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INTERPOLANT-BASED TRANSITION RELATION APPROXIMATION [*]

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Abstract. In predicate abstraction, exact image computation is problematic, requiring in the worst case an exponential number of calls to a decision procedure. For this reason, software model checkers typically use a weak approximation of the image. This can result in a failure to prove a property, even given an adequate set of predicates. We present an interpolant-based method for strengthening the abstract transition relation in case of such failures. This approach guarantees convergence given an adequate set of predicates, without requiring an exact image computation. We show empirically that the method converges more rapidly than an earlier method based on counterexample analysis.

1. Introduction

Predicate abstraction [GS97] is a technique commonly used in software model checking in which an infinite-state system is represented abstractly by a finite-state system whose states are the truth valuations of a chosen set of predicates. The reachable state set of the abstract system corresponds to the strongest inductive invariant of the infinite-state system expressible as a Boolean combination of the given predicates.

The primary computational difficulty of predicate abstraction is the *abstract image* computation. That is, given a set of predicate states (perhaps represented symbolically) we wish to compute the set of predicate states reachable from this set in one step of the abstract system. This can be done by enumerating the predicate states, using a suitable decision procedure to determine whether each state is reachable in one step. However, since the number of decision procedure calls is exponential in the number of predicates, this approach is practical only for small predicates sets. For this reason, software model checkers, such as Slam [BR02] and Blast [HJMS02] typically use weak approximations of the abstract image. For example, the Cartesian image approximation is the strongest cube over the predicates that is implied at the next time. This approximation loses all information about predicates that are neither deterministically true nor deterministically false at the next time. Perhaps surprisingly, some properties of large programs, such as operating system device

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drivers, can be verified with this weak approximation [BR02, HJMM04]. Unfortunately, as we will observe, this approach fails to verify properties of even very simple programs, if the properties relate to data stored in arrays.

This paper introduces an approach to approximating the transition relation of a system using Craig interpolants derived from proofs of bounded model checking instances. These interpolants are formulas that capture the information about the transition relation of the system that was deduced in proving the property in a bounded sense. Thus, the transition relation approximation we obtain is tailored to the property we are trying to prove. Moreover, it is a formula over only state-holding variables. Hence, for abstract models produced by predicate abstraction, the approximate transition relation is a purely propositional formula, even though the original transition relation is characterized by a first-order formula. Thus, we can apply well-developed Boolean image computation methods to the approximate system, eliminating the need for a decision procedure in the image computation. By iteratively refining the approximate transition relation we can guarantee convergence, in the sense that whenever the chosen predicates are adequate to prove the property, the approximate transition relation is eventually strong enough to prove the property.

The reader should bear in mind that there are two kinds of abstraction occurring here. The first is *predicate abstraction*, which produces an abstract transition system whose state-holding variables are propositional. The second is *transition relation approximation*, which weakens the abstract transition formula, yielding a purely propositional approximate transition formula. To avoid confusion, we will always refer to the former as *abstraction*, and the latter as *approximation*. The techniques presented here produce an exact reachability result *for the abstract model*. However, we may still fail to prove unreachability if an inadequate set of predicates is chosen for the abstraction.

After beginning with an overview of closely related work (Section 2), we introduce some notation and definitions related to modelling infinite-state systems symbolically and briefly describe the method of deriving interpolants from proofs (Section 3). Next, we introduce the basic method of transition relation approximation using interpolants (Section 4). In the following section, we discuss a number of optimizations of this basic method that are particular to software verification, including a new method to strengthen interpolants that makes convergence more rapid (Section 7). Finally, we present an experimental presents an experimental comparison of the interpolation method with the Das and Dill method (Section 8).

2. Related work

The chief alternative to iterative approximation is to produce an exact propositional characterization of the abstract transition relation. For example the method of [LBC03] uses small-domain techniques to translate a first-order transition formula into a propositional one that is equisatisfiable over the state-holding predicates. However, this translation introduces a large number of auxiliary Boolean variables, making it impractical to use BDD-based methods for image computation. Though SAT-base Boolean quantifier elimination methods can be used, the effect is still essentially to enumerate the states in the image. By contrast, the interpolation-based method produces an approximate transition relation with no auxiliary Boolean variables, allowing efficient use of BDD-based methods.

The most closely related method is that of Das and Dill [DD01]. This method analyzes abstract counterexamples (sequences of predicate states), refining the transition relation approximation in such a way as to rule out infeasible transitions. This method is effective, but has the disadvantage that it uses a specific counterexample and does not consider the property being verified. Thus it can easily generate refinements not relevant to the property. The interpolation-based method does not use abstract counterexamples. Rather, it generates facts relevant to proving the given property in a bounded sense. Thus, it tends to generate more relevant refinements, and as a result converges more rapidly.

In [HJMM04], interpolants are used to choose new predicates to refine a predicate abstraction. Here, we use interpolants to refine an approximation of the abstract transition relation for a given set of predicates.

3. Preliminaries

Let S be a first-order signature, consisting of individual variables and uninterpreted n-ary functional and propositional constants. A *state formula* is a first-order formula over S, (which may include various interpreted symbols, such as $=$ and $+$). We can think of a state formula ϕ as representing a set of states, namely, the set of first-order models of ϕ. We will express the proposition that an interpretation σ over S models ϕ by $\phi[\sigma]$.

We also assume a first-order signature S', disjoint from S, and containing for every symbol $s \in S$, a unique symbol s' of the same type. For any formula or term ϕ over S, we will use ϕ' to represent the result of replacing every occurrence of a symbol s in ϕ with s'. Similarly, for any interpretation σ over S, we will denote by σ' the interpretation over S' such that $\sigma' s' = \sigma s$. A *transition formula* is a first-order formula over $S \cup S'$. We think of a transition formula T as representing a set of state pairs, namely the set of pairs (σ_1, σ_2), such that $\sigma_1 \cup \sigma_2'$ models T. Will will express the proposition that $\sigma_1 \cup \sigma_2'$ models T by $T[\sigma_1, \sigma_2]$.

The *strongest postcondition* of a state formula ϕ with respect to transition formula T, denoted $\mathrm{sp}_T(\phi)$, is the strongest proposition ψ such that $\phi \wedge T$ implies ψ'. We will also refer to this as the *image* of ϕ with respect to T. Similarly, the *weakest precondition* of a state formula ϕ with respect to transition formula T, denoted $\mathrm{wp}_T(\phi)$ is the weakest proposition ψ such that $\psi \wedge T$ implies ϕ'.

A *transition system* is a pair (I, T), where I is a state formula and T is a transition formula. Given a state formula ψ, we will say that ψ is *k-reachable* in (I, T) when there exists a sequence of states $\sigma_0, \ldots, \sigma_k$, such that $I[\sigma_0]$ and for all $0 \leq i < k$, $T[\sigma_i, \sigma_{i+1}]$, and $\psi[\sigma_k]$. Further, ψ is *reachable* in (I, T) if it is k-reachable for some k. We will say that ϕ is an *invariant* of (I, T) when $\neg\phi$ is not reachable in (I, T). A state formula ϕ is an *inductive invariant* of (I, T) when I implies ϕ and $\mathrm{sp}_T(\phi)$ implies ϕ (note that an inductive invariant is trivially an invariant).

3.1. Bounded model checking.

The fact that ψ is k-reachable in (I, T) can be expressed symbolically. For any symbol s, and natural number i, we will use the notation $s^{\langle i \rangle}$ to represent the symbol s with i primes added. Thus, $s^{\langle 3 \rangle}$ is s'''. A symbol with i primes will be used to represent the value of that symbol at time i. We also extend this notation to formulas. Thus, the formula $\phi^{\langle i \rangle}$ is the result of adding i primes to every uninterpreted symbol in ϕ.

Now, assuming T is total, the state formula ψ is k-reachable in (I, T) exactly when this formula is consistent:

$$I^{\langle 0 \rangle} \wedge T^{\langle 0 \rangle} \wedge \cdots T^{\langle k-1 \rangle} \wedge \psi^{\langle k \rangle}$$

We will refer to this as a *bounded model checking* formula [BCCZ99], since by testing satisfiability of such formulas, we can determine the reachability of a given condition within a bounded number of steps.

3.2. **Interpolants from proofs.** Given a pair of formulas (A, B), such that $A \wedge B$ is inconsistent, an *interpolant* for (A, B) is a formula $\hat{A}$ with the following properties:

- A implies $\hat{A}$,
- $\hat{A} \wedge B$ is unsatisfiable, and
- $\hat{A}$ refers only to the common symbols of A and B.

Here, "symbols" excludes symbols such as $\wedge$ and $=$ that are part of the logic itself. Craig showed that for first-order formulas, an interpolant always exists for inconsistent formulas [Cra57]. Of more practical interest is that, for certain proof systems, an interpolant can be derived from a refutation of $A \wedge B$ in linear time. For example, a purely propositional refutation of $A \wedge B$ using the resolution rule can be translated to an interpolant in the form of a Boolean circuit having the same structure as the proof [Kra97, Pud97].

In [McM05] it is shown that linear-size interpolants can be derived from refutations in a first-order theory with uninterpreted function symbols and linear arithmetic. This translation has the property that whenever A and B are quantifier-free, the derived interpolant $\hat{A}$ is also quantifier-free.[1] We will exploit this property in the sequel.

Heuristically, the chief advantage of interpolants derived from refutations is that they capture the facts that the prover derived about A in showing that A is inconsistent with B. Thus, if the prover tends to ignore irrelevant facts and focus on relevant ones, we can think of interpolation as a way of filtering out irrelevant information from A.

For the purposes of this paper, we must extend the notion of interpolant slightly. That is, given an indexed set of formulas $A = \{a_1, \ldots, a_n\}$ such that $\bigwedge A$ is inconsistent, a *symmetric interpolant* for A is an indexed set of formulas $\hat{A} = \{\hat{a}_1, \ldots, \hat{a}_n\}$ such that each a_i implies $\hat{a}_i$, and $\bigwedge \hat{A}$ is inconsistent, and each $\hat{a}_i$ is over the symbols common to a_i and $A \setminus a_i$. We can construct a symmetric interpolant for A from a refutation of $\bigwedge A$ by simply letting $\hat{a}_i$ be the interpolant derived from the given refutation for the pair $(a_i, \bigwedge A \setminus a_i)$. As long as all the individual interpolants are derived *from the same proof*, we are guaranteed that their conjunction is inconsistent. In the sequel, if $\hat{A}$ is a symmetric interpolant for A, and the elements of A are not explicitly indexed, we will use the notation $\hat{A}(a_i)$ to refer to $\hat{a}_i$.

4. Transition relation approximation

Because of the expense of image computation in symbolic model checking, it is often beneficial to abstract the transition relation before model checking, removing information that is not relevant to the property to be proved. Some examples of techniques for this purpose are [CGJ$^+$00, MA03].

In this paper, we introduce a method of approximating the transition relation using bounded model checking and symmetric interpolation. Given a transition system (I, T) and a state formula ψ that we wish to prove unreachable, we will use interpolation to

[1] Note that the Craig theorem does not guarantee the existence of quantifier-free interpolants. In general this depends on the choice of interpreted symbols in the logic.

```
$\hat{T} \leftarrow$ TRUE
**repeat**
    if $\psi$ unreachable in $(I, \hat{T})$, return "unreachable"
    else, if $\psi$ reachable in $k$ steps in $(I, \hat{T})$
        $A \leftarrow \{I^{\langle 0\rangle}, T^{\langle 0\rangle}, \ldots, T^{\langle k-1\rangle}, \psi^{\langle k\rangle}\}$
        if $\bigwedge A$ satisfiable, return "reachable in $k$ steps"
        else
            $\hat{A} \leftarrow$ ITP$(A)$
            $\hat{T} \leftarrow \hat{T} \wedge \bigwedge_{i=0}^{k-1} (\hat{A}(T^{\langle i\rangle}))^{\langle -i\rangle}$
**end repeat**
```

Figure 1: Interpolation-based transition approximation loop. Here, ITP is a function that computes a symmetric interpolant for a set of formulas.

refine an approximation $\hat{T}$ of the transition relation T, such that T implies $\hat{T}$. The initial approximation is just $\hat{T} = $ TRUE.

We begin the refinement loop by attempting to verify the unreachabilty of ψ in the approximate system $(I, \hat{T})$, using an appropriate model checking algorithm. If ψ is found to be unreachable in $(I, \hat{T})$, we know it is unreachable in the stronger system (I, T). Suppose, on the other hand that ψ is found to be k-reachable in $(I, \hat{T})$. It may be that in fact ψ is k-reachable in (I, T), or it may be that $\hat{T}$ is simply too weak an approximation to refute this. To find out, we will use bounded model checking.

That is, we construct the following set of formulas:

$$A \doteq \{I^{\langle 0\rangle}, T^{\langle 0\rangle}, \ldots, T^{\langle k-1\rangle}, \psi^{\langle k\rangle}\}$$

Note that $\bigwedge A$ is exactly the bounded model checking formula that characterizes k-reachability of ψ in (I, T). We use a decision procedure to determine satisfiability of $\bigwedge A$. If it is satisfiable, ψ is reachable and we are done. If not, we obtain from the decision procedure a refutation of $\bigwedge A$. From this, we extract a symmetric interpolant $\hat{A}$. Notice that for each i in $0 \ldots k-1$, $\hat{A}(T^{\langle i\rangle})$ is a formula implied by $T^{\langle i\rangle}$, the transition formula shifted to time i. Let us shift these formulas back to time 0, thus converting them to transition formulas. That is, for $i = 0 \ldots k-1$, let:

$$\hat{T}_i \doteq (\hat{A}(T^{\langle i\rangle}))^{\langle -i\rangle}$$

where we use $\phi^{\langle -i\rangle}$ to denote removal of i primes from ϕ, when feasible. We will call these formulas the *transition interpolants*. From the properties of symmetric interpolants, we know the bounded model checking formula

$$I_0 \wedge \hat{T}_0^{\langle 0\rangle} \wedge \cdots \hat{T}_{k-1}^{\langle k-1\rangle} \wedge \psi_k$$

is unsatisfiable. Thus we know that the conjunction of the transition interpolants $\bigwedge_i \hat{T}_i$ admits no path of k steps from I to ψ. We now compute a refined approximation $\dot{T} \doteq \hat{T} \wedge \bigwedge_i \hat{T}_i$. This becomes our approximation $\hat{T}$ in the next iteration of the loop. This procedure is summarized in Figure 1. Notice that at each iteration, the refined approximation $\dot{T}$ is strictly stronger than $\hat{T}$, since $\hat{T}$ allows a counterexample of k steps, but $\dot{T}$ does not. Thus, for finite-state systems, the loop must terminate. This is simply because we cannot strengthen a formula with a finite number of models infinitely.

The approximate transition formula $\hat{T}$ has two principle advantages over T. First, it contains only facts about the transition relation that were derived by the prover in resolving the bounded model checking problem. Thus it is in some sense an abstraction of T relative to ψ. Second, $\hat{T}$ contains only state-holding symbols. We will say that a symbol $s \in S$ is *state-holding* in (I, T) when s occurs in I, or s' occurs in T. In the bounded model checking formula, the only symbols in common between $T^{\langle i \rangle}$ and the remainder of the formula are of the form $s^{\langle i \rangle}$ or $s^{\langle i+1 \rangle}$, where s is state-holding. Thus, the transition interpolants $\hat{T}_i$ contain only state-holding symbols and their primed versions.

The elimination of the non-state-holding symbols by interpolation has two potential benefits. First, in hardware verification there are usually many non-state-holding symbols representing inputs of the system. These symbols contribute substantially to the cost of the image computation in symbolic model checking. Second, for this paper, the chief benefit is in the case when the state-holding symbols are all propositional (*i.e.*, they are propositional constants). In this case, even if the transition relation T is a first-order formula, the approximation $\hat{T}$ is a propositional formula. The individual variables and function symbols are eliminated by interpolation. Thus we can apply well-developed Boolean methods for symbolic model checking to the approximate system. In the next section, we will apply this approach to predicate abstraction.

5. Application to predicate abstraction

Predicate abstraction [GS97] is a technique commonly used in software model checking in which the state of an infinite-state system is represented abstractly by the truth values of a chosen set of predicates P. The method computes the strongest inductive invariant of the system expressible as a Boolean combination of these predicates.

Let us fix a concrete transition system (I, T) and a finite set of state formulas P that we will refer to simply as “the predicates”. We assume a finite set $V \subset S$ of uninterpreted propositional symbols not occurring in I or T. The set V consists of a symbol v_p for every predicate $p \in P$. We will construct an abstract transition system $(\bar{I}, \bar{T})$ whose states are the minterms over V. To relate the abstract and concrete systems, we define a concretization function γ. Given a formula over V, γ replaces every occurrence of a symbol v_p with the corresponding predicate p. Thus, if ϕ is a Boolean combination over V, $\gamma(\phi)$ is the same combination of the corresponding predicates in P.

For the sake of simplicity, we assume that the initial condition I is a Boolean combination of the predicates. Thus we choose $\bar{I}$ so that $\gamma(\bar{I}) = I$. We define the abstract transition relation $\bar{T}$ such that, for any two minterms $s, t \in 2^V$, we have $\bar{T}[s, t]$ exactly when $\gamma(s) \wedge T \wedge \gamma(t)'$ is consistent. In other words, there is a transition from abstract state s to abstract state t exactly when there is a transition from a concrete state satisfying $\gamma(s)$ to a concrete state satisfying $\gamma(t)$.

We can easily show by induction on the number of steps that if a formula ψ over V is unreachable in $(\bar{I}, \bar{T})$ then $\gamma(\psi)$ is unreachable in (I, T) (though the converse does not hold). To allow us to check whether a given ψ is in fact reachable in the abstract system, we can express the abstract transition relation symbolically [LBC03]. The abstract transition relation can be expressed as

$$\bar{T} \doteq \left(\left(\bigwedge_{p \in P} (v_p \iff p) \right) \wedge T \wedge \left(\bigwedge_{p \in P} (p' \iff v_p') \right) \right) \downarrow (V \cup V')$$

where $Q \downarrow W$ denotes the "hiding" of non-W symbols in Q by renaming them to fresh symbols in S. Hiding the concrete symbols in this way takes the place of existential quantification. Notice that, under this definition, the state-holding symbols of $(\bar{I}, \bar{T})$ are exactly V. Moreover, for any two minterms $s, t \in 2^V$, the formula $s \wedge \bar{T} \wedge t'$ is consistent exactly when $\gamma(s) \wedge T \wedge \gamma(t)'$ is consistent. Thus, $\bar{T}$ characterizes exactly the transitions of our abstract system.

To determine whether ψ is reachable in this system using the standard "symbolic" approach, we would compute the reachable states R of the system as the limit of the following recurrence:

$$\begin{array}{rcl} R_0 & \doteq & \bar{I} \\ R_{i+1} & \doteq & R_i \vee \mathrm{sp}_{\bar{T}}(R_i) \end{array}$$

The difficulty here is to compute the image $\mathrm{sp}_{\bar{T}}$. We cannot apply standard propositional methods for image computation, since the transition formula $\bar{T}$ is not propositional. We can compute $\mathrm{sp}_{\bar{T}}(\phi)$ as the disjunction of all the minterms $s \in 2^V$ such that $\phi \wedge \bar{T} \wedge s'$ is consistent. However, this is quite expensive in practice, since it requires an exponential number of calls to a theorem prover. In [LBC03], this is avoided by translating $\bar{T}$ into a propositional formula that is equisatisfiable with $\bar{T}$ over $V \cup V'$. This makes it possible to use well developed Boolean image computation methods to compute the abstract strongest postcondition. Nonetheless, because the translation introduces a large number of free propositional variables, the standard approaches to image computation using Binary Decision Diagrams (BDD's) were found to be inefficient. Alternative methods based on enumerating the satisfiable assignments using a SAT solver were found to be more effective, at least for small numbers of predicates. However, this method is still essentially enumerative. Its primary advantage is that information learned by the solver during the generation of one satisfying assignment can be reused in the next iteration.

Here, rather than attempting to compute images exactly in the abstract system, we will simply observe that state-holding symbols of the abstraction $(\bar{I}, \bar{T})$ are all propositional. Thus, the interpolation-based transition relation approximation method of the previous section reduces the transition relation to a purely propositional formula. Moreover, it does this without introducing extraneous Boolean variables. Thus, we can apply standard BDD-based model checking methods to the approximated system $(I, \hat{T})$ without concern that non-state-holding Boolean variables will cause a combinatorial explosion. Finally, termination of the approximation loop is guaranteed because the abstract state space is finite.

6. Software model checking

In model checking sequential deterministic programs, we can make some significant optimizations in the above method.

6.1. **Path-based approximation.**

The first optimization is to treat the program counter explicitly, rather than modeling it as a symbolic variable. The main advantage of this is that it will allow us to apply bounded model checking only to particular program paths (*i.e.*, sequences of program locations) rather than to the program as a whole.

We will say that a *program* Π is a pair (L, R), where L is a finite set of *locations*, and R is a finite set of *operations*. An operation is a triple (l, T, l') where T is a transition formula, $l \in L$ is the entry location of the statement, and $l' \in L$ is the exit location of the statement.

A *path* of program Π from location $l_0 \in L$ to location $l_k \in L$ is a sequence $\pi \in R^{k-1}$, of the form $(l_0, T_0, l_1)(l_1, T_1, l_2) \cdots (l_{k-1}, T_{k-1}, l_k)$. We say that the path is *feasible* when there exists a sequence of states $\sigma_0 \cdots \sigma_k$ such that, for all $0 \leq i < k$, we have $T_i[\sigma_i, \sigma_{i+1}]$. The reachability problem is to determine whether program Π has a feasible path from a given initial location l_0 to a given final location l_f.

As in the previous section, we assume a fixed set of predicates P, and a corresponding set of uninterpreted propositional symbols V. Using these, we construct an abstract program $\bar{\Pi} = (L, \bar{R})$. For any operation $r = (l, T, l')$, let the abstract operation $\bar{r}$ be $(l, \bar{T}, l')$, where, as before

$$\bar{T} \doteq \left(\left(\bigwedge_{p \in P}(v_p \iff p)\right) \wedge T \wedge \left(\bigwedge_{p \in P}(p' \iff v'_p)\right)\right) \downarrow (V \cup V')$$

The abstract operation set $\bar{R}$ is then $\{\bar{r} \mid r \in R\}$. We can easily show that if a path $r_0 \cdots r_{k-1}$ is feasible, then the corresponding abstract path $\bar{r}_0 \cdots \bar{r}_{k-1}$ is also feasible. Thus if a given location l_f is unreachable from l_0 in the abstract program, it is unreachable from l_0 in the concrete program.

Now we can apply the interpolation-based approximation approach to programs. We will build an approximate program $\hat{\Pi} = (L, \hat{R})$, where $\hat{R}$ consists of an operation $\hat{r} = (l, \hat{T}, l')$ for every $\bar{r} = (l, \bar{T}, l')$ in $\bar{R}$, such that $\bar{T}$ implies $\hat{T}$, and $\hat{T}$ is over $V \cup V'$. Initially, every $\hat{T}$ is just TRUE.

At every step of the iteration, we use standard model checking methods to determine whether the approximation $\hat{\Pi}$ has a feasible path from l_0 to l_f. We can do this because the transition formulas $\hat{T}$ are all propositional. If there is no such path, then l_f is not reachable in the concrete program and we are done. Suppose on the other hand that there is such a path $\hat{\pi} = \hat{\pi}_0 \cdots \hat{\pi}_{k-1}$. Let $\bar{\pi} = \bar{\pi}_0 \cdots \bar{\pi}_{k-1}$ be the corresponding path of $\bar{\Pi}$. We can construct a bounded model checking formula to determine the feasibility of this path. Using the notation $T(r)$ to denote the T component of an operation r, let

$$A \doteq \{T(\bar{\pi}_i)^{\langle i \rangle} \mid i \in 0 \ldots k-1\}$$

The conjunction $\bigwedge A$ is consistent exactly when the abstract path $\bar{\pi}$ is feasible. Thus, if $\bigwedge A$ is consistent, the abstraction does not prove unreachability of l_f and we are done. If it is inconsistent, we construct a symmetric interpolant $\hat{A}$ for A. We extract transition interpolants as follows:

$$\hat{T}_i \doteq (\hat{A}(T(\bar{\pi}_i)^{\langle i \rangle}))^{\langle -i \rangle}$$

Each of these is implied by the $T(\bar{\pi}_i)$, the transition formula of the corresponding abstract operation. We now strengthen our approximate program $\hat{\Pi}$ using these transition interpolants. That is, for each abstract operation $\bar{r} \in \bar{R}$, the refined approximation is $\dot{r} = (l, T(\dot{r}), l')$ where

$$T(\dot{r}) \doteq T(\hat{r}) \wedge \left(\bigwedge\{\hat{T}_i \mid \bar{\pi}_i = \bar{r},\ i \in 0 \ldots k-1\}\right)$$

In other words, we constrain each approximate operation $\hat{r}$ by the set of transition interpolants for the occurrences of $\bar{r}$ in the abstract path $\bar{\pi}$. The refined approximate program is thus $(L, \dot{R})$, where $\dot{R} = \{\dot{r} \mid \bar{r} \in \bar{R}\}$. From the interpolant properties, we can easily show that the refined approximate program does not admit a feasible path corresponding to $\bar{\pi}$.

We continue in this manner until either the model checker determines that the approximate program $\hat{\Pi}$ has no feasible path from l_0 to l_f, or until bounded model checking determines that the abstract program $\bar{\Pi}$ does have such a feasible path. This process must

statement	transition interpolant
$a[x] \leftarrow y$	$(x = z)' \Rightarrow (a[z] = y)'$
$y \leftarrow y + 1$	$(a[z] = y \Rightarrow (a[z] = y - 1)') \wedge ((x = z)' \Rightarrow x = z)$
assume $z = x$	$(a[z] = y - 1 \Rightarrow (a[z] = y - 1)') \wedge x = z$
assume $a[z] \neq y - 1$	$a[z] \neq y - 1$

Figure 2: An infeasible program path, with transition interpolants. The statement "assume ϕ" is a guard. It aborts when ϕ is false. In the transition interpolants, we have replaced v_p with p for clarity, but in fact these formulas are over $V \cup V'$.

terminate, since at each step $\hat{\Pi}$ is strengthened, and we cannot strengthen a finite set of propositional formulas infinitely.

The advantage of this approach, relative to that of section 4, is that the bounded model checking formula $\bigwedge A$ only relates to a single program path. In practice, the refutation of a single path using a decision procedure is considerably less costly than the refutation of all possible paths of a given length.

As an example of using interpolation to compute an approximate program, Figure 2 shows a small program with one path, which happens to be infeasible. The method of [HJMM04] chooses the predicates $x = z$, $a[z] = y$ and $a[z] = y - 1$ to represent the abstract state space. Next to each operation in the path is shown the transition interpolant $\hat{T}_i$ that was obtained for that operation. Note that each transition interpolant is implied by the semantics of the corresponding statement, and that collectively the transition interpolants rule out the program path (the reader might wish to verify this). Moreover, the transition interpolant for the first statement, $a[x] \leftarrow y$, is $x = z \Rightarrow a[z] = y$. This is a disjunction and therefore cannot be inferred by predicate image techniques that use the Cartesian or Boolean programs approximations. In fact, the Blast model checker cannot rule out this program path. However, using transition interpolants, we obtain a transition relation approximation that proves the program has no feasible path from beginning to end.

6.2. **Modeling with weakest precondition.** A further optimization that we can use in the case of deterministic programs is that we can express the abstract transition formulas $\bar{T}$ in terms of the weakest precondition operator. That is, if T is deterministic, the abstract transition formula $\bar{T}$ is satisfiability equivalent over $V \cup V'$ to:

$$\left(\bigwedge_{p \in P} (v_p \iff p)\right) \wedge \neg \mathrm{wp}_T(\textsc{False}) \wedge \left(\bigwedge_{p \in P} (v'_p \iff \mathrm{wp}_T(p))\right)$$

Thus, if we can symbolically compute the weakest precondition operator for the operations in our programming language, we can use this formula in place of $\bar{T}$ as the abstract transition formula. In this way, the abstract transition formula is localized to just those program variables that are related in some way to predicates P. In particular, if π is an assignment to a program variable not occurring in P, then we will have $v'_p \iff p$, for every predicate in P.

6.3. **A hybrid approach.** We can combine transition interpolants with other methods of approximating the transition relation or the image. For example, given a set of propositions V, the *strongest Cartesian postcondition* $\mathrm{scp}_T(\phi)$ of a formula ϕ with respect to a transition formula T is the strongest cube ψ over V such that $\phi \wedge T$ implies ψ' (a cube is a conjunction of literals). In computing the image of a state formula ϕ with respect to an operation $\hat{r}$ of the approximate program, we can strengthen the result by conjoining it with the strongest Cartesian postcondition with respect to the corresponding abstract operation $\bar{r}$. Thus, the *hybrid image* of ϕ with respect to transition $\hat{r}$ is:

$$\mathrm{hi}_{\hat{r}}(\phi) \doteq \mathrm{sp}_{T(\hat{r})}(\phi) \wedge \mathrm{scp}_{T(\bar{r})}(\phi)$$

This set is still an over-approximation of the exact abstract image $\mathrm{sp}_{T(\bar{r})}(\phi)$, so it is sound to use the hybrid image in the reachability computation. This may result in fewer iterations of the refinement loop.

7. Computing strong interpolants

In preliminary tests of the method, we found that transition interpolants derived from proofs by the method of [McM03] were often unnecessarily weak. For example, we might obtain $(p \wedge q) \Rightarrow (p' \wedge q')$ when the stronger $(p \Rightarrow p') \wedge (q \Rightarrow q')$ could be proved. This slowed convergence substantially. The experiments presented in this paper use a modified version of the method of [McM03] which incorporates technique of strengthening the interpolant obtained from a resolution proof . It is important in practice to compute strong transition interpolants, to reduce the number of refinement iterations needed to compute the transiton relation approximation.

As an example of this, notice that in Figure 2, the transition interpolant for the second step is a conjunction of disjunctions:

$$(a[z] = y \Rightarrow (a[z] = y - 1)') \wedge ((x = z)' \Rightarrow x = z)$$

However, other valid interpolants are possible. For example, we might have obtained a weaker version:

$$(x = z)' \Rightarrow (x = z \wedge (a[z] = y \Rightarrow (a[z] = y - 1)'))$$

This formula has been weakened by pulling one disjunction outside of the conjunction, though it is still sufficient to rule out this particular program path. The stronger interpolant has the advantage that it may be more useful in ruling out other program paths in a more complex program. Unfortunately, either of these interpolants might be obtained in practice, depending on the exact order of resolution steps generated by the prover. The order of resolution steps generated by a SAT solver depends on the order in which implications are propagated by the Boolean constraint propagation (BCP) procedure, and is quite arbitrary. Thus, it is useful in practice to try to adjust the proof before computing an interpolant, in such a way that a stronger interpolant results.

To understand this process in detail, it is necessary to understand the process of generating interpolants from resolution proofs, as described in [McM03]. A full treatment of this subject is beyond the scope of this paper. However, to gain some intuition about the problem, it is only necessary to know two things about such interpolants. First, the interpolant for (A, B) is a Boolean circuit whose structure mirrors the structure of the resolution proof that refutes $A \wedge B$. Second, resolutions on local atoms (those not occurring in B) generate "or" gates, while resolutions on global atoms (those occurring in B) generate "and" gates. Thus, if we want to generate a strong interpolant formula, it would be best to move the local resolutions toward the hypotheses of the proof, and the global resolutions toward the conclusion. This effectively moves the "or" gates toward the inputs of the interpolant circuit, and the "and" gates toward the output, thus strengthening the interpolant.

We will think of a refutation proof by resolution as a DAG (V, E), in which the vertices V are clauses. Each root of the DAG is a hypothesis of the proof, and the unique leaf is the empty clause (representing "false"). Each non-root vertex v has exactly two parents, which we will denote v^+ and v^-, and a *pivot variable* $\textsc{Pivot}(v)$. The proof is *valid* when, for every non-root vertex v, v^+ has the form $\textsc{Pivot}(v) \vee \Theta_1$ and v^- has the form $\neg\textsc{Pivot}(v) \vee \Theta_2$ and $v = \Theta_1 \vee \Theta_2$ (that is, each derived clause is the result of resolving its two parents on variable $\textsc{Pivot}(v)$). Figuratively speaking, each literal in a hypothesis flows down the DAG until it is annihilated by resolution with its negation. As an example, Figure 3a shows a simple resolution refutation whose hypotheses are $p \vee q$, $\neg p$ and $\neg q$. In this case we resolve first on p, then on q.

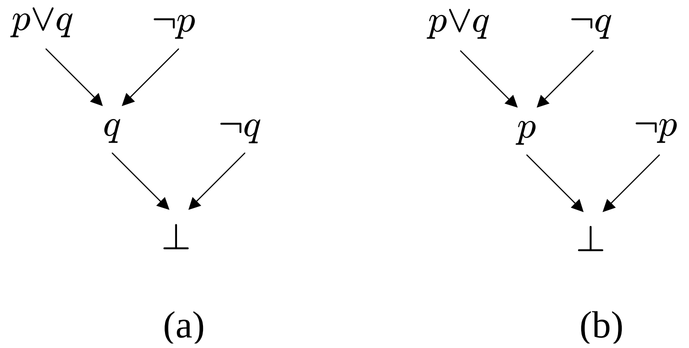


Figure 3: Exchanging the order of two resolution steps

The order of two consecutive resolution steps in a proof can always be reversed, though possibly at the cost of expanding the proof. For example, Figure 3b shows the result of reversing the order of resolution in our simple example, so that we resolve first on q, then on p. This is an example of a generally valid graph transformation on resultion proofs, depicted in Figure 4. In the figure, a box containing a variable p denotes the result of resolving its parents on p. The left parent of a vertex v is v^+, while the right parent is v^-. This transformation is valid when q occurs in v_1, but not in v_2. The reader can easily verify that the right graph is a valid proof when the left one is. Note that the vertex marked * on the left hand side may have other successors in the graph that are not pictured. In this case we cannot delete this vertex from the proof when applying the transformation. Thus, applying the transformation may increase the size of the proof graph by one vertex.

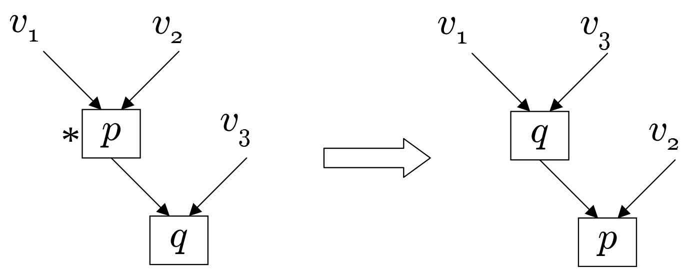


Figure 4: Proof transformation valid when v_1 contains q, but v_2 does not.

Figure 5 shows the case when both the clauses v_1 and v_2 contain q. In this case, we obtain two resolutions on q. Again, the original resolution on p, marked *, may or may not be deleted, depending on whether it has additional successors. Both these transformation are symmetric with respect to polarity. Thus, we obtain similar transformations by reversing the antecedents of either resolution step on the left-hand side.

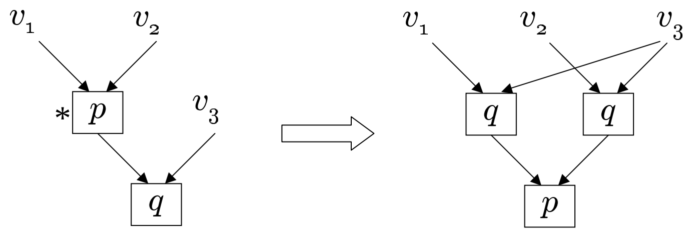


Figure 5: Proof transformation valid when both v_1 and v_2 contain q.

By applying these graph transformations systematically, we can in principle move all of the resolutions on local atoms to the top of the proof, and all the resolutions on global atoms to the bottom. This would result in an interpolant in conjunctive normal form (CNF). However, it may also result in an exponential expansion of the proof. Instead, we will take a limited approach that keeps the interpolant linear in the size of the original resolution proof, but may not yield an interpolant in CNF.

First, we must first take into account that the proof is a DAG and not a tree. As noted above, raising resolution on q above a resolution on p could result in the loss of shared structure, if the latter has more than one successor in the graph. To prevent this, we never apply the reversal transformations when the upper resolution (on p) has multiple successors. We will say a vertex is *passable* if it is not a root, and if it has exactly one successor. We traverse the proof in some topological order, from antecedents to consequents. Each time we encounter a resolution step v on a local atom q, we raise this resolution step by repeatedly exchanging it with one of its parents, until neither of its parents is passable. This procedure is shown in pseudocode in Figure 6.

Note that the procedure RAISE, if q occurs in both v'^{+} and v'^{-}, then we create two new resolutions on q, while removing only one (this corresponds to the transformation of Figure 5). Thus, the size of the proof increases. However, the final number of resolutions on q is no more than the number of occurrences of q in the the original proof. Thus, the number of resolutions we obtain after raising all the resolutions on local atoms is linear in the size of the original proof (if we measure it by the number of literals it contains). As a result the interpolant we obtain from the rewritten proof is still linear in size of the original proof (though it may be quadratically larger than the interpolant derived from the original proof).

Also note that in procedure RAISE, it may be possible to raise a given resolution v over either v^{+} or v^{-}. We have arbitrarily chosen v^{+} in this case, though it may be, for example, that the heavier of the two proof branches would be heuristically the better choice. This choice occurs rarely, however, in proofs generated by SAT solvers. These proofs tend to consist of long chains of resolutions in which one of the two antecedents are hypotheses. These chains are the result of Boolean constraint propagation. The unusual case in which both antecedents are derived clauses are typically the result of the SAT solver backtracking out of a decision. Thus in most cases, only one choice is possible.

```
function $\textsc{Resolve}(q, v_1, v_2)$
    if $q$ does not occur in $v_1$ then return $v_2$
    else if $\neg q$ does not occur in $v_2$ then return $v_1$
    else
        let $r$ be the resolvent of $v_1$ and $v_2$ on $q$
        add $r$ to the proof, with $r^+ = v_1$ and $r^- = v_2$
        $\textsc{Raise}(r)$
        return $r$

procedure $\textsc{Raise}(v)$
    let $q = \textsc{Pivot}(v)$
    if $v^+$ is passable then
        let $v' = v^+$ and $p = \textsc{Pivot}(v')$
        set $v^+ \leftarrow \textsc{Resolve}(q, {v'}^+, v^-)$
        set $v^- \leftarrow \textsc{Resolve}(q, {v'}^-, v^-)$
        remove vertex $v'$ and set $\textsc{Pivot}(v) \leftarrow p$
    else if $v^-$ is passable then
        let $v' = v^-$ and $p = \textsc{Pivot}(v')$
        set $v^+ \leftarrow \textsc{Resolve}(q, v^+, {v'}^+)$
        set $v^- \leftarrow \textsc{Resolve}(q, v^+, {v'}^-)$
        remove vertex $v'$ and set $\textsc{Pivot}(v) \leftarrow p$

procedure $\textsc{TransformProof}$
    let $c_1, \ldots, c_n$ be a topological sort of the non-root proof vertices
    for $i = 1 \ldots n$ do
        if $\textsc{Pivot}(c_i)$ is local (occurs only in $A$) then $\textsc{Raise}(c_i)$
```

Figure 6: Proof transformation procedure

There are two reasons why, after apply the transformation procedure, we may still have global resolutions above local resolutions (and thus "and" gates inside "or" gates in the interpolant). Most obviously, the proof may not have been a tree, and thus raising some local resolution may have been blocked because neither parent was passable. The other reason is that when we raise resolution on q above one antecedent, we also raise the proof of the other antecedent. This may itself contain global resolutions (though as mentioned, in most cases the other antecedent is a hypothesis) We might imagine continuing by raising each resulting resolution on q above its other antecedent. However, the resulting loss of structure sharing would cause an exponential expansion in the proof DAG. In practice, we have found that the limited transformation procedure outlined figure 6 resutls in an interpolant in CNF most of the time, producing a substantial improvement in the performance of interpolation-based refinement over the basic procedure of [McM03].

8. Experiments

We now experimentally compare the method of the previous section with a method due to Das and Dill [DD01]. This method refines an approximate transition relation by analyzing counterexamples from the approximate system to infer a refinement that rules out each counterexample. More precisely, a *counterexample* of the approximate program $(L, \hat{R})$ is an alternating sequence $\pi = \sigma_0 \hat{r}_0 \sigma_1 \cdots \hat{r}_{k-1} \sigma_k$, where each σ_i is a minterm over V, each $\hat{r}_i$ is an operation in $\hat{R}$, $l(r_0) = l_0$, $l'(r_{k-1}) = l_f$, and for all $0 \leq i < k$, we have $T(\hat{r}_i)[\sigma_i, \sigma_{i+1}]$. This induces a set of *transition minterms*, $t_i = \sigma_i \wedge \sigma'_{i+1}$, for $0 \leq i < k$. Note that each t_i is by definition consistent with $T(\hat{r}_i)$.

To refine the approximate program, we test each t_i for consistency with the corresponding abstract transition formula $T(\bar{r}_i)$. If it is inconsistent, the counterexample is false (due to over-approximation). Using an incremental decision procedure, we then greedily remove literals from t_i that can be removed while retaining inconsistency with $T(\bar{r}_i)$. The result is a minimal (but not minimum) cube that is inconsistent with $T(\bar{r}_i)$. The negation of this cube is implied by $T(\bar{r}_i)$, so we use it to strengthen corresponding approximate transition formula $T(\hat{r}_i)$. Since more than one transition minterm may be inconsistent, we may refine several approximate operations in this way (however if none are inconsistent, we have found a true counterexample of the abstraction).

Both approximation refinement procedures are embedded as subroutines of the Blast software model checker. Whenever the model checker finds a path from an initial state to a failure state in the approximate program, it calls the refinement procedure. If refinement fails because the abstraction does not prove the property, the procedure of [HJMM04] is used to add predicates to the abstraction. Since both refinement methods are embedded in the same model checking procedure and use the same decision procedure, we can obtain a fairly direct comparison.

Our benchmarks are a set of C programs with assertions embedded to test properties relating to the contents of arrays.[2] Some of these programs were written expressly as tests. Others were obtained by adding assertions to a sample device driver for the Linux operating system from a textbook [RC01]. Most of the properties are true. None of the properties can be verified or refuted by Blast without using a refinement procedure, due to its use of the Cartesian image.

Figure 7 shows a comparison in terms of run time (on a 3GHz Intel Xeon processor) and number of refinement steps. The latter includes refinement steps that fail, causing predicates to be added. Run time includes model checking, refinement, and predicate selection. Each point represents a single benchmark problem. The X axis represents the Das/Dill method and the Y axis the interpolation-based method. Points below the heavy diagonal represent wins for the interpolation method, while points below the light diagonal represent improvements of an order of magnitude (note in one case a run-time improvement of two orders of magnitude is obtained). Figure 8 shows the same comparison with the hybrid image computation. Here, the reduction in number of refinement steps is less pronounced, since less information must be learned by refinement.

The lower number of refinement steps required by interpolation method is easily explained. The Das/Dill method uses a specific counterexample and does not consider the property being verified. Thus it can easily generate refinements not relevant to proving the property. The interpolation procedure considers only the program path, and generates facts relevant to proving the property for that path. Thus, it tends to generate more relevant refinements, and as a result it converges in fewer refinements.

[2] Available at `http://www-cad.eecs.berkeley.edu/~kenmcmil/cav05data.tar.gz`

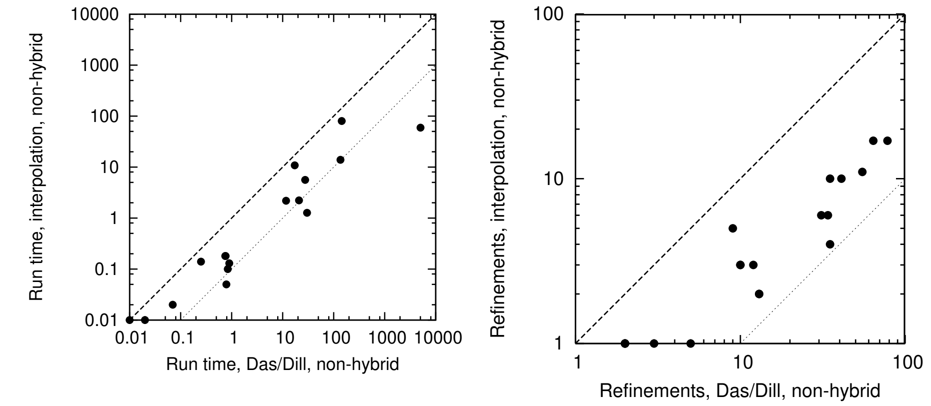


Figure 7: Comparison of the Das/Dill and interpolation-based methods as to run time and number of refinement steps.

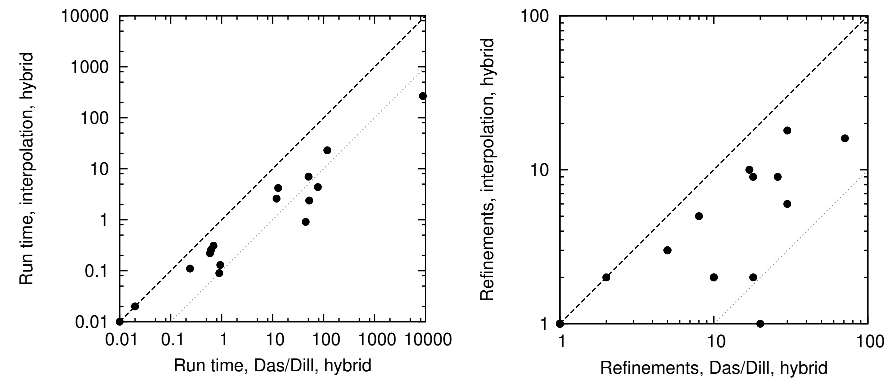


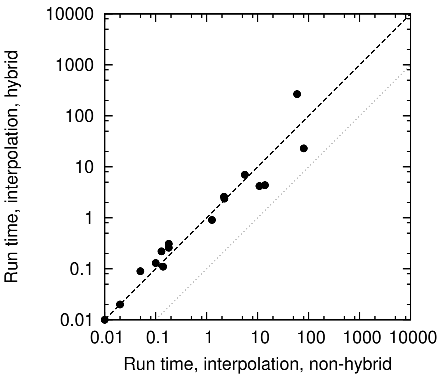
Figure 8: Comparison of the Das/Dill and interpolation-based refinement methods, using the hybrid image.

Figure 9 compares the performance the interpolation-based method with and without hybrid image computation. Though the hybrid method can reduce the number of refinement steps, it sometimes increases the run time due to the cost of computing the Cartesian image using a decision procedure.

9. Conclusions

We have described a method that combines bounded model checking and interpolation to approximate the transition relation of a system with respect to a given safety property. The method is extensible to liveness properties of finite-state systems, in the same manner as the method of [MA03]. When used with predicate abstraction, the method eliminates the individual variables and function symbols from the approximate transition formula, leaving it in a propositional form. Unlike the method of [LBC03], it does this without introducing extraneous Boolean variables. Thus, we can apply standard symbolic model checking methods to the approximate system.

For a set of benchmark programs, the method was found to converge more rapidly than the counterexample-based method of Das and Dill, primarily due to the prover's ability



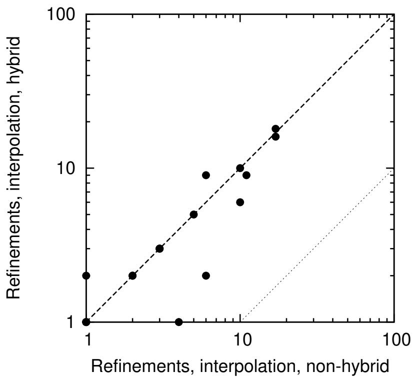


Figure 9: Comparison of the interpolation-based refinement methods, without and with hybrid image.

to focus the proof, and therefore the refinements, on facts relevant to the property. The benchmark programs used here are small (the largest being a sample device driver from a textbook), and the benchmark set contains only 19 problems. Thus we cannot draw broad conclusions about the applicability of the method. However, the experiments do show a potential to speed the convergence of transition relation refinement for real programs. Our hope is that this will make it easier to model check data-oriented rather than control-oriented properties of software.

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