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Internal stellar rotation and orbital period modulation in close binary systems

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ABSTRACT

The orbital period modulation, observed in close binary systems with late-type secondary stars, is considered in the framework of a general model that allows us to test the hypothesis proposed by Applegate. It relates the orbital period variation to the modulation of the gravitational quadrupole moment of their magnetically active secondary stars produced by angular momentum exchanges within their convective envelopes. By considering the case of RS CVn binary systems, it is found that the surface angular velocity variation of the secondary component required by Applegate's hypothesis is between 4 and 12 per cent, i.e. too large to be compatible with the observations and that the kinetic energy dissipated in its convection zone ranges from 4 to 43 times that supplied by the stellar luminosity along one cycle of the orbital period modulation. Similar results are obtained for other classes of close binary systems by applying a scaling relationship based on a simplified internal structure model. The effect of rapid rotation is briefly discussed finding that it is unlikely that the rotational quenching of the turbulent viscosity may solve the discrepancy. Therefore, the hypothesis proposed by Applegate is not adequate to explain the orbital period modulation of close binary systems with a late-type secondary.

Key words: MHD – stars: activity – binaries: close – stars: magnetic fields – stars: rotation.

1 INTRODUCTION

The long-term monitoring of close binary systems with late-type secondary components shows that in general their orbital period is not constant but it is modulated around its mean value on time-scales of decades. Specifically, in Algols and RS Canum Venaticorum systems, the relative amplitude of the orbital period modulation is of the order of $\Delta P/P \sim (1\text{–}3) \times 10^{-5}$ with cycles of 30–50 yr, whereas in cataclysmic variables and W UMa systems $\Delta P/P \sim 1 \times 10^{-6}$ with cycles of 5–30 yr (cf. e.g. Hall 1989; Lanza & Rodonò 1999, and references therein).

RS CVn binary systems represent the best case to investigate this intriguing phenomenon because the complication due to mass transfer is absent given their detached nature and light–time effects due to a third companion can usually be excluded (cf. van Buren 1986; Frasca & Lanza 2005).

RS CVn systems have typical orbital periods between 1 and 15 d and consist of late-type main sequence or subgiant components. The rapid rotation enforced by tidal synchronization and the deep convection zones of their component stars promote a vigorous dynamo action that manifests itself in a high level of solar-like magnetic activity (e.g. Rodonò 1992; Guinan & Gimenez 1993; Strassmeier et al. 1993; García-Alvarez et al. 2003, and reference therein).

Applegate (1992) reviewed the previous theoretical models for the orbital period modulation and proposed a new hypothesis to explain the phenomenon. It relates the orbital period modulation to the operation of a hydromagnetic dynamo in the convection zone of the active components of close binary systems. More precisely, Applegate's hypothesis assumes that a few per cent of the internal angular momentum of the active component is cyclically exchanged between an inner and an outer convective shell due to a varying internal magnetic torque versus the activity cycle phase. This affects the oblateness and the gravitational quadrupole moment of the active star, which oscillates around its mean value. When the quadrupole moment is maximum, the companion star feels a stronger gravitational force, so that it is forced to move closer and faster around the barycentre of the system, thus attaining the minimum orbital period. On the other hand, when the quadrupole moment is minimum, the orbital period attains its maximum value.

In a recent paper, Lanza (2005) presented a detailed model for the angular momentum exchanges within the convection zone of a rapidly rotating star under the assumption that the angular velocity is constant on cylindrical surfaces co-axial with the rotation axis. That analysis led to the rejection of the Applegate hypothesis in the case of RS CVn binary systems because the required angular velocity variations are one or two orders of magnitude larger than

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the upper limits set by the observations and the mechanical energy dissipated in the turbulent convection zone during one cycle of the modulation exceeds that supplied by the stellar luminosity.

In the present paper, the approach of Lanza (2005) is extended to the general case of an internal angular velocity that depends on both radius and latitude. The results reinforce the conclusion of the previous analysis, showing that the mechanism proposed by Applegate is not adequate to explain the orbital period modulation of RS CVn binary systems in particular and of close binary systems with a late-type secondary in general.

2 THE MODEL

2.1 Hypotheses and basic equations

We consider an inertial reference frame with its origin at the barycentre of the active component star and the z-axis in the direction of the stellar rotation axis. A spherical polar coordinate system (r, θ, φ) is assumed, where r is the distance from the origin, θ is the colatitude measured from the North Pole and φ is the azimuthal angle. We assume that all the variables do not depend on φ. As a matter of fact, the tidal deformation due to the companion star introduces a dependence on the azimuthal angle, but it can be neglected in the analysis of Applegate's model because it does not produce any variation of the gravitational quadrupole moment that affects the orbital motion in a time-dependent fashion (cf. Applegate 1992; Lanza, Rodonò & Rosner 1998).

In order to treat the hydrodynamics of a turbulent convection zone, a mean-field approach is adopted. In particular, the velocity $\boldsymbol{V} = \boldsymbol{v} + \boldsymbol{v}'$, where $\boldsymbol{v}$ is the mean velocity and $\boldsymbol{v}'$ is its fluctuation with respect to the mean value at a given point and time. The mean velocity field is assumed to be that arising from stellar rotation $\boldsymbol{v} = (0, 0, v_{\varphi})$, with $v_{\varphi} = v_{\varphi}(r, \theta, t)$, where t is the time, i.e. the meridional circulation is neglected. Moreover, we neglect the density fluctuations (the so-called anelastic approximation) and assume that the density ρ depends only on r.

The equations of mass continuity and angular momentum conservation in the mean-field approach can be found in, e.g., Lanza (2005) [cf. equation (1) and equations (2), (3) and (4), respectively]. For the sake of simplicity, we shall not consider the energy equation in detail and require only that the mechanical energy dissipated by the mean flow in the turbulent convection zone does not exceed some fraction, say 10 per cent, of the energy supplied by the stellar luminosity along one cycle of the orbital period modulation.

Starting from the equation for the angular momentum conservation, the equation for the angular velocity $\omega \equiv v_{\varphi}/(r \sin\theta)$ can be written as

$$\frac{\partial \omega}{\partial t} - \frac{1}{\rho r^4}\frac{\partial}{\partial r}\left(r^4 \eta_{\mathrm{t}} \frac{\partial \omega}{\partial r}\right) - \frac{\eta_{\mathrm{t}}}{\rho r^2}\frac{1}{(1-\mu^2)}\frac{\partial}{\partial \mu}\left[(1-\mu^2)^2 \frac{\partial \omega}{\partial \mu}\right] = S, \quad (1)$$

where η_{t} is the turbulent dynamical viscosity (assumed to be a function of r only), $\mu \equiv \cos\theta$ and S is a source term given by

$$S = -\frac{\nabla \cdot \boldsymbol{\tau}}{\rho r^2 (1-\mu^2)}, \quad (2)$$

where $\boldsymbol{\tau}$ is a vector whose components are

$$\tau_i = r \sin\theta \left[\boldsymbol{\Lambda}_{i\varphi} + \frac{1}{\tilde{\mu}}(B_i B_{\varphi} + \boldsymbol{M}_{i\varphi})\right], \quad (3)$$

where $\boldsymbol{\Lambda}_{ik}$ is the Reynolds stress tensor, $\tilde{\mu}$ is the magnetic permeability, $\boldsymbol{B}$ is the mean magnetic field and $\boldsymbol{M}_{ik}$ is the Maxwell stress tensor arising from the correlation of the magnetic field fluctuations (see Lanza 2005, equation 4). Equations (2) and (3) account for the angular momentum transfer by the Reynolds stresses and the magnetic torques. We solve equation (1) with the stress-free boundary conditions

$$\left(\frac{\partial \omega}{\partial r}\right)_{r_{\mathrm{b}}, R} = 0, \quad (4)$$

where r_{b} is the radius at the base of the stellar convection zone and R is the radius of the star. This ensures that the angular momentum flux outside the convection zone vanishes, i.e. the total angular momentum of the convection zone is conserved.

2.2 Solution of the angular momentum equation

Following Kopal (1978), we seek a solution of equation (1) with the boundary conditions (4) of form

$$\omega(r, \mu, t) = \sum_{n=0}^{\infty} \alpha_n(t) \zeta_n(r) P_n^{(1,1)}(\mu), \quad (5)$$

where the functions α and ζ will be specified below and $P_n^{(1,1)}(\mu)$ are Jacobian polynomials (cf. Smirnov 1964a). They are the finite solutions of the equation

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)^2 \frac{\mathrm{d}P_n^{(1,1)}}{\mathrm{d}\mu}\right] + n(n+3)(1-\mu^2) P_n^{(1,1)} = 0 \quad (6)$$

in the interval $-1 \leqslant \mu \leqslant 1$, including its ends. They form a complete and orthogonal set with respect to the weight function $(1 - \mu^2)$ in the interval $-1 \leqslant \mu \leqslant 1$. In order to solve equation (1), we also develop the source term S in a similar fashion:

$$S = \sum_{n=0}^{\infty} \beta_n(t) \zeta_n(r) P_n^{(1,1)}(\mu). \quad (7)$$

We substitute equations (5) and (7) into equation (1), apply equation (6) and the orthogonality of the Jacobian polynomials and separate the variables, putting the functions that depend on the time t on the left-hand side and those that depend on the radial coordinate r on the right-hand side. Thus, we obtain for each n

$$\frac{\dot{\alpha}_n - \beta_n}{\alpha_n} = \frac{1}{\zeta_n}\left[\frac{1}{\rho r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 \eta_{\mathrm{t}} \zeta_n'\right) - n(n+3)\frac{\eta_{\mathrm{t}}}{\rho r^2}\zeta_n\right], \quad (8)$$

where the dot indicates the derivative with respect to the time and the prime the derivative with respect to the radius. Since the left-hand side of equation (8) depends only on the time while the right-hand side depends only on the radius, the two sides must be equal to the same constant, say, $-\lambda_n$. Therefore, we obtain two equations

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \lambda_n \alpha_n = \beta_n, \quad (9)$$

whose solution is

$$\alpha_n(t) = \exp(-\lambda_n t) \int_0^t \beta_n(t') \exp(\lambda_n t')\, \mathrm{d}t' + \alpha_n(0), \quad (10)$$

and

$$\frac{1}{\rho r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 \eta_{\mathrm{t}} \zeta_n'\right) - n(n+3)\frac{\eta_{\mathrm{t}}}{\rho r^2}\zeta_n + \lambda_n \zeta_n = 0, \quad (11)$$

with the boundary conditions (4), i.e.

$$\zeta_n'(r) = 0 \quad \text{at} \quad r = r_{\mathrm{b}},\ R. \quad (12)$$

In other words, the function ζ_n is a solution of the Sturm–Liouville problem defined by equation (11) with the boundary conditions (12) in the interval $r_b \leqslant r \leqslant R$ (cf. e.g. Morse & Feshbach 1953). We indicate by ζ_{nk} the eigenfunction corresponding to the kth eigenvalue λ_{nk}. The eigenfunctions ζ_{nk}, for a fixed n, form a complete and orthogonal set of functions in the interval $[r_b, R]$ with respect to the weight function ρr^4 that does not depend on n. We recall from the theory of the Sturm–Liouville problem that the eigenvalues verify the inequality, $\lambda_{n0} < \lambda_{n1} < \cdots < \lambda_{nk} < \lambda_{nk+1} < \cdots$, and the eigenfunction ζ_{nk} has k nodes in the interval $[r_b, R]$ for each n. For $n = 0$, the first eigenvalue corresponding to the eigenfunction ζ_{00} is zero and the eigenfunction vanishes at all points in $[r_b, R]$, as it is evident by integrating both sides of equation (11) in the same interval, applying the boundary conditions (12) and considering that ζ_{00} has no nodes. For $n > 0$, all the eigenvalues λ_{n0} are positive, as can be derived by integrating both sides of equation (11) in the interval $[r_b, R]$, applying the boundary conditions (12) and considering that ζ_{n0} has no nodes. In view of the inequality given above, all the eigenvalues λ_{nk} are then positive for $n \geqslant 1$. Moreover, it is possible to prove that $\lambda_{n'k} \geqslant \lambda_{nk}$ if $n' > n$ (cf. Smirnov 1964b).

In consideration of these results, the general solution for the angular velocity can be written in the form

$$\omega(r, \mu, t) = \sum_k \sum_{n=0}^{\infty} \alpha_{nk}(t)\zeta_{nk}(r)P_n^{(1,1)}(\mu), \tag{13}$$

where the summation over n is for fixed k and the functions α_{nk} and β_{nk} depend also on the order k of the radial eigenfunction. Thanks to the orthogonality of the Jacobian polynomials and of the eigenfunctions ζ_{nk}, we find

$$\beta_{nk} = E_{nk} \int_{r_b}^{R} \int_{-1}^{1} \rho r^4 S(r, \mu, t)\zeta_{nk} P_n^{(1,1)} \cdot (1 - \mu^2)\,\mathrm{d}\mu\,\mathrm{d}r, \tag{14}$$

where E_{nk} is a normalization factor depending on n and k.

The solution of our problem is now complete: given the source term S, we can compute the functions β_{nk} from equation (14) and solve for the α_{nk} functions from equation (10) by making use of the initial conditions.

2.3 Kinetic energy variation and dissipation

It is convenient to measure the variation of the kinetic energy of rotation with respect to the state of minimum mechanical energy. It corresponds to a rigid rotation with a total angular momentum equal to the total angular momentum of the stellar convection zone. If the angular velocity of such a reference state is Ω_0, the angular velocity Ω at a given position and time can be written as: $\Omega(r, \mu, t) = \Omega_0 + \omega(r, \mu, t)$, where ω is the deviation from the rigid rotation state and is given by the solution of the angular momentum equation discussed above. The variation of the total kinetic energy of rotation with respect to the rigidly rotating state can be written as: $\Delta\mathcal{T} = \sum_k \sum_n \Delta\mathcal{T}_{nk}$, where the contribution of each term of the series in equation (13) is

$$\Delta\mathcal{T}_{nk} = \frac{8\pi(n+1)}{(2n+3)(n+2)}\,\alpha_{nk}^2(t) \int_{r_b}^{R} \rho r^4 \zeta_{nk}^2\,\mathrm{d}r. \tag{15}$$

The numerical factor in front of the integral comes from the normalization of the Jacobian polynomials.

According to Applegate's hypothesis, some angular momentum is exchanged back and forth between an inner and an outer shell in the stellar convection zone and this leads to a periodic change of the kinetic energy of the convection zone itself. This process is not reversible because convective turbulence produces energy dissipation whenever angular velocity gradients are present. A fraction of the kinetic energy is injected into the turbulent Kolmogorov cascade and is eventually dissipated at the length-scales at which molecular viscosity becomes important. The amount of kinetic energy dissipated per unit time is (cf. e.g. Landau & Lifshitz 1959; Chandrasekhar 1961)

$$\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} = -\int_{V_c} \eta_t r^2 (1 - \mu^2) \left[\left(\frac{\partial\omega}{\partial r}\right)^2 + \frac{1-\mu^2}{r^2}\left(\frac{\partial\omega}{\partial\mu}\right)^2 \right] \mathrm{d}V, \tag{16}$$

where V_c is the volume of the convection zone to which the integration is extended. Since $P_n^{(1,1)} \propto (\mathrm{d}P_{n+1})/(\mathrm{d}\mu)$, where P_{n+1} is the Legendre polynomial of degree $n+1$ (cf. e.g. Abramowitz & Stegun 1965; Kopal 1978; Gradshteyn & Ryzhik 1994), we can apply the results of Higgins & Kopal (1968) to compute the total kinetic energy dissipation. Taking into account the properties of the radial eigenfunctions ζ_{nk}, the expression can be further simplified, yielding

$$\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} = -2\sum_k \sum_n \lambda_{nk}\Delta\mathcal{T}_{nk}. \tag{17}$$

This implies that the rate of kinetic energy dissipation is the sum of the dissipation rates associated with each term of equation (13). The extremal properties of the radial eigenfunctions (cf. e.g. Morse & Feshbach 1953) imply that the minimum dissipated energy for a given kinetic energy variation is obtained when the radial profile of the angular velocity variation is proportional to the eigenfunction with the lowest angular degree and radial order, i.e. $n = 0$ and $k = 1$.

2.4 Variation of the gravitational quadrupole moment

The variation of the gravitational quadrupole moment with respect to the reference state of rigid rotation with angular velocity Ω_0 can be computed by means of the approach devised by Ulrich & Hawkins (1981). Specifically, the differential equation for the variation of the dimensional quadrupole potential $\Delta\Phi_{12}(r, t)$ is

$$\begin{aligned}&\frac{\partial^2(\Delta\Phi_{12})}{\partial r^2} + \frac{2}{r}\frac{\partial(\Delta\Phi_{12})}{\partial r} - \frac{6}{r^2}(\Delta\Phi_{12})\\ &\quad= \frac{4\pi r^2}{m(r)}\left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)(\Delta\Phi_{12}) - \frac{\partial}{\partial r}(r^2 b_2\rho) - r\rho a_2\right],\end{aligned} \tag{18}$$

where $m(r)$ is the mass of the star inside radius r and the functions b_2 and a_2 have been introduced by Lebovitz (1970). If the deviation from the reference state is small, i.e. $|\omega| \ll \Omega_0$, they are given by

$$b_2(r, t) = \frac{5}{6}a_0(r, t) + \frac{1}{3}a_2(r, t), \tag{19}$$

where

$$a_0(r, t) \simeq \Omega_0 \int_{-1}^{1} \omega(r, \mu, t)(1 - \mu^2)\,\mathrm{d}\mu,$$

$$a_2(r, t) \simeq \frac{5}{2}\Omega_0 \int_{-1}^{1} \omega(r, \mu, t)(3\mu^2 - 1)(1 - \mu^2)\,\mathrm{d}\mu. \tag{20}$$

If the angular velocity is expressed as a series of the kind of equation (13), we find

$$a_2(r, t) \simeq \Omega_0 \sum_k \left[\frac{12}{7}\alpha_{2k}(t)\zeta_{2k}(r) - \frac{4}{3}\alpha_{0k}(t)\zeta_{0k}(r)\right],$$

$$b_2(r, t) \simeq \Omega_0 \sum_k \left[\frac{2}{3}\alpha_{0k}(t)\zeta_{0k}(r) + \frac{4}{7}\alpha_{2k}(t)\zeta_{2k}(r)\right], \tag{21}$$

that is, only the eigenfunctions of angular degrees $n = 0$ and $n = 2$ are relevant for the quadrupole moment variation in the considered linear approximation.

The solution of equation (18) must match continuously the outer gravitational potential at the stellar surface $r = R$ at any time t:

$$\Delta\Phi'_{12}(R, t) + 3\Delta\Phi_{12}(R, t)/R = 0, \tag{22}$$

while close to the centre $\Delta\Phi_{12}(r, t) = C\, r^2$. In order to solve equation (18) with the appropriate boundary conditions, we apply a shooting method by integrating outward from the centre of the star, varying the trial constant C until equation (22) is satisfied. The dimensional quadrupole moment variation is given by $\Delta Q = -R^3\, \Delta\Phi_{12}(R)/3G$, where G is the gravitation constant. The relative orbital period variation with respect to the period corresponding to rigid stellar rotation with angular velocity Ω_0 is

$$\frac{\Delta P}{P} = -9\frac{\Delta Q}{Ma^2}, \tag{23}$$

where M is the mass of the active component star and a is the semimajor axis of the relative orbit (cf. e.g. Applegate 1992).

Since the angular velocity variation is small, the variation of the quadrupole moment ΔQ can be expressed as the sum of the variations due to each of the terms in equation (21), i.e.

$$\Delta Q = \sum_k \left(\Delta Q_{0k} + \Delta Q_{2k}\right), \tag{24}$$

where ΔQ_{0k} and ΔQ_{2k} are the variations produced by the angular velocity perturbations proportional to ζ_{0k} and ζ_{2k}, respectively.

2.5 Scaling relationships

It is interesting to derive some scaling relationships for the eigenfunctions and the eigenvalues of equation (11) that can be used to compute the scaling of the gravitational quadrupole moment and energy dissipation rate with the fundamental stellar parameters. This can be done by adopting a composite polytrope approximation to describe the internal structure of the active component star, i.e. by assuming that it consists of a polytrope of polytropic index $n_{\mathrm{p}} = 3$ in its radiative core (that extends between $r = 0$ and $r = r_{\mathrm{b}}$) and of a polytrope of index $n_{\mathrm{p}} = 3/2$ in its convective envelope (i.e. ranging from $r = r_{\mathrm{b}}$ to $r = R$; see Rappaport, Verbunt & Joss 1983). The turbulent viscosity is computed according to the mixing-length approximation: $\eta_{\mathrm{t}} = \frac{1}{3}\alpha_{\mathrm{ml}}\rho v_{\mathrm{c}} H_{\mathrm{p}}$, where α_{ml} is the ratio between the mixing length and the local pressure scale height H_{p} and v_{c} is the convective velocity that we estimate by assuming that convection transports the whole stellar luminosity L through the convective envelope $v_{\mathrm{c}} = [\alpha_{\mathrm{ml}} L/(40\pi r^2\rho)]^{1/3}$ (Kippenhahn & Weigert 1990).

After some lengthy calculations, we find the equation for the radial eigenfunctions expressed in terms of the dimensionless length ξ, i.e. the independent variable of the Lane–Emden equation for the composite polytrope (see Rappaport et al. 1983, for the definition of ξ and of the other variables appearing in the composite polytrope model)

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi^{10/3}\theta_2^2(\theta_2')^{-1}\zeta'_{nk}\right] - n(n+3)\xi^{4/3}\theta_2^2(\theta_2')^{-1}\zeta_{nk} + (K\lambda_{nk})\xi^4\theta_2^{3/2}\zeta_{nk} = 0, \tag{25}$$

with the boundary conditions $\zeta'_{nk}(\xi) = 0$, for $\xi = \xi_{2i}, \xi_{2\mathrm{s}}$, where ξ_{2i} and $\xi_{2\mathrm{s}}$ correspond to the base of the convective envelope and the surface of the star, respectively, $\theta_2(\xi)$ is the solution of the Lane–Emden equation in the convective envelope as defined in Rappaport et al. (1983) and $K = (15/2)\,(10)^{1/3}\alpha_{\mathrm{ml}}^{-4/3}M^{1/3}R^{2/3}L^{-1/3}\,\xi_{2\mathrm{s}}^{-4/3}(\theta'_{2\mathrm{s}})^{-1/3}$ depends on the stellar parameters. It is important to note that $\xi_{2i}, \xi_{2\mathrm{s}}$ and $\theta'_{2\mathrm{s}}$ depend only on the ratio r_{b}/R and thus they are constant for stellar models with the same relative depth of the convection zone. Equation (25) predicts that the eigenvalues should scale as K^{-1}. Precise calculations made for main-sequence models with masses of 0.5, 0.7 and 0.9 $\mathrm{M}_{\odot}$[1] show that such a proportionality is indeed reproduced with an accuracy between 30 and 50 per cent for the first eigenvalue λ_{01}.

By applying the same considerations to the equation for the gravitational quadrupole moment, we find that $\Delta\Phi_{12}(R) \propto R^2\,\xi_{2\mathrm{s}}^{-2}$, i.e. $\Delta\Phi_{12}(R)$ scales as R^2 for fixed r_{b}/R. Finally, for fixed r_{b}/R, the ratio between the power dissipated by the perturbation with radial dependence proportional to ζ_{nk} and the stellar luminosity is found to scale as

$$\frac{1}{L}\left(\frac{\mathrm{d}\Delta\mathcal{T}_{nk}}{\mathrm{d}t}\right) \propto \alpha_{\mathrm{ml}}^{4/3}M^{8/3}R^{-6}T_{\mathrm{eff}}^{-8/3}\left(\frac{R}{a}\right)^{-4}\left(\frac{\Delta P}{P}\right)^2, \tag{26}$$

where T_{eff} is the effective temperature of the star.

3 APPLICATION TO A STELLAR MODEL

The above theory can be applied to the internal structure model of a typical active component star in a RS CVn system. Specifically, we shall consider the model introduced in Lanza (2005). It refers to a star with solar chemical abundances and a mass of $M = 1.3\,\mathrm{M}_{\odot}$ that has been evolved for 4.583 Gyr up to a radius of $R = 4.047\,\mathrm{R}_{\odot}$ and a luminosity of $L = 8.298\,\mathrm{L}_{\odot} = 3.20 \times 10^{27}$ W. Its effective temperature is 4871 K. The base of the convection zone is located at a fractionary radius $r_{\mathrm{b}}/R = 0.181$ or, in terms of the mass coordinate, at $m_{\mathrm{b}}/M = 0.240$. The turbulent viscosity has been computed using the mixing-length theory as explained in Lanza (2005) to which the reader is referred for more details on the model.

The perturbation of the angular velocity with respect to the rigidly rotating reference state can be assumed symmetric with respect to the stellar equator which implies that only the terms with an even angular degree n appear in equation (13). For our purposes, it suffices to consider only the eigenfunctions with $n = 0$ and $n = 2$, because the quadrupole moment variation depends only on them.

The first six eigenvalues of equation (11) for $n = 0$ with the boundary conditions (12) are computed by means of a shooting method (Press et al. 1992) and are listed in the second column of Table 1, while the first four eigenfunctions are plotted in Fig. 1. A useful check of the accuracy of their calculation is provided by the constraint that the corresponding total angular momentum variation must be zero. The first seven eigenvalues of equation (11) for $n = 2$ with the boundary conditions (12) are listed in the second column of Table 2, while the first four eigenfunctions are plotted in Fig. 2. Note that the functions ζ_{2k} are not constrained by the conservation of the total angular momentum since $\int_{-1}^{1} P_2^{(1,1)}(\mu)(1-\mu^2)\,\mathrm{d}\mu = 0$ in any case.

According to the general theory in Section 2, the smallest non-vanishing eigenvalue is λ_{20} which corresponds to the longest characteristic time-scale for angular momentum transfer under the action of the turbulent viscosity. Since $\lambda_{20}^{-1} \approx 0.86$ yr, the time-scale for the angular momentum transfer is much smaller than the typical cycle

[1] Stellar structure models were obtained from the Dartmouth Stellar Evolution web server (http://stellar.dartmouth.edu/) that allows us to run a specialized version of the stellar evolution code introduced by Chaboyer et al. (2001) and Guenther et al. (1992).

Table 1. Eigenvalue λ_{0k}, quadrupole moment variation $|\Delta Q_{0k}|$, relative orbital period variation $(\Delta P/P)_{0k}$, maximum kinetic energy variation $\Delta \mathcal{T}_{0k}$ and maximum power $P_{\mathrm{diss\,0k}} = 2\lambda_{0k}\Delta\mathcal{T}_{0k}$ dissipated along a cycle of the quadrupole moment variation versus the order k of the radial eigenfunction ζ_{0k}. The amplitude of ζ_{0k} is assumed to be 1 per cent of the unperturbed stellar angular velocity at the base of the stellar convection zone (i.e. at $r = r_{\mathrm{b}}$) for all the values of k.

Radial order k	Eigenvalue λ_{0k} (s^{-1})	$\lvert\Delta Q_{0k}\rvert$ (kg m^2)	$(\Delta P/P)_{0k}$	$\Delta\mathcal{T}_{0k}$ (J)	$P_{\mathrm{diss\,0k}}$ (W)
1	5.2650×10^{-8}	1.16×10^{44}	3.18×10^{-6}	2.499×10^{34}	2.631×10^{27}
2	1.4148×10^{-7}	1.20×10^{43}	3.29×10^{-7}	1.279×10^{34}	3.618×10^{27}
3	2.6676×10^{-7}	7.90×10^{42}	2.16×10^{-7}	8.943×10^{33}	4.771×10^{27}
4	4.3022×10^{-7}	1.49×10^{42}	4.09×10^{-8}	7.392×10^{33}	6.361×10^{27}
5	6.3260×10^{-7}	1.42×10^{42}	3.88×10^{-8}	6.657×10^{33}	8.423×10^{27}
6	8.7443×10^{-7}	4.85×10^{41}	1.33×10^{-8}	6.258×10^{33}	1.094×10^{28}

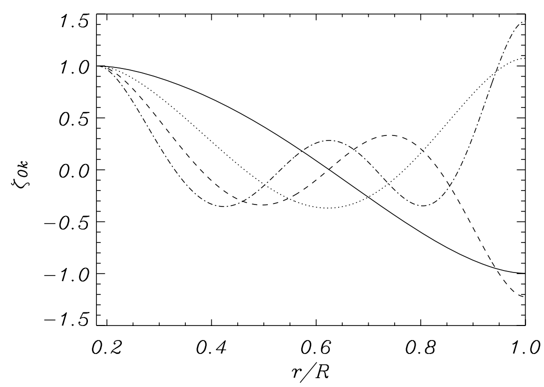

Figure 1. The first four eigenfunctions of the Sturm–Liouville problem defined by equation (11) for $n = 0$ with the boundary conditions (12) versus the fractionary radius in the case of our stellar interior model. They have been normalized to the unity at the base of the convective envelope, i.e. at $r = r_{\mathrm{b}}$. Different linestyles refer to eigenfunctions of different radial order, i.e. $k = 1$: solid line; $k = 2$: dotted line; $k = 3$: dashed line and $k = 4$: dot–dashed line.

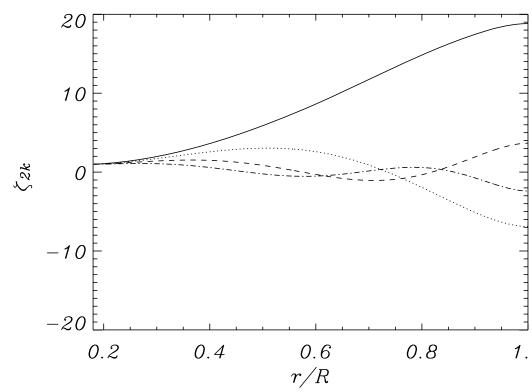

Figure 2. The first four eigenfunctions of the Sturm–Liouville problem defined by equation (11) for $n = 2$ with the boundary conditions (12) versus the fractionary radius in the case of our stellar interior model. They have been normalized to the unity at the base of the convective envelope, i.e. at $r = r_{\mathrm{b}}$. Different linestyles refer to eigenfunctions of different radial order, i.e. $k = 0$: solid line; $k = 1$: dotted line; $k = 2$: dashed line and $k = 3$: dot–dashed line.

of the orbital period modulation and equation (9) gives $\alpha_{nk} \simeq \lambda_{nk}^{-1}\,\beta_{nk}$.

In order to evaluate the quadrupole moment variation and the associated kinetic energy change, we assume that the angular velocity of the unperturbed state is $\Omega_0 = 2.569 \times 10^{-5}\ \mathrm{s}^{-1}$, i.e. that of the active component of the very active system HR 1099, already considered by Frasca & Lanza (2005) and Lanza (2005) because of the remarkable amplitude of orbital period variation. The amplitude of the angular velocity perturbation is fixed at 0.01 Ω_0 at $r = r_{\mathrm{b}}$ in all the cases. The absolute value of the quadrupole moment variation $|\Delta Q_{nk}|$, the corresponding orbital period change $(\Delta P/P)_{nk}$, the kinetic energy change $\Delta\mathcal{T}_{nk}$, as given by equation (15), and the maximum dissipated power $2\lambda_{nk}\delta\mathcal{T}_{nk}$ for the eigenfunction ζ_{nk} with $n = 0$ and $n = 2$ are listed versus the radial order k in Tables 1 and 2, in the columns from the third to the sixth, respectively. The orbital period change is derived from equation (23) in the case of $a = 4R$.

Table 2. Eigenvalue λ_{2k}, quadrupole moment variation $|\Delta Q_{2k}|$, relative orbital period variation $(\Delta P/P)_{2k}$, maximum kinetic energy variation $\Delta \mathcal{T}_{2k}$ and maximum power $P_{\mathrm{diss\,2k}} = 2\lambda_{2k}\Delta\mathcal{T}_{2k}$ dissipated along a cycle of the quadrupole moment variation versus the order k of the radial eigenfunction ζ_{2k}. The amplitude of ζ_{2k} is assumed to be 1 per cent of the unperturbed stellar angular velocity at the base of the stellar convection zone (i.e. at $r = r_{\mathrm{b}}$) for all the values of k.

Radial order k	Eigenvalue λ_{2k} (s^{-1})	$\lvert\Delta Q_{2k}\rvert$ (kg m^2)	$(\Delta P/P)_{2k}$	$\Delta\mathcal{T}_{2k}$ (J)	$P_{\mathrm{diss\,2k}}$ (W)
0	3.6629×10^{-8}	1.36×10^{45}	3.71×10^{-5}	6.817×10^{36}	4.994×10^{29}
1	1.2072×10^{-7}	2.54×10^{43}	6.97×10^{-7}	4.001×10^{35}	9.661×10^{28}
2	2.3426×10^{-7}	1.43×10^{43}	3.93×10^{-7}	6.288×10^{34}	2.946×10^{28}
3	3.7513×10^{-7}	5.32×10^{41}	1.46×10^{-8}	1.729×10^{34}	1.297×10^{28}
4	5.4318×10^{-7}	1.03×10^{42}	2.83×10^{-8}	7.553×10^{33}	8.206×10^{27}
5	7.4285×10^{-7}	7.36×10^{40}	2.01×10^{-9}	4.795×10^{33}	7.124×10^{27}
6	9.7979×10^{-7}	2.53×10^{41}	6.93×10^{-9}	3.876×10^{33}	7.594×10^{27}

The kinetic energy variation and the dissipated power decrease with increasing k for the eigenfunctions ζ_{2k}. This is due to the fact that their amplitude decreases with increasing k, as shown in Fig. 2. We note that the time average of the dissipated power is $(\sqrt{2})^{-1}$ times the maximum value listed in Tables 1 and 2 in the case that the variation of the angular velocity is sinusoidal versus time.

It is interesting to note that for $n = 0$ the quadrupole moment variation associated with the eigenfunction of radial order $k = 1$ dominates over the variations associated with the eigenfunctions of higher order by at least a factor of 10, while for $n = 2$ the variation associated with $k = 0$ dominates over those with $k \geqslant 1$ by at least two orders of magnitude. This is due to the fact that ζ_{01} has only one node and one sign change, whereas ζ_{0k} for $k > 1$ has k nodes and sign changes that average out the effects of the angular velocity perturbation on the gravitational quadrupole moment. Similarly, the variation associated with ζ_{20} is the largest because it has no nodes in the interval $[r_b, R]$. Therefore, we can neglect the quadrupole moment variations associated with the eigenfunctions with $k \geqslant 2$ for $n = 0$ and with $k \geqslant 1$ for $n = 2$ and write

$$\Delta Q \simeq \Delta Q_{01} + \Delta Q_{20}. \tag{27}$$

Assuming that only the eigenfunctions ζ_{01} and ζ_{20} are responsible for the angular velocity variation, the amplitude of the angular velocity change at the surface of the star $\Delta\omega_s$ can be written as

$$\frac{\Delta\omega_s}{\Omega_0} = f_{01}\left(\frac{\Delta P}{P}\right)_{01} + f_{20}\left(\frac{\Delta P}{P}\right)_{20}, \tag{28}$$

where $(\Delta P/P)_{01}$ and $(\Delta P/P)_{20}$ are the orbital period changes associated with ζ_{01} and ζ_{20}, respectively. The coefficients $f_{01} = 3.14 \times 10^3$ and $f_{20} = 5.12 \times 10^3$ have been computed by making use of the data in Tables 1 and 2 and of the values of the respective eigenfunctions at the base and the surface of the stellar convection zone plotted in Figs 1 and 2.

Since the orbital period changes are linearly related with ζ_{01} and ζ_{20}, while the kinetic energy dissipation rates scale with the square of ζ_{01} and ζ_{20}, we can express the total dissipated power as

$$P_{\rm diss} = g_{01}\left(\frac{\Delta P}{P}\right)^2_{01} + g_{20}\left(\frac{\Delta P}{P}\right)^2_{20}, \tag{29}$$

where the coefficients $g_{01} = 2.602 \times 10^{38}$ W and $g_{20} = 3.628 \times 10^{38}$ W are derived from the data in Tables 1 and 2, respectively. For a given orbital period variation $\Delta P/P = (\Delta P/P)_{01} + (\Delta P/P)_{20}$, the minimum dissipated power is

$$P_{\rm diss\,min} = \frac{g_{01}g_{20}}{g_{01}+g_{20}}\left(\frac{\Delta P}{P}\right)^2 = 1.515 \times 10^{38}\left(\frac{\Delta P}{P}\right)^2 \text{ W}. \tag{30}$$

The associated angular velocity variation derived from equation (28) is

$$\frac{\Delta\omega_s}{\Omega_0} = \left(\frac{f_{01}g_{20} + f_{20}g_{01}}{g_{01}+g_{20}}\right)\left(\frac{\Delta P}{P}\right) = 3.97 \times 10^3\left(\frac{\Delta P}{P}\right). \tag{31}$$

The typical relative amplitude of the orbital period variations in RS CVn systems is $(1 - 3) \times 10^{-5}$ which implies an angular velocity variation of 4–12 per cent, i.e. one or two orders of magnitude larger than the variations inferred from the observations (cf. Donati, Collier Cameron & Petit 2003; Lanza & Rodonò 2004, and references therein). The variation of the kinetic energy of rotation ranges from 1.6×10^{35} to 1.5×10^{36} J with respect to the unperturbed rigid rotation. The minimum dissipated power ranges between 1.5×10^{28} and 1.4×10^{29} W, i.e. between 4 and 43 times the stellar luminosity. The discrepancy is particularly remarkable for HR 1099 for which an orbital period change as large as $\approx 9 \times 10^{-5}$ has been observed (Frasca & Lanza 2005) implying an energy dissipation rate up to $\sim$430 times the stellar luminosity.

Comparing our kinetic energy variations with those computed on the basis of the simplified model of Applegate (cf. his equations 27 and 28), we find that our values are larger by a factor of $\sim$3–4 for the case of a unperturbed rigid rotation. What makes our dissipation rates so high are the short dissipation time-scales that are proportional to the inverse of the eigenvalues, i.e. of the order of $\approx$0.5 yr or shorter (cf. equation 17 and Tables 1 and 2). On the other hand, Applegate considered time-scales of dissipation that are comparable with the duration of the orbital period cycles, i.e. at least 30–100 times longer. This assumption is wrong, except in the case that the turbulent kinematic viscosity is overestimated in our model by a comparable factor (see Section 4).

It is interesting to note that for short period RS CVn systems, i.e. with an orbital period smaller than 1 d, the observed orbital period variations have a typical amplitude of $\Delta P/P \sim 10^{-6}$ (e.g. Lanza & Rodonò 1999). Their active components are late-type main-sequence stars with a radius somewhat smaller than 1 $R_\odot$ and a mass between $\sim$0.5 and $\sim$1 $M_\odot$. The ratio (R/a) is similar to that of classic RS CVn systems. Hence, by applying equation (26), we conclude that the ratio between the dissipated power and the stellar luminosity is even larger in short period than in classic RS CVn systems because of the dependence on the factor $R^{-6}(\Delta P/P)^2$.

In conclusion, the hypothesis proposed by Applegate requires too large surface angular velocity variations and too much energy to be supported by the stellar luminosity in the case of classic as well as short period RS CVn systems. If we apply equation (26) to cataclysmic variables, the ratio of the dissipated power to the luminosity of the secondary star is found to be approximately comparable to that of classic RS CVn systems, again too large to be sustained by stellar luminosity, unless turbulent viscosity is significantly quenched due to very fast rotation.

4 DISCUSSION AND CONCLUSIONS

A general model for the angular momentum transfer within a turbulent stellar convection zone has been introduced and applied to a quantitative discussion of Applegate's hypothesis in the case of RS CVn active binaries. It is interesting to note that the separation of the variables applied to solve the equation for the angular velocity perturbation holds in the more general case in which the turbulent viscosity η_t can be expressed as the product of a function of the radial coordinate r by a function of the colatitude θ. In this case, the Jacobian polynomials can no longer be used to express the angular dependence of the solution and appropriate eigenfunctions must be computed.

In our model, the angular velocity exchanges are confined to the convection zone because the stellar dynamo is assumed to operate within its boundaries. The penetration depth of the oscillation of the angular velocity into the radiative core is of the order of $\delta \simeq \sqrt{2\nu/\sigma_{\rm cycle}}$, where $\sigma_{\rm cycle}$ is the frequency of the angular velocity variation and ν is the kinematic viscosity in the radiative core that is several orders of magnitude smaller than the turbulent viscosity in the convection zone. Therefore, the variation of the angular velocity in the radiative core is confined to a thin layer and has a negligible effect on the quadrupole moment variation.

An important point is that the estimate of the turbulent viscosity based on the mixing-length theory may not be correct for a rapidly rotating stellar convection zone. A pronounced quenching of the viscosity is indeed expected (cf. Stevenson 1979; Kichatinov, Pipin & Rüdiger 1994) that may reduce significantly the ratio of the dissipated power to the stellar luminosity. However, a reduction by at least two orders of magnitude is needed to solve the energy dissipation problem (cf. Lanza 2005). Moreover, the problem related to the large amplitude of the surface angular velocity variation predicted by our model remains, unless the change of η_t produces a significant variation of the radial profile of the eigenfunctions ζ_{nk}.

A superposition of different modes can be invoked to reduce the amplitude of the surface variations, but this would lead to a remarkable increase of the dissipated power making this explanation unlikely. In any case, the large angular velocity changes required within the convection zone may lead to a violation of the Rayleigh criterion on the angular momentum distribution making the perturbed rotation regime dynamically unstable (cf. Lanza 2005).

In conclusion, the results found by Lanza (2005), in the case of an angular velocity constant along cylindrical surfaces co-axial with the rotation axis, are confirmed in the general case of an angular velocity perturbation that depends on both radius and colatitude. This implies that Applegate's hypothesis must be rejected in the case of RS CVn binary systems. Considerations based on the scaling relationship (26) show that the hypothesis leads to the same energy balance problem in the case of the secondary components of cataclysmic variables and related systems. However, a detailed analysis of the orbital period variation in such systems deserves a dedicated study given the complication that arises from fast rotation, mass transfer and angular momentum loss through magnetic braking of their secondary components (e.g. Warner 1995; Brinkworth et al. 2006; Hussain et al. 2006).

It is important to note that the interpretation of the orbital period modulation of close binary systems in terms of an oscillation of the gravitational quadrupole moment of their magnetically active component is, however, not excluded. The anisotropic Lorentz force due to an internal magnetic field may produce such a variation, as discussed by, e.g. Lanza et al. (1998), Lanza & Rodonò (1999), Rüdiger et al. (2002) and Lanza & Rodonò (2004).

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