noted in [Kracht, 1990a].

Lemma 6.1.1 *If $\mathcal{G} \overset{\subset}{\to} \mathcal{H}$ and if $\gamma : \mathbb{P} \longrightarrow A$ and $\delta : \mathbb{P} \longrightarrow B$ are such that $\delta(p) = \gamma(p) \cap h = \gamma(p) \cap \gamma(Q)$ then*

$$\langle \mathcal{G}, \gamma\rangle \models s \epsilon P\downarrow Q \Leftrightarrow s \in h \text{ and } \langle \mathcal{H}, \delta\rangle \models s \epsilon P \dashv$$

It is easy to see that if $\mathcal{G}$ is differentiated (tight, refined) then so is any subframe $\mathcal{H}$ of $\mathcal{G}$. However, if $\mathcal{G}_+$ is generated by $\{a_i : i \in k\}$ then it is not necessarily the case that $\{a_i \cap h : i \in k\}$ generates $\mathcal{H}_+$. $\mathcal{H}_+$ may even not be k-generated at all. This is easily demonstrated with $\mathcal{H} := \mathcal{G}^{>d}$ the subframe of points of depth $\not\leq d$.

Now if $\langle \mathcal{G}, \gamma, \iota\rangle \models \forall w(w \epsilon P \equiv \eta(w))$ and $\langle \mathcal{G}, \gamma, \iota\rangle \models \forall w(w \epsilon Q \equiv \xi(w))$ then $\langle \mathcal{G}, \gamma, \iota\rangle \models \forall w(w \epsilon (P \downarrow Q). \equiv .\eta \downarrow \xi(w))$ for a suitably defined localization $\eta \downarrow \xi$ of η by ξ. This is accomplished by the definitions

$$
\begin{array}{llcl}
(\downarrow v) & x \epsilon p\downarrow\xi & \leftrightharpoons & x \epsilon p \;\&\; \xi(x) \\
(\downarrow =) & x = y\downarrow\xi & \leftrightharpoons & x = y \;\&\; \xi(x) \;\&\; \xi(y) \\
(\downarrow \lhd) & x \lhd y\downarrow\xi & \leftrightharpoons & x \lhd y \;\&\; \xi(x) \;\&\; \xi(y) \\
(\downarrow \&) & (\eta_1 \;\&\; \eta_2)\downarrow\xi & \leftrightharpoons & (\eta_1\downarrow\xi) \;\&\; (\eta_2\downarrow\xi) \\
(\downarrow \sim) & (\sim \eta)\downarrow\xi & \leftrightharpoons & \xi \;\&\; \sim(\eta\downarrow\xi) \\
(\downarrow \exists) & (\exists y \rhd x)\eta\downarrow\xi & \leftrightharpoons & \xi(x) \;\&\; (\exists y \rhd x)(\xi(y) \;\&\; \eta\downarrow\xi)
\end{array}
$$

Notice that while η might be any e-formula, ξ has to contain exactly one free variable. Observing the equivalences $(d1), (d\wedge), (d\neg)$ and $(d\Box)$ we can now localize any e-formula η to a set of points having an e-definable property ξ. We read "$\eta\downarrow\xi$" as "η within ξ".

Theorem 6.1.2 (Localization) *Let $\mathcal{G} \overset{\subset}{\to} \mathcal{H}$ be frames and $\langle \mathcal{G}, \gamma, \iota\rangle$ be a triple. Suppose that there is an e-formula $\xi(w)$ which defines h in $\langle \mathcal{G}, \gamma, \iota\rangle$ i.e.*

(i) $\langle \mathcal{G}, \gamma, \iota\rangle \models \xi(s)$ iff $s \in h$
(ii) $im(\iota) \subseteq h$.

Then if $\delta = \gamma \cap h$ *we have*

$$\langle \mathcal{G}, \gamma, \iota \rangle \models (\eta {\downarrow} \xi)(s) \text{ iff } s \in h \text{ and } \langle \mathcal{H}, \delta, \iota \rangle \models \eta(s).$$

In particular, if $\langle \mathcal{G}, \gamma, \iota \rangle \models \forall w(\xi(w) \equiv x \in Q)$ *then*

$$\langle \mathcal{G}, \gamma, \iota \rangle \models (\forall w)[(w \in P){\downarrow}(w \in Q) \equiv w \in (P {\downarrow} Q))]. \dashv$$

6.2 Depth defined

Let f be a Kripke-frame. $s \in f$ is said to be of **depth** 1 – in symbols $d_f(s) = 1$—if for all $t \in \mathrm{Tr}_f(s) : s \in \mathrm{Tr}_f(t)$. s is said to be of **depth** $d + 1$ if for all $t \in \mathrm{Tr}_f(s)$ either $s \in \mathrm{Tr}_f(t)$ or t is of depth $\leq d$, but s is not itself of depth $\leq d$. A frame f is of **depth** d if it contains a point of depth d but no point of depth $> d$. We write $d(f) = d$ in that case. The world-concepts "is of depth d" or "is of depth $\leq$ d" are in general not elementary. But if f is *weakly transitive*, that is, if for some $k \in \omega$

$$f \models (\forall x \rhd^{k+1} w)(w \lhd^{(k)} x)$$

the definitions reduce to

$$\begin{array}{llll} (dp1) & d_f(w) = 1 & \Leftrightarrow & (\forall x \rhd^{(k)} w)(x \lhd^{(k)} w) \\ (dp+) & d_f(w) \leq d + 1 & \Leftrightarrow & (\forall x \rhd^{(k)} w)(x \lhd^{(k)} w \vee d_f(x) \leq d). \end{array}$$

If f is transitive this simplifies to

$$\begin{array}{lll} d_f(w) = 1 & \Leftrightarrow & (\forall x \rhd w)(x \lhd w) \\ d_f(w) \leq d + 1 & \Leftrightarrow & (\forall x \rhd w)(x \lhd w \vee d_f(x) \leq d). \end{array}$$

Say that x is a **weak successor** of w if $w \lhd^{(1)} x$ i.e. $w \lhd x$ or $w = x$ and that x is a **strict successor** of w if $w \lhd x$ but $x \ntriangleleft w$. In transitive frames we have $s \in \mathrm{Tr}_f(t)$ iff s is a weak successor of t and if t is of depth d then s is a strict successor iff s is of depth $< d$. In proofs by induction on the depth of points it is mostly useful to start the induction with 0 rather than 1. Since there are no points of depth 0 we then have $d_f(w) \leq 1 \Leftrightarrow (\forall x \rhd w)(d_f(x) \leq 0 \vee x \lhd w) \Leftrightarrow (\forall x \rhd w)(f \vee x \lhd w) \Leftrightarrow (\forall x \rhd w)(x \lhd w)$

as required. Note that $d_f(w) > d$ is an existential and negative formula and $d_f(w) \leq d$ therefore universal and positive. It follows that $[\![d_f(w) \leq d]\!]$ is internally definable in the class **R4** of transitive refined frames by Proposition 2.6.1.

Now define $\Delta_d = \Delta_n[p_0, \ldots, p_{d-1}]$ by

$$\begin{array}{lll} (\Delta 0) & \Delta_0 & = \top \\ (\Delta +) & \Delta_{d+1} & = p_d \wedge \Diamond(\Box\neg p_d \wedge \Delta_d). \end{array}$$

Proposition 6.2.1 *Δ_n describes $[\![d_f(w) \nleq d]\!]$ in the class of refined transitive frames.*

Proof. For $n = 0$ this is clear. For the induction note that $\langle p_d, \Box\neg p_d\rangle$ describes $w_1 \ntriangleleft w_0$ so that $\langle p_d, \Box\neg p_d \wedge \Delta_d\rangle$ describes $w_1 \ntriangleleft w_0 \& d_f(w_1) \nleq d$ and finally $\langle p_d \wedge \Diamond(\Box\neg p_d \wedge \Delta_d)\rangle$ describes $(\exists w_1 \rhd w_0)(w_1 \ntriangleleft w_0 \& d_f(w_1) \nleq d) \equiv \sim (\forall w_1 \rhd w_0)(w_1 \lhd w_0 \vee d_f(w_1) \leq d) \equiv \sim d_f(w_0) \leq d + 1$. ⊣A final note. If f is transitive, a set $C \subseteq f$ is called a **cluster** if either $C = \{x\}$ and $x \ntriangleleft x$ or $C = \{s : t \lhd s \lhd t\}$ for some t. With respect to $\lhd$, clusters can be treated as points. So we write $C \lhd D$ if $s \lhd t$ for some $s \in C$ and $t \in D$ iff $s \lhd t$ for all $s \in C, t \in D$. A point is of depth d in f iff all points in the cluster of that point are of depth d.

6.3 The Structure of Finitely Generated K4-Frames

The study of describable concepts is strongly connected with the study of the structure of modal algebras. In the field of K4 there has been great progress through the works of [Segerberg, 1971], [Maksimova, 1975], [Blok, 1976] and [Fine, 1985]. Many very powerful completeness results can be derived from the structure theory of finitely generated K4-frames developed in these papers. We will reproduce the results here insofar as we will need them later on. Our exposition has been inspired largely by [Fine, 1985].

Say that w is of **finite depth** in f iff $d_f(w) = k$ for some $k \in \omega$. The subframe which consists of all points of depth d is denoted by $\mathcal{G}^d$ and the subframe consisting of all points of finite depth is denoted by $\mathcal{G}^{<\omega}$. Analoguously, $\mathcal{G}^{\leq d}$ and $\mathcal{G}^{>d}$ are defined. By definition, $\mathcal{G}^{<\omega} \rightarrowtail \mathcal{G}$. Call $\mathcal{G}$ **top-heavy** if every point of infinite depth has a successor of depth n for every $n \in \omega$. In the sequel we will prove the following results:

Theorem 6.3.1 (Fine) *For every transitive frame $\mathcal{G}$ there is a p-morphism $\mathcal{G} \twoheadrightarrow \mathcal{H}$ onto a top-heavy frame.*

Theorem 6.3.2 (Fine) *If $\mathcal{G}$ is finitely generated, refined and transitive then $\mathcal{G}$ is top-heavy.*

Theorem 6.3.3 (Segerberg) *$\mathcal{F}_{K4}(n)^{<\omega}$ is atomic.*

6.4 Blocks and Nets

From now on we will work over a fixed set $\mathbb{P}_k = \{p_i : i \in k\}$ of proposition letters. The question we want to address here is how to characterize a frame $\langle g, A\rangle$ which is transitive, k-generated and refined. To this end we will assume that $\gamma : \mathbb{P}_k \longrightarrow A$ is such that $[\gamma] = A$, that is, the elements $\gamma(p_i)$, $i \in k$, generate A. A pair $\langle \mathcal{G}, \gamma\rangle$ where $A = \mathcal{G}_+$ which satisfies all these conditions will henceforth be called k-**good**. Under those conditions a world-concept $[\![\alpha]\!]$ is describable in $\mathcal{G}$ if there is a proposition P with pvar$(P) \subseteq \mathbb{P}_k$ such that $\langle \mathcal{G}, \gamma, \iota\rangle \models \forall w(w \epsilon P \equiv \alpha(w))$. If P happens to describe α in all refined, transitive frames under the condition that $[\gamma] = A$, we write "$\ulcorner\alpha\urcorner = P$" and say that α **translates** into P. For example $\alpha = (\exists x \rhd w) x \epsilon p_i$ translates into $\Diamond p_i$. "$\ulcorner\alpha\urcorner = P$" is not in any sense of the definition the same as "α corresponds to P in $\mathbf{X}$" in some suitable class $\mathbf{X}$. However, there is a way to interpret this as a correspondency property. Add to $\mathcal{L}^e$ the constants $\mathbf{p}_i$, $i \in k$, and define $\mathbf{Gd}_k$ to be the class of k-good frames $\langle \mathcal{G}, \sigma\rangle$, where $\sigma : \mathbf{p}_i \mapsto a_i$. Replacing p_i in P by $\mathbf{p}_i$ we get a constant formula $\mathbf{P}$. Now α translates into P iff $[\![\alpha]\!]$ corresponds to $\mathbf{P}$ in the class $\mathbf{Gd}_k$.

A nonempty collection B of (not necessarily e-definable) world-concepts is called a **block** if the members of B are mutually exclusive, that is, a world can satisfy at most one of the properties of B. The members of a block are called **categories** and a world w is of the (B-)category β if β holds at w. The fact that the categories are exclusive ensures that a world has a unique B-category. B is called **complete** if every world always has a category. Say that B is **internal** if each category has a translation. If B is internal and $\ulcorner\alpha\urcorner = P_\beta$ for $\beta \in B$ then B is equivalent to the block $\{w \epsilon P_\beta : \beta \in B\}$. We will not distinguish between B and the latter nor the set $\{P_\beta : \beta \in B\}$. So, a set of mutually exclusive modal formulae is likewise called a block. For complete internal blocks the values of the $\ulcorner\beta\urcorner$ are the atoms of the boolean subalgebra they generate in A. Now say that an internal block B is a **net** if for all $\beta \in B$ there is a *finite* $N_\beta \subseteq B$ such that $(\exists x \rhd w)\beta(x)$ is equivalent to $\bigvee\langle\alpha(w) : \alpha \in N_\beta\rangle$. In other words, if B is complete B is a net iff the translations $\ulcorner\beta\urcorner$ are the atoms of the modal subalgebra they generate in A.

With $B \neq \emptyset$ a block we say that a point in $\mathcal{G}$ is of (B-)**span** $N, N \subseteq B$, if every successor is of a category from N and for every $\beta \in N$ there is a successor of category β. If a point has no successor, its span is therefore the empty set. We say that a point is of **minimal** (B-)**span** if either it has no successor or else every successor has the same span. Equivalently, a point is of minimal span if every α in the span of that point is in the span of all of its successors. A point without successors is always of minimal span. This leads to the following definitions.

$$
\begin{array}{lcl}
\mathrm{cat}(w) = \beta & \leftrightharpoons & \beta(w) \\
\beta \in \mathrm{span}(w) & \leftrightharpoons & (\exists x \rhd w)\beta(x) \\
N \subseteq \mathrm{span}(w) & \leftrightharpoons & \&\langle \alpha \in \mathrm{span}(w) \mid \alpha \in N\rangle \\
\mathrm{span}(w) \subseteq N & \leftrightharpoons & \&\langle \alpha \notin \mathrm{span}(w) : \alpha \notin N\rangle \\
\mathrm{mspan}(w) & \leftrightharpoons & \&\langle \beta \in \mathrm{span}(w) \supset (\forall x \rhd w)\beta \in \mathrm{span}(x) \mid \beta \in B\rangle
\end{array}
$$

If B consists of e-definable concepts, all the definitions are sound and yield e-definable world-concepts provided that the occurring conjunctions are finite. If B is internal and finite, all of the above concepts are internal and translate straightforwardly. For example

$$
\begin{array}{lcl}
\ulcorner\mathrm{span}(w) = N\urcorner & = & \bigwedge\langle \Diamond\ulcorner\mathrm{cat}(w) = \beta\urcorner : \beta \in N\rangle \\
 & & \wedge \bigwedge\langle \neg\Diamond\ulcorner\mathrm{cat}(w) = \alpha\urcorner : \alpha \notin N\rangle \\
\ulcorner\mathrm{mspan}(w)\urcorner & = & \bigwedge\langle \Diamond\ulcorner\mathrm{cat}(w) = \beta\urcorner \to \Box\Diamond\ulcorner\mathrm{cat}(w) = \beta\urcorner : \beta \in B\rangle
\end{array}
$$

One might wonder about the necessity of introducing such a roundabout terminology when it seems that there is a much simpler way to express the same facts. But this is not quite true. Granted, the same can be expressed *somehow* using a more conventional terminology as, say in [Fine, 1974c; Fine, 1985], but this one has been tailored to be very close to ordinary language and therefore things can be stated with more ease. The situation is similar in computing. While everything that can be programmed in Pascal can in principle be programmed in Assembler very few people actually do write programs in Assembler —for quite obvious reasons. And so, although things have been proved in more conventional ways this does not per se exclude searching for new tools which essentially do the same job. We ask no more of the reader than to read on and see for himself whether or not our terminology succeeds in making things easier to manipulate. What will certainly become clear is that we have found a way to unify numerous proofs into a single system which makes the methods used less ad hoc.

6.5 Points of Depth One

Let us now fix the block $\mathrm{At} = \{w \epsilon P_S : S \subseteq k\}$ with $P_S = \bigwedge\langle p_i : i \in S\rangle \wedge \bigwedge\langle \neg p_j : j \notin S\rangle$ and let $\ell = 2^k$. The At-category of a point s is called the **atomic** category of s.

Lemma 6.5.1 *Let $\langle \mathcal{H}, \gamma\rangle$ be k-good. Then*

$$(\dagger) \qquad \langle \mathcal{H}, \gamma\rangle \models (\forall w)(d_{\mathcal{H}}(w) = 1 \equiv \mathrm{mspan}(w))$$

. Or, equivalently, $\ulcorner d_{\mathcal{H}}(w) = 1\urcorner = \bigwedge\langle \Diamond\ulcorner \mathrm{cat}(w) = \alpha\urcorner \to \Box\Diamond\ulcorner \mathrm{cat}(w) = \alpha\urcorner : \alpha \in \mathrm{At}\rangle$*. For two points s, t of depth 1 $s \lhd t$ iff $t \lhd s$ iff s and t are of equal, nonzero atomic span and $s = t$ iff s and t are of equal atomic span and equal atomic category. Moreover, every point in $\mathcal{H}$ has a weak successor of depth 1. For every point s of depth 1 there is a formula Q_s of modal degree 2 satisfying $\gamma(Q_s) = \{s\}$. Finally, $\sharp\mathcal{H}^1 \leq \sharp\mathcal{F}_{K4}(k)^1 = \ell(2^{\ell-1} + 1), \ell = 2^k$.*

Proof. Let h° be the subframe of points of minimal span in $\mathcal{H}$ and $\mathcal{H}^\circ = \mathcal{H} \cap h^\circ$. We have $\mathcal{H}^1 \rightarrowtail \mathcal{H}^\circ \rightarrowtail \mathcal{H}$. For $N \subseteq \mathrm{At}$ let h°_N be the subframe of h° of points of span N. If $s, t \in h^\circ$ and $s \lhd t$ then $s \in h^\circ_N \Leftrightarrow t \in h^\circ_N$ and so $h^\circ = \oplus\langle h^\circ_N : N \subseteq \mathrm{At}\rangle$. Each set h°_N is internal and thus $B^\circ \twoheadrightarrow B^\circ_N := B^\circ \cap h^\circ_N$ is a projection for each N showing $\mathcal{H}^\circ = \oplus\langle \mathcal{H}^\circ_N : N \subseteq \mathrm{At}\rangle$. Now we have

$$\mathcal{H}^\circ_N \models \Diamond\ulcorner \mathrm{cat}(w) = \alpha\urcorner \leftrightarrow \ulcorner \alpha \in N\urcorner$$

which means that $\mathcal{H}^\circ_N \models \Diamond\ulcorner \mathrm{cat}(w) = \alpha\urcorner$ if $\alpha \in N$ and $\mathcal{H}^\circ_N \models \neg\Diamond\ulcorner \mathrm{cat}(w) = \alpha\urcorner$ if $\alpha \notin N$. Thus At is complete and a net in $\mathcal{H}^\circ_N$. Consequently, since $\mathcal{H}^\circ_N$ is refined, the value $\gamma(P_S)$ is empty or a singleton for every S, which means that the points are discriminated by their atomic category. It is easily seen now that $s \lhd t$ iff $t \lhd s$ iff s and t are of equal, nonzero At-span, and that $s = t$ if in addition both have the same atom. Therefore $\mathcal{H}^\circ = \mathcal{H}^1$ which shows (†) of Lemma 6.5.1. Now given a point s of depth 1, At-category α and At-span N, let $Q_s = \ulcorner d_{\mathcal{H}}(s) = 1\urcorner \wedge \ulcorner \text{At-span}(w) = N\urcorner \wedge \ulcorner \text{cat}(w) = \alpha\urcorner$. Then $\gamma(Q_s) = \{s\}$ and $\text{dg}(Q_s) = 2$. Finally, we have that $\sharp h^\circ_N = \sharp N$ for $N \neq \emptyset$ and $\sharp^\circ_\emptyset \leq \ell$. There are $\begin{pmatrix} \ell \\ n \end{pmatrix}$ subsets of cardinality $n = \sharp N$ and so

$$\begin{aligned} \sharp h^1 &= \sharp h^1_\emptyset + \textstyle\sum_{\emptyset \neq N} \sharp h^1_N = \sum_{n=1}^{\ell} n \begin{pmatrix} \ell \\ n \end{pmatrix} + \ell = \ell + \ell \times \sum_{n=0}^{\ell-1} \begin{pmatrix} \ell - 1 \\ n \end{pmatrix} \\ &= 2^k(2^{2^k-1} + 1). \end{aligned}$$

Moreover, this bound is exact for $\mathcal{F}^{K4}(k)$. ⊣

Remark. The number of clusters of depth 1 is bounded by $\sharp h^1_\emptyset + \sum_{n \neq \emptyset} \begin{pmatrix} \ell \\ n \end{pmatrix} = 2^k + (2^{2^k} - 1)$.

6.6 Points of Finite Depth

Suppose that the points of depth $\leq d$ have already been described. Let $\text{Dp}^{\leq d}$, with $\delta(d) = \sharp\,\text{Dp}^{\leq d}$ and assume that all points of depth $\leq d$ are discriminated by their $\text{Dp}^{\leq d}$-category in a k-good frame. So, $\text{Dp}^{\leq d}$ is a block. Moreover, $\text{Dp}^{\leq d}$ is complete and a net in $\mathcal{H}^{\leq d}$. From the previous discussion we get that $\delta(1) = (2^{2^k} - 1) + 2^k$. Consider now the subframe $\mathcal{H}^{>d} = \langle h^{>d}, B^{>d}\rangle$. The algebra $B^{>d}$ is generated by $\gamma(p_i) \cap h^{>d}$ and $\gamma(\ulcorner \text{Dp}^{\leq d}\text{-span}(w) = N \& d_{\mathcal{H}}(w) \not\leq d\urcorner)$. This is seen as follows. We have $B = B^{>d} \otimes B^{\leq d}$ as boolean algebras. $B^{\leq d}$ is atomic with atom set $\gamma(\ulcorner \delta(w)\urcorner)$, $\delta \in \text{Dp}^{\leq d}$. To show that $\Sigma = \{\gamma(p_i) \cap h^{>d} : i \in k\} \cup \{\gamma(\ulcorner \text{Dp}^d\text{-span}(w) = M \& d_{\mathcal{H}}(w) \not\leq d\urcorner) \cap h^{>d} : M \subseteq \text{Dp}^d\}$ generates $B^{>d}$ as a modal algebra it suffices to prove that every element of B is the sum of an element of $B^{\leq d}$ plus an element generated from Σ in $B^{>d}$. This is shown by induction. For $a = \gamma(p_i), i \in k$, this is immediately clear. The induction steps for $\cap$ and $\setminus$ are unproblematic. Now let $a = \blacklozenge b$ and $b = b^+ \cup b^-$ with $b^- \in B^{\leq d}$ and $b^+ \in B^{>d}$ generated from Σ. Then $a = \blacklozenge b^+ \cup \blacklozenge b^- = (\blacklozenge b^+ \cap h^{>d}) \cup (\blacklozenge b^+ \cap h^{\leq d}) \cup (\blacklozenge b^- \cap h^{>d}) \cup (\blacklozenge b^- \cap h^{\leq d})$. We have $\blacklozenge b^+ \cap h^{\leq d} = 0$ and $\blacklozenge b^- \cap h^{\leq d} \in B^{\leq d}$. Since $b^+ = b^+ \cap h^{>d}$, $\blacklozenge b^+ \cap h^{>d}$ is generated from Σ. It remains to investigate $\blacklozenge b^- \cap h^{>d}$. Without restriction we can assume that b^- is the sum of clusters

$\gamma(\ulcorner\delta\urcorner)$, $\delta \in N \subseteq \mathrm{Dp}^{\leq d}$. Then $\blacklozenge b^- = \bigvee\langle\gamma(\ulcorner\mathrm{Dp}^{\leq d}\text{-}\mathrm{span}(w) = M\urcorner : M \supseteq N\rangle$; and so $\blacklozenge b^- \cap h^{>d} = \bigvee\langle\gamma(\ulcorner\mathrm{Dp}^{d}\text{-}\mathrm{span}(w) = M \& d_{\mathcal{H}}(w) \nleq d\urcorner) : M \supseteq N\rangle$, which is generated from Σ in $B^{>d}$.

Let $\mathcal{H}^\circ$ be the subframe of points which are of minimal atomic span within being of minimal $\mathrm{Dp}^{\leq d}$-span within not being of depth $\leq d$. We then have $\mathcal{H}^{d+1} \rightarrowtail \mathcal{H}^\circ \rightarrowtail \mathcal{H}^{>d}$. Thus, $B^{>d}$ is generated by Σ and consequently B° is generated by $\gamma(p_i) \cap h^\circ$ and $\gamma(\ulcorner\mathrm{Dp}^{\leq d}\text{-}\mathrm{span}(w) = M\urcorner \wedge \ulcorner d_{\mathcal{H}}(w) \nleq d\urcorner) \cap h^\circ$. If $M \subseteq \mathrm{Dp}^d$ then let $\mathcal{H}^\circ_M$ denote the subframe of the points of $\mathrm{Dp}^{\leq d}$-span M in $\mathcal{H}^\circ$. As above we get that $\mathcal{H}^\circ = \oplus\langle\mathcal{H}^\circ_M : M \subseteq \mathrm{Dp}^d\rangle$. Since $\gamma(\ulcorner\mathrm{Dp}^{\leq d}\text{-}\mathrm{span}(w) = M'\urcorner \wedge \ulcorner d_{\mathcal{H}}(w) \nleq d\urcorner) \cap h^\circ_M = \emptyset$ if $M' \neq M$ and $= 1$ if $M' = M$ we know that $\mathcal{H}^\circ_M$ is generated by $\gamma(p_i) \cap h^\circ_M$. Now we perform the same argument as in Lemma 6.5.1. We have $\mathcal{H}^\circ_M = \oplus\langle\mathcal{H}^\circ_{M,N} : N \subseteq \mathrm{Dp}\rangle$. Now each $\mathcal{H}^\circ_{M,N}$ is a cluster with the exception of $N = \emptyset$ which is a direct coproduct of $\leq \ell := 2^k$ clusters. This proves that all points of $\mathcal{H}^\circ$ are of depth $d+1$ and so $\mathcal{H}^\circ \rightarrowtail \mathcal{H}^{d+1}$. Moreover, for two points s, t of $\mathcal{H}^\circ$ $s \lhd t$ iff s and t are of equal, nonzero At- and $\mathrm{Dp}^{\leq d}$-span, and $s = t$ if they are of equal At- and $\mathrm{Dp}^{\leq d}$-span and of equal At-category. There are $2^{\delta(d)} - 1$ nonempty subsets of $\mathrm{Dp}^{\leq d}$. Thus for $M \neq \emptyset$, $\sharp h^{d+1}_M \leq \ell(2^{\ell-1} + 1)$. If $M = \emptyset$, $\sharp h^{d+1}_M = \ell$. Together we get $\sharp h^{d+1} \leq \ell(2^\ell - 1)(2^{\delta(d)} - 1) + \ell$. Finally, $\delta(d+1)$ is determined as follows. For $M \neq \emptyset$, there are $\leq (2^{\delta(d)} - 1)(2^\ell - 1)$ clusters and for $M = \emptyset$ there are $\leq \ell$. So $\delta(d+1) = (2^{\delta(d)} - 1)(2^\ell - 1) + \ell$. To sum up we have the

Theorem 6.6.1 *Let $\langle\mathcal{H}, \gamma\rangle$ be k-good. Then $\mathcal{H}^{<\omega} \rightarrowtail \mathcal{H}$ is a generated subframe and $\mathcal{H}^{<\omega}$ is atomic. Every point of depth $\nleq d$ has a successor of depth d. Moreover, $\sharp\mathcal{H}^d \leq 2^k(2^{2^k-1} + 1)(2^{\delta(d)} - 1) + 2^k$ and $\delta(d)$ is computed recursively by*

$$\begin{aligned} \delta(1) &= (2^{2^k} - 1) + 2^k \\ \delta(d+1) &= (2^{2^k} - 1)(2^{\delta(d)} - 1) + 2^k. \end{aligned}$$

These bounds are not exact. Finally, given a point s of depth d there is a formula Q_s of modal degree $2^{d+1} - 2$ satisfying $\gamma(Q_s) = \{s\}$.

Proof. Only the last claim has to be verified. Q_s is defined by induction on the depth of s. Let $f(d)$ be the modal degree of Q_s for s of depth d. Then $f(1) = 2$. Let now $d = d_{\mathcal{H}}(s) > 1$. Suppose s is of At-category α, of At-span M and of $\mathrm{Dp}^{<d}$-span N. Define

$Q_s = \ulcorner \text{At-cat}(w) = \alpha \urcorner \wedge \ulcorner \text{At-span}(w) = M \urcorner \wedge \ulcorner \text{Dp}^{<d}\text{-span}(w) = N \urcorner \wedge \ulcorner \text{At-mspan}(w) \downarrow \text{Dp}^{<d}\text{-mspan}(w) \downarrow d_{\mathcal{H}}(w) \nless d \urcorner$. Then $\gamma(Q_s) = \{s\}$ and $\text{dg}(Q_s) = max\{0, 1, f(d) + 1, 2 + 2 + f(d) + f(d)\} = 2 + 2f(d)$. Thus $f(d+1) = 2 + 2f(d)$. $f : d \mapsto 2^{d+1} - 2$ is readily checked to be a solution of this equation. ⊣

The Theorems 6.3.1- 6.3.3 are now easily proved. For Theorem 6.3.2 is a direct consequence of Theorem 6.6.1 and Theorem 6.3.1 follows with $\mathcal{H} = \mathcal{G}/\equiv$. Theorem 6.3.3 is true because $\mathcal{F}_{K4}(k)$ is refined. It would have been possible to start the induction in § 6.5 with $d = 0$ which would have been much easier. Theorem 6.6.1 then covers the case $d = 1$ as well. For expository purposes we have chosen not to do it this way.

6.7 Some Consequences

This structure theory for finitely generated $K4$-frames has an immediate consequence:

Theorem 6.7.1 (Segerberg) *The variety of Λ-algebras is locally finite. Consequently, every logic containing $K4.J_d := K4(\neg\Delta_{d+1})$ has the finite model property.*

Proof. If $\Lambda \supseteq K4(\neg\Delta_{d+1})$ then $\mathcal{F}_\Lambda(k) \rightarrowtail \mathcal{F}_{K4}(k)^{\leq d}$. The latter is a finite frame. ⊣

Corollary 6.7.2 (Maximova) *Every logic containing $S4.J_d = S4(\neg\Delta_{d+1})$ has the finite model property.* ⊣

Incidentally, $S4.J_d$ is a splitting of $S4$ by the logic of the chain consisting of $d+1$ points. Call a logic Λ **tabular** if $\Lambda = \text{Th}(f)$ for a finite frame f.

Theorem 6.7.3 *A logic containing $K4$ is tabular iff it is of finite codimension in the lattice of normal extensions of $K4$.*

Proof. The implication from left to right is trivial. For the other direction observe that the logics $\text{Th}(\mathcal{F}_\Lambda(k))$ form a downgoing chain with limit Λ. Therefore, $\Lambda = \mathcal{F}_\Lambda(k)$ for some $k \in \omega$. Likewise, $\text{Th}(\mathcal{F}_\Lambda(k)^{\leq d})$ form a downgoing chain of logics for $d \in \omega$. Hence $\Lambda = \text{Th}(\mathcal{F}_\Lambda(k)^{\leq d})$ for some $d \in \omega$. The latter is a finite frame. Hence Λ is tabular. ⊣

6.8 Quasi-Maximal Points

Now we are going to prepare a number of results which are needed in order to prove that all subframe logics have the finite model property as first discovered in [Fine, 1985]. The method we use is the one that is implicit in [Fine, 1985] and [Zakharyaschev, 1987]. The propositions with variables in $\mathbb{P}_k$ and of modal degree $\leq n$ form a boolean algebra $\mathrm{Fml}(k, n)$ whose atom set we denote by $\mathrm{Mo} := \mathrm{Mo}(k, n)$. The Mo-category of a point is called its (k, n)**-molecule** or simply its **molecule**. We have the block $\mathrm{Mo} = \{w \epsilon A : A \in \mathrm{Mo}(k, n)\}$. Now if $V \subseteq g$ is any set in a frame we say that s is **maximal** in V if $s \in V$ and every strict successor of s is outside of V. Let V^μ denote the set of maximal points of V. Assume that g is a finite Kripke-frame. Given a valuation γ, say that s is Mo-maximal or simply **maximal** if there is a $A \in \mathrm{Mo}(k, n)$ such that $s \in \gamma(A)^\mu$. The interest in maximal points lies in a theorem, which is quite easy to prove. Denote by g^μ the subframe of maximal points. Then $\langle g, \gamma \rangle \models P \Leftrightarrow \langle g^\mu, \gamma \rangle \models P$ for all $P \in \mathrm{Fml}(k, n)$. This fact is the driving force behind [Fine, 1985]. Unfortunately, as one cannot always work within finite frames, one has to battle with two problems. First, the sets V^μ are not necessarily internal; and second, the set V^μ may be empty even though V itself is not, and consequently the theorem does not hold any longer. There are various possibilities to avoid these problems. [Fine, 1985] overcomes them by working in reduced frames. [Kracht, 1990b] on the other hand has found a way to prove the same results using finite frames only. Here we will show a third way. The essence lies in defining sets of points which are not necessarily maximal but *quasi maximal*. These sets do not suffer from the deficiencies of the sets of maximal points but nevertheless a similar theorem can be proved with respect to them (Lemma 6.8.3). The advantage over [Kracht, 1990b] is that we can also prove results on completeness rather than finite model property. The purpose of this section is to define the block Qm of quasi-maximal points by induction on the layer and then to prove some results which prepare the completeness proofs of Chapter 7. We start by defining the *criticality* of a point in Mo:

Definition 6.8.1 *For each $t \in \mathcal{G}$ there is a maximum number $c(t)$ such that there is a chain $t = t_0 \lhd t_1 \lhd \ldots \lhd t_{c(t)-1}$ such that for each i, t_{i+1} has lesser* Mo*-span than t_i. This number is called the **molecular criticality of** t.*

The criticality of a point can also be defined by induction. Say that s is **Mo-external** for α if no weak successor of s satisfying α is of category equal to s. If α has a translation then

$$(cr) \quad \ulcorner ext(\alpha)(w)\urcorner = \bigwedge\langle A \to \Box^{(1)}(\ulcorner\alpha\urcorner \to \neg A) : A \in \mathrm{Mo}(k,n)\rangle$$

No point is of critality 0 and a point is of criticality $c+1$ iff it is of minimal Mo-span in the set of points external for criticality $\leq c$. So let "crit(w) $= c$" denote the concept "w is of molecular criticality c". Then "crit(w) $\leq c$" denotes the concept "w is of molecular criticality $\leq c$" and $\mathrm{crit}(w) \leq c \equiv \bigvee\langle \mathrm{crit}(b)(w) : b \leq c\rangle$. Then

$$\begin{array}{lll} (cr0) & \ulcorner\mathrm{crit}(w) = 0\urcorner & \leftrightharpoons \quad \bot \\ (cr+) & \ulcorner\mathrm{crit}(w) = c+1\urcorner & \leftrightharpoons \quad \ulcorner\mathrm{Mo}\text{-}\mathrm{mspan}(w)\urcorner\downarrow\ulcorner ext(\mathrm{crit}(v) \leq c)(w)\urcorner \end{array}$$

Definition 6.8.2 *Let* $\mathrm{Qm}^{\leq d}$ *denote the block of quasi-maximal points of layer* $\leq d$*. Then* $\mathrm{Qm}^{\leq 0} = \emptyset$*. A point* s *in* $\mathcal{G}$ *is then said to be* ***quasi-maximal of layer d+1*** *if* s *is of minimal*

$Qm^{\leq d}$*-span within being of criticality* $d+1$.

Clearly, quasi maximality of layer $d+1$ is definable and we have

$$\mathrm{qm}(d+1)(w) = \qquad Qm^{\leq d}\text{-}\mathrm{mspan}(w)\downarrow\mathrm{crit}(w) = d+1$$

Now an inductive definition of Qm runs as follows.

$$\begin{array}{lll} \mathrm{Qm}^0 & = & \emptyset \\ \mathrm{Qm}^{\leq d} & = & \bigcup\langle \mathrm{Qm}^c : c \leq d\rangle \\ \mathrm{Qm}^{d+1} & = & \{\mathrm{Mo}\text{-}\mathrm{cat}(w) = \beta \;\&\; \mathrm{Mo}\text{-}\mathrm{span}(w) = M \;\&\; \mathrm{Qm}^{\leq d}\text{-}\mathrm{span}(w) = N \\ & & \;\&\; \mathrm{qm}(d+1)(w) : \beta \in \mathrm{Mo}, M \subseteq \mathrm{Mo}, N \subseteq \mathrm{Qm}^{\leq d}\} \end{array}$$

Denote the subframe of quasi-maximal points of $\mathcal{G}$ by $\mathcal{G}^q$. We suppress the explicit mentioning of the fact that $\mathcal{G}^q$ is dependent on γ. Also, if s a point, we denote by s^q a weak quasi-maximal successor of same molecular category. Given a pair $\langle\mathcal{G},\gamma\rangle$ where $\mathcal{G} = \langle g, A\rangle$ and $dom(\gamma) \supseteq \mathbb{P}_k$ we let $\mathcal{Q} = \langle g^q, [\gamma \upharpoonright \mathbb{P}_k]\rangle$ be the subframe generated by $\gamma(p_i), i \in k$. Finally, denote the refinement of $\mathcal{Q}$ by $\mathcal{R}(\langle\mathcal{G},\gamma\rangle)$ or simply by $\mathcal{R}$. Two points are identified by the refinement map $r : \mathcal{Q} \twoheadrightarrow \mathcal{R}$ iff they are of equal Qm-category. We call $\mathcal{R}$ the (k,n)**-reduct** of $\mathcal{G}$ with respect to γ. γ induces a natural valuation on $\mathcal{R}$ which is also denoted by γ. Thus Qm is a net in $\mathcal{R}$ and all the points of $\mathcal{R}$ are discriminated by their Qm-category.

Now the next lemma tells us that the concepts "molecular category in $\mathcal{G}$" and "molecular category in $\mathcal{G}^q$" coincide, and as a consequence also the concepts "molecular span in $\mathcal{G}$" and "molecular span in $\mathcal{G}^q$" and the concepts "molecular criticality c in $\mathcal{G}$"and "molecular criticality c in $\mathcal{G}^q$".

Lemma 6.8.3 *For all $s^q \in \mathcal{G}^q$ and $P \in \mathrm{Fml}(k, n)$*

$$\langle \mathcal{G}^q, \gamma, s^q \rangle \models P \Leftrightarrow \langle \mathcal{G}, \gamma, s^q \rangle \models P.$$

In particular, this holds for all $A \in \mathrm{Mo}(k, n)$. Thus, s^q is of molecular category A in $\langle \mathcal{G}, \gamma \rangle$ iff s^q is of molecular category A in $\langle \mathcal{G}^q, \gamma \rangle$.

Proof. By induction on P. The critical step is $P = \Diamond Q$. Let therefore $\langle \mathcal{G}, \gamma, s^q \rangle \models \Diamond Q$. Then for some $t \rhd s^q$ $\langle \mathcal{G}, \gamma, s^q \rangle \models Q$. There is a quasi-maximal weak successor t^q of t which is of equal molecular category and thus $\langle \mathcal{G}, \gamma, t^q \rangle \models Q$. But now $s^q \lhd t^q$, whence $\langle \mathcal{G}^q, \gamma, s^q \rangle \models \Diamond Q$. ⊣

Now we show that $\mathcal{R}$ is of bounded depth irrespective of $\langle \mathcal{G}, \gamma \rangle$. First note to this end that the criticality of a point can never exceed $\sharp\,\mathrm{Mo}(k, n)$. Then

Proposition 6.8.4 *s^q is of criticality c in $\mathcal{G}$ iff s^q is of criticality c in $\mathcal{G}^q$ iff $r(s^q)$ is of depth c in $\mathcal{R}$.*

Proof. The statement is proved by induction on c. The case $c = 0$ is easily settled. Now, by Lemma 6.8.3 we have $\langle \mathcal{R}, \gamma, r(s^q) \rangle \models P \Leftrightarrow \langle \mathcal{G}^q, \gamma, s^q \rangle \models P \Leftrightarrow \langle \mathcal{G}, \gamma, s^q \rangle \models P$ for all $P \in \mathrm{Fml}(k, n)$. Then let s^q be of criticality $c + 1$. We have to show that s^q is of minimal atomic span within being of minimal $\mathrm{Dp}^{\leq c}$-span within being of depth $c + 1$. To see this, take a successor $t^q \rhd s^q$. If t^q is of depth $\leq c$ then by IH, t^q is also of criticality $\leq c$. If t^q is of lesser $\mathrm{Dp}^{\leq c}$-span but of depth $\nleq c$ then by IH, t^q is of lesser $\mathrm{Qm}^{\leq c}$-span but also of depth $\nleq c$; but by IH t^q is not of criticality $\leq c$, and so s^q is of criticality $> c + 1$. Similarly, t^q cannot be of lesser atomic span but identical $\mathrm{Dp}^{\leq c}$-span and depth $\nleq c$ because again s^q would be of criticality $> c + 1$. Reversely, suppose that s^q is of minimal atomic span within being of minimal $\mathrm{Dp}^{\leq c}$-span within not being of depth $\leq c$ in $\mathcal{R}$. Then $s^q \lhd t^q$ implies either that t^q is of depth $\leq c$ in which case by IH t^q is also of criticality $\leq c$ or of depth $\not\leq c$, same $\mathrm{Dp}^{\leq c}$-criticality and same atomic span as s^q and thus of equal molecular span. Thus s^q is of criticality $c + 1$. ⊣

Theorem 6.8.5 *Let $\mathcal{R}$ be the (k,n)-reduct of $\mathcal{G}$ w.r.t. γ. Then the following holds:*

(i) *Every point of $\mathcal{R}$ is of depth $\leq \sharp\mathrm{Mo}(k,n)$. Moreover, if $\lambda = \sharp\mathrm{Mo}(k,n)$ then $\sharp\mathcal{R} \leq \sharp\mathcal{F}_{K4}(k)^{\leq\lambda} < \omega$.*

(ii) *For every $P \in \mathrm{Fml}(k,n)$ with $\langle\mathcal{G},\gamma\rangle \nvDash \neg P$ there is a $s^q \in \mathcal{R}$ with $\langle\mathcal{R},\gamma,s^q\rangle \models P$.*

(iii) *$\mathcal{R}$ is the (k,n)-reduct of $\mathcal{R}$ w.r.t. γ.*

(iv) *For every point $s \in \mathcal{R}_\sharp$ there is a formula QM_s such that $\langle\mathcal{G},\gamma,x\rangle \models \mathrm{QM}_s$ iff s is quasi-maximal and $r(x) = s$. Then $\langle\mathcal{R},\gamma,x\rangle \models \mathrm{QM}_s$ iff $x = s$. For s of depth d, $\mathrm{dg}(\mathrm{QM}_s) = d(n+4) + \frac{d(d+1)}{2}$. An overall bound for the modal degree of the QM_s is given by $\mu(k,n) = f(\sharp\mathrm{Mo}(k,n))$.*

Proof. (i) No point can have criticality $> \sharp\mathrm{Mo}(k,n)$.
(ii) Take s such that $\langle\mathcal{G},\gamma,s\rangle \models P$. Then $\langle\mathcal{G}^q,\gamma,s^q\rangle \models P$ and therefore $\langle\mathcal{Q},\gamma,s^q\rangle \models P$ whence $\langle\mathcal{R},\gamma,r(s^q)\rangle \models P$.
(iii) Any point of $\mathcal{R}$ is quasi-maximal and thus $\mathcal{R}^q = \mathcal{R}$. Since $\mathcal{R}^q$ is generated by $\gamma \restriction \mathbb{P}_k$ and also refined, $\mathcal{R}$ is it's own (k,n)-reduct w.r.t. γ.
(iv) Let s be of depth $d+1$ in $\mathcal{R}$, of Mo-category β and Mo-span M. Define $\mathrm{QM}_s = \ulcorner\mathrm{Mo\text{-}cat}(w) = \beta\urcorner \wedge \ulcorner\mathrm{Mo\text{-}span}(w) = M\urcorner \wedge \ulcorner\mathrm{Qm}^{\leq d}\text{-}\mathrm{mspan}(w) \downarrow \mathrm{crit}(w) = d+1\urcorner$. The formulas QM_s are of bounded modal degree for any d. It can be computed that if $m(d)$ is the modal degree of QM_s as constructed in the text for points of criticality d then $f(0) = 0$ and $f(d+1) = f(d) + n + d + 4$. It is checked that $f(d) = d(n+4) + \frac{d(d+1)}{2}$. ⊣

Zakhar'yashchev [87] obtains a similar result for $S4$-frames using a slightly more economical technique. Instead of taking the block $\{w \epsilon A : A \in \mathrm{Mo}(k,n)\}$ with $\mathrm{Mo}(k,n)$ the atom set of $\mathrm{Fml}(k,n)$ we may chose the block $\mathrm{Mo}(P) = \{w \epsilon A : A \in \mathrm{Mo}(P)\}$ where $\mathrm{Mo}(P)$ is the set of atoms of the boolean algebra generated by all subformulas of P. In principle, there is no difference between these two approaches.

Lemma 6.8.6 *Let $\mathcal{G}$ be a frame and $\gamma, \widetilde{\gamma}$ be valuations such that*

(i) *All quasi-maximal points of $\langle\mathcal{G},\gamma\rangle$ are of equal molecular category in $\langle\mathcal{G},\gamma\rangle$ and $\langle\mathcal{G},\widetilde{\gamma}\rangle$.*

(ii) *All points are of equal molecular span in $\langle\mathcal{G},\gamma\rangle$ and $\langle\mathcal{G},\widetilde{\gamma}\rangle$.*

then s is quasi-maximal in $\langle\mathcal{G},\gamma\rangle$ iff s is quasi-maximal in $\langle\mathcal{G},\widetilde{\gamma}\rangle$ and $\mathcal{R}\langle\mathcal{G},\gamma\rangle \cong \mathcal{R}\langle\mathcal{G},\widetilde{\gamma}\rangle$.

Proof. Clearly, all points are of equal criticality in $\langle\mathcal{G},\gamma\rangle$ and $\langle\mathcal{G},\widetilde{\gamma}\rangle$. By induction on d we now prove

(iii)$_d$ s is quasi-maximal of layer d in $\langle\mathcal{G},\gamma\rangle$ iff s is quasi-maximal of layer d in $\langle\mathcal{G},\widetilde{\gamma}\rangle$.

(iv)$_d$ All points are of equal $\mathrm{Qm}^{\leq d}$-span in $\langle\mathcal{G},\gamma\rangle$ and $\langle\mathcal{G},\widetilde{\gamma}\rangle$.

The case $d = 0$ is trivial. Now suppose s is quasi-maximal of layer $d + 1$ in $\langle\mathcal{G},\gamma\rangle$. Then if $s \lhd t$, either t is of criticality $\leq d$ in $\langle\mathcal{G},\gamma\rangle$ and therefore in $\langle\mathcal{G},\widetilde{\gamma}\rangle$ or t is of same $\mathrm{Qm}^{\leq d}$-span as s in $\langle\mathcal{G},\gamma\rangle$ and by (iv)$_d$ of same $\mathrm{Qm}^{\leq d}$-span as s in $\langle\mathcal{G},\widetilde{\gamma}\rangle$. Similarly, if s is quasi-maximal of layer $d + 1$ in $\langle\mathcal{G},\widetilde{\gamma}\rangle$, it is quasi-maximal of layer $d + 1$ in $\langle\mathcal{G},\gamma\rangle$. This shows (iii)$_{d+1}$.

Now let s be of $\mathrm{Qm}^{\leq d+1}$-span M in $\langle\mathcal{G},\gamma\rangle$ and $s \lhd t$. If t is of category β, molecular span M, $\mathrm{Qm}^{\leq d}$-span N and quasi-maximality of layer $d + 1$ in $\langle\mathcal{G},\gamma\rangle$. Then t is of Qm-category β, molecular span M, $\mathrm{Qm}^{\leq d}$-span N and quasi-maximality $d + 1$ of layer $d + 1$ in $\langle\mathcal{G},\widetilde{\gamma}\rangle$ by (i), (ii), (iv)$_d$ and (iii)$_{d+1}$. So, s is at least of $\mathrm{Qm}^{\leq d+1}$-span M in $\langle\mathcal{G},\widetilde{\gamma}\rangle$. Reversing the argument shows that if s is of $\mathrm{Qm}^{\leq d+1}$-span M in $\langle\mathcal{G},\widetilde{\gamma}\rangle$ then it is of $\mathrm{Qm}^{\leq d+1}$-span at least M in $\langle\mathcal{G},\gamma\rangle$ and this shows (iv)$_{d+1}$. ⊣

6.9 Logics of finite width

The structure theory of finitely generated K4-frames can be pushed even further in the case of logics of finite width. A logic is said to be of **finite width** if it contains $K.I_\ell$ for some $\ell \in \omega$ (see page **??**). A Kripke-frame f is said to be of width ℓ iff $f \models \mathrm{wd}(\ell)$ with

$$\mathrm{wd}(\ell) \leftrightharpoons (\forall v)(\forall w_0 \rhd v)\ldots(\forall w_\ell \rhd v) \bigvee \langle w_i \lhd^{(1)} w_j : i \neq j\rangle$$

This formula is universal and positive and therefore the logic $\mathsf{K4}.I_\ell$ of frames of width ℓ is **R**-persistent. The surprising fact about transitive frames of finite width is that the depth function can be extended to a total function on such frames. This of course requires a definition of depth for limit ordinals. Here is a definition which works in fact uniformly for all ordinals. We let $d_f(s) = \lambda$ if for all $0 < \mu < \lambda$ there is a successor of depth μ and every successor of depth μ is strict.

Theorem 6.9.1 *Suppose that $\langle \mathcal{G}, \gamma \rangle$ is k-good and of finite width. Then $d_{\mathcal{G}}$ can be extended over $\mathcal{G}$. Consequently, $\mathcal{G}$ contains no infinite ascending chain of points.*

Proof. It remains to be seen that the methods of Theorem 6.6.1 can be applied to all ordinal numbers. Suppose therefore that we have defined depth for all ordinals $< \lambda$ and that $\mathcal{G}^{<\lambda} \neq \mathcal{G}$. We have to show that every point not in $\mathcal{G}^{<\lambda}$ has a weak successor of depth λ. The crunch lies in the fact that when λ is infinite, the block $\mathrm{Dp}^{<\lambda}$ is infinite as well and in the general case we cannot find points of minimal $\mathrm{Dp}^{<\lambda}$-span within not being of depth $< \lambda$. But, following an idea of [Fine, 1974c], we can prove this for frames of finite width. From there the rest easily follows. So, pick a point $w_0 \notin \mathcal{G}^{<\lambda}$. Suppose that no weak successor of w_0 outside of $\mathcal{G}^{<\lambda}$ has minimal $\mathrm{Dp}^{<\lambda}$-span within not being of depth $< \lambda$. Then we can find a chain of points $\langle w_i : i \in \omega \rangle$ such that $w_i \lhd w_{i+1}$ and the $\mathrm{Dp}^{<\lambda}$-span of w_{i+1} is properly included in the $\mathrm{Dp}^{<\lambda}$-span of w_i. Pick u_i from the difference of these two sets, that is, let $u_i \in \mathrm{Dp}^{<\lambda}\text{-}\mathrm{span}(w_i) - \mathrm{Dp}^{<\lambda}\text{-}\mathrm{span}(w_{i+1})$. The sequence $\langle u_i : i \in \omega \rangle$ is non-descending, that is, contains no infinite antichain, and therefore contains an ascending subchain, contrary to the choice of the u_i as being of depth $< \lambda$. Would there not be such an ascending subchain, then every u_i has a weak successor in the set $M \subseteq \{u_i : i \in \omega\}$ of $\lhd$-maximal points of $\{u_i : i \in \omega\}$. Now M is a subset of a union of incomparable clusters of $\mathcal{G}$. But each cluster has $\leq 2^k$ points and there are at most ℓ such clusters since they all succeed w_0. Thus M is finite. But each $u \in M$ has only finitely many predecessors, as $\langle u_i : i \in \omega \rangle$ is non-descending. Hence $\{u_i : i \in \omega\}$ is finite. Contradiction. ⊣

In contrast to Theorem 6.6.1 the sets defined here are not necessarily internal. Neither are the sets of points of depth $< \lambda$. The argument used at the end of the proof deserves special attention. We can use it to prove something quite strong about these frames. Say that a set C of points is a **cover** for S if every point of S is either in C or has a strict successor in C. A frame $\mathcal{G}$ has the **finite cover property** if every set has a finite cover. Neither the set V nor the cover are required to be internal. Then using the same argument once more we can show that if $\mathcal{G}$ is a finitely generated refined frame of finite width, $\mathcal{G}$ has the finite cover property. Just take any set S and let $S^\mu = \{s : (\forall t \rhd s)(s \in S \Rightarrow d_{\mathcal{G}}(t) = d_{\mathcal{G}}(s)\}$. As before, S^μ is the union of subsets of incomparable clusters and therefore finite.

Theorem 6.9.2 *Let Λ be of finite width. Then any finitely generated refined Λ-frame has the finite cover property.* $\dashv$

Now say that a point s is **eliminable** if for all $a \in \mathcal{G}_+$ with $s \in a$, s has a strict successor in a. Say that $\mathcal{G}$ is **reduced** if no point is eliminable. Let g_r denote the set of non-eliminable points in $\mathcal{G}$. Put $\mathcal{G}_r = \langle g_r, \{a \cap g_r : a \in A\}\rangle$. This is the same construction as for subframes but as g_r is not likely to be internal, $\mathcal{G}_r$ need not be a subframe of $\mathcal{G}$. If $\mathcal{G}$ is finitely generated and refined, so is $\mathcal{G}_r$. This is clear for refinedness; that the reduced frame is also finitely generated is proved by noting that $a \mapsto g_r \cap a$ is a homomorphism. This we see as follows. It is certainly a boolean homomorphism; so it is left to check the modal operator. Since $\blacklozenge_r a = g_r \cap \blacklozenge a$ is the diamond operator in $\mathcal{G}_r$, we have to prove $g_r \cap \blacklozenge a = g_r \cap \blacklozenge(a \cap g_r)$ for all a. Thus let $s \in g_r \cap \blacklozenge a$; then s has a successor in a which in turn has a noneliminable weak successor in a whence $s \in g_r \cap \blacklozenge(a \cap g_r)$. Conversely, if $s \in g_r \cap \blacklozenge(a \cap g_r)$ then $s \in g_r \cap \blacklozenge a$. We denote the class of finitely generated, refined and reduced frames by **Rrf**.

Theorem 6.9.3 (Fine) *Suppose that $\Lambda \supseteq$ K4. Then any point s in $\mathcal{F}_\Lambda(k)$ has a non-eliminable weak successor for any a.* ($\dashv$)

If Λ is of finite width then this follows from the finite cover property, for then a is finitely covered. Under such circumstances we can make use of the following proposition.

Proposition 6.9.4 *Suppose that every point s has non-eliminable successor in every internal set. Then for $s_r \in \mathcal{G}_r$*

$$\langle \mathcal{G}_r, \gamma, s_r\rangle \models P \Leftrightarrow \langle \mathcal{G}, \gamma, s\rangle \models P$$

This finishes the proof. $\dashv$

Corollary 6.9.5 (Fine) *Every logic containing K4 is **Rrf**-complete.* $\dashv$

Generally, finitely generated refined frames are atomic only in the layers of finite depth. But again we are helped by finite width.

Theorem 6.9.6 *Let $\mathcal{G}$ be finitely generated and refined. Then $\mathcal{G}_r$ is atomic and if g_r is rooted the sets $g_r^\lambda, g_r^{\geq\lambda}, g_r^{<\lambda}$ etc. are internal.*

Proof. Take a point v. Since $\mathcal{G}_r$ is reduced we find a a such that v is maximal in a. Now let F be a finite cover for $\{t : v \ntriangleleft^{(1)} t\}$. For each $t \in F$ there is a b_t such that $v \in b_t$ but $t \notin b_t$ and a c_t such that $v \in \blacksquare c_t$ but $t \notin c_t$. We prove that if $e = a \cap \bigcap\langle b_t \cap \blacksquare c_t : t \in F\rangle$, e is the cluster containing a. Since this cluster is finite and $\mathcal{G}$ differentiated, $\{v\}$ is then indeed internal. Thus supose $u \in e$ and $u \neq v$. We have to show $v \triangleleft u \triangleleft v$. If $v \ntriangleleft u$ then u has a weak successor $t \in F$. If $u = t$ then $u \notin b_t$ whence $u \notin e$; but if $u \triangleleft t$ then $u \in \blacklozenge \setminus c_t = \setminus\blacksquare c_t$ whence again $u \notin e$. Thus $v \triangleleft u$. But if $v \triangleleft u$ and $u \ntriangleleft v$ then $u \notin a$.

If $\mathcal{G}$ is rooted the sets g_r^λ are finite and therefore internal. It is easy to see that $g_r^{\geq\lambda}$ and $g_r^{<\lambda}$ are also internal. ⊣

It is worthwile reflecting on the difference between g_r^λ being an internal set and $[\![d_{\mathcal{G}}(w) = \lambda]\!]$ having a translation in the class of finitely generated reduced frames. For while for each reduced frame $\mathcal{G}_r$ and each valuation γ there is a formula P such that $\gamma(P) = g_r^\lambda$ there is no guarantee that a single formula can be selected uniformly for all frames and valuations. In view of the fact that the logics $\mathsf{K4}.I_\ell$ have the finite model property this is not surprising. For if the world-concept "is of depth ω" has a translation P, this formula will pick out all points of depth ω in reduced k-good frames. This of course means that there can be no finite model for P based on a reduced k-good frame.

Corollary 6.9.7 *If Λ is of finite width, $\mathcal{F}_\Lambda(k)_r$ is atomic.* ⊣

Chapter 7

Logics Containing K4 with and without F.M.P.

7.1 Subframe Logics

Call a logic Λ a **subframe logic** if the class of Λ-frames is W°-closed i.e. closed under subframes. Examples of subframe logics are FINE-splittings of K. It is straightforward to check that if Λ is subframe logic, so is any FINE-splitting Λ_g of Λ. In the case of transitive subframe logics, it turns out that the subframe logics are exactly the FINE-splittings of **K4**. First of all, a transitive subframe logic Λ has the finite model property. For if $\Lambda \supseteq \mathsf{K4}$ is a subframe logic and $\langle \mathcal{G}, \gamma, s \rangle \models P$ is a model for $P \in F(k, n)$ based on a Λ-frame $\mathcal{G}$, then according to Theorem 6.8.5(ii) there is a s^q such that $\langle \mathcal{R}, \gamma, s^q \rangle \models P$, if $\mathcal{R}$ denotes the (k,n)-reduct of $\mathcal{G}$. $\mathcal{R}$ is finite and a p-morphic image of subframe of $\mathcal{G}$. Therefore $\mathcal{R}$ is a Λ-frame. Finally, if we let $\Lambda_o = \bigsqcup \langle \mathsf{K4}_g : g \notin \mathrm{Fr}(\Lambda), g$ rooted and finite$\rangle$ then Λ_o is a subframe logic and therefore has the fmp. Consequently, as Λ and Λ_o have the same finite models, they must be equal. This proves

Theorem 7.1.1 (Fine) *A logic containing $\mathsf{K4}$ is a subframe logic iff it is an iterated FINE-splitting of $\mathsf{K4}$. Moreover, all subframe logics have fmp.* $\dashv$

In addition, if Λ is a subframe logic and $P \in F(k, n)$ is consistent with Λ then there exists a model $\langle \mathcal{G}, \gamma, s \rangle \models P$ containing at most $\sharp F_{K4}(k)^{\leq \lambda}$ points, where $\lambda = \sharp A(k, n)$. The size of the smallest model can therefore bound from above by a recursive function as shown in the previous chapter. It is tempting to conclude that since such an a priori bound to the size of a model can be given therefore all subframe logics are decidable. Such a conclusion is not immediate as is shown in [Urquhart, 1981]. On the other hand, the proof given in [Fine, 1985] that there are $2^{\aleph_0}$ subframe logics is also incorrect. It is therefore still an open question whether all subframe logics are decidable. We believe that the answer is positive but have found no means of proving it. We will present a partial solution to this question. Following [Kruskal, 1960] we say that a relation $\preccurlyeq$ is a **well-partial order (wpo)** if it is a partial order without infinite, strictly descending chains such that every set N of mutually incomparable elements is finite. On the set of finite rooted and transitive Kripke-frames we define $\preccurlyeq$ by $f \preccurlyeq g \Leftrightarrow g$ subreduces to f. Clearly, $\preccurlyeq$ is a partial order without infinite strictly descending chains. If we can prove that $\preccurlyeq$ is a wpo we can show that every subframe logic $K4_M$ can be finitely axiomatized. It follows that all subframe logics are finitely axiomatizable and thus decidable. For if $K4_M$ is a FINE-splitting of $K4$ then letting $N \subseteq M$ to be the set of $\preccurlyeq$-minimal elements of M it is easily seen that $K4_M = K4_N$. Moreover, all frames of N are mutually incomparable. Therefore, N is finite and $K4_M$ finitely axiomatizable. Furthermore, if all subframe logics are decidable then $\preccurlyeq$ is a well partial order. For if not, there is an infinite set N of mutually incomparable frames and therefore $2^{\aleph_0}$ subframe logics.

Theorem 7.1.2 *The following are equivalent*

(i) $\preccurlyeq$ *is a well partial order.*

(ii) *All subframe logics are finitely axiomatizable.*

(iii) *All subframe logics are decidable.* $\dashv$

We can show for a restricted class of frames that $\preccurlyeq$ is a wpo. With every transitive frame g we can associate a partial order $g^\flat = \langle g^\flat, \leq\rangle$ by letting $g^\flat$ to be the set of clusters of g and $C \leq D$ iff $C = D$ or $(\forall s \in C)(\forall t \in D)(s \lhd t)$. Next we define an indexing function $\iota : g^\flat \to \omega$ by letting $\iota(C) = \sharp\{s | (\exists t \in C)(s \lhd t \lhd s)\}$. Finally, define a partial order $\preccurlyeq$ on the natural numbers by $m \preccurlyeq n \Leftrightarrow m = n = 0$ or $1 \leq m \leq n$ and define $\tau(g) = \langle g^\flat, \iota\rangle$. It is not hard to see that $\epsilon : g \rightarrowtail h$ iff ϵ embeds $\tau(g)$ into $\tau(h)$ as a partial-order-over-$\langle\omega, \preccurlyeq\rangle$, that is, such that $\iota(C) \preccurlyeq \iota(\epsilon(C))$. Say that g is a **quasi-tree** if $g^\flat$ is a tree. Denote the set of finite quasi-trees by Q. Quasi-trees are equivalent to trees-over-$\langle\omega, \preccurlyeq\rangle$ in the sense of

[Kruskal, 1960]. Now, by the famous result of Kruskal obtained in [Kruskal, 1960], since $\langle\omega, \preccurlyeq\rangle$ is wpo, so is $\langle Q, \subseteq\rangle$, the space of trees-over-$\langle\omega, \preccurlyeq\rangle$. Therefore any subframe logics axiomatized by a set of finite quasi-trees is finitely axiomatizable. Since linear frames are quasi-trees, we get an interesting corollary, first noted in [Fine, 1971]. If $\Lambda = \mathsf{S4.3}/g$ is a splitting of $\mathsf{S4.3} = \mathsf{S4}.I_1$ by g then $\Lambda = \mathsf{S4.3}_g$—and is therefore a subframe logic, since $g \subseteq h$ iff there is a p-morphism $p : h \twoheadrightarrow g$ onto g. So, for any $\Lambda \supseteq \mathsf{S4.3}$ and any rooted frame $\Lambda/g = \Lambda_g$. Therefore, all splittings $\mathsf{S4.3}/N$ of $\mathsf{S4.3}$ by a set N of finite frames have fmp since they are FINE-splittings. Now if $\Lambda \supseteq \mathsf{S4.3}$, let $\Lambda_o = \mathsf{S4.3}/N$ with $N = \{g : g \notin \mathrm{Fr}(\Lambda), g$ one-generated and finite$\}$. Then since $\Lambda_o \subseteq \Lambda$ and Λ_o has fmp and shares all the finite models with Λ, $\Lambda = \Lambda_o$.

Theorem 7.1.3 *All logics containing S4.3 have fmp, are finitely axiomatizable and decidable.* $\dashv$

This holds even for nonnormal extensions of S4.3 since by a result of [Segerberg, 1975] there are none. A final note. [Fine, 1985] defines the notion of a *descendant* of a finite frame f and proves that a finite frame g subreduces to f iff a descendant of f is embeddable in g. Since f has only finitely many descendants, this implies that $Sf(f)$ is elementary in **F4**.

7.2 Homogenization of Models

If $p : \mathcal{G} \twoheadrightarrow \mathcal{H}$ and $\gamma, \widetilde{\gamma} : \mathbb{P}_k \to \mathcal{G}_+$ are such that $\widetilde{\gamma}$ admits p and for all $P \in F(k,n)$ $\langle\mathcal{H}, \widetilde{\gamma}\rangle \nvDash P \Leftrightarrow \langle\mathcal{G}, \gamma\rangle \nvDash P$ then $\langle\mathcal{H}, \widetilde{\gamma}\rangle$ is called a **homogenization** of $\langle\mathcal{G}, \gamma\rangle$. If in addition $\mathcal{R}\langle\mathcal{G}, \gamma\rangle \cong \mathcal{R}\langle\mathcal{G}, \widetilde{\gamma}\rangle$ then $\langle\mathcal{H}, \widetilde{\gamma}\rangle$ is called an **exact homogenisation** of $\langle\mathcal{G}, \gamma\rangle$. In that case $\mathcal{R}\langle\mathcal{H}, \widetilde{\gamma}\rangle \cong \mathcal{R}\langle\mathcal{G}, \gamma\rangle$. $\langle\mathcal{G}, \gamma\rangle$ is **homogenized** if it admits no nontrivial homogenization. There are two important useful exact homogenizations of a model $\langle\mathcal{G}, \gamma, s\rangle \models P$. First, if $\mathrm{var}(P) = \mathbb{P}_k$ and $p \notin \mathbb{P}_k$, then the value of $\gamma(p)$ can be redefined so that $\widetilde{\gamma}(p) = \widetilde{\gamma}(p_0)$. This clearly does not affect the structure of the (k,n)-reduct. Second, the refinement map $p : \mathcal{G} \twoheadrightarrow \mathcal{G}/\equiv$ is admissible for any valuation and does also not alter the structure of the (k,n)-reduct.

Lemma 7.2.1 (Exact Homogenization) *Suppose that $p : \mathcal{G} \twoheadrightarrow \mathcal{H}$ is such that if a fibre $p^{-1}(s)$ contains a quasi-maximal point then every point in $p^{-1}(s)$ has a weak quasi-*

maximal successor and all quasi-maximal points in $p^{-1}(s)$ are of equal Qm-category. Then there is a $\widetilde{\gamma}$ for which p is admissible and $\mathcal{R}\langle\mathcal{G},\gamma\rangle \cong \mathcal{R}\langle\mathcal{G},\widetilde{\gamma}\rangle$ and a t such that $\langle\mathcal{G},\widetilde{\gamma},t\rangle \models P$.

Proof. Let $e : \mathcal{H}_\sharp \rightharpoonup \mathcal{G}_\sharp$ be a section of p. Define $\widetilde{\gamma}$ by $x \in \widetilde{\gamma}(p_i) \Leftrightarrow ep(x)^q \in \gamma(p_i)$. For every point s we let s^+ denote a weak quasi-maximal successor in the fibre containing $ep(s)^q$. Such a point always exists. For $ep(s)^q$ is a weak quasi-maximal successor of $ep(s)$ and since $ep(s)$ is in the same fibre as s, s has a weak successor t in the fibre of $ep(s)^q$ since p is a p-morphism. By assumption, t has a weak quasi-maximal successor within the same fibre. It is easily checked that s and s^+ are of equal H-span. By induction on the complexity of Q we show that for all $Q \in F(k,n)$:

(i) $\langle\mathcal{G},\gamma,s^q\rangle \models Q \Leftrightarrow \langle\mathcal{G},\widetilde{\gamma},s^q\rangle \models Q$

(ii) $\langle\mathcal{G},\widetilde{\gamma},s\rangle \models Q \Leftrightarrow \langle\mathcal{G},\widetilde{\gamma},s^+\rangle \models Q$

The start with $Q = p_i$ is straightforward for (i). For (ii) observe that $s \in \widetilde{\gamma}(p_i) \Leftrightarrow ep(s)^q \in \gamma(p_i) \Leftrightarrow ep(s)^q \in \widetilde{\gamma}(p_i)$. The induction steps for $\neg$ and $\wedge$ are unproblematic. Now let $Q = \Diamond P$. If $\langle\mathcal{G},\gamma,s^q\rangle \models \Diamond P$ then for some $t^q \rhd s^q$ $\langle\mathcal{G},\gamma,t^q\rangle \models P$ and by IH $\langle\mathcal{G},\widetilde{\gamma},t^q\rangle \models P$ and so $\langle\mathcal{G},\widetilde{\gamma},s^q\rangle \models \Diamond P$. But if $\langle\mathcal{G},\widetilde{\gamma},s^q\rangle \models \Diamond P$ then for some $t \rhd s^q$ $\langle\mathcal{G},\widetilde{\gamma},t\rangle \models P$ and by (ii) $\langle\mathcal{G},\widetilde{\gamma},t^+\rangle \models P$; whence $\langle\mathcal{G},\gamma,t^+\rangle \models P$ by IH and finally $\langle\mathcal{G},\gamma,s^q\rangle \models \Diamond P$. To prove (ii), only the direction from left to right is needed. Suppose therefore $\langle\mathcal{G},\widetilde{\gamma},s\rangle \models \Diamond P$. Then for some $t \rhd s$ $\langle\mathcal{G},\widetilde{\gamma},t\rangle \models P$ and so $\langle\mathcal{G},\widetilde{\gamma},t^+\rangle \models P$ by IH. If t^+ is of Mo-category i in $\langle\mathcal{G},\gamma\rangle$ then i is in the Mo-span of s and since s and s^+ are of equal Mo-span in $\langle\mathcal{G},\gamma\rangle$ i is in the Mo-span of s^+; thus there is a quasi-maximal point $u^q \rhd s^+$ of category i. Then t^+ and u^q are of equal category. Now $\langle\mathcal{G},\widetilde{\gamma},t^+\rangle \models P$ implies $\langle\mathcal{G},\gamma,t^+\rangle \models P$ by IH and $\langle\mathcal{G},\gamma,u^q\rangle \models P$ by definition of u^q and finally $\langle\mathcal{G},\widetilde{\gamma},u^q\rangle \models P$ by (i). Hence $\langle\mathcal{G},\widetilde{\gamma},s^+\rangle \models \Diamond P$. ⊣

Another important case of homogenization is provided when $\mathcal{G}^q \overset{\subset}{\to} \mathcal{H} \overset{\subset}{\to} \mathcal{G}$.

Lemma 7.2.2 *Let $\mathcal{G}^q \overset{\subset}{\to} \mathcal{H} \overset{\subset}{\to} \mathcal{G}$. Then for all $s \in \mathcal{H}_\sharp$ and $Q \in Fm(k,n)$:*

$$\langle\mathcal{H},\gamma,s\rangle \models Q \Leftrightarrow \langle\mathcal{G},\gamma,s\rangle \models Q.$$

This is easily shown by induction on Q. The only critical step is $Q = \Diamond P$. There $\langle\mathcal{H},\gamma,s\rangle \models \Diamond P$ implies $\langle\mathcal{H},\gamma,t^q\rangle \models P$ for some successor which is quasi-maximal in

$\langle \mathcal{G}, \gamma \rangle$. Then $\langle \mathcal{G}, \gamma, t^q \rangle \models P$ and thus $\langle \mathcal{G}, \gamma, s \rangle \models \Diamond P$. This lemma can be strengthened. Call $\mathcal{H} \overset{\subset}{\to} \mathcal{G}$ **qm-covered** if every point $s \in \mathcal{H}_\sharp$ has a weak successor $s^q \in \mathcal{H}_\sharp$ which is quasi-maximal in $\langle \mathcal{G}, \gamma \rangle$ and whose Mo-category in $\langle \mathcal{G}, \gamma \rangle$ is the same as the Mo-category of s in $\langle \mathcal{G}, \gamma \rangle$. Then $\langle \mathcal{H}, \gamma, s \rangle \models Q \Leftrightarrow \langle \mathcal{G}, \gamma, s \rangle \models Q$ for all $s \in \mathcal{H}_\sharp$. Then s has the same Mo-category and the same Mo-span in $\langle \mathcal{G}, \gamma \rangle$ and $\langle \mathcal{H}, \gamma \rangle$.

Theorem 7.2.3 *Suppose $\mathcal{H} \overset{\subset}{\to} \mathcal{G}$ is qm-covered in $\langle \mathcal{G}, \gamma \rangle$ and that there is a p-morphism $p : \mathcal{G} \twoheadrightarrow \mathcal{H}$. Then $\langle \mathcal{H}, \gamma \rangle$ is an exact homogenization of $\langle \mathcal{G}, \gamma \rangle$.* $\dashv$

This is effectively the same as "dropping" from $\mathcal{G}$ the points which are not in $\mathcal{H}$, a technique which is applied with great skill in [Fine, 1974c] and [Fine, 1985]. If $\mathcal{G}$ is finite there is a least qm-covered subframe, the subframe $\mathcal{G}^\mu$ of maximal points. This leads us to the following results.

Corollary 7.2.4 *For any pair $\langle \mathcal{G}, \gamma \rangle$ where $\mathcal{G}$ is finite there is an exact homogenization $\langle \mathcal{H}, \widetilde{\gamma} \rangle$ of $\langle \mathcal{G}, \gamma \rangle$ in which every cluster of cardinality > 1 contains only maximal points.* $\dashv$

Corollary 7.2.5 *For any pair $\langle \mathcal{G}, \gamma \rangle$ where $\mathcal{G}$ is finite there is an exact homogenization $\langle \mathcal{H}, \widetilde{\gamma} \rangle$ such that if $s \lhd t$ is such that $t \ntriangleleft s$ and $(\forall u \rhd s)(u \lhd s \vee t \rhd u)$ then s is maximal.* $\dashv$

The two theorems state that whenever there is a model $\langle \mathcal{G}, \gamma, s \rangle \models P$ for a formula $P \in Fm(k, n)$ then there is a model for P based on a frame $\mathcal{H} \in \mathrm{C}(\mathcal{G})$ which has nontrivial clusters only if they consist entirely of maximal points which are not of equal category; and which has a point s immediately preceding t and seeing only points which t sees except they are within its cluster only if s is maximal. Thus the described configurations can in some sense be made very rare with a suitabe homogenization which does not disturb the structure of the (k, n)-reduct.

7.3 Logics of finite width once again

In the first part of his essay on extensions of $\mathsf{K4}$, Fine shows that all transitive logics of finite width are complete. This theorem is a result of a combination of the structure theory for frames of finite width developed in § 6.9 and the homogenization technique. It is a beautiful example of the power of homogenization.

We start with a pair $\langle \mathcal{G}, \gamma \rangle$ and its (k,n)-reduct $\mathcal{R}$. We then have the block $Qm := \{w \,\epsilon\, QM_s | s \in \mathcal{R}_\sharp\}$.

Lemma 7.3.1 *Let $\mathcal{G}$ be a frame and γ a valuation on $\mathcal{G}$. For each set $K \subseteq \mathcal{R}_\sharp$ for which the set of points of* Qm-*span K is not empty we pick a quasi-maximal point s_K from this very set. Then we define a new valuation $\widetilde{\gamma}$ as follows.*

$$\begin{array}{lll} (i) & \textit{If } s = s^q: & s \in \widetilde{\gamma}(p) \leftrightharpoons s \in \gamma(p) \\ (ii) & \textit{If } s \neq s^q: & s \in \widetilde{\gamma}(p) \leftrightharpoons s_K \in \gamma(p) \end{array}$$

Then $\langle \mathcal{G}, \widetilde{\gamma} \rangle$ is a homogenization of $\langle \mathcal{G}, \gamma \rangle$.

Proof. Define a function $s \mapsto \widetilde{s}$ by letting $\widetilde{s} = s$ in case s is quasi-maximal and else let $\widetilde{s}$ be a quasi-maximal successor of s of same Mo-category as s_K. Such a point always exists since the Mo-category of s_K is in the Mo-span of s_K and therefore in the Mo-span of s. (Remember that the Qm-span of s and s_K are equal; a fortiori, the H-span of s and s_K are equal.) It is easily checked that $s \in \widetilde{\gamma}(p) \Leftrightarrow \widetilde{s} \in \gamma(p)$. Now we show by induction that for all $P \in \mathrm{Fml}(k, n)$ $s \in \widetilde{\gamma}(P) \Leftrightarrow \widetilde{s} \in \gamma(P)$. The steps for $\wedge$ and $\neg$ are trivial. For the $\Diamond$-step observe first that $s \in \gamma(\Diamond Q) \Leftrightarrow \widetilde{s} \in \gamma(\Diamond Q)$ since s and $\widetilde{s}$ are of equal Qm-span and therefore also of equal Mo-span. Now suppose $s \in \widetilde{\gamma}(\Diamond Q)$. Then for some successor t $t \in \widetilde{\gamma}(Q)$ whence $\widetilde{t} \in \gamma(Q)$ and so $s \in \gamma(\Diamond Q)$ since $s \lhd t \lhd^{(1)} \widetilde{t}$. Consequently $\widetilde{s} \in \gamma(\Diamond Q)$. Conversely, if $\widetilde{s} \in \gamma(\Diamond Q)$ then there is a quasi-maximal successor t with $t \in \gamma(Q)$. Then, as $\widetilde{t} = t$, $\widetilde{\widetilde{t}} = \widetilde{t} \in \gamma(Q)$ whence by IH $\widetilde{t} \in \widetilde{\gamma}(Q)$ and so $s \in \widetilde{\gamma}(\Diamond Q)$. ⊣

In contrast to the proof given in Fine [74a] we do not require that $\mathcal{G}$ is finitely covered.

Theorem 7.3.2 (Fine) *Every transitive logic of finite width is complete.*

Proof. In the light of Corollary 6.9.5 it suffices to show that all transitive logics of finite width are **Rrf**-persistent. So let Λ be such a logic and $\mathcal{G}$ a finitely generated refined and reduced frame and $\langle (\mathcal{G}_\sharp)^\sharp, \gamma, s \rangle \models P$ for some γ, s. Then since $\mathcal{G}$ has the finite cover property the set of quasi-maximal points is finitely covered by the set M of maximal points. Now construct $\widetilde{\gamma}$ as in Lemma 7.3.1. It is enough to show that each $\widetilde{\gamma}(p)$ is internal in $\mathcal{G}$; for then $\langle (\mathcal{G}_\sharp)^\sharp, \gamma, s \rangle \models P$ implies $\langle (\mathcal{G}_\sharp)^\sharp, \widetilde{\gamma}, s \rangle \models P$ by Lemma 7.3.1 and so $\langle \mathcal{G}, \widetilde{\gamma}, s \rangle \models P$. But

$$\widetilde{\gamma}(p) = \gamma(p) \cap M. \cup . \bigcup \langle \gamma(\ulcorner Qm\text{-}\operatorname{span}(w) = K \urcorner) - M | s_K \in \gamma(p) \rangle$$

Since M is finite and the union is finite, $\widetilde{\gamma}(p)$ is a boolean combination of internal sets by Theorem 6.9.6 and therefore internal. $\dashv$

7.4 Logics of Tightness Two

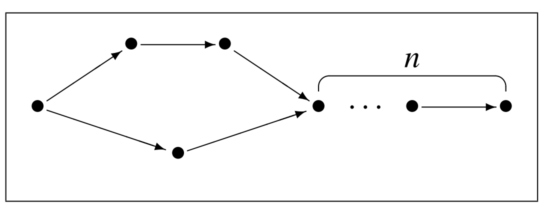
A transitive logic Λ is said to be of **tightness** m if there is an $n \in \omega$ such that $\mathcal{G}$ is a Λ-frame iff $\mathcal{G} \models ti^m(n)$, where

$$\begin{aligned} ti^m(n) \leftrightharpoons\ & (\forall v)(\forall w_0 \rhd v)(\forall w_1 \rhd w_0)\ldots(\forall w_{m-1} \rhd w_{m-2})(\forall y \rhd v)[d_{\mathcal{G}}(w_{m-1}) > n \\ & \& d_{\mathcal{G}}(y) > n. \supset . \bigvee\langle w_{i+1} \lhd w_i : i \in m-1\rangle \vee \bigvee\langle w_i \lhd y \vee y \lhd w_i : i \in m\rangle]. \end{aligned}$$

It can be checked that $ti^m(n)$ is a universal and positive formula and therefore the logic $K.Ti^m(n)$ of frames satisfying $ti^m(n)$ is **R**-persistent and a subframe logic. The condition $ti^m(n)$ can be rephrased as follows; call $\overline{w} = \langle w_i : i \in m\rangle$ a **strict chain** if $w_i \lhd w_{i+1} \ntrianglelefteq w_i$ for all $i \in m-1$ and say that $\overline{w}$ is of depth $> n$ if w_{m-1} is of depth $> n$. Say that $\overline{w}$ is **parallel** to y if for all $i \in m$ $y \ntrianglelefteq w_i \ntrianglelefteq y$. Then $\mathcal{G}$ satisfies $ti^m(n)$ iff for all $s \in \mathcal{G}_\sharp$ $\mathrm{Tr}_{\mathcal{G}}(s)$ does not contain a strict chain of m points of depth $> n$ parallel to a point of depth $> n$.

We are especially interested in the logics $\mathsf{S4}.Ti^2(n)$. It is readily seen that $\mathsf{S4}.Ti^2(n) = \mathsf{S4}_{ti(2,n)} = \mathsf{S4}/ti(2,n)$ for $n > 0$ and $\mathsf{S4}.Ti^2(0) = \mathsf{S4}_{ti(2,0)} = \mathsf{S4}/\{ti(2,0), ti(2,1)\}$.

$ti(2,n)$



$\mathsf{S4}.Ti^2(n)$-frames have the property that for all $s, t \rhd r$ $d_{\mathcal{G}}(s) > d_{\mathcal{G}}(t) > n$ implies $s \lhd t$. The reason is that otherwise there is a point $t' \lhd t$ of depth $d_{\mathcal{G}}(t) + 1$. Clearly, $s \ntrianglelefteq t'$ but $t' \lhd s$ can also not hold, for $d_{\mathcal{G}}(t') \leq d_{\mathcal{G}}(s)$, so in that case also $s \lhd t'$ and thus $s \lhd t$. So $\langle t', t\rangle$ is a chain of depth $> n$ parallel to s.

Theorem 7.4.1 *Every logic containing $\mathsf{S4}.Ti^2(n)$ has fmp. In addition, effective bounds can be given for the size of a model.*

Proof. Let $\Lambda \supseteq \mathsf{S4}.Ti^2(n)$, let $\mathcal{G}$ be a rooted Λ-frame, k-generated and refined. Fix k and n and let $P \in Fm(k,n)$. We have to show that if P is consistent with Λ, it has a finite model. For every set of points $N \subseteq \mathcal{R}\langle\mathcal{G},\gamma\rangle$ we define

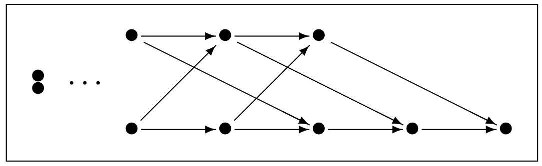
$$A_N = \bigwedge\langle\Diamond QM_s : s \in N\rangle \wedge \bigwedge\langle\neg\Diamond QM_s : s \notin N\rangle \wedge \ulcorner d_{\mathcal{G}}(w) > n\urcorner$$

With $g = \mathcal{G}_\sharp$ let $h = g^{\leq n} \cup \{s_N : N \subseteq \mathcal{R}\langle\mathcal{G},\gamma\rangle\}$ and $a \lhd_h b$ iff $a, b \in g^{\leq n}$ and $a \lhd_g b$ or $b \in g^{\leq n}$ but $a \notin g^{\leq n}$ or $a = s_N, b = s_M$ and $N \supseteq M$. Define a map $p : g \rightharpoonup h$ by $p(x) = x$ if $x \in g^{\leq n}$ and $p(x) = s_N$ for N such that $\langle\mathcal{G},\gamma,x\rangle \models A_N$. We show that p is a p-morphism. (i) If $x \lhd y$ then $p(x) \lhd p(y)$. For if $d_g(x) \leq n$ or $d_g(y) \leq n$ this is trivially satisfied. But if $d_g(x), d_g(y) > n$ then $x \models A_M, y \models A_N$ and $M \supseteq N$. Thus $p(x) = s_M \lhd s_N = p(y)$. (ii) If $p(x) \lhd u$ we have to find a y such that $p(y) = u$. If $d_h(p(x)) \leq n$, then $y = u$. Likewise if $d_h(y) \leq n$ and $d_h(x) > n$. This leaves the case where $d_h(x), d_h(y) > n$. Then $u = s_N$ and $p(x) = s_M$. By definition of $\lhd_h$ we have $N \subseteq M$. If $N = M$ we can let $y = x$. But if $N \subsetneq M$ for any $y \in p^{-1}(u)$ we have $x \lhd y$. For $d_g(y) > d_g(x)$ would imply $y \lhd x$ and so $M \subseteq N$ and $d_g(y) = d_g(x)$ is incompatible with $N \subsetneq M$. So $d_g(y) < d_g(x)$ and consequently $x \lhd y$.

The conditions of the Exact Homogenization Lemma are satisfied. We have $p : \mathcal{G} \twoheadrightarrow h^\sharp$ and if $x, y \in p^{-1}(s)$ are quasi-maximal, then either $d_{\mathcal{G}}(x) \leq n$ and $x = y$ or x and y are of equal Qm-category. So we get a $\widetilde{\gamma}$ and a t with $\langle\mathcal{G},\widetilde{\gamma},t\rangle \models P$. Then $\langle h^\sharp,\widetilde{\gamma},p(t)\rangle \models P$. And $h^\sharp$ is finite and of depth $< \sharp A(\sharp\,\mathrm{var}(P), \mathrm{dg}(P)) + n$. Hence we know that an upper bound for a finite model of minimal size for P can be given in advance. ⊣

For logics containing $\mathsf{S4}.Ti^2(0)$ note that $f \in \mathrm{CW}(g)$ is equivalent to $f \in \mathrm{W}^\circ(g)$ which is an elementary concept. Consequently, every extension of $\mathsf{S4}.Ti^2(0)$ determines an elementary class of frames. Theorem 7.4.1 is a substantial strengthening with respect to fmp of Theorem 7.1.3. However, since $\mathsf{S4}.Ti^2(2) \subseteq \mathsf{S4}_3$ (see Theorem 6.7.2) and the latter has $2^{\aleph_0}$ extensions ([Fine, 1974a]) not all extensions of $\mathsf{S4}.Ti^2(2)$ can be decidable. In addition, there is an extension of $\mathsf{S4}.J_3$ which has fmp, is recursively axiomatizable but not decidable ([Kracht, 1989]), solving a problem by [Urquhart, 1981]. It is not true that all logics of tightness 3 have fmp. A counterexample is given by the logic determined by the frame f.

f



Let us note in passing that we believe that Th(f) *bounds* the finite model property. Λ is defined to **bound** a property P if (i) Λ is not POST-complete and (ii) every proper extension of Λ has P while Λ lacks it. In [Schumm, 1981] it is shown that there exists a logic which bounds fmp, but no specific example is constructed. Th(f) is a likely candidate for such a logic. We lack a strict proof but it is not hard to see that if $g \in$CW(f) then either Th(g) = Th(f) or Th(g) has the finite model property.

7.5 Scattered Sketches

Definition 7.5.1 *Let* $\mathbf{X}$ *be a class of* ***K4****-frames. A sketch* Σ *is called* ***scattered*** *in* $\mathbf{X}$ *if for all* $k, n \in \omega$ *there is a finite set* $S = S(k, n)$ *of substitutions* $\sigma : \{p_s : s \in \mathbb{W}\} \to \mathcal{L}^i(\mathbb{P}_k)$ *such that if* $\mathcal{G} \in \mathbf{X}$ *and* $\langle \mathcal{G}, \gamma \rangle \models \{\sigma(\Box^{(1)}\Sigma^\sharp \to \neg p_r) : \sigma \in S\}$ *then there is an exact homogenization* $\langle \mathcal{H}, \widetilde{\gamma} \rangle$ *of* $\langle \mathcal{G}, \gamma \rangle$ *such that* $\mathcal{H}$ *omits* Σ*. We say in this case that* S ***forces*** *the omission of* Σ *from* $\langle \mathcal{G}, \gamma \rangle$*.*

If $\mathbf{X}$ = CW($\mathbf{X}$) it suffices to show that there is a finite set $S = S(k, n)$ which forces the omission of Σ from $\langle \mathcal{G}, \gamma \rangle$ if only $\langle \mathcal{G}, \gamma \rangle$ is k-good. For if $\langle \mathcal{H}, \delta \rangle$ is an arbitrary pair and $\langle \mathcal{H}, \delta \rangle \models \{\sigma(\Box^{(1)}\Sigma^\sharp \to \neg p_r) : \sigma \in S\}$ with $\mathcal{H} = \langle h, B \rangle$ then by defining $\gamma = \delta \restriction \mathbb{P}_k$, $A = [\gamma]$ and $\mathcal{G} = \langle h, A \rangle / \equiv$ we have that $\langle \mathcal{G}, \gamma \rangle$ is k-good and for every formula based on variables from $\mathbb{P}_k$, $\langle \mathcal{G}, \gamma \rangle \models P \Leftrightarrow \langle \mathcal{H}, \delta \rangle \models P$. Moreover, $\mathcal{G} \in$WC($\mathcal{H}$) $\subseteq \mathbf{X}$. So $\langle \mathcal{G}, \gamma \rangle \models \{\sigma(\Box^{(1)}\Sigma^\sharp \to \neg p_r) : \sigma \in S\}$ and since S forces the omission of Σ from $\langle \mathcal{G}, \gamma \rangle$ there is an exact homogenization $\langle \mathcal{K}, \epsilon \rangle$ omitting Σ. $\langle \mathcal{K}, \epsilon \rangle$ is an exact homogenization of $\langle \mathcal{H}, \delta \rangle$.

Theorem 7.5.2 *If* Σ *is scattered in* $\mathbf{X}$ *and* $\mathbf{X}$ = CW*(*$\mathbf{X}$*) then if* $\Lambda \supseteq$ ***K4*** *is* $\mathbf{X}$*-complete, so is* Λ_Σ*.*

Proof. Suppose P is Λ_Σ-consistent. Assume that P is based on variables from $\mathbb{P}_k$ and is of modal degree n. Then there is a finite set S which forces the omission of Σ. Now let

$P^\sharp := \{\sigma(\Box^{(1)}(\Box^{(1)}\Sigma^\sharp \to \neg p_r)) : \sigma \in S\}$. $P; P^\sharp$ is Λ_Σ-consistent and therefore Λ-consistent; and since it is finite, it has a model $\langle \mathcal{G}, \gamma, s\rangle \models P; P^\sharp$. Take $\mathcal{K} = \mathrm{Tr}_{\mathcal{G}}(s)$. Then $\mathcal{K} \in \mathbf{X}$ and $\langle \mathcal{K}, \gamma\rangle \models P^\sharp$ as well as $\langle \mathcal{K}, \gamma, s\rangle \models P$. There is an exact homogenization $\langle \mathcal{H}, \widetilde{\gamma}\rangle$ such that $\mathcal{H}$ omits Σ. Then $\mathcal{H} \nvDash \neg P$ and $\mathcal{H} \in \mathbf{X}$. ⊣

Clearly, if Σ is scattered in $\mathbf{X}$ then Σ is scattered in any subclass $\mathbf{Y}$ of $\mathbf{X}$ and if $\mathbf{Y} = \mathrm{CW}(\mathbf{Y})$ then Λ_Σ is $\mathbf{Y}$-complete if only Λ is $\mathbf{Y}$-complete. Thus the larger the the class in which Σ is scattered the better. Preferably we therefore want to have that Σ is scattered in **G4**, the class of transitive K4-frames.

Call a frame f **cycle-free** if $f \models \Box^m\bot$ for some $m \in \omega$. Call f **solid** if for all clusters C, D which are distinct there exists a cluster T of depth 1 such that either $C \lhd T, D \ntriangleleft T$ or $C \ntriangleleft T, D \lhd T$. Say that f is **fat** if every cluster is of cardinality > 1 and **meager** if every cluster is of cardinality 1. In [Kracht, 1990a] the following result is proved.

Theorem 7.5.3 *Suppose that f is a finite, rooted and transitive frame. If $r \rightarrowtail f$ is a generated cycle-free subframe such that the complement $f \setminus r$ of r in f is solid, then the frame-sketch* $\mathrm{Sp}(f)$ *is scattered in* ***F4****, the class of finite, transitive Kripke-frames.* (⊣)

The proof given there does with minor alterations prove that such frame-sketches are scattered in **G4**. We will show this in some specific cases to give the reader a flavour of the methods involved.

Theorem 7.5.4 *Suppose that f is rooted, transitive and cycle-free. Then* $\mathrm{Sp}(f)$ *is scattered in* ***G4*** *and elementary in* ***G4****.*

Proof. Let $\langle \mathcal{G}, \gamma\rangle$ be k-good. Suppose that there is a $\mathcal{H} \rightarrowtail \mathcal{G}$ such that $p : \mathcal{H} \twoheadrightarrow f^\sharp$. Then $\mathcal{H}$ is cycle-free and of depth $d(f)$. So we have $\mathcal{H} = h^\sharp$ for $h \subseteq \mathcal{G}_\sharp$ and $\langle h, \gamma\rangle$ is refined. Thus for every $s \in f$ there exists a Q^γ_s of modal degree $\leq 2^{d(f)+1} - 2$ such that $\gamma(Q^\gamma_s) = \{s\}$. Hence, if we let S to be set of substitutions $\sigma : \{p_s : s \in f\} \to \mathcal{L}^i(\mathbb{P}_k)$ such that $\mathrm{dg}(\sigma(p_s)) \leq 2^{d(f)+1} - 2$ then if $\langle \mathcal{G}, \gamma\rangle$ is k-good, S forces $f^\sharp \notin \mathrm{CW}(\mathcal{G})$. That $\mathrm{Sp}(f)$ is elementary in **G4** is shown in [Benthem, 1989]. ⊣

Theorem 7.5.5 *If f is of depth 1,* $\mathrm{Sp}(f)$ *is scattered in* ***G4****. Moreover,* $\mathrm{Sp}(f)$ *is elementary in* $\boldsymbol{Rf4}^\bullet$*.*

Proof. Let f be a cluster of n points and $\langle \mathcal{G}, \gamma\rangle$ be k-good. If $f^\sharp \in \mathrm{CW}(\mathcal{G})$ then $\mathcal{G}$ contains a cluster C of depth 1 and cardinality $\geq m$. Let $p : C \twoheadrightarrow f$ be a p-morphism. Then there are formulae $Q_s, s \in f$ of modal degree ≤ 2 such that $\gamma(Q_s) = p^{-1}(s)$. Let then S be the set of substitutions such that $\sigma(p_s) \leq 2$. S is finite and if $\langle \mathcal{G}, \gamma\rangle$ is k-good S forces $f^\sharp \notin \mathrm{CW}(\mathcal{G})$. To see that $\mathrm{Sp}(f)$ is elementary in $\mathbf{Rf4}^\bullet$ note that all frames in that class are top-heavy and therefore $\mathcal{G} \in \mathbf{Rf4}^\bullet$ omits $\mathrm{Sp}(f)$ iff it contains no final cluster isomorphic to f. ⊣

In particular, if Λ is complete or has fmp then $\Lambda.1 = \Lambda/\boxed{\bullet}$ is complete or has fmp.

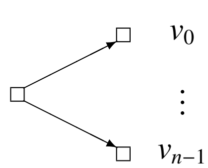
Theorem 7.5.6 *If f is meager and solid then* $\mathrm{Sp}(f)$ *is scattered in* ***G4****.*

Proof. Suppose that $\langle \mathcal{G}, \gamma\rangle$ is k-good and that for some $\mathcal{H} \rightarrowtail \mathcal{G}$ there is a $p : \mathcal{H} \twoheadrightarrow f^\sharp$. For a point of depth 1 in f there is a formula R_s of degree ≤ 2 such that $\mathcal{H}^1_\sharp \cap \gamma(R_s) = p^{-1}(s)$. then define for $s \in f$

$$\begin{array}{ll} Q^\gamma_s = R_s \wedge \Box\bot & \text{if } s \text{ has no successor in } f \\ Q^\gamma_s = \bigwedge\langle \Diamond R_t : s \lhd t\rangle \wedge \bigwedge\langle \neg\Diamond R_t : s \ntriangleleft t\rangle & \text{else.} \end{array}$$

Then $\mathrm{dg}(Q^\gamma_s) \leq 3$ and $\gamma(Q^\gamma_s) \cap \mathcal{H}_\sharp = p^{-1}(s)$. Let S be the set of all substitutions such that $\sigma(p_s) \leq 3$. S forces $f^\sharp \notin \mathrm{CW}(\mathcal{G})$. ⊣

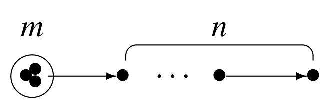
The same proof can be used for sketches $\Sigma = \langle \mathbb{W}, \prec, r; \alpha\rangle$ where $\langle \mathbb{W}, \prec, r\rangle$ is a properly branching tree and $\alpha = \bigwedge\langle d_f(v) = 1 : v \text{ is a leaf in } \langle \mathbb{W}, \prec, r\rangle\rangle$. Such sketches are therefore also scattered in **G4**. In addition, these sketches are elementary in $\mathbf{Rf4}^\bullet$. The most prominent example is $B(2)$ where $B(n)$ is defined by



$$\alpha \leftrightharpoons \&\langle d_f(v_i) = 1 : i \in n\rangle$$

We have $\mathsf{K4}_{B(2)} = \mathsf{K4.2}$. Consequently, if $\Lambda \supseteq \mathsf{K4}$ is complete (has fmp), $\Lambda.2$ is complete (has fmp).

Let $\Gamma(m,n)$ be the following sketch. Let $\mathbb{W} = \{v_i : i \in m+n\}$, $v_i \prec v_{i+1}, i \in m+n-1$, and $\alpha \leftrightharpoons v_{n-1} \lhd v_0 \& v_n \not\lhd v_0 \&.\& \langle v_j \not\lhd v_{j-1} : n \in j \in m+n \rangle$. In a Kripke-frame, $\Gamma(m,n)$ can be realized iff there is a cluster of cardinality $\geq n$ and depth $> m$. We get that $\mathsf{S4.Dum}$ is the logic $S4_{\Gamma(2,1)}$ and $\mathsf{Grz} = \mathsf{S4}_{\Gamma(2,0)}$. The next theorem tells us that adding the axioms Dum or Grz preserves fmp for $\Lambda \supseteq S4$. One might be tempted to conclude that Dum and Grz preserve fmp for for $\mathsf{K4}$. But this is not the case since they do not exactly correspond with the sketch omission logic in the nonreflexive case. For Grz is known to imply reflexivity and so $\Lambda_{\Gamma(2,0)} \neq \Lambda.\mathsf{Grz}$ unless $\Lambda \supseteq \mathsf{S4}$. Since it is not clear whether $p \to \Diamond p$ preserves completeness, we cannot conclude the more general theorem that Grz preserves completeness for $\mathsf{K4}$. Incidentally, if $f(m,n)$ is the frame pictured below, we get that for any $\Lambda \supseteq \mathsf{S4}$, $m > 0$, $\Lambda_{\Gamma(m,n)} = \Lambda_{f(m,n)} = \Lambda/f(m,n)$ and $\Lambda_{\Gamma(m,0)} = \Lambda_{f(m,0)} = \Lambda/\{f(m,0), f(m,1)\}$.

$f(m,n)$ 

Theorem 7.5.7 *$\Gamma(m,n)$ is scattered and elementary in **F4**.*

Proof. Let $\langle g, \gamma\rangle$ be k-good. Suppose there is a cluster C of depth $> m$ and cardinality $\geq n$. It is clear by Corollary 7.2.4 that it suffices to force that this cluster contains a non-maximal point. So suppose that there are $\geq n$ maximal points in C and let $s_i, i \in n$, be among them. Then there is a strict chain of points $s_{n-1} \lhd s_n \lhd \ldots \lhd s_{m+n-1}$ such that $d_g(s_i) = m+n-i$. For $i \geq m$ there are formulas Q_i^γ such that $\gamma(Q_i^\gamma) = \{s_i\}$ and $\mathrm{dg}(Q_i^\gamma) \leq 2^{m+1} - 2$. Now for each $s_i, i \in n$, there is a formula QM_i such that $s_j \in QM_i \Leftrightarrow i = j$. Take $Q_i^\gamma = QM_i \wedge \Diamond QM_n$ for $i \in n$. Define $\sigma_\gamma : p(s(i)) \mapsto Q_i^\gamma$. Since there is an upper bound on these formulae, there is a finite set S of substitutions forcing that no cluster of depth $> m$ can consist of more than $n-1$ maximal points. ⊣

Theorem 7.5.8 *Suppose f is fat and linear. Then $Sf(f)$ is scattered and elementary in **F4**.*

Proof. Let $\langle \mathcal{G}, \gamma\rangle$ be finite and k-good. If there is a finite S which forces $f \notin \mathrm{CW}^\circ(\mathcal{G}^\mu)$ then $f \notin \mathrm{CW}^\circ(\langle \mathcal{H}, \gamma\rangle)$ for a suitable exact homogenization (see Corollary 7.2.4). But suppose $f \in \mathrm{CW}^\circ(\mathcal{G}^\mu)$. Then also $f \in \mathrm{CW}^\circ\mathcal{R}\langle \mathcal{G}, \gamma\rangle$ since the map $\mathcal{G}^\mu \twoheadrightarrow \mathcal{R}\langle \mathcal{G}, \gamma\rangle$ is depth discriminating and injective on each cluster. Finally, $f \in \mathrm{CW}^{\circ}\mathcal{R}\langle\mathcal{G},\gamma\rangle$ implies $f \in \mathrm{W}^{\circ}\mathcal{R}\langle\mathcal{G},\gamma\rangle$. If $f \overset{\subset}{\to} \mathcal{R}\langle\mathcal{G},\gamma\rangle$ then for each $s \in f$ we find a Q^{γ}_{s} such that $\langle\mathcal{R}\langle\mathcal{G},\gamma,\rangle,\gamma\rangle \models Q^{\gamma}_{s} \Leftrightarrow t = s$. Moreover, $\mathrm{dg}(Q^{\gamma}_{s}) \leq \mu(k,n)$. So if S is the set of substitutions with $\mathrm{dg}(\sigma(p_s)) \leq \mu(k,n)$ then S forces $f \notin \mathrm{W}^{\circ}\mathcal{R}\langle\mathcal{G},\gamma\rangle$ and hence $f \notin \mathrm{CW}^{\circ}(\mathcal{H})$ for a suitable exact homogenization of $\langle\mathcal{G},\gamma\rangle$. $\dashv$

Even though the homogenization has proven to be a powerful tool, it is still not sophisticated enough to yield this result which we believe to be correct, namely that $\mathrm{Sp}(f)$ and $Sf(f)$ are both scattered in **F4** for fat frames. Another problem is whether a sketch which is elementary in **X** is also scattered in **X**. Finally, does $p \to \Diamond p = Sf(\boxed{\mathsf{X}})$ have any preservation properties? We know that $\Lambda_{\boxed{\mathsf{x}}} = \Lambda/\{\boxed{\mathsf{X}}, \boxed{\mathsf{X}\longrightarrow\bullet}\}$ ([Rautenberg, 1979]) and that $\mathrm{Sp}(\boxed{\mathsf{X}})$ is scattered in **G4**, so this question reduces to the same problem for $\mathrm{Sp}(\boxed{\mathsf{X}\longrightarrow\bullet})$. We conjecture that $\boxed{\mathsf{X}\longrightarrow\bullet}$ is not scattered even in **F4**, but that it is scattered in **G4.3**.

7.6 More Preservation Properties

Let A be a property of logics. Let us say that a logic Λ **preserves A beyond** Θ if, whenever Ξ contains Θ and has A, so does $\Xi \sqcup \Lambda$. And let us say that a sketch Σ **preserves A beyond** Θ if Θ_Σ preserves A beyond Θ. Since for all $\Xi \supseteq \Theta$ $\Xi_\Sigma = \Xi \sqcup \Theta_\Sigma$, Theorem 7.5.2 can be rephrased as saying that if Σ is scattered in $\mathbf{X} \subseteq \mathbf{G4}$ then Σ preserves **X**-completeness beyond $\mathsf{K4}$. The results proved in the previous section do, however, prove more than that. Let us agree to call a sketch **extra scattered** if there is a recursive function $f(k,n)$ such that for all $k,n \in \omega$ the set of all substitutions $\sigma : \{p_s | s \in \mathbb{W}\} \to \mathcal{L}^i(\mathbb{P}_k)$ with $\mathrm{dg}(\sigma(p_s)) \leq f(k,n)$, $s \in \mathbb{W}$, which forces the omission of Σ from any model based on an **X**-frame. Then in the previous section we have not only shown that the various sketches are scattered but also that they are extra scattered so that various other properties follow.

Theorem 7.6.1 *Suppose that Σ is extra scattered in **X**. Then Σ preserves "is **X**-complete and decidable" beyond $\mathsf{K4}$.*

Proof. If Σ is extra scattered there exists for every P a computable finite set $P^\sharp$ such that

$$\vdash_{\Lambda_\Sigma} P \Leftrightarrow \models_{\mathbf{X}_\Sigma} P \Leftrightarrow \models_{\mathbf{X}} P; P^\sharp \Leftrightarrow \vdash_{\Lambda} P; P^\sharp$$

The right hand side is decidable; consequently $\vdash_{\Lambda_\Sigma} P$ is decidable. ⊣

Theorem 7.6.2 *Suppose that Σ is extra scattered in $\boldsymbol{X} \subseteq \boldsymbol{F4}$. Then Σ preserves " has a priori bounds for models" beyond $\mathsf{K4}$.*

Proof. Suppose that there is a function g such that for every formula P which is Λ-consistent there is a Λ-model for P based on no more than $g(P)$ points. Then let $h(P) := g(P; P^\sharp)$. Then h is an upper bound on the size of Λ_Σ-models. ⊣

The bounds which can be given on the basis of this theorem are of course mostly much higher than necessary, specifically in the case of subframe logics where we have already given much better estimates.

These preservation results are not only interesting because of their unusual nature; for they do not directly prove the properties of a specific logic but rather show how they are transferred from another logic. They are also interesting because the methods involved are purely constructive and in the case of fmp also finitistic. This is particularly useful in the case of fmp. Most logics are known to have fmp by showing how to extract a finite model from the canonical frame. But the canonical frame cannot be effectively constructed in the general case and is also mostly not finite. So for a constructivist canonical models are non-existent. Tableau methods on the other hand are constructive but tableau systems have been developed only for very few logics. This is the point where our techniques become useful. For if $\Lambda \supseteq \mathsf{K4}$ has fmp and we have a tableau system for Λ and if finally Σ is extra scattered in **F4**, then there is a tableau system for Λ_Σ. Simply start a tableau for P by throwing in the set $P^\sharp$ and then apply the rules of the Λ-tableau. If the tableau closes then $P^\sharp \vdash_\Lambda \neg P$, whence $\vdash_{\Lambda_\Sigma} \neg P$; but if the tableau does not close there is a finite Λ-model for $P; P^\sharp$ constructed from the tableau. From this we can construct a finite Λ_Σ-model for P. Of course, constructivity does not depend on tableaux and other ways of constructing models are just as good.

Theorem 7.6.3 *Suppose that Σ is extra scattered in $\boldsymbol{X}$. Then Σ preserves "decidability plus fmp can be proved constructively".* ⊣

Corollary 7.6.4 *If $\mathbb{S}$ is a finite set of sketches extra scattered in $\boldsymbol{F4}$ then there is a construtive and finitistic proof that $\mathsf{K4}_{\mathbb{S}}$ has fmp and is decidable. Thus it is constructively valid that a finitely axiomatizable subframe logic has fmp and is decidable.* ⊣

For a proof it suffices to prove that $\mathsf{K4}$ has these properties. For example [Rautenberg, 1983] proves this. Moreover, as [Kracht, 1990b] points out, there is a constructive way to show that all subframe logics have fmp. For if $\mathbb{N}$ is a set of finite frames and $P \in F(k, n)$ a formula, then let $\mathbb{N}^P := \{f \in \mathbb{N} | \sharp f \leq \sharp\mathcal{F}_{\mathbf{K4}}^{\leq \sharp A(k,n)}\}$. Then we have $K4_{\mathbb{N}} \nvdash \neg P \Leftrightarrow \mathsf{K4}_{\mathbb{N}^P} \nvdash \neg P$ by the bounds for subframe logics established earlier. Although we further believe that all subframe logics are finitely axiomatizable and therefore decidable we believe that this is *not* constructively valid.

Symbols

$\rightarrow$, 16
$\rightarrowtail$, 16
$\twoheadrightarrow$, 16
$\otimes$, $\oplus$, 16
$\leftrightharpoons$, 16

$\mathcal{L}^i$, 21
$\mathbb{P}_\omega$, 21
$\mathrm{dg}(\cdot)$, 21
$\overline{\gamma}$, 22
$[\gamma]$, 23
$\mathrm{Th}(\cdots)$, 23
$\lhd$, 23
$\mathrm{Tr}(\cdot)$, 24
$\mathrm{Fr}(\cdot)$, 24
C, W, $\mathrm{U\small{E}}$, $\mathrm{C\small{P}}$, 24
$(\cdot)_+$, 24
$(\cdots)^\sharp$, 25
$\mathcal{L}^e$, 25
$\mathbb{W}_\omega$, 25
ϵ, 25
t, f, $\sim$, &, $\supset$, $\equiv$, 25
$\mathcal{L}^f$, 26
$(\exists w \rhd v)$, $(\forall w \rhd v)$, 26
$v \lhd^n w$, 26
F, 28
$\mathrm{Fr}_\Lambda(\cdot)$, 28
$s/\equiv$, 29

$[\![\alpha]\!]$, 33

$\star$, 38
$\mathrm{Tr}^k(\cdot)$, 41

$C \underset{\mathrm{X}}{\leftrightsquigarrow} D$, 43

Σ^e, 49

${}^{\flat(\cdot)}$, 59
$\langle \mathbb{W}, \prec, r; \alpha \rangle$, 60
$\Sigma^\sharp$, 60
$\mathcal{V}_\Sigma$, 61
Λ_Σ, 62
$\overset{\subset}{\rightarrow}$, 63
$\mathrm{SF}(\cdot)$, 63
Λ_h, 64
Λ/f, 65
$\mathrm{Sp}(\cdot)$, 65
$D(f, \Delta)$, 65

$\downarrow$, 68
$d_f(\cdot)$, 69
Δ_n, 70
$(\cdot)^d$, 70
$\mathbb{P}_k$, 71
$\ulcorner \alpha \urcorner = P$, 71
$\mathrm{Dp}^{\leq d}$, 74
$\delta(k)$, 75
$\mathrm{Mo}(k, n)$, 77
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$(\cdot)^q$, 78
$\mathcal{R}$, 78

QM_s, 80
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Bibliography

[Bellissima, 1988] Fabio Bellissima. On the lattice of extensions of the modal logics $K.Alt_n$. *Archive for Mathematical Logic*, 27:107–114, 1988.

[Bellissima, 1990] Fabio Bellissima. Post complete and 0–axiomatizable modal logics. *Annals of Pure and Applied Logic*, 47, 1990.

[Benthem, 1976] Johan van Benthem. Modal reduction principles. *The Journal of Symbolic Logic*, 41:301–312, 1976.

[Benthem, 1983] Johan van Benthem. *Modal and Classical Logic*. Bibliopolis, 1983.

[Benthem, 1984] Johan van Benthem. Correspondence theory. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 167–247. Reidel, 1984.

[Benthem, 1989] Johan van Benthem. Notes on Modal Definability. *Notre Dame Journal of Formal Logic*, 39:20–39, 1989.

[Blok, 1976] Wim Blok. *Varieties of interior algebras*. PhD thesis, University of Amsterdam, 1976.

[Blok, 1978] W. J. Blok. On the degree of incompleteness of modal logics. *Bulletin of the Section of Logic of the Polish Academy of Sciences*, 7:167 – 175, 1978.

[Bull and Segerberg, 1984] Robert A. Bull and Krister Segerberg. Basic modal logic. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 1–88. Reidel, 1984.

[Bull, 1966] Robert A. Bull. That all normal extensions of S4.3 have the finite model property. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 12:341–344, 1966.

[Fine, 1971] Kit Fine. The logics containing S4.3. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 17:371–376, 1971.

[Fine, 1974a] Kit Fine. An ascending chain of S4 logics. *Theoria*, 40:110–116, 1974.

[Fine, 1974b] Kit Fine. An incomplete logic containing S4. *Theoria*, 40:23–29, 1974.

[Fine, 1974c] Kit Fine. Logics containing K4, Part I. *The Journal of Symbolic Logic*, 39:229–237, 1974.

[Fine, 1975] Kit Fine. Some connections between elementary and modal logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 15–31. North–Holland, 1975.

[Fine, 1985] Kit Fine. Logics containing K4, Part II. *The Journal of Symbolic Logic*, 50:619–651, 1985.

[Gabbay, 70] Dov M. Gabbay. Selective filtration in modal logic. *Theoria*, 30:323–330, 70.

[Goldblatt, 1975] Robert I. Goldblatt. First–order definability in modal logic. *The Journal of Symbolic Logic*, 40:35–40, 1975.

[Goldblatt, 1976] Robert I. Goldblatt. Metamathematics of modal logics, Part I. *Reports on Mathematical Logic*, 6:41–78, 1976.

[Goldblatt, 1989] Robert I. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 38:173–241, 1989.

[Goldblatt, 1991] Robert I. Goldblatt. The McKinsey axiom is not canonical. *The Journal of Symbolic Logic*, 56:554 – 562, 1991.

[Hintikka, 1962] Jaakko Hintikka. *Knowledge and Belief. An Introduction into the logic of the two notions*. Cornell University Press, Ithaca, 1962.

[Jónsson and Tarski, 1951] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 73:891–939, 1951.

[Kracht, 1989] Marcus Kracht. Logics of width2 lacking the finite model property. Manuscript, 1989.

[Kracht, 1990a] Marcus Kracht. An almost general splitting theorem for modal logic. *Studia Logica*, 49:455–470, 1990.

[Kracht, 1990b] Marcus Kracht. That fine-splittings are definable. Manuscript, 1990.

[Kripke, 1963] Saul Kripke. Semantic analysis of modal logic, Part I. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[Kruskal, 1960] J. B. Kruskal. Well–quasi–orderings, the Tree Theorem and Vazsonyi's Conjecture. *Transactions of the American Mathematical Society*, 95:210–225, 1960.

[Lachlan, 1974] A. H. Lachlan. A note on Thomason's refined structures for tense logics. *Theoria*, 40:117–120, 1974.

[Maksimova, 1975] Larisa L. Maksimova. Pretabular extensions of Lewis S4. *Algebra i Logica*, 14:28–55, 1975.

[McKinsey and Tarski, 1944] J. C. C. McKinsey and Alfred Tarski. The algebra of topology. *Annals of Mathematics*, 45:141 – 191, 1944.

[Rautenberg, 1977] Wolfgang Rautenberg. Der Verband der normalen verzweigten Modallogiken. *Mathematische Zeitschrift*, 156:123–140, 1977.

[Rautenberg, 1979] Wolfgang Rautenberg. *Klassische und nichtklassische Aussagenlogik*. Vieweg, Braunschweig/Wiesbaden, 1979.

[Rautenberg, 1980] Wolfgang Rautenberg. Splitting lattices of logics. *Archiv für Mathematische Logik*, 20:155–159, 1980.

[Rautenberg, 1983] Wolfgang Rautenberg. Modal tableau calculi and interpolation. *Journal of Philosophical Logic*, 12:403–423, 1983.

[Sahlqvist, 1975] Hendrik Sahlqvist. First and second order semantics for modal logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 15–31. North-Holland, 1975.

[Sambin and Vaccaro, 1988] Giovanni Sambin and Virginia Vaccaro. Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37:249–296, 1988.

[Schumm, 1981] George Schumm. Bounded properties in modal logic. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 27:197–200, 1981.

[Segerberg, 1971] Krister Segerberg. An essay in classical modal logic. Mimeograph, Uppsala, 1971.

[Segerberg, 1975] Krister Segerberg. That all extensions of S4.3 are normal. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 194–196. North-Holland, 1975.

[Thomason, 1972] S. K. Thomason. Semantic analysis of tense logic. *The Journal of Symbolic Logic*, 37:150–158, 1972.

[Thomason, 1974] S. K. Thomason. An incompleteness theorem in modal logic. *Theoria*, 40:30–34, 1974.

[Urquhart, 1981] Alasdair Urquhart. Decidability and the finite model property. *Journal of Philosophical Logic*, 10:367–370, 1981.

[Vakarelov, 1988] Dimiter Vakarelov. Modal logics for knowledge representation systems. Technical Report SU/LAL/7, Laboratory for Applied Logic, University of Sofia, 1988.

[Zakharyaschev, 1987] Michael Zakharyaschev. Syntax and semantics of modal logics containing S4. *Algebra i Logika*, 27:659–689, 1987.

Interne Definierbarkeit und Vollständigkeit in der Modalen Logik

Es sei **X** eine Klasse von verallgemeinerten Kripke-Strukturen und α eine Eigenschaft von Welten. Wir sagen, α sei in **X** *intern definierbar*, falls es eine modale Aussage P gibt derart, daß die Eigenschaft α auf eine Welt einer Struktur aus **X** genau dann zutrifft, wenn diese P unter jeder Belegung erfüllt. Ebenso sagen wir, eine n-stellige Relation ρ sei in **X** *intern definierbar* falls es ein n-Tupel $\langle P_i : i \in n \rangle$ gibt derart, daß ρ auf ein n-Tupel $\langle w_i : i \in n \rangle$ von Welten einer Struktur aus **X** genau dann zutrifft, wenn zugleich die w_i jeweils P_i unter jeder Belegung erfüllen. Die vorliegende Dissertation beschäftigt sich sowohl mit der Charakterisierung intern definierbarer Eigenschaften und Relationen in Abhängigkeit von der Klasse **X** als auch mit den verschiedenen Anwendungen der Theorie der internen Definierbarkeit. Diese Theorie vereinigt nämlich Vollständigkeitstheorie und Korrespondenztheorie auf natürliche Weise und baut so einen Bogen über diese beiden “Säulen der Weisheit in der Modallogik” (van Benthem). Denn einerseits ist die Theorie der intern definierbaren *elementaren* Relationen nichts anderes als die Korrespondenztheorie; andererseits lassen sich fast alle Vollständigkeitsbeweise uniform auf das Problem zurückführen, bestimmte Eigenschaften von Welten in bestimmten Klassen von verallgemeinerten Kripke-Strukturen intern zu definieren. Ein zusätzlicher Gewinn unserer Methoden ist es, daß die meisten Beweise konstruktiv sind und daß wir die endliche Modelleigenschaft beweisen können, ohne einen Umweg über unendliche Modelle und speziell kanonische Modelle gehen zu müssen, und daß wir obere Schranken für die Größe von Modellen angeben können.

Lebenslauf

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