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Internal Energy, Specific Heat and Correlation Function of the Bond-Random Ising Model

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Gauge transformations of the random Ising model are shown to be useful in obtaining rigorous results on thermodynamic quantities. In a restricted region of the phase diagram, we obtain the exact value of the internal energy, rigorous upper bound to the specific heat and a few rigorous relations concerning the correlation functions. In particular it is pointed out that the phase boundary between the paramagnetic and ferromagnetic phases has a singular shape if the spin glass phase does not exist.

§ 1. Introduction

Several years ago Toulouse[1] pointed out the importance of a gauge symmetry in characterizing the spin glass phase of random spin systems. Since then many papers appeared to examine the effects of gauge symmetries and frustrations. Most of them are centered on approximate (numerical or phenomenological, for instance) treatments, and explicit usefulness of gauge transformations and frustrations in deriving rigorous results was only recently pointed out.[2)~5)] The aim of the present paper is to generalize and unify some of the rigorous results reported in the literature;[2)~4)] gauge transformations will be shown to be useful in obtaining rigorous results on thermodynamic functions such as the internal energy, specific heat and correlation functions. Also we give a refinement of the correlation inequality recently proved by Horiguchi and Morita.[6)] By using the refined relation we show that the phase boundary between the paramagnetic and ferromagnetic phases should have a singular shape in the absence of the spin glass phase.

In the next section gauge transformations are defined for the bond-random Ising model. We utilize the transformations in § 3 to derive rigorous results on physical quantities. Here the usefulness of gauge transformations is explicitly recognized. In § 4 we present a refinement of the correlation inequality of Horiguchi and Morita. The physical significance of the inequality is discussed in detail. A few remarks are made in the last section.

§ 2. Gauge transformation

Our model Hamiltonian is

$$H = -\sum J_{ij} S_i S_j \,, \qquad (S_i = \pm 1) \tag{1}$$

where the interaction J_{ij} is randomly distributed at each bond with the probability function $P(J_{ij})$. Throughout the paper, $\sum$ (and $\prod$) without subscript denotes summation (and product) over all interacting pairs on an arbitrary lattice. Physical observables are averaged over $P(J_{ij})$. For instance the free energy (divided by $-kT$) is written as

$$F=\int_{-\infty}^{\infty}\prod dJ_{ij}P(J_{ij})\ln \Sigma_S \exp(\beta \sum J_{ij}S_iS_j).$$

To define a gauge transformation, it is convenient to rewrite the configurational average (integration over J_{ij}) by restricting J_{ij} to the positive range and introducing a new variable $\tau_{ij}(=\pm 1)$:[5)]

$$F=\int_0^{\infty}\prod dJ_{ij}Q(J_{ij})\Sigma_\tau \exp(\sum \tau_{ij}A_{ij})\ln \Sigma_S \exp(\beta \sum \tau_{ij}J_{ij}S_iS_j), \qquad (2)$$

where

$$Q(J_{ij})=\{P(J_{ij})+P(-J_{ij})\}/2\cosh A_{ij}$$

and

$$\exp(-2A_{ij})=P(-J_{ij})/P(J_{ij}). \qquad (3)$$

Although the integration is restricted to the positive range, the summation over $\tau_{ij}(=\pm 1)$ covers the negative values of interactions. The factor $\exp(\tau A)$ is inserted in Eq. (2) to guarantee the relative weight of positive and negative bonds by Eq. (3). We define a gauge transformation as a change of variables, $S_i\to S_i\sigma_i$ and $\tau_{ij}\to\sigma_i\sigma_j\tau_{ij}$, where $\{\sigma\}$ is a set of arbitrarily fixed Ising variables. By the transformation the states of up ($S_i=+1$) and down ($S_i=-1$) spins are interchanged if $\sigma_i=-1$ and unchanged if $\sigma_i=+1$. Equivalently we may say that the standard of coordinate at each site is changed, and this is the reason we call the transformation "a gauge transformation". Since the summations over τ and S in Eq. (2) are performed over ± 1, a gauge transformation never affects the value of Eq. (2). Furthermore, the set $\{\sigma\}$ can be chosen arbitrarily. Therefore we may sum over all possible configurations of $\{\sigma\}$ and then divide by 2^N (the total number of possible configurations $\{\sigma\}$). Thus we have

$$F=\int_0^{\infty}\prod dJ_{ij}Q(J_{ij})2^{-N}\Sigma_\tau\Sigma_\sigma \exp(\sum \tau_{ij}A_{ij}\sigma_i\sigma_j)$$
$$\times\ln \Sigma_S \exp(\beta \sum \tau_{ij}J_{ij}S_iS_j). \qquad (4)$$

The expression (4) of the free energy indicates that the relative probability weight of bond configurations is given by $Q(J)$ times the partition function of the random Ising model (1) with the effective interaction $\tau_{ij}A_{ij}$. This observation yields numerous rigorous results on the thermodynamic functions as will be demonstrated in the next section.

§ 3. Physical quantities

3.1. *Internal energy*

By using the expression (4) of the free energy, we can rigorously calculate the internal energy in a subspace of the phase diagram.

The internal energy is essentially given by the derivative of Eq. (4) by β. By differentiating Eq. (4) we readily find that, if A_{ij} is equal to βJ_{ij}, the expectation value of the internal energy is reduced to a simple form

$$[\langle E\rangle]=-(\partial/\partial\beta)F$$
$$=-\int_0^\infty \prod dJ_{ij}Q(J_{ij})2^{-N}\Sigma_\tau(\partial/\partial\beta)\Sigma_S \exp(\beta \sum \tau_{ij}J_{ij}S_iS_j). \quad (5)$$

By summing over τ first and then over S, we can easily obtain

$$[\langle E\rangle]=-\sum \int_0^\infty dJ_{ij}\{P(J_{ij})+P(-J_{ij})\}J_{ij}\tanh\beta J_{ij}\,. \quad (6)$$

The expression on the right-hand side of Eq. (6) can further be simplified. By using the relation $A_{ij}=BJ_{ij}$, under which we have derived Eq. (6), we can verify that $\tanh\beta J_{ij}$ in the above integrand is equal to $\{P(J_{ij})-P(-J_{ij})\}/\{P(J_{ij})+P(-J_{ij})\}$ (see Eq. (3)). Thus Eq. (6) is rewritten as

$$[\langle E\rangle]=-\sum \int_0^\infty dJ_{ij}\{P(J_{ij})-P(-J_{ij})\}J_{ij}$$
$$=-\sum \int_{-\infty}^\infty dJ_{ij}J_{ij}P(J_{ij})$$
$$=-\sum[J_{ij}]=-\frac{1}{2}zN[J_{ij}], \quad (7)$$

where $zN/2$ is the total number of interacting pairs.

We give here a comment on the condition $A_{ij}=\beta J_{ij}$. According to Eq. (3), this relation is equivalently written as $\exp(-2\beta J_{ij})=P(-J_{ij})/P(J_{ij})$. This condition is satisfied if the distribution function has the form

$$P(J_{ij})=\exp(aJ_{ij})f(J_{ij}) \quad (8)$$

with $a=\beta$ and $f(J_{ij})=f(-J_{ij})$. The probability distribution (8) should originally be independent of the temperature, but we have to set $a=\beta$ to obtain Eq. (7). The constraint $a=\beta$ reduces one degree of freedom in the parameter space of the system and therefore it restricts us to a subspace of the phase diagram.

Let us give a few examples. First, consider the binary distribution $P(J_{ij})=p\delta(J_{ij}-J)+(1-p)\delta(J_{ij}+J)$. This distribution function can easily be rewritten into a form of Eq. (8). The parameter a in Eq. (8) is now K_p/J where K_p is defined by $\exp(-2K_p)=(1-p)/p$. Therefore the condition $\alpha=\beta$ for Eq. (7) to be valid is $K_p/J=\beta$, or $\exp(-2\beta J)=(1-p)/p$. This relation represents a curve in the T-p phase diagram (Fig. 1). Thus, according to Eq. (7), we obtain the exact value of the internal energy on the line $\exp(-2\beta J)=(1-p)/p$ as $[\langle E\rangle]=-zN(2p-1)/2$. It should be noted that this expression of the internal energy is free of singularities although the line $\exp(-2\beta J)=(1-p)/p$ clearly intersects the phase boundary (see Fig. 1). This is not necessarily a contradiction as will be demonstrated later.

If the distribution is Gaussian $P(J_{ij})=\exp\{-(J_{ij}-J_0)^2/2J^2\}/\sqrt{2\pi}J$, the condition $\alpha=\beta$ is verified to be represented as $\beta J^2=J_0$. This relation again gives a curve in the two-dimensional phase diagram (Fig. 2). On this curve the internal

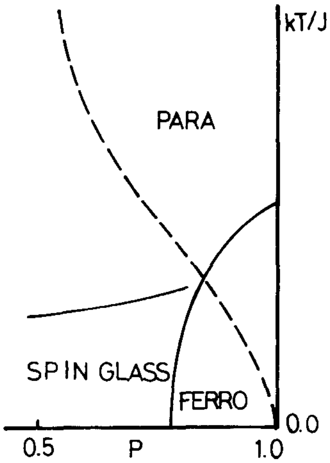

Fig. 1. The relation between the expected phase diagram of the $\pm J$ model and the line $\exp(-2\beta J)=(1-p)/p$ (dashed curve).

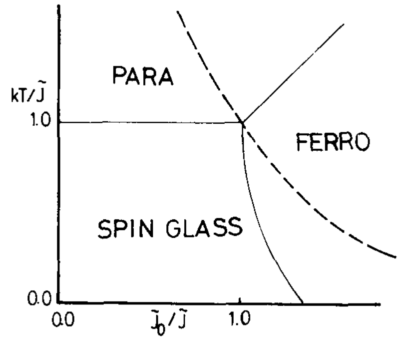

Fig. 2. The relation between the phase diagram of the long-ranged Gaussian model and the line $\beta\tilde{J}^2=\tilde{J}_0$ (dashed curve).

energy is $[\langle E\rangle]=-zNJ_0/2$ as is evident from Eq. (7). The long-range limit of the interaction is of particular interest because Sherrington and Kirkpatrick[7] solved it exactly (although some problematic limiting procedures were adopted). In the long-range limit we should scale the parameters as $J_0=\tilde{J}_0/N$, $J=\tilde{J}/\sqrt{N}$ and $z=N-1$ in order to guarantee the extensivity of physical quantities. The constraint $\beta J^2=J_0$ becomes $\beta\tilde{J}^2=\tilde{J}_0$, which is the dashed line in Fig. 2. On this line we have the internal energy as $[\langle E\rangle]=-N\tilde{J}_0/2$. For comparison with the solution of Sherrington and Kirkpatrick, we note that the relation $q=m$ (q and m are order parameters defined in their paper) holds on the line $\beta\tilde{J}^2=\tilde{J}_0$ as can be seen from the self-consistent equations given by Sherrington and Kirkpatrick. It is then straightforward to verify from $q=m$ that the internal energy is in fact given by $-N\tilde{J}_0/2$ on the line $\beta\tilde{J}^2=\tilde{J}_0$ (see their equation (16)).

Another example is provided by the diluted binary distribution[4] $P(J_{ij}) = p\delta(J_{ij}-J)+q\delta(J_{ij}+J)+r\delta(J_{ij})$. The constraint $\beta J_{ij}=A_{ij}$ (or $a=\beta$) is verified to take the form $\exp(-2\beta J)=q/p$. The internal energy is $[\langle E\rangle] = -zNJ(p-q)/2$. This is just the same as Horiguchi's expression.[4]

We remark that Eq. (7) is free of singularities. But this absence of singularities never implies the absence of phase boundary across the subspace defined by the constraint $a=\beta$. For instance, in the long-ranged Gaussian model, the line $\beta\tilde{J}^2=\tilde{J}_0$ (which is equivalent to $a=\beta$) is across the tricritical point where the paramagnetic, ferromagnetic and spin glass phases meet. However the internal energy $-N\tilde{J}_0/2$ is not singular at the tricritical point. A cancellation of singularities occurs when we confine ourselves to the line $\beta\tilde{J}^2=\tilde{J}_0$. Therefore, in general, we cannot assert absence of phase boundary though Eq. (7) is regular.

The examples presented above may give an impression that the formula (7) is of no use because the internal energy (7) is not singular and it fails to provide information on phase transition. Nevertheless we mention here two excuses for devoting so much space to the calculation of the non-singular internal energy: (i) Anyway the internal energy is obtained exactly for a general lattice. The result (7) would serve to check the validity of numerical calculations and full exact solutions in the future. (ii) An analogous way of consideration can be applied to calculate other physical quantities and derive more explicitly useful results (see the following sections).

3.2. *Specific heat*

The specific heat is the mean fluctuation of the internal energy. We are not able to calculate it exactly but it is possible to evaluate the upper bound and give a condition for the specific heat to be finite. Let us note that

$$kT^2[\langle C\rangle]=[\langle E^2\rangle]-[\langle E\rangle^2]. \tag{9}$$

It is easy to verify that, if $\beta J_{ij}=A_{ij}$ (or $a=\beta$), the first term on the right-hand side of Eq. (9) can be rigorously calculated in the same way as we did in deriving Eq. (7). This term $[\langle E^2\rangle]$ is given by replacing $\partial/\partial\beta$ in Eq. (5) by $\partial^2/\partial\beta^2$ and changing the overall sign of Eq. (5). There are no difficulties in summing over τ first and S in the resulting relation and then performing the differentiation $\partial^2/\partial\beta^2$. As for $[\langle E\rangle^2]$, we cannot carry through the calculation of it because there remains the partition function in the denominator in an equation corresponding to Eq. (5) even if $\beta J_{ij}=A_{ij}$ (or $a=\beta$). Hence we cannot sum over τ, which prevents us from calculating $[\langle E\rangle^2]$ exactly.

However, if we replace the term $[\langle E\rangle^2]$ of Eq. (9) by $[\langle E\rangle]^2$ and use the inequality $[\langle E\rangle^2]\geqq[\langle E\rangle]^2$, we have

$$kT^2[\langle C\rangle]\leqq[\langle E^2\rangle]-[\langle E\rangle]^2 .$$

In this equation both terms on the right-hand side are explicitly evaluated if $\beta J_{ij} = A_{ij}$ to yield (in the same way as we did from Eq. (5) to Eq. (7))

$$[\langle E^2\rangle]=\frac{1}{2}zN[J_{ij}^2]+\frac{1}{2}zN\left(\frac{1}{2}zN-1\right)[J_{ij}]^2$$

and

$$[\langle E\rangle]^2=\left(\frac{1}{2}zN[J_{ij}]\right)^2 .$$

Therefore the upper bound is

$$kT^2[\langle C\rangle]\leqq\frac{1}{2}zN\{[J_{ij}^2]-[J_{ij}]^2\}. \tag{10}$$

According to Eq. (10), all of the three distribution functions presented in the previous subsection have finite specific heats in the subspace $\beta J_{ij}=A_{ij}$ (or $\alpha=\beta$): The upper bounds for the binary, diluted binary and Gaussian distributions are

$$kT^2[\langle C\rangle]\leqq J^2 zN \operatorname{sech}^2 \beta J/2<+\infty ,$$

$$kT^2[\langle C\rangle]\leqq J^2 zN(4pq+pr+qr)/2<+\infty$$

and

$$kT^2[\langle C\rangle]\leqq zNJ^2/2<+\infty ,$$

respectively. We note here that the subspace $\alpha=\beta$, on which we have proved the finiteness of the specific heat, intersects the phase boundary (see Figs. 1 and 2). It means that the singularity of the specific heat at the phase boundary, if any, should be a weak one. For example, if the dashed curve in Fig. 1 intersects the paramagnetic-ferromagnetic phase boundary, the specific heat is not divergent at the intersection even when it diverges in the pure limit ($p=1$). The two-dimensional random Ising model is believed to have no spin glass phase[8)] and hence the above argument applies, that is, the curve $\exp(-2\beta J)=(1-p)/p$ should intersect the para-ferro boundary. This implies that the critical exponent α of the specific heat is dependent on p in two dimensions: $\alpha=0$ (log) if $p=1$ and $\alpha<0$ at the intersection (where $p<1$).

3.3. *Correlation function*

We can also prove a few rigorous relations on the correlation functions.

The first one is Griffiths' inequality (as was derived by Horiguchi for the diluted binary distribution[4)]):

$$[\langle S^X\rangle]\geqq 0 , \qquad (\text{if } \beta J_{ij}=A_{ij})$$

where S^X denotes an arbitrary product of spin variables. This inequality implies that the constraint $\beta J_{ij}=A_{ij}$ restricts us to a region where the interactions are rather ferromagnetic (see Figs. 1 and 2). The proof is simple. The argument which led us to Eq. (2) is applicable also to the correlation function to yield

$$[\langle S^X\rangle]=\int_0^\infty \prod dJ_{ij}Q(J_{ij})\Sigma_\tau \exp(\sum \tau_{ij}A_{ij})\frac{\Sigma_S S^X \exp(\beta\sum \tau_{ij}J_{ij}S_iS_j)}{\Sigma_S \exp(\beta\sum \tau_{ij}J_{ij}S_iS_j)}\,. \quad (11)$$

If we perform the gauge transformation here and sum over all $\{\sigma\}$, we find

$$[\langle S^X\rangle]=\int_0^\infty \prod dJ_{ij}Q(J_{ij})2^{-N}\Sigma_\tau\Sigma_\sigma\sigma^X \exp(\sum \tau_{ij}A_{ij}\sigma_i\sigma_j)\langle S^X\rangle_{\beta J_{ij}\tau_{ij}}$$

$$=\int_0^\infty \prod dJ_{ij}Q(J_{ij})2^{-N}\Sigma_\tau\Sigma_\sigma \exp(\sum \tau_{ij}A_{ij}\sigma_i\sigma_j)\langle \sigma^X\rangle_{A_{ij}\tau_{ij}}\langle S^X\rangle_{\beta J_{ij}\tau_{ij}}\,, \quad (12)$$

where the angular brackets represent the thermal average with the prescribed interactions. The final expression above can be replaced by

$$\int_0^\infty \prod dJ_{ij}Q(J_{ij})\Sigma_\tau \exp(\sum \tau_{ij}A_{ij})\langle \sigma^X\rangle_{A_{ij}\tau_{ij}}\langle S^X\rangle_{\beta J_{ij}\tau_{ij}}\,, \quad (13)$$

because gauge transformations are shown to bring Eq. (13) back into Eq. (12). Thus $[\langle S^X\rangle]$ is given by the configurational average of $\langle\sigma^X\rangle\times\langle S^X\rangle$. Since both σ and S are Ising variables, we find, if $\beta J_{ij}=A_{ij}$,

$$\langle \sigma^X\rangle_{A_{ij}\tau_{ij}}=\langle S^X\rangle_{\beta J_{ij}\tau_{ij}}\,,$$

and therefore

$$[\langle S^X\rangle]=[\langle S^X\rangle^2]\geqq 0\,. \qquad (\text{if } \beta J_{ij}=A_{ij})$$

Another interesting relation is

$$\left[\frac{1}{\langle S_X\rangle}\right]=1\,, \qquad (\text{if } \beta J_{ij}=A_{ij})$$

where S^X is again a product of arbitrary spin variables. For proof, we start with a relation corresponding to Eq. (11):

$$\left[\frac{1}{\langle S^X\rangle}\right]=\int_0^\infty \prod dJ_{ij}Q(J_{ij})\Sigma_\tau \exp(\sum \tau_{ij}A_{ij})\frac{\Sigma_S \exp(\beta\sum \tau_{ij}J_{ij}S_iS_j)}{\Sigma_S S^X \exp(\beta\sum \tau_{ij}J_{ij}S_iS_j)}\,.$$

Gauge transformations bring this into

$$\int_0^\infty \prod dJ_{ij}Q(J_{ij})\Sigma_\tau\sigma^X \exp(\sum \tau_{ij}A_{ij}\sigma_i\sigma_j)\frac{1}{\langle S^X\rangle}\,,$$

where the relation $\sigma^X=1/\sigma^X$ (since $(\sigma^X)^2=1$) has been used. Now we may sum over all $\{\sigma\}$, divide by 2^N and then put $\beta J_{ij}=A_{ij}$, as usual. It is found that

$$\left[\frac{1}{\langle S^X\rangle}\right]=\left[\frac{\langle\sigma^X\rangle}{\langle S^X\rangle}\right]=1\,. \qquad (\text{if } \beta J_{ij}=A_{ij})$$

We should remark that the above relation does not imply the absence of phase boundaries across the subspace $A_{ij}=\beta J_{ij}$. The reason is the same as that for the internal energy.

§ 4. Refinement of Horiguchi and Morita's inequality

A correlation inequality was recently proved by Horiguchi and Morita[6] for classical random spin systems. Their inequality restricts the region where the ferromagnetic phase can exist in the phase diagram. We present here a greatly simplified derivation of their result and a refinement of the inequality. Our refined relation gives far more information on the phase diagram than one expects from its simplicity. To avoid inessential complications, we restrict ourselves to the random Ising model with the binary distribution function $P(J_{ij})=p\delta(J_{ij}-J)+(1-p)\delta(J_{ij}+J)$. Only the expectation value of a single site spin $[\langle S_0\rangle]$ is explicitly considered. Some of these restrictions are easily removed. We comment on the Gaussian distribution later.

Suppose that a random Ising model with the binary distribution function is on a finite set of sites Λ:

$$H=-J\sum \tau_{ij}S_iS_j \qquad (J>0,\ \tau_{ij},\ S_i=\pm 1) \tag{14}$$

with the boundary condition that $S_i=+1$ at all boundary sites. We call this boundary condition B hereafter. Let us define the spontaneous magnetization of this system by

$$m(\beta J,\ p)=\lim_{N\to\infty}[\langle S_0\rangle], \tag{15}$$

where N is the number of sites in Λ and 0 is an arbitrarily chosen site in Λ. Rigorously speaking, the definition (14) of the spontaneous magnetization is not proved to be equivalent to the ordinary definition ($h\to +0$ after $N\to\infty$ with free or periodic boundary). However, physically, our special boundary B picks out one of the two degenerate states brought about by the ferromagnetic symmetry breakdown (if any) at low temperatures. And in this sense the condition B is equivalent to the limiting procedures $h\to +0$ after $N\to\infty$ with free or periodic boundary. Moreover, it is possible to show[9] that the ordinary spontaneous magnetization is not vanishing if Eq. (15) is non-zero. Therefore we may adopt Eq. (15) as an alternative definition of the spontaneous magnetization.

Horiguchi and Morita's inequality is

$$0\leqq|m(\beta J,\ p)|\leqq m(K_p,\ 1) \tag{16}$$

with K_p defined by $\exp(-2K_p)=(1-p)/p$. Our refined relation to be proved below is

$$0 \leqq |m(\beta J, p)| \leqq g(K_p, p), \tag{17}$$

where $g(K_p, p)$ is defined by confining the following function $g(\beta J, p)$:

$$g(\beta J, p) \equiv \lim_{N\to\infty} [|\langle S_0 \rangle|] \quad \text{(with the boundary } B\text{)} \tag{18}$$

to the subspace $\beta J = K_p$ (or $\exp(-2\beta J) = (1-p)/p$, the dashed line in Fig. 1). A relation similar to Eq. (17) is already found by Horiguchi and Morita in the course of their proof of the inequality (16), but they did not notice the importance of the relation (17).

It is quite easy to prove Eqs. (16) and (17). The expectation value of a single site spin is written as, according to Eq. (13),

$$[\langle S_0 \rangle_{\beta J}] = [\langle \sigma_0 \rangle_{K_p} \langle S_0 \rangle_{\beta J}], \tag{19}$$

where the subscripts βJ and K_p denote the absolute value of the interaction strength at which the thermal average is evaluated. In Eq. (19), $\langle \sigma_0 \rangle$ is evaluated at K_p because A_{ij} in Eq. (13) is equal to K_p for the binary distribution (see Eq. (3)). The boundary condition B for $\langle S_0 \rangle$ is inherited by $\langle \sigma_0 \rangle$ since the variables $\{\sigma\}$ are introduced by gauge transformations; to leave the Hamiltonian (14) invariant, we should choose $\sigma_i = +1$ at the boundary. From Eq. (19) it follows that

$$0 \leqq |[\langle S_0 \rangle_{\beta J}]| \leqq [|\langle \sigma_0 \rangle_{K_p}|],$$

the thermodynamic limit of which is exactly Eq. (17). Equation (16) follows from Eq. (17) because $g(K_p, p) \leqq m(K_p, 1)$,[13] as was pointed out by Horiguchi and Morita.[6]

We remark that $g(\beta J, p)$ as defined by Eq. (18) is not the order parameter of the ROP (random ordered phase) introduced by Ueno and Oguchi.[10] Although the expression (18) is quite similar to the ROP order parameter,[10] the boundary condition (or the direction of the ordering field) is quite different. The boundary condition B in Eq. (18) is physically equivalent to a uniform external field while the ROP order parameter is defined under a "random field" applied at each site along one of the directions which the spin favors in the ground-state.

To demonstrate the usefulness of the relation (17), it is convenient to interprete Eq. (17) in the following way. In the paramagnetic phase the expectation value of a single site spin vanishes in the thermodynamic limit for any boundary conditions. Since the function g in Eq. (17) is the expectation value of S_0, $g(K_p, p)$ vanishes when the line $\beta J = K_p$ (or $\exp(-2\beta J) = (1-p)/p$) is in the paramagnetic phase. Equation (17) is thus understood to provide a criterion to determine whether a point $C(\beta J, p)$ in the phase diagram is in the ferromagnetic region or not; if the line $\beta J = K_p$ is in the paramagnetic phase at the same p as the point C, then $m=0$ at C irrespective of βJ. This criterion is more conveniently restated in the following way. If the line $\beta J = K_p$ intersects the phase boundary between the paramagnetic and ordered (ferromagnetic or spin glass) phases at $p = p_0$, then the line $\beta J = K_p$ lies in the paramagnetic phase for $1/2 < p < p_0$ by topological considerations (see Fig. 1). It readily follows that the spontaneous magnetization is absent for $1/2 < p < p_0$. A few interesting examples are given below.

Let us suppose that the spin glass phase does not exist at finite temperatures. The random Ising model on the square lattice is expected to have this property.[8] If we denote the boundary curve (between the ferromagnetic and paramagnetic phases) by $T = T(p)$, it can be proved that $dT(p)/dp = \infty$ at $p = p_0$ (Figs. 3(a) and (b)) or the boundary $T = T(p)$ is non-differentiable at $p = p_0$ (Fig. 3(c)); otherwise (if $dT(p)/dp$ exists at $p = p_0$ and is finite there) the ferromagnetic long-range order m becomes non-vanishing for $p < p_0$ (Figs. 3(d) and (e)), which is a contradiction. If $dT(p)/dp$ diverges at p_0, two possibilities exist on the shape of phase boundary. First, the curve $T = T(p)$ itself may have no singularities (Fig. 3(a)). In this case, for p fixed slightly greater than p_0, the system experiences two phase transitions, as the temperature is lowered from infinity, from $m = 0$ to $m \neq 0$ and then to $m = 0$ again. This means that the paramagnetic state is more stable than the ferromagnetic one at low temperatures. The second possibility is

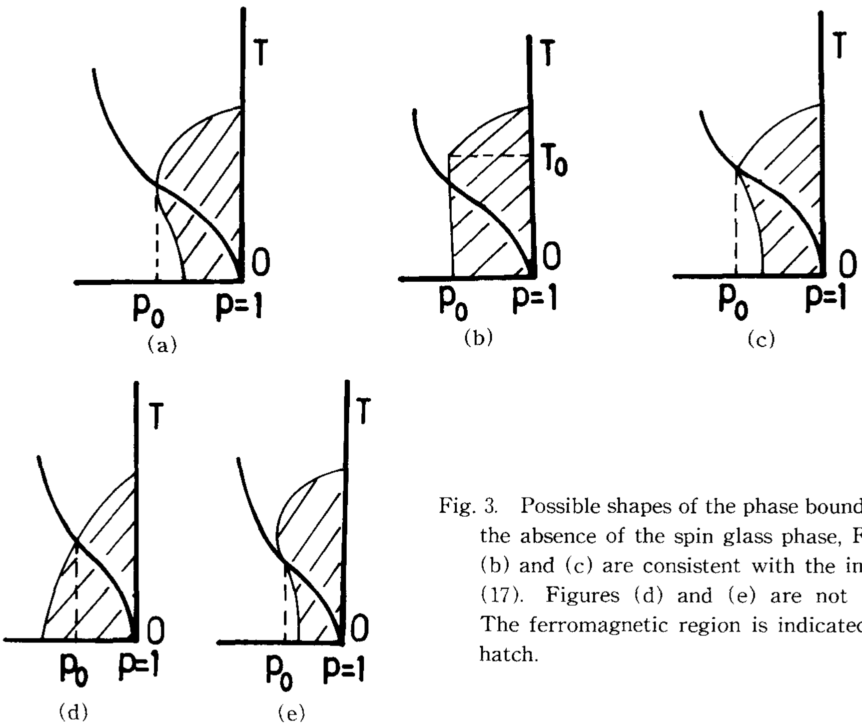

Fig. 3. Possible shapes of the phase boundary. In the absence of the spin glass phase, Figs. (a), (b) and (c) are consistent with the inequality (17). Figures (d) and (e) are not allowed. The ferromagnetic region is indicated by the hatch.

illustrated in Fig. 3(b); the phase boundary may have a singular point at ($p=p_0$, $T=T_0$) below which ($T<T_0$) the boundary is a vertical line. If this is the case, the paramagnetic phase is no more stable than the ferromagnetic one at low temperatures.

If the spin glass phase exists, Eq. (17) provides restrictions on the shape of the phase diagram in a somewhat different manner. First, if the line $\beta J=K_p$ intersects the phase boundary of the ferromagnetic and paramagnetic phases, the same argument as before is applied to prove the vertical slope (or non-differentiability) of the boundary curve at the intersection of the boundary and the line $\beta J=K_p$. In this case the boundary between the ferromagnetic and spin glass phases should also lie in the region $p>p_0$ because $m=0$ when $p\geqq p_0$ (Fig. 4).

If the curve $\beta J=K_p$ intersects the boundary of the spin glass and paramagnetic phases at $p=p_0$ (Fig. 5), then the ferromagnetic phase cannot exist for $1/2<p<p_0$. A slightly stronger restriction may be given in this case. The function $g(K_p, p)$ in Eq. (17) is defined as the thermodynamic limit of the expectation value of σ_0 under the boundary condition B. Under the condition B, all spins surrounding the system are fixed to up states, which correspond to applying an infinitesimal external field on the free boundary system (see Ref. 9)). On the other hand a uniform external field may not be the relevant ordering field of the spin glass state.[11] If this is the case, $g(K_p, p)$ vanishes even in the spin glass phase because g is not the correct order parameter of the spin glass phase. Consequently we can apply the relation (17) up to the phase boundary between the spin glass and ferromagnetic phases. The same argument as before applies to the spin glass-ferromagnetic boundary and it is concluded that this boundary curve should be locally vertical or non-differentiable at the intersection of the curve $\beta J=K_p$ and the phase boundary.

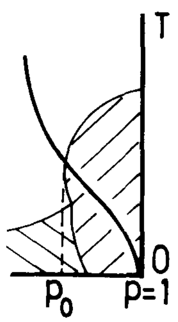

Fig. 4. If the curve $\beta J=K_p$ intersects the para-ferro boundary, the boundary curve is singular at the intersection even in the presence of the spin glass phase.

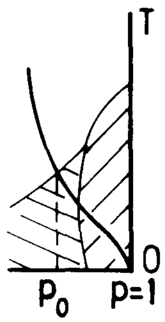

Fig. 5. If the curve $\beta J=K_p$ intersects the spin glass-paramagnetic boundary, no singularities in the boundary curve are predicted.

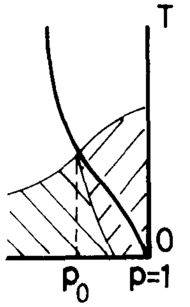

Fig. 6. If the tricritical point is on the curve $\beta J = K_p$, the spin glass-ferromagnetic boundary is in the region $p \geqq p_0$.

The last possibility is that the tricritical point (where the ferromagnetic, paramagnetic and spin glass phases meet) is on the line $\beta J = K_p$ (see Fig. 6). If this is the case, we can predict that the boundary curve $T = T(p)$ between the ferromagnetic and spin glass phases has a negative slope $dT(p)/dp < 0$ as is evident in Fig. 6.

Similar arguments apply to other distribution functions (e.g., the Gaussian distribution) and we find analogous results. We only write here a relation corresponding to Eq. (17) in the case of the Gaussian distribution:

$$0 \leqq |m(J_0/J,\, kT/J)| \leqq g(J_0/J,\, J/J_0),$$

where the spontaneous magnetization is regarded as a function of J_0/J and kT/J (see Fig. 2). The function g is defined in the same way as in Eq. (18) and is originally a function of J_0/J and kT/J. In the above inequality the argument kT/J of g is set equal to J/J_0.

§ 5. Remarks

All of the above results apply only to the Ising model. The reason is that the variables $\{\sigma\}$ of the gauge transformations are of the Ising type. It is possible to apply the same method as above to the planar model, for instance, by modifying the interaction as

$$H = -J \sum \cos(\theta_i - \theta_j + \chi_{ij}), \tag{20}$$

where χ_{ij} is a continuous and quenched random variable $(-\pi < \chi \leqq \pi)$. Although some people are interested in this type of random system,[12] we do not discuss the application of our method to the planar model (20); this type of interaction (20) does not seem to be realized in spin systems experimentally.

An effort to interpret the constraint $\beta J_{ij} = A_{ij}$ physically is desired to be done. The formula (7) for the internal energy looks like that of a non-interacting system. A deeper insight of its physical implication will be useful.

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