Internal solitary waves

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Abstract

Internal solitary waves are an ubiquitous feature of the coastal ocean and atmospheric boundary layer. We will review the use of the variable coefficient Korteweg de Vries equation, and the extended Korteweg de-Vries equation (that is, with an extra cubic nonlinear term), to model these waves. We will describe both the adiabatic theories for slowly-varying solitary waves, and the results from numerical simulations. Particular emphasis will be placed on the consequences when the coefficients of either of the nonlinear terms undergoes a sign change, which may lead to a radical and non-adiabatic change of the wave- form.

1 Introduction

Solitary waves are finite-amplitude waves of permanent form which owe their existence to a balance between nonlinear wave-steepening processes and linear wave dispersion. Typically, they consist of a single isolated wave, whose speed is an increasing function of the amplitude. They are ubiquitous, and in particular internal solitary waves are a commonly occuring feature in the stratified flows of coastal seas, fjords and lakes (see, for instance [1-6]) and in the atmospheric boundary layer (see, for instance [7-9]) Moreover, solitary waves are notable, not only because of their widespread occurrence, but also because they can be described by certain generic nonlinear wave equations which are either integrable, or close to integrability. The most notable example in this context is the now famous Korteweg-de Vries equation, which will figure prominently in this brief article.

Our aim here is to describe appropriate model evolution equations to describe internal solitary waves, and indicate briefly some of their more salient properties. In the next section we will introduce a canonical model equation, which can be systematically derived from the complete fluid equations of motion for an invis-

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cid, incompressible, density-stratified, fluid, with boundary conditions appropriate to an oceanic situation. This equation in its simplest form is the well-known Korteweg-de Vries (KdV) equation with its familiar solitary wave solution, but importantly, in order to account for the large amplitudes sometimes observed, we extend this model here to the extended Korteweg-de Vries (eKdV) equation which contains both quadratic and cubic nonlinearity, and describe its solitary wave solutions. Further, in the third section, in order to allow for the effects of a variable background environment, we describe a further extension to variable-coefficient extended Korteweg-de Vries equation. In general this model equation needs to be solved numerically, but to give some insight into the nature of the solutions, we present a particular class of asymptotic solutions describing a slowly-varying solitary wave.

2 Asymptotic model evolution equations

Let us consider an inviscid, incompressible fluid which is bounded above by a free surface and below by a flat rigid boundary. Suppose that the flow is two-dimensional and can be described by the spatial coordinates (x, z) where x is horizontal and z is vertical. This configuration is appropriate for the modelling of internal solitary waves in coastal seas, and also in straits, fjords or lakes provided that the effect of lateral boundaries can be ignored. The extensions to this basic model needed to incorporate the effects of a horizontally variable background state, and will be described later. Further extensions to take account of dissipation and lateral effects are discussed in [3,4].

In the basic state the fluid has density $\rho_0(z)$, pressure $p_0(z)$ (such that $p_{0z} = -g\rho_0$) and a horizontal shear flow $u_0(z)$ in the $x-$direction. Then, in standard notation, the equations of motion relative to this basic state are

$$\rho_0(u_t+u_0u_x+wu_{0z})+p_x = -(\rho_0+\rho)(uu_x+wu_z)-\rho(u_t+u_0u_x+wu_{0z}), \quad (1a)$$

$$p_z + g\rho = -(\rho_0 + \rho)(w_t + u_0w_x + uw_x + ww_z) \quad (1b)$$

$$g(\rho_t + u_0\rho_x) - \rho_0N^2w = -g(u\rho_x + w\rho_z) \quad (1c)$$

$$u_x + w_z = 0 \quad (1d)$$

Here $(u_0 + u, w)$ are the velocity components in the (x, z) directions, $\rho_0 + \rho$ is the density, $p_0 + p$ is the pressure and t is time. $N(z)$ is the buoyancy frequency, defined by

$$\rho_0N^2 = -g\rho_{0z} \quad (2)$$

The boundary conditions are

$$w = 0, \text{ at } z = -h \quad (3a)$$

$$p_0 + p = 0, \text{ at } z = \eta, \quad (3b)$$

and

$$\eta_t + u_0\eta_x + u\eta_x = w, \text{ at } z = \eta. \quad (3c)$$

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Here, the fluid has undisturbed constant depth h, and η is the displacement of the free surface from its undisturbed position $z = 0$.

To describe internal solitary waves we seek solutions of small amplitude and long wavelength. Then the dominant balance is obtained by equating to zero the terms on the left-hand side of (1a-d) (and treating the boundary conditions in a similar way) to obtain a set of equations describing linear long wave theory. We will use the vertical particle displacement ζ as the primary dependent variable, where

$$\zeta_t + u_0\zeta_x + u\zeta_x + w\zeta_z = w. \tag{4}$$

Note that it then follows that the perturbation density field is given by $\rho = \rho_0(z - \zeta) - \rho_0(z) \approx \rho_0 N^2\zeta$ as $\zeta \to 0$, where we have assumed that as $x \to -\infty$, the density field relaxes to its basic state.

Linear long wave theory is now obtained by omitting the right-hand side of equations (1a-d), and simultaneously linearising boundary conditions (3b,c). Solutions are sought in the form

$$\zeta = A(x - ct)\phi(z), \tag{5}$$

while the remaining dependent variables are then given by analogous expressions. Here c is the linear long wave speed, and the modal function $\phi(z)$ is defined by the boundary-value problem,

$$\{\rho_0(c - u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0, \text{ in } -h < z < 0, \tag{6a}$$

$$\phi = 0 \text{ at } z = -h, \tag{6b}$$

$$\text{and} \qquad (c - u_0)^2\phi_z = g\phi \text{ at } z = 0. \tag{6c}$$

Typically, the boundary-value problem (6a-c) defines an infinite sequence of modes, $\phi_n^{\pm}(z),\ n = 0, 1, 2, \dots$, with corresponding speeds $c_n^{\pm}$. Here, the superscript "$\pm$" indicates waves with $c_n^+ > \max u_0(z)$ and $c_n^- < \min u_0(z)$ respectively. We shall confine our attention to these regular modes, and consider only stable shear flows.

It can now be shown that, within the context of linear long wave theory, any localised initial disturbance will evolve into a set of outwardly propagating modes, each given by an expression of the form (5). Assuming thats the speeds $c_n^{\pm}$ of each mode are sufficiently distinct, it is sufficient for large times to consider just a single mode. Henceforth, we shall omit the indices and assume that the relevant mode (usually n=1) has speed c, amplitude A and modal function $\phi(z)$. Then, as time increases, we expect the hitherto neglected nonlinear terms to have an effect, and to cause wave steepening. However, this is opposed by the terms representing linear wave dispersion, also neglected in the linear long wave theory. A balance between these two effects emerges as time increases and the well-known outcome is the Korteweg-de Vries (KdV) equation, or a related equation, for the wave amplitude.

The formal derivation of the evolution equation requires the introduction of the small parameters, α and ϵ, respectively characterising the wave amplitude and dispersion. A KdV balance requires $\alpha = \epsilon^2$, with a corresponding timescale of ϵ^{-3}.

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The asymptotic analysis required is well understood (see e.g. [3,4,10-13], so we shall give only a brief outline here. We introduce the scaled variables

$$\tau = \epsilon\alpha t\ ,\ \ \theta = \epsilon(x - ct) \tag{7}$$

and then let

$$\zeta = \alpha A(\theta,\tau)\phi(z) + \alpha^2\zeta_2 + \ldots, \tag{8}$$

with similar expressions analogous to (8) for the other dependent variables. At leading order, we get the linear long wave theory for the modal function $\phi(z)$ and the speed c, defined by (6a-c). Note that since the modal equation is homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. Then, at the next order, we obtain the equation for ζ_2,

$$\{\rho_0(c-u_0)^2\zeta_{2\theta z}\}_z + \rho_0 N^2\zeta_{2\theta} = M_2,\ \text{ in }\ -h < z < 0, \tag{9a}$$

$$\zeta_{2\theta} = 0,\ \text{ at }\ z = -h, \tag{9b}$$

$$\rho_0(c-u_0)^2\zeta_{2\theta z} - \rho_0 g\zeta_{2\theta} = N_2,\ \text{ at }\ z = 0. \tag{9c}$$

Here the inhomogeneous terms M_2, N_2 are known in terms of $A(\theta,\tau)$ and $\phi(z)$, and are given by

$$M_2 = 2\{\rho_0(c-u_0)\phi_z\}_z A_\tau + 3\{\rho_0(c-u_0)^2\phi_z^2\}_z AA_\theta - \rho_0(c-u_0)^2\phi A_{\theta\theta\theta}, \tag{10a}$$

$$N_2 = 2\{\rho_0(c-u_0)\phi_z\}A_\tau + 3\{\rho_0(c-u_0)^2\phi_z^2\}AA_\theta. \tag{10b}$$

The left-hand side of the equations (9a-c) is identical to the equations defining the modal function (i.e. (6a-c)), and hence can be solved only if a certain compatibility condition is satisfied, given by,

$$\int_{-h}^{0} M_2\phi\, dz = [N_2\phi]_{z=0} \tag{11}$$

Note that the solution for ζ_2 contains a term $A_2\phi(z)$ where the amplitude A_2 is left undetermined at this stage.

Substituting the expressions (10a,b) into (11) we obtain the required evolution equation for A, namely the KdV equation

$$A_\tau + \mu AA_\theta + \lambda A_{\theta\theta\theta} = 0. \tag{12}$$

Here, the coefficients μ and λ are given by

$$I\mu = 3\int_{-h}^{0} \rho_0(c-u_0)^2\phi_z^3\, dz, \tag{13a}$$

$$I\lambda = \int_{-h}^{0} \rho_0(c-u_0)^2\phi^2\, dz, \tag{13b}$$

where

$$I = 2\int_{-h}^{0} \rho_0(c-u_0)\phi_z^2\,dz. \tag{13c}$$

Confining attention to waves propagating to the right, so that $c > u_M = \max u_0(z)$, we see that I and λ are always positive. Further, if we normalise the first internal modal function $\phi(z)$ so that it is positive at its extremal point, then it is readily shown that for the usual situation of a near-surface pycnocline, μ is negative for this first internal mode. However, in general μ can take either sign, and in some special situations may even be zero. Explicit evaluation of the coefficients μ and λ requires knowledge of the modal function, and hence they are usually evaluated numerically.

Proceeding to the next highest order will yield an equation set analogous to (9a-c) for ζ_3, whose compatibility condition then determines an evolution equation for the second-order amplitude A_2. We shall not give details here (see [14]), but note that using the transformation $A + \alpha A_2 \to A$, and then combining the KdV equation (12) with the evolution equation for A_2 will lead to a higher-order KdV equation for A of the general form.

$$A_\tau + \mu A A_\theta + \lambda A_{\theta\theta\theta}$$

$$+\alpha(\nu A^2 A_\theta + \beta A_{\theta\theta\theta\theta\theta} + \gamma A A_{\theta\theta\theta} + \delta A_\theta A_{\theta\theta}) = 0. \tag{14}$$

However, we must now point out that this higher-order Korteweg-de Vries equation (14) is, strictly speaking, an asymptotic result valid when α is sufficiently small, and is most likely to be useful when the coefficient μ of the quadratic nonlinear term is small (e.g. $O(\epsilon)$ where we recall that $\alpha = \epsilon^2$). In this situation, the cubic nonlinear term in the higher-order KdV equation is the most important higher-order term, and (14) reduces to the extended KdV equation,

$$A_\tau + \mu A A_\theta + \nu A^2 A_\theta + \lambda A_{\theta\theta\theta} = 0. \tag{15}$$

Here, since $\mu \approx 0$, a rescaling has been used, in which μ is $0(\epsilon)$, and A is replaced with A/ϵ. In effect the amplitude parameter is ϵ in place of ϵ^2.

Both of the evolution equations, viz. the KdV equation (12) and the extended KdV equation (15) are exactly integrable, (see, for instance, [15]), with the consequence that the initial-value problem with a localised initial condition is exactly solvable. The most important implication of this integrability for the KdV equation is that an arbitrary initial disturbance will evolve into a finite number (S) of solitary waves (called solitons in this context) and an oscillatory decaying tail. This, together with the robust stability properties of solitary waves, explains why internal solitary waves are so commonly observed. Note that because solitary waves typically have speeds which increase with the wave amplitude, the S waves are rank-ordered by amplitude as $t \to \infty$. Also, to produce solitary waves at all, the initial disturbance should have the correct polarity (e.g. $\mu\int_{-\infty}^{\infty} A(\theta,0)d\theta > 0$ for the case of the KdV equation (12)). Note that, in applications the initial condition $A(\theta, 0)$ for the evolution equation is found by first solving the linear long wave

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equations, and then identifying the mode of interest. Thus $A(\theta, 0)$ is given by (5), which in turn is a reduction from the actual initial conditions.

It follows from the proceeding discussion that in describing the solution of the evolution equations, the most important step is to determine the solitary wave solutions. For the KdV equation (12) this is given by

$$A = a\,\mathrm{sech}^2\beta(\theta - V\tau), \tag{16a}$$

$$\text{where} \qquad V = \frac{1}{3}\mu a = 4\lambda\beta^2. \tag{16b}$$

Note that the speed V is for the phase variable θ, and the actual total speed is $c + \alpha V$. Since the dispersion coefficient λ is always positive for right-going waves, it follows that these solitary waves are always supercritical ($V > 0$), and are waves of elevation or depression according as $\mu \gtrless 0$. We also see that β^{-1} is proportional to $|a|^{-\frac{1}{2}}$, and hence the larger waves are not only faster, but narrower.

For the extended KdV equation (15) the corresponding solitary wave is given by

$$A = \frac{a}{b + (1-b)\cosh^2\beta(\theta - V\tau)}, \tag{17a}$$

$$\text{where} \qquad V = \frac{1}{3}a\left(\mu + \frac{1}{2}\nu a\right) = 4\lambda\beta^2\,, \qquad b = \frac{-\nu a}{(\mu + \nu a)}. \tag{17b}$$

There are two cases to consider. If $\nu < 0$, then there is a single family of solutions such that $0 < b < 1$ and $\mu a > 0$. As b increases from 0 to 1, the amplitude $|a|$ increases from 0 to a maximum of $|\mu/\nu|$, while the speed V also increases from 0 to a maximum of $-\mu^2/6\nu$. In the limiting case when $b \to 1$ the solution (17a) describes the so-called "table-top" solitary wave, which has a flat crest of amplitude $a_m = -\mu/\gamma$ and is terminated at each end by bore-like solutions. In the case $\mu/\epsilon = 0$ there are no exact solitary wave solutions of (15) when $\nu < 0$, but instead there is the travelling bore solution $\epsilon A = a\tanh\beta(\theta - V\tau)$ where $V = \frac{1}{3}\nu a^2 = -2\lambda\beta^2$. Note here that the amplitude of the travelling bore is a free parameter, and that the speed $V < 0$.

For the case when $\nu > 0$, $b < 0$ and there are two families of solitary waves. One is defined by $-1 < b < 0$, has $\mu a > 0$, and as b decreases from 0 to -1, the amplitude a increases from 0 to ∞, while the speed V also increases from 0 to ∞. The other is defined by $-\infty < b < -1$, has $\mu a < 0$ and, as b increases from $-\infty$ to -1, the amplitude $|a|$ increases from $-2|\mu|/\nu$ to ∞. In this case solitary waves exist if $\mu = 0$ ($b = -1$) and are given by

$$A = a\,\mathrm{sech}\,2\beta(\theta - V\tau), \tag{18a}$$

$$\text{where} \qquad V = \frac{1}{6}\nu a^2 = 4\lambda\beta^2 \tag{18b}$$

On the other hand, as $b \to -\infty$, $\beta \to 0$ and the solitary wave (17a) reduces to an algebraic soliton.

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3 Variable background

The KdV equation (12) is the basic model for the situation studied in Section 2, when the flow is unidirectional, and the background state is horizontally uniform. Our purpose now is to extend this basic model to situations where there is a variable background environment. This can arise due to a variable depth $h(X)$, or due to horizontal variability in the basic density $\rho_0(X;z)$ and horizontal velocity field $u_0(X;z)$ where $X = \epsilon\alpha x$. Here, for simplicity, we are considering the situation when the background variability is unidirectional and in the flow direction. The scaling indicates that we are assuming that the background varies on a length scale which is much greater than that of the solitary waves, but is comparable to the length scale over which the wave field evolves. The modal functions $\phi(X;z)$ are again defined by (6a-c), but now depend parametrically on X, and hence so does the wave speed $c(X)$. An asymptotic expansion analogous to (8) is then introduced, but the variables τ and θ in (7) are here replaced by

$$s = \int_0^X \frac{dX'}{c(X')}, \qquad \psi = \frac{1}{\alpha}(s - \tau), \tag{19}$$

where we recall that τ is defined by (7). The amplitude $A(s,\psi)$ can then be shown to satisfy the variable-coefficient extended KdV equation [3-4,16],

$$A_s + \frac{\sigma_s}{2\sigma}A + \frac{\mu}{c}AA_\psi + \frac{\nu}{c}A^2A_\psi + \frac{\lambda}{c^3}A_{\psi\psi\psi} + \;=\; 0\,, \tag{20}$$

which thus replaces (15). Here the coefficients $\mu(X)$, $\lambda(X)$ are defined by (13a,b), and $\sigma(X) = c^2 I$, whose significance is that σA^2 is a measure of the wave action flux in the X-direction, and is a conserved quantity.

In general the gKdV equation (20) must be solved numerically. However, to gain insight into the expected behaviour of the solitary wave solutions, it is useful to consider the asymptotic construction of the slowly-varying solitary wave solution, in which it is assumed that the background variability and the dissipative effects are sufficiently weak that a solitary wave is able to maintain its structure over long distances. In this case a multi-scale perturbation technique [17-21] can be used in which the leading term is

$$A = A_0 = \frac{a}{b + (1-b)\cosh^2\beta(\theta - \int^\theta V\,d\tau)}, \tag{21a}$$

$$\text{where} \qquad V = \frac{1}{3}a\left(\mu + \frac{1}{2}\nu a\right) = 4\lambda\beta^2, \qquad b = \frac{-\nu a}{(2\mu + \nu a)}. \tag{21b}$$

Here the wave amplitude $a(s)$, and hence also $V(s), \beta(s)$, are slowly- varying functions of s. Their variation is most readily determined by noting that (20) possesses the conservation law,

$$\frac{\partial}{\partial s}\int_{-\infty}^{\infty} \sigma A^2 d\psi \;=\; 0\,, \tag{22}$$

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which expresses conservation of wave action flux. Substitution of (31a) into (32) gives

$$P_0 = \int_{-\infty}^{\infty} \sigma A_0^2 d\psi = \text{constant}. \qquad (23)$$

The integral can be evaluated explicitly in terms of the coefficients in the gKdV equation (20) and the solitary wave parameters, and then on using the relations (21b) this is an equation for $a(s)$ (see [19-21], and note for instance that for the KdV equation, i.e. $\nu = 0$ in (20), $P_0 = 2\sigma a^2/3\beta$). However, although the slowly-varying solitary wave conserves wave action flux it cannot simultaneously satisfy the law for conservation of mass,

$$\frac{\partial}{\partial s}\int_{-\infty}^{\infty} \sqrt{\sigma} A d\psi \;=\; 0\,, \qquad (24)$$

Instead, it is accompanied by a trailing shelf of small amplitude but long length scale, whose amplitude A_- at the rear of the solitary wave is determined from (24) and is given by

$$\sqrt{\sigma} A_- \;=\; -\frac{\partial}{\partial s} M_0\,, \qquad \text{where} \qquad M_0 = \int_{-\infty}^{\infty} \sqrt{\sigma} A_0 d\psi\,. \qquad (25)$$

Like P_0 the solitary wave mass M_0 can be evaluated explicitly (see [21-23] and note that in the KdV case $M_0 = \sqrt{\sigma} 2a/\beta$. When the coefficients λ, μ, σ and c are known explicitly as functions of s, the expressions in (24) and (25) can also be evaluated explicitly. However, usually these coefficients, being determined *inter alia* from the modal functions, are known only numerically, and hence $a(s)$ and $A_-(s)$ can also only be obtained numerically.

A detailed analysis of the formula (23) shows that situations of particular interest occurs at the critical locations when the coefficients of the nonlinear terms in (15), $\mu(s)$ or $\nu(s)$, are zero. In the oceanic environment there may be several such critical locations as the waves propagate shoreward over a decreasing ocean depth h. For instance μ is typically negative in deep water, but positive in shallow water (here we consider waves propagating to the right so that $I(13c) > 0$ and then the dispersive coefficient λ (13b) is always positive (13b)). It follows from (17b) that the solitary wave is usually a wave of depression when $\mu < 0$, but a wave of elevation when $\mu > 0$. The issue then arises as to how the solitary will behave as $\mu \to 0$ (i.e. as the wave approaches a critical location), and in particular, as to whether a solitary wave of depression can be converted into one or more solitary waves of elevation as the critical location is traversed. This problem has been intensively studied (see, for instance [21-23] and the references therein), and the solution depends on how rapidly the coefficient μ changes sign. If μ passes through zero rapidly compared to the local width of the solitary wave, then the solitary wave is destroyed, and converted into an oscillatory wavetrain. On the other hand, if μ changes sufficient slowly that the formula (23) holds as the critical location is approached, then the outcome depends on the sign of the

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coefficient ν of the cubic nonlinear term. If $\nu < 0$, then as the solitary wave amplitude decreases, the amplitude of the trailing shelf, which has the opposite polarity, grows indefinitely until a point is reached just prior to the critical location where the slowly-varying solitary wave asymptotic theory fails. A combination of this trailing shelf and the distortion of the solitary wave itself then provide the appropriate "initial" condition for one or more solitary waves of the opposite polarity to emerge as the critical location is traversed. But if $\nu > 0$, then the solitary wave can successfully traverse the critical location, but typically continues to decrease in amplitude and may eventually transform into a breather.

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