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Internal symmetry in bilinear elastoplasticity

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Abstract

Internal symmetry of a constitutive model of bilinear elastoplasticity (i.e. linear elasticity combined with linear kinematic hardening–softening plasticity) is investigated. First the model is analyzed and synthesized so that a two-phase two-stage linear representation of the constitutive model is obtained. The underlying structure of the representation is found to be Minkowski spacetime, in which the augmented active states admit of a Lorentz group of transformations in the on phase. The kinematic rule of the model renders the transformation group inhomogeneous, resulting in a larger group—the proper orthochronous Poincaré group. © 1998 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A constitutive law is said to possess internal symmetry if it retains the form of a certain expression for constitutive phenomena after some changes in the point of view from which the phenomena are observed. The changes or transformations made to the constitutive law which leave the form unchanged in the effects are naturally linked with the invariance of conserved quantities. A procedure for parameter estimation (and model identification) with effective utilization of the invariance property along the experimental path will be more capable of capturing key features during elastoplastic deformation, and a numerical algorithm which preserves symmetry in time marching will have long-term stability and much improved efficiency and accuracy. So the issue of internal symmetries in constitutive laws of plasticity is not only important in its own right, but will also find applications to computational plasticity and to experimental research and industrial testing practice.

Recently, Hong and Liu [1] considered a constitutive model of perfect elastoplasticity, revealing that the model possesses two kinds of internal symmetries, characterized by the translation group $T(n)$ in the off (or elastic) phase and by the projective proper orthochronous Lorentz group $PSO_o(n, 1)$ in the on (or elastoplastic) phase, and has symmetry switching between the two groups dictated by the control input. The perfect elastoplasticity model is the simplest to use, but being linearly elastic and perfectly plastic it cannot

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predict the Bauschinger effect which is observed experimentally in most metals under reversed cyclic testing. The Bauschinger effect refers to a particular type of directional anisotropy in stress space induced by plastic deformation—an initial plastic deformation of one direction reduces the subsequent yield strength in the opposite direction. To model the Bauschinger effect, Prager [2] suggested a linear kinematic hardening rule. Kinematic hardening is the hypothesis that the yield hypersurface translates as a rigid body in the stress space during the plastic deformation, maintaining the size, shape and orientation of the yield hypersurface in the subsequent plastic deforming [3,4].

In this paper let us limit our scope to linear kinematic hardening–softening and extend the symmetry study to a constitutive model of bilinear elastoplasticity, investigating the influence of linear kinematic hardening–softening on the structure of the underlying vector space and on the transformation group of symmetries. Here *bilinear* elastoplasticity is used to abbreviate the combination of *linear* elasticity with *linear* kinematic hardening–softening plasticity.

It may be interesting to name a few application areas of the bilinear elastoplasticity model [of equations (1)–(8) below]. It is well known that the model (of dimension $n = 5$) has been used to describe the deviatoric part of the stress versus strain-rate relation of a three-dimensional elastoplastic material. Less known is that the model (of dimension $n = 1$ or 2 or more) has been used in the analyses of isolation systems of buildings and equipment in recent years (see, e.g. [5] or [6]), where the model describing the relation between the restoring force and the relative velocity of the two end-plates of a seismic isolator was combined with the equation of motion to simulate the hysteretic motion and dissipation capacity of the isolator. Fig. 1 depicts the restoring force-relative displacement curves for a uni-directional isolator (the model of dimension $n = 1$).

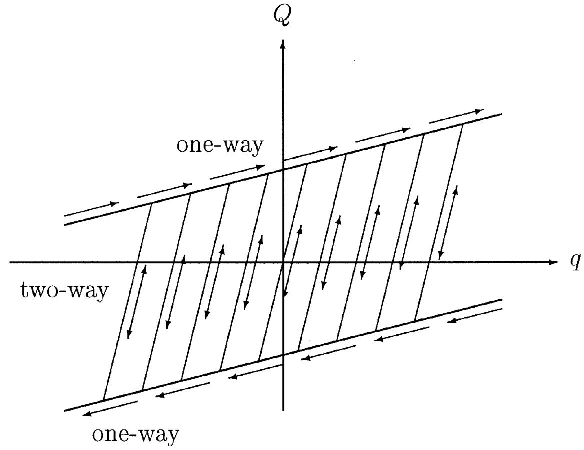

Fig. 1. The restoring force–relative displacement curves of a unit-directional seismic isolator, showing elastic deformation taking place at infinitely many *two-way* line segments while elastoplastic flow occurring at two *one-way* lines.

2. Bilinear elastoplasticity

Since our aim is to reveal symmetry in a constitutive model, we need an appropriate setting to make the presentation clearer and simpler; therefore, we choose to postulate the constitutive model directly in Euclidean vector form as in the following:

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^e + \dot{\mathbf{q}}^p, \tag{1}$$

$$\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_b, \tag{2}$$

$$\dot{\mathbf{Q}} = k_e \dot{\mathbf{q}}^e, \tag{3}$$

$$\mathbf{Q}_a \dot{q}_0^a = Q_a^0 \dot{\mathbf{q}}^p, \tag{4}$$

$$\dot{\mathbf{Q}}_b = k_p \dot{\mathbf{q}}^p, \tag{5}$$

$$\|\mathbf{Q}_a\| \leqslant Q_a^0, \tag{6}$$

$$\dot{q}_0^a \geqslant 0, \tag{7}$$

$$\|\mathbf{Q}_a\| \dot{q}_0^a = Q_a^0 \dot{q}_0^a, \tag{8}$$

in which the generalized elastic modulus k_e, the generalized kinematic modulus k_p, and the generalized yield stress Q_a^0 are the only three property constants needed in the model, with the limitations that

$$k_e > 0, \quad k_p > -k_e, \quad Q_a^0 > 0.$$

Table 1
Generalized kinematic modulus k_p and hardening–softening factor β

Property constants	Range	Hardening plasticity	Perfect plasticity	Softening plasticity
k_p	$\infty > k_p > -k_e$	$\infty > k_p > 0$	$k_p = 0$	$0 > k_p > -k_e$
β	$0 < \beta < \infty$	$0 < \beta < 1$	$\beta = 1$	$1 < \beta < \infty$

Depending upon the value of k_p, the model is capable of treating hardening, perfect plasticity, and softening, as shown in Table 1.

The bold faced symbols $\mathbf{q}$, $\mathbf{q}^e$, $\mathbf{q}^p$, $\mathbf{Q}$, $\mathbf{Q}_a$ and $\mathbf{Q}_b$ are the (Euclidean) (column) vectors of generalized strain, generalized elastic strain, generalized plastic strain, generalized stress, generalized active stress and generalized back stress, respectively, all with n components, whereas q_0^a is a scalar called the equivalent generalized plastic strain, with $Q_a^0 \dot{q}_0^a$ being the (specific) power of dissipation. The generalized strain vector $\mathbf{q}$ and the generalized strain rate vector $\dot{\mathbf{q}}$ are assumed to be so small that no account is taken of the geometric non-linearity effect (including spinning) and the rate effect as well as the inertia effect.

Here the norm of a vector, say $\mathbf{Q} = \mathrm{col}(Q^1, Q^2, \ldots, Q^n) = \mathrm{col}(Q_1, Q_2, \ldots, Q_n)$, in n-dimensional Euclidean space endowed with the Euclidean metric $\mathbf{I}_n$ is denoted by $\|\mathbf{Q}\| := \sqrt{\mathbf{Q}^t\mathbf{Q}} = \sqrt{\sum_{i=1}^n Q_i Q^i}$, where a superscript t indicates the transpose and $\mathbf{I}_n$ is the identity tensor of order n. All $\mathbf{q}$, $\mathbf{q}^e$, $\mathbf{q}^p$, $\mathbf{Q}$, $\mathbf{Q}_a$, $\mathbf{Q}_b$ and q_0^a are functions of one and the same independent variable, which in most cases is taken either as the ordinary time or as the arc length of a control path; however, for convenience, the independent variable no matter what it is will be simply called (the external) time and given the symbol t. A superimposed dot denotes differentiation with respect to time, that is d/dt.

The mechanical-element model displayed in Fig. 2 may help explain the meanings of Eqs. (1)–(8). It is easy to comprehend and appreciate Eqs. (1)–(7): Eq. (1) decomposes the generalized strain rate $\dot{\mathbf{q}}$ into an elastic part and a plastic part; Eq. (2) decomposes the generalized stress $\mathbf{Q}$ into an active part and a back (or kinematic or translational) part; Eq. (3) gives a linear law for the elastic part; Eq. (4) is an associated plastic-flow rule; Eq. (5) is a

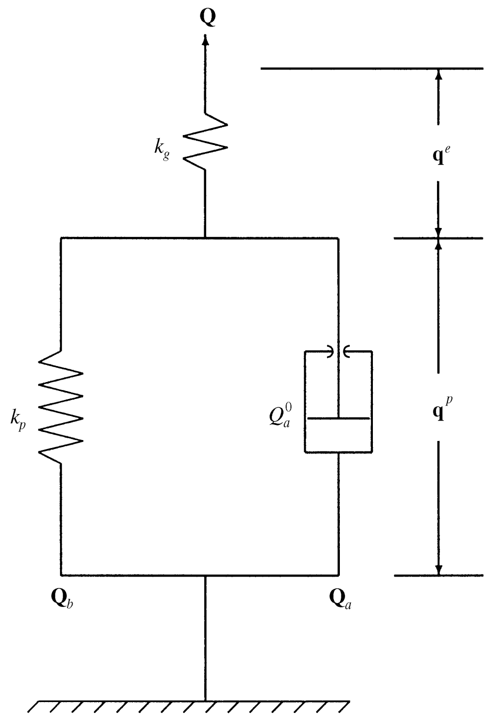

Fig. 2. Schematic drawing showing the connection of mechanical elements—two springs and one slide-damper—for the model of bilinear elastoplasticity.

linear kinematic hardening–softening rule (often known as Prager's rule); Eq. (6) specifies an admissible range of generalized active stresses (thus, $\|\mathbf{Q}_a\| = Q_a^0$ is the yield hypersphere with center $\mathbf{Q}_b$); and Eq. (7) or $Q_a^0 \dot{q}_0^a \geqslant 0$ requires the non-negativity of the (specific) power of dissipation. But Eq. (8) may need more explanations: With the aid of the two inequalities (6) and (7), it simply requires $\dot{q}_0^a$ be frozen if $\|\mathbf{Q}_a\| < Q_a^0$, so that $\dot{q}_0^a = 0$ drastically reduces Eqs. (1), (3) and (4) to $\dot{\mathbf{Q}} = k_e\dot{\mathbf{q}}$, and Eq. (5) to $\dot{\mathbf{Q}}_b = 0$. The complementary trios

(6)–(8) furnish the model with two phases, as to e analyzed in the next section.

3. Two phases

Since once dt is factored out the differential equations (1), (3)–(5), (7) and (8) become incremental equations independent of time, the flow model represented by axioms (1)–(8) restrict itself to *time-independent* elastoplasticity; therefore, we are mostly concerned with *paths* rather than histories in the present paper. We require a path as a continuous curve with piecewise continuous tangent vectors. From Eqs. (4), (7) and (8) and $Q_a^0 > 0$, it is not difficult to prove that

$$\dot{q}_0^a = \|\dot{\mathbf{q}}^p\|, \tag{9}$$

which indicates that q_0^a is the arc length of a path in the generalized plastic strain space.

From Eqs. (1)–(3) and (5) it follows that

$$\dot{\mathbf{Q}}_a + k_p\dot{\mathbf{q}}^p = k_e(\dot{\mathbf{q}} - \dot{\mathbf{q}}^p). \tag{10}$$

Inserting Eq. (4) into Eq. (10) we have

$$\dot{\mathbf{Q}}_a + \frac{k_e + k_p}{Q_a^0}\dot{q}_0^a\mathbf{Q}_a = k_e\dot{\mathbf{q}}, \tag{11}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_a^0\mathbf{Q}_a) = k_e X_a^0\dot{\mathbf{q}}, \tag{12}$$

where

$$X_a^0 := \exp\left(\frac{q_0^a}{\beta q_y}\right) \tag{13}$$

with the generalized yield strain

$$q_y := \frac{Q_a^0}{k_e} \tag{14}$$

and the hardening–softening factor

$$\beta := \frac{k_e}{k_e + k_p}. \tag{15}$$

The ranges of values of β as well as k_p are shown in Table 1.

The inner product of $\mathbf{Q}_a$ with Eq. (11) is

$$(\mathbf{Q}_a)^t\dot{\mathbf{Q}}_a + \frac{k_e + k_p}{Q_a^0}\dot{q}_0^a(\mathbf{Q}_a)^t\mathbf{Q}_a = k_e(\mathbf{Q}_a)^t\dot{\mathbf{q}}, \tag{16}$$

which, due to the constancy of Q_a^0, asserts the statement

$$\|\mathbf{Q}_a\| = Q_a^0 \Rightarrow \beta(\mathbf{Q}_a)^t\dot{\mathbf{q}} = Q_a^0\dot{q}_0^a. \tag{17}$$

Since $Q_a^0 > 0$ and $\beta > 0$, the statement

$$\|\mathbf{Q}_a\| = Q_a^0 \Rightarrow \{(\mathbf{Q}_a)^t\dot{\mathbf{q}} > 0 \Leftrightarrow \dot{q}_0^a > 0\} \tag{18}$$

is true. Thus

$$\{\|\mathbf{Q}_a\| = Q_a^0 \text{ and } (\mathbf{Q}_a)^t\dot{\mathbf{q}} > 0\} \Rightarrow \dot{q}_0^a > 0. \tag{19}$$

On the other hand, if $\dot{q}_0^a > 0$, axiom (8) assures $\|\mathbf{Q}_a\| = Q_a^0$, which together with Eq. (18) asserts that

$$\dot{q}_0^a > 0 \Rightarrow \{\|\mathbf{Q}_a\| = Q_a^0 \text{ and } (\mathbf{Q}_a)^t\dot{\mathbf{q}} > 0\}. \tag{20}$$

Statements (19) and (20) tell us that the yield condition $\|\mathbf{Q}_a\| = Q_a^0$ and the straining condition $(\mathbf{Q}_a)^t\dot{\mathbf{q}} > 0$ are sufficient and necessary for plastic irreversibility $\dot{q}_0^a > 0$.

In view of Eqs. (6) and (7) and of Eqs. (9) and (17), statements (19) and (20) are logically equivalent to the following on–off switching criteria for the mechanism of plasticity:

$$\dot{q}_0^a = \|\dot{\mathbf{q}}^p\| = \begin{cases} \dfrac{\beta}{Q_a^0}(\mathbf{Q}_a)^t\dot{\mathbf{q}} > 0 & \text{if } \|\mathbf{Q}_a\| = Q_a^0 \text{ and } (\mathbf{Q}_a)^t\dot{\mathbf{q}} > 0, \\ 0 & \text{if } \|\mathbf{Q}_a\| < Q_a^0 \text{ or } (\mathbf{Q}_a)^t\dot{\mathbf{q}} \leqslant 0. \end{cases} \tag{21}$$

Based on the criteria and the complementary trios (6)–(8), the model of elastoplasticity has two phases (and just two phases): the ON phase in which $\dot{q}_0^a > 0$ and $\|\mathbf{Q}_a\| = Q_a^0$ and the OFF phase in which $\dot{q}_0^a = 0$ and $\|\mathbf{Q}_a\| \leqslant Q_a^0$. In the on phase the mechanism of plasticity is on so that the model exhibits elastoplastic behavior, which is irreversible, while in the off phase the mechanism of plasticity is off so that the model responds elastically and reversibly.

4. Homogeneous coordinates

Let us normalize the generalized active stress $\mathbf{Q}_a$, the generalized back stress $\mathbf{Q}_b$, and the generalized stress $\mathbf{Q}$ with respect to the generalized yield stress Q_a^0, and then consider their homogeneous coordinates:

$$\mathbf{X}_a = \begin{bmatrix} \mathbf{X}_a^s \\ X_a^0 \end{bmatrix} = \begin{bmatrix} X_a^1 \\ X_a^2 \\ \vdots \\ X_a^n \\ X_a^0 \end{bmatrix} := \frac{\exp\left(\dfrac{q_0^a}{\beta q_y}\right)}{Q_a^0} \begin{bmatrix} Q_a^1 \\ Q_a^2 \\ \vdots \\ Q_a^n \\ Q_a^0 \end{bmatrix} = \frac{\exp\left(\dfrac{q_0^a}{\beta q_y}\right)}{Q_a^0} \begin{bmatrix} \mathbf{Q}_a \\ Q_a^0 \end{bmatrix}, \tag{22}$$

$$\mathbf{X}_b = \mathrm{col}(X_b^1, X_b^2, \ldots, X_b^n, X_b^0) := \begin{bmatrix} \dfrac{\mathbf{Q}_b}{Q_a^0} \\ 0 \end{bmatrix}, \tag{23}$$

$$\mathbf{X} = \mathrm{col}(X^1, X^2, \ldots, X^n, X^0) := \mathbf{X}_a + \mathbf{X}_b = \begin{bmatrix} X_a^0 \dfrac{\mathbf{Q}_a}{Q_a^0} + \dfrac{\mathbf{Q}_b}{Q_a^0} \\ X_a^0 \end{bmatrix}. \tag{24}$$

Here and henceforth the index s refers to the totality of the "internal space" coordinates, while the index 0 refers to the "internal time" coordinate. Together the homogeneous coordinates will be utilized to describe the intrinsic properties of the constitutive model in its "internal spacetime."

For convenience $\mathbf{X}_a$, $\mathbf{X}_b$, and $\mathbf{X}$ are thus deemed as $(n+1)$-dimensional vectors, $\mathbf{X}_a$ being called the augmented active state vector, $\mathbf{X}_b$ the augmented back state vector, $\mathbf{X}$ the augmented state vector. Then the constitutive model postulated in the state space of $(Q^1, Q^2, \ldots, Q^n)$ may be translated into a model in the augmented state space of $(X^1, X^2, \ldots, X^n, X^0)$, thus Eqs. (2), (12),[1] (8), (6), (7) and (5) become successively

$$\mathbf{X} = \mathbf{X}_a + \mathbf{X}_b, \tag{25}$$

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a \end{bmatrix} \dot{\mathbf{X}}_a = \frac{1}{q_y} \begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \mathbf{X}_a, \tag{26}$$

$$(\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a \leqslant 0, \tag{27}$$

$$\dot{X}_a^0 \geqslant 0, \tag{28}$$

$$\dot{\mathbf{X}}_b = (1-\beta)\frac{\dot{X}_a^0}{(X_a^0)^2} \begin{bmatrix} \mathbf{X}_a^s \\ 0 \end{bmatrix}, \tag{29}$$

in terms of the Minkowski metric

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_{ss} & \mathbf{g}_{s0} \\ \mathbf{g}_{0s} & g_{00} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix}. \tag{30}$$

The $(n+1)$-dimensional vector space of augmented state $\mathbf{X}$ endowed with the Minkowski metric $\mathbf{g}$ is referred to as *Minkowski spacetime* $\mathbb{M}^{n+1}$.

Regarding Eqs. (6) and (27), we may further distinguish two correspondences:

$$\|\mathbf{Q}_a\| = Q_a^0 \Leftrightarrow (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a = 0, \tag{31}$$

$$\|\mathbf{Q}_a\| < Q_a^0 \Leftrightarrow (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a < 0. \tag{32}$$

That is, a generalized stress state $\mathbf{Q}$ on the yield hypersphere $\|\mathbf{Q}-\mathbf{Q}_b\| = Q_a^0$ with center at $\mathbf{Q}_b$ in the state space of $(Q^1, Q^2, \ldots, Q^n)$ corresponds to an augmented state $\mathbf{X}$ on the right circular cone $\{\mathbf{X} \mid (\mathbf{X}-\mathbf{X}_b)^{\mathrm{t}}\mathbf{g}(\mathbf{X}-\mathbf{X}_b) = 0\}$ emanating from the event point $\mathbf{X}_b$ in Minkowski spacetime of $(X^1, X^2, \ldots, X^n, X^0)$, henceforth abbreviated to *the cone*, whereas a $\mathbf{Q}$ within the yield hypersphere corresponds to an $\mathbf{X}$ in the interior $\{\mathbf{X} \mid (\mathbf{X}-\mathbf{X}_b)^{\mathrm{t}}\mathbf{g}(\mathbf{X}-\mathbf{X}_b) < 0\}$ of the cone. The exterior $\{\mathbf{X} \mid (\mathbf{X}-\mathbf{X}_b)^{\mathrm{t}}\mathbf{g}(\mathbf{X}-\mathbf{X}_b) > 0\}$ of the cone is uninhabitable since $\|\mathbf{Q}-\mathbf{Q}_b\| > Q_a^0$ is forbidden according to axiom (6). Even though it admits an infinite number of Riemannian metrics, the yield hypersphere $\mathbb{S}^{n-1}$ in the state space of $(Q^1, Q^2, \ldots, Q^n)$ does not admit a Minkowskian metric, nor does the cylinder in the enlarged space of $(Q^1, Q^2, \ldots, Q^n, X^0)$. It is the cone in the augmented state space of $(X^1, X^2, \ldots, X^n, X^0)$ which admits the Minkowski metric.

Taking the Euclidean inner product of Eq. (12) with $\mathbf{Q}_a X_a^0/(Q_a^0)^2$, substituting Eq. (22), and

[1] Recall that Eq. (12) is obtained by combining Eqs. (1)–(5).

considering the positivity of Q_a^0, k_e and X_a^0, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] = \frac{(X_a^0)^2 k_e}{(Q_a^0)^2}(\mathbf{Q}_a)^{\mathrm{t}}\dot{\mathbf{q}},$$

and, therefore,

$$(\mathbf{Q}_a)^{\mathrm{t}}\dot{\mathbf{q}} > 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \tag{33}$$

$$(\mathbf{Q}_a)^{\mathrm{t}}\dot{\mathbf{q}} \leqslant 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0. \tag{34}$$

Hence in the augmented state space what corresponds to the yield condition $\|\mathbf{Q}_a\| = Q_a^0$ is the cone condition $(\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a = 0$ and what corresponds to the straining condition $(\mathbf{Q}_a)^{\mathrm{t}}\dot{\mathbf{q}} > 0$ is the growing "internal space" radial coordinate condition $(\mathrm{d}/\mathrm{d}t)$ $[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0$.

5. Two-phase two-stage linear representation

In view of Eqs. (22) and (31)–(34), the on–off switching criteria (21) become

$$\dot{X}_a^0 = \begin{cases} (\mathbf{X}_a^s)^{\mathrm{t}}\dfrac{\dot{\mathbf{q}}}{q_y} > 0 \\ \text{if } (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a = 0 \text{ and } \dfrac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \\ 0 \\ \text{if } (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a < 0 \text{ or } \dfrac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0, \end{cases} \tag{35}$$

in the augmented state space. Expressing Eq. (12) in terms of the homogeneous coordinates (22) and arranging them and Eq. (35) together in matrix form, we obtain a linear system for augmented active states

$$\dot{\mathbf{X}}_a = \mathbf{A}\mathbf{X}_a \tag{36}$$

with the control tensor

$$\mathbf{A} := \frac{1}{q_y}\begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^{\mathrm{t}} & 0 \end{bmatrix}$$

$$\text{if } (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \tag{37a}$$

$$\mathbf{A} := \frac{1}{q_y}\begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}$$

$$\text{if } (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a < 0 \text{ or } \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0. \tag{37b}$$

The last row of $\mathbf{A}$ in the off phase is full of zeros since X_a^0 is frozen in the off phase.

Addition of Eqs. (36) and (29) gives

$$\dot{\mathbf{X}} := \begin{bmatrix} (1-\beta)\dfrac{\dot{X}_a^0}{(X_a^0)^2}\mathbf{I}_n & \dot{\mathbf{q}}/q_y \\ \dot{\mathbf{q}}^{\mathrm{t}}/q_y & 0 \end{bmatrix}\mathbf{X}_a$$

$$\text{if } (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \tag{38a}$$

$$\dot{\mathbf{X}} := \begin{bmatrix} \mathbf{0}_{n\times n} & \dot{\mathbf{q}}/q_y \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}\mathbf{X}_a$$

$$\text{if } (\mathbf{X}_a)^{\mathrm{t}}\mathbf{g}\mathbf{X}_a < 0 \text{ or } \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{t}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0. \tag{38b}$$

Hence, besides the distinction between the on phase and the off phase, there are two stages of calculations: first solving Eq. (36) for the augmented active state $\mathbf{X}_a$ and then integrating Eq. (38) to get the augmented state $\mathbf{X}$. Note that in each phase and each stage the relevant equations are linear. So Eqs. (36) and (38) constitute a *two-phase two-stage linear* representation of the constitutive model (1)–(8).

6. Non-time-like paths

In this section we study paths in the augmented state space. Criteria (21) ensure that

$$\dot{q}_0^a(\mathbf{Q}_a)^{\mathrm{t}}\dot{\mathbf{Q}}_a = 0 \tag{39}$$

no matter whether in the on or in the off phase. In view of Eqs. (15), (4), (9) and (7), and of the positivity of Q_a^0, β and k_e, Eqs. (11) and (39) become, respectively,

$$\frac{\beta}{k_e}\dot{\mathbf{Q}}_a + \dot{\mathbf{q}}^p = \beta\dot{\mathbf{q}}, \tag{40}$$

$$(\dot{\mathbf{Q}}_a)^{\mathrm{t}}\dot{\mathbf{q}}^p = 0, \tag{41}$$

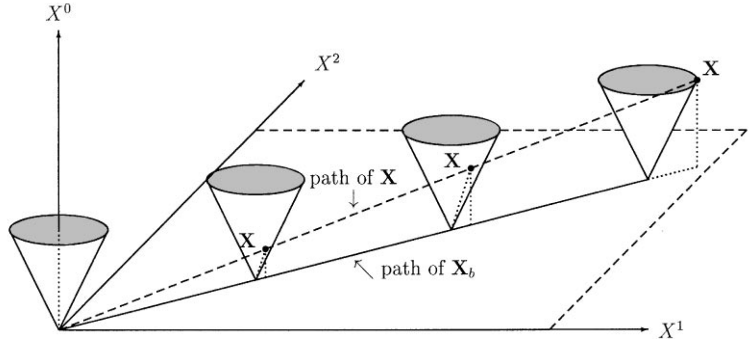

Fig. 3. The internal spacetime (X^1, X^2, X^0) of the model of dimension $n = 2$, in which the cone is moving in the direction of the internal space projection of the active part $\mathbf{X}_a$, the vertex $\mathbf{X}_b$ of the cone remaining in the internal space (X^1, X^2) and the axis of the cone being always in the direction of the internal time X^0.

which together yield

$$0 \leqslant \dot{q}_0^a = \|\dot{\mathbf{q}}^p\| \leqslant \beta \|\dot{\mathbf{q}}\| \tag{42}$$

due to the Pythagorean theorem. It tells us that the minimum and maximum values of the (specific) dissipation power $\dot{\Lambda} = Q_a^0 \dot{q}_0^a$ an admissible path in the state space may discharge are zero and $\beta Q_a^0 \|\dot{\mathbf{q}}\|$, respectively.

What does this important observation (42) imply for a path in the Minkowski spacetime of augmented states? From Eqs. (42), (7) and (13) it follows that

$$(X_a^0)^2 \dot{\mathbf{q}}^t \dot{\mathbf{q}} - (q_y \dot{X}_a^0)^2 \geqslant 0. \tag{43}$$

Inserting Eqs. (12), (22) and (30) into inequalities (42), we obtain

$$(\dot{\mathbf{X}}_a)^t \mathbf{g} \dot{\mathbf{X}}_a \geqslant 0, \tag{44}$$

or

$$(\mathrm{d}\mathbf{X}_a)^t \mathbf{g}\, \mathrm{d}\mathbf{X}_a \geqslant 0, \tag{45}$$

which requires that *any curve* $\{\mathbf{X}(t') | t_i < t' \leqslant t\}$ *in the internal spacetime of (bilinear) elastoplasticity be a non-time-like path of Minkowski spacetime*, where t is the current time and t_i is an initial time.

The vector $\mathbf{X}(t) - \mathbf{X}(t_i)$ and the path $\{\mathbf{X}(t') | t_i < t' \leqslant t\}$ are said to be future-pointing if $X^0(t) > X^0(t_i)$ strictly. Therefore, the solution to Eq. (36) with (37a) and to (38a) can be viewed as a future-pointing space-like or null path on the cone $\{\mathbf{X} | (\mathbf{X} - \mathbf{X}_b)^t \mathbf{g}(\mathbf{X} - \mathbf{X}_b) = 0\}$, which is endowed with the Minkowski metrix $\mathbf{g}$, while the solution to Eq. (36) with (37b) and to (38b) is a space-like path on a closed disc $\mathbb{D}^n$ (that is the closed ball $\mathbb{B}^n$) of simultaneity $\{\mathbf{X} | (\mathbf{X} - \mathbf{X}_b)^t \mathbf{g}(\mathbf{X} - \mathbf{X}_b) \leqslant 0 \text{ and } \dot{X}_0 = 0\}$, which is endowed with the Euclidean metric $\mathbf{I}_n$. It is worth stressing that the interior of the cone is sliced into stacking discs of simultaneity tagged with different values of X^0; therefore, an admissible augmented stress can be reached either along paths in the discs of simultaneity when in the off phase or/and along the future-pointing space-like or null paths on the cone when in the on phase. According to Eq. (29), the cone moves in the direction of the "internal space" projection of the augmented active state vector $\mathbf{X}_a$ in the on phase and remains frozen in the off phase. A geometrical visualization of the cone and its motion in the augmented state space of (X^1, X^2, X^0) is displayed in Fig. 3 for the model of dimension $n = 2$.

7. Transformation of augmented states

The solution of Eq. (36) may be expressed in the following transition formula from the augmented active state $\mathbf{X}_a(t_1)$ at time t_1 to the augmented active state $\mathbf{X}_a(t)$ at time t:

$$\mathbf{X}_a(t) = [\mathbf{G}(t)\ \mathbf{G}^{-1}(t_1)]\mathbf{X}_a(t_1), \tag{46}$$

in which $\mathbf{G}(t)$ is the fundamental solution of Eq. (36), that is a transformation tensor (represented by a square matrix of order $(n+1)$ containing the mixed components) satisfying

$$\dot{\mathbf{G}}(t) = \mathbf{A}(t)\mathbf{G}(t), \tag{47}$$

$$\mathbf{G}(0) = \mathbf{I}_{n+1}. \tag{48}$$

Furthermore, substituting Eq. (25) into equation (46) yields

$$\mathbf{X}(t) = [\mathbf{G}(t)\,\mathbf{G}^{-1}(t_1)]\mathbf{X}(t_1) + (\mathbf{X}_b(t) - [\mathbf{G}(t)\,\mathbf{G}^{-1}(t_1)]\mathbf{X}_b(t_1)). \tag{49}$$

From Eqs. (24) and (28) it follows that

$$X^0(t) \geqslant X^0(t') \geqslant X^0(t_i) \quad \forall t \geqslant t' \geqslant t_i. \tag{50}$$

All the relations (46)–(50) are applicable to any time interval which is exclusively in the on phase or exclusively in the off phase.

From the foregoing discussion it is clear that the transformation tensor $\mathbf{G}$ plays a fundamental role in the elastoplastic model. Indeed, a further study on it will reveal internal symmetry of the model as to be presented in the following two sections.

8. Internal symmetry in the on phase

We first study the transformation tensor of the on phase. Denote by I_{on} an open, maximal, continuous time interval during which the mechanism of plasticity is on exclusively. From Eqs. (37a) and (30) it is easy to verify that the control tensor $\mathbf{A}$ in the on phase satisfies

$$\mathbf{A}^{\text{t}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{51}$$

By Eqs. (51) and (47) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{G}^{\text{t}}(t)\,\mathbf{g}\mathbf{G}(t)] = \mathbf{0}.$$

At $t = 0$, $\mathbf{G}^{\text{t}}(t)\mathbf{g}\,\mathbf{G}(t) = \mathbf{I}^{\text{t}}_{n+1}\mathbf{g}\,\mathbf{I}_{n+1} = \mathbf{g}$ from Eq. (48); thus, we prove that

$$\mathbf{G}^{\text{t}}(t)\,\mathbf{g}\,\mathbf{G}(t) = \mathbf{g} \tag{52}$$

for all $t \in I_{\text{on}}$. Take determinants of both sides of the above equation, getting

$$(\det \mathbf{G})^2 = 1, \tag{53}$$

so that $\mathbf{G}$ is invertible. The 00th component of the tensorial Eq. (52) is $\sum_{i=1}^{n}(G^i_0)^2 - (G^0_0)^2 = -1$, from which

$$(G^0_0)^2 \geqslant 1. \tag{54}$$

Here G^i_j, $i, j = 1, 2, \ldots, n, 0,$ is the ijth mixed component of the tensor $\mathbf{G}$. Since $\det \mathbf{G} = -1$ or $G^0_0 \leqslant -1$ would violate Eq. (48), it turns out that

$$\det \mathbf{G} = 1, \tag{55}$$

$$G^0_0 \geqslant 1. \tag{56}$$

In summary, $\mathbf{G}$ has the three characteristic properties explicitly expressed by Eqs. (52), (55) and (56).

Thereby the on-phase control tensor $\mathbf{A}$ is an element of the real Lie algebra $so(n, 1)$ and generates the on-phase transformation $\mathbf{G}$, which is thus an element of the proper orthochronous Lorentz group $SO_o(n, 1)$, see, e.g. [7]. So the function $\mathbf{G}(t)$ of time $t \in I_{\text{on}}$ may be viewed as a connected path of the Lorentz group.

In this way the first term on the right-hand side of Eq. (49) indicates an action of the proper orthochronous Lorentz group, and the second term is an element of the translation group. The semi-direct product of the translation group $T(n+1)$ and the proper orthochronous Lorentz group $SO_o(n, 1)$ is known as the inhomogeneous proper orthochronous Lorentz group $ISO_o(n, 1)$, or called the orthochronous Poincaré group $SE_o(n, 1)$. In view of Eqs. (22)–(24) it is concluded that the internal symmetry in the on phase is characterized by a projective realization of $ISO_o(n, 1)$.

From Eqs. (24) and $(35)_1$, $\dot{X}^0 > 0$ strictly when the mechanism of plasticity is on; hence,

$$X^0(t) > X^0(t_1) \quad \forall t > t_1,\ t, t_1 \in I_{\text{on}}, \tag{57}$$

which means that in the sense of irreversibility there exists future-pointing time-orientation from the augmented states $\mathbf{X}(t_1)$ to $\mathbf{X}(t)$. Moreover, such time-orientation is a causal one, because the augmented state transition Eq. (49) and inequality (57) establish a *causality relation* between the two augmented states $\mathbf{X}(t_1)$ and $\mathbf{X}(t)$ in the sense that the preceding augmented state $\mathbf{X}(t_1)$ influences the following augmented state $\mathbf{X}(t)$ according to Eq. (49). Accordingly, the augmented state $\mathbf{X}(t_1)$ chronologically and causally precedes the augmented stress $\mathbf{X}(t)$. This is indeed a common property for all models with inherent symmetry of the proper orthochronous Poincaré group.

In order to derive a product formula for $ISO_o(n, 1)$ it is more convenient to embed $ISO_o(n, 1)$ in a special linear group $SL(n+2, \mathbb{R})$; thus Eq. (49) becomes

$$\begin{bmatrix} \mathbf{X}(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{G}(t) & \mathbf{X}_b(t) \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{G}(t_1) & \mathbf{X}_b(t_1) \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} \mathbf{X}(t_1) \\ 1 \end{bmatrix}. \tag{58}$$

It is easy to solve for the inverses

$$\begin{bmatrix} \mathbf{G} & \mathbf{X}_b \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & -\mathbf{G}^{-1}\mathbf{X}_b \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix}$$

and (by Eq. (52))

$$\mathbf{G}^{-1} = \mathbf{g}\mathbf{G}^{\mathrm{t}}\mathbf{g}. \tag{59}$$

The transformation matrix of the group $SL(n+2, \mathbb{R})$ may be further split as follows:

$$\begin{bmatrix} \mathbf{G}(t) & \mathbf{X}_b(t) \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n+1} & \mathbf{X}_b(t) \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{G}(t) & \mathbf{0}_{(n+1)\times 1} \\ \mathbf{0}_{1\times(n+1)} & 1 \end{bmatrix}, \tag{60}$$

or denoted as

$$(\mathbf{T}(t)|\mathbf{L}(t)) = \mathbf{T}(t)\,\mathbf{L}(t), \tag{61}$$

where the group action $(\mathbf{T}(t)|\mathbf{L}(t))$ has the following algebraic properties:

$$\mathbf{T}(t_1)\,\mathbf{T}(t_2) = \mathbf{T}(t_2)\mathbf{T}(t_1), \tag{62}$$

$$(\mathbf{T}(t_2)|\mathbf{L}(t_2))\,(\mathbf{T}(t_1)|\mathbf{L}(t_1)) = (\mathbf{T}(t_2)\,\mathbf{L}(t_2)\,\mathbf{T}(t_1)\,\mathbf{L}^{-1}(t_2)|\,\mathbf{L}(t_2)\,\mathbf{L}(t_1)), \tag{63}$$

for all $t_1, t_2 \in I_{\mathrm{on}}$. The former indicates that $\mathbf{T}(t)$ forms an invariant subgroup of the Poincaré group, and the latter shows out the reason why the Poincaré group is a semi-direct product of the translation group and the Lorentz group. Both the groups of $\mathbf{T}$(t) and $\mathbf{L}(t)$ are non-compact, so the Poincaré group is also non-compact. As the group of $\mathbf{T}(t)$ is an abelian subgroup, the Poincaré group is not semi-simple.

9. Internal symmetry in the off phase

Contrary to the on-phase transformation, the off-phase transformation is very simple. We recall that Eqs. (46)–(50) are applicable to the off phase and readily find that

$$\mathbf{G}(t) = \begin{bmatrix} \mathbf{I}_n & \dfrac{\mathbf{q}(t)}{q_y} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix}, \quad \mathbf{X}_b(t) = \mathbf{X}_b(t_1),$$

$$X^0(t) = X^0(t_1). \tag{64}$$

Thus, by Eq. (49)

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{I}_n & \dfrac{\mathbf{q}(t) - \mathbf{q}(t_1)}{q_y} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \mathbf{X}(t_1), \tag{65}$$

which is valid $\forall\ t, t_1 \in I_{\mathrm{off}}$, where I_{off} is an open, maximal, continuous time interval during which the mechanism of plasticity is off exclusively.

Even such an off-phase transformation exists and is invertible; it is no longer an element of the Lorentz group because such $\mathbf{G}$ does not satisfy equation (52) although $\det \mathbf{G} = 1$ remains to hold and $G^0_0 = 1$. It belongs to a translation group $T(n+1)$. In view of Eqs. (22)–(24) it is concluded that the internal symmetry in the off phase is characterized by the translation group $T(n)$ on the closed ball $\mathbb{B}^n$ of Euclidean space $\mathbb{E}^n$. The change from a transformation of a Poincaré group in the on phase to a non-Lorentzian transformation in the off phase indicates that internal symmetry switches from one kind to another, and vice versa.

10. Concluding remarks

It has been found that the bilinear elastoplasticity model of Eqs. (1)–(8) possesses two kinds of internal symmetries—$T(n)$ in the off phase and a projective realization of the proper orthochronous Poincaré group $ISO_o(n,1)$ in the on phase—and has symmetry switching between the two groups. The switching is dictated by the input generalized stain rate $\dot{\mathbf{q}}$ or equivalently the control tensor $\mathbf{A}$ according to the on–off switching criteria (21) or (35).

In the course of investigation the concepts of two phases and homogeneous coordinates have helped render such a highly non-linear model in the two-phase two-stage linear representation, only from which the internal spacetime of the model and the intrinsic structures of symmetry groups could be detected.

The internal symmetries revealed here for the two phases may be utilized to devise group conserving schemes—numerical algorithms specifically designed to preserve symmetry at all time steps without iterations. The schemes are expected to have long-term stability and high efficiency and accuracy. Although this issue is not pursued here, we would like to call attention to its potential importance in computational plasticity.

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