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INTERPOLATION AND DUALITY OF GENERALIZED GRAND MORREY SPACES ON QUASI-METRIC MEASURE SPACES

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Abstract. Let $\theta \in (0,1)$, $\lambda \in [0,1)$ and $p, p_0, p_1 \in (1,\infty]$ be such that $(1-\theta)/p_0 + \theta/p_1 = 1/p$, and let $\varphi, \varphi_0, \varphi_1$ be some admissible functions such that $\varphi, \varphi_0^{p/p_0}$ and φ_1^{p/p_1} are equivalent. We first prove that, via the $\pm$ interpolation method, the interpolation $\langle L^{p_0),\lambda}_{\varphi_0}(\mathcal{X}), L^{p_1),\lambda}_{\varphi_1}(\mathcal{X}), \theta\rangle$ of two generalized grand Morrey spaces on a quasi-metric measure space $\mathcal{X}$ is the generalized grand Morrey space $L^{p),\lambda}_{\varphi}(\mathcal{X})$. Then, by using block functions, we also find a predual space of the generalized grand Morrey space. These results are new even for generalized grand Lebesgue spaces.

Keywords: grand Lebesgue space; grand Morrey space; Gagliardo-Peetre method; quasi-metric measure space; Calderón product; predual space; $\pm$ interpolation method

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1. Introduction

It is known that the grand Lebesgue spaces were introduced by Iwaniec and Sbordone in [18] in 1992 to study the integrability of the Jacobian determinant of an order preserving mapping from a bounded domain $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$. From then on, the grand Lebesgue spaces and their generalized versions have attracted a lot of attention and found several applications in various areas of analysis, such as partial differential equations and harmonic analysis; see, for example, [36], [15], [19], [13], [7], [10], [11], [12], [9], [8], [6], [3] and references therein.

To recall the definition of grand Lebesgue spaces, let $(\mathcal{X}, d, \mu)$ be a *quasi-metric measure space*, which means that $\mathcal{X}$ is a nonempty set, d a *quasi-metric* (that is, for

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all $x, y, z \in \mathcal{X}$, we have $d(x, y) \in [0, \infty)$, $d(x, y) = d(y, x)$ and $d(x, y) \leqslant K[d(x, z) + d(z, y)]$, where $K \in [1, \infty)$ is a constant independent of x, y, z) and μ a *nonnegative measure*. Assume further $\mu(\mathcal{X}) < \infty$. Recall that, for all $p \in (1, \infty)$, the *generalized grand Lebesgue space* $L_\varphi^{p)}(\mathcal{X})$ with an increasing weight function $\varphi\colon (0, \infty) \to (0, 1]$ is defined as the space of all measurable functions f on $\mathcal{X}$ such that

$$\|f\|_{L_\varphi^{p)}(\mathcal{X})} := \inf_{\varepsilon \in (0, p-1)} \varphi(\varepsilon) \left[\int_{\mathcal{X}} |f(x)|^{p-\varepsilon} \, \mathrm{d}\mu(x) \right]^{1/(p-\varepsilon)} < \infty.$$

When $\varphi(\varepsilon) := \varepsilon^{p-\varepsilon}$, $\mathcal{X}$ is a bounded subset of $\mathbb{R}^n$ and μ is the restriction of the Lebesgue measure on $\mathcal{X}$, the space $L_\varphi^{p)}(\mathcal{X})$ goes back to the grand Lebesgue space introduced in [18]. The name "grand Lebesgue space" comes from the continuous embedding

$$L^p(\mathcal{X}) \subset L_\varphi^{p)}(\mathcal{X}) \subset L^{p-\varepsilon}(\mathcal{X}) \quad \text{for all } \varepsilon \in (0, p).$$

The grand Lebesgue spaces are known to be rearrangement-invariant Banach function spaces. Some properties of these spaces, including real interpolation and duality, were studied in [7], [9].

On the other hand, the study of Morrey spaces can be traced to Morrey's work on the regularity problems of solutions to partial differential equations in 1938 (see [28]). As a natural extension of Lebesgue spaces, Morrey spaces have found lots of applications in partial differential equations, harmonic analysis and potential analysis; we refer, for example, to [1], [2], [34], [35] for some recent works. In 2009, Meskhi [25], [26] introduced the grand version of Morrey spaces and studied the boundedness of the maximal operator, Calderón-Zygmund operators and Riesz potentials on these spaces. Later Ye in [38] obtained the boundedness of commutators of singular and potential operators on grand Morrey spaces on spaces of homogeneous type in the sense of Coifman and Weiss. In 2013, Kokilashvili, Meskhi and Rafeiro in [21], [20], [22] further introduced and studied the generalized grand Morrey spaces in a general setting of quasi-metric measure spaces; see also [27], [30], [14] for more results on grand Morrey spaces.

The main purpose of this paper is to study the interpolation and dual properties of generalized grand Morrey spaces. To recall their definitions, we first need the following weight class.

Definition 1.1. Let $\mathcal{W}$ be the class of all functions $\varphi\colon (0, \infty) \to (0, 1]$ which are nondecreasing and there exist constants $1 < C_1 \leqslant C_2 < \infty$ such that $C_1\varphi(2^{-k}) \leqslant \varphi(2^{-k+1}) \leqslant C_2\varphi(2^{-k})$ for all $k \in \mathbb{N}$.

Definition 1.2. Let $p \in (1, \infty]$, $\lambda \in [0, 1)$ and $\varphi \in \mathcal{W}$. The *generalized grand Morrey space* $L_\varphi^{p),\lambda}(\mathcal{X})$ is defined as the space of all measurable functions f on $\mathcal{X}$

such that

$$\|f\|_{L^{p),\lambda}_{\varphi}(\mathcal{X})} = \sup_{\varepsilon\in(0,p-1)} \varphi(\varepsilon) \sup_{B} \left(\frac{1}{[\mu(B)]^{\lambda}} \int_{B} |f(x)|^{p-\varepsilon} \, \mathrm{d}\mu(x) \right)^{1/(p-\varepsilon)} < \infty,$$

where the second supermum is take over all balls B in $\mathcal{X}$.

Obviously, when $\lambda = 0$, the generalized grand Morrey space $L^{p),0}_{\varphi}(\mathcal{X})$ coincides with the generalized grand Lebesgue space considered in [6]. On the other hand, when $p = \infty$, it is easy to see that $L^{p),0}_{\varphi}(\mathcal{X})$ goes back to $L^{\infty}(\mathcal{X})$.

The study of interpolation on classical Morrey spaces started with Stampacchia in [37], Campanato and Murthy in [5] and Peetre in [31]. In 1990's, Ruiz and Vega in [33] and Blasco, Ruiz and Vega in [4] showed that in general Morrey spaces have no interpolation properties. Lemarié-Rieusset in [23] further pointed out explicitly that Morrey spaces have no interpolation properties if the parameters λ of two Morrey spaces are different. Very recently, it was proved in [24] (see also [39]) that, via the $\pm$ interpolation method, the interpolation space of two Morrey spaces with the same λ on quasi-metric measure space is also a Morrey space. In view of this, it is natural to ask whether we can interpolate generalized grand Morrey spaces as well. Indeed, the first main result of this paper reads as follows.

Theorem 1.3. *Let $\theta \in (0,1)$, $\lambda \in [0,1)$ and $p, p_0, p_1 \in (1,\infty]$ be such that $(1-\theta)/p_0 + \theta/p_1 = 1/p$. Assume further that φ, φ_0, $\varphi_1 \in \mathcal{W}$ so that φ, φ_0^{p/p_0} and φ_1^{p/p_1} are equivalent. Then*

$$\langle L^{p_0),\lambda}_{\varphi_0}(\mathcal{X}), L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})\rangle_{\theta} = L^{p),\lambda}_{\varphi}(\mathcal{X})^{\circ}$$

and

$$\langle L^{p_0),\lambda}_{\varphi_0}(\mathcal{X}), L^{p_1),\lambda}_{\varphi_1}(\mathcal{X}), \theta\rangle = L^{p),\lambda}_{\varphi}(\mathcal{X}),$$

where $L^{p),\lambda}_{\varphi}(\mathcal{X})^{\circ}$ denotes the closure of $L^{p_0),\lambda}_{\varphi_0}(\mathcal{X}) \cap L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})$ in $L^{p),\lambda}_{\varphi}(\mathcal{X})$.

Here, for any quasi-Banach spaces X_0 and X_1, $\langle X_0, X_1\rangle_{\theta}$ and $\langle X_0, X_1, \theta\rangle$ denote the Gagliardo-Peetre interpolation method introduced in [29], [32] and the $\pm$ interpolation method in [17], [16], respectively. The definitions of these interpolation notions are given in Section 2 in details. We also remark that these interpolation properties of Theorem 1.3 are new even when $\lambda = 0$, i.e., for the generalized grand Lebesgue spaces.

As a corollary of Theorem 1.3, we have the following interpolation property of linear operators on generalized grand Morrey spaces.

Corollary 1.4. *Let all notation be as in Theorem* 1.3, *and let* (X_0, X_1) *be a couple of quasi-Banach spaces.*

(i) *If a linear operator* T *is bounded from* $L^{p_j),\lambda}_{\varphi_j}(\mathcal{X})$ *to* A_j *with operator norms* M_j, $j \in \{0,1\}$, *then* T *is also bounded from* $L^{p),\lambda}_{\varphi}(\mathcal{X})$ *to* $\langle A_0, A_1, \theta\rangle$ *with the operator norm not greater than a positive constant multiple of* $M_0^{1-\theta}M_1^{\theta}$.

(ii) *If a linear operator* T *is bounded from* A_j *to* $L^{p_j),\lambda}_{\varphi_j}(\mathcal{X})$ *with operator norms* M_j, $j \in \{0,1\}$, *then* T *is also bounded from* $\langle X_0, X_1, \theta\rangle$ *with the operator norm not greater than a positive constant multiple of* $M_0^{1-\theta}M_1^{\theta}$.

The second aim of this paper is to determine the predual space of generalized grand Morrey spaces in spirit of classical Morrey spaces. The desired predual spaces are described via the following blocks.

Definition 1.5. Let $p \in (1,\infty)$, $\lambda \in [0,1)$ and $\varphi \in \mathcal{W}$. A measurable function b on $\mathcal{X}$ is called a (p', λ, φ)-*block* if b is supported on a ball B and satisfies

$$\inf_{\varepsilon\in(0,p-1)}[\varphi(\varepsilon)]^{-1}\left(\int_B |b(x)|^{(p-\varepsilon)'}\,\mathrm{d}\mu(x)\right)^{1/(p-\varepsilon)'}[\mu(B)]^{\lambda/(p-\varepsilon)} \leqslant 1.$$

Next we define the following block spaces.

Definition 1.6. Let $p \in (1,\infty)$, $\lambda \in [0,1)$ and $\varphi \in \mathcal{W}$. The *block space* $B^{p',\lambda}_{\varphi}(\mathcal{X})$ is defined to be the collection of all measurable functions f which can be represented as $f = \sum_i t_i b_i$ almost everywhere, where $\{t_i\}_i \in l^1$ and $\{b_i\}_i$ is a sequence of (p', λ, φ)-blocks. Moreover, let

$$\|f\|_{B^{p',\lambda}_{\varphi}(\mathcal{X})} := \inf\left\{\|\{t_i\}_i\|_{l^1} :\ f = \sum_i t_i b_i\right\},$$

where the infimum is taken over all possible decompositions of f.

The second main result of this paper reads as follows.

Theorem 1.7. *Let* $p \in (1,\infty)$, $\lambda \in [0,1)$ *and* $\varphi \in \mathcal{W}$. *Then the dual space of* $B^{p',\lambda}_{\varphi}(\mathcal{X})$ *is* $L^{p),\lambda}_{\varphi}(\mathcal{X})$ *in the following sense: for any* $g \in L^{p),\lambda}_{\varphi}(\mathcal{X})$, *the functional* $\int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)$ *induces a bounded linear functional on* $B^{p',\lambda}_{\varphi}(\mathcal{X})$; *conversely, for any* $L \in (B^{p',\lambda}_{\varphi}(\mathcal{X}))^*$, *there exists a* $g \in L^{p),\lambda}_{\varphi}(\mathcal{X})$ *such that* $L(f) = \int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)$ *for all* $f \in B^{p',\lambda}_{\varphi}(\mathcal{X})$.

We point out that Theorem 1.7 is also new when $\lambda = 0$, i.e., for generalized grand Lebesgue spaces. Recall that in [7], Fiorenza introduced the small Lebesgue spaces on subsets of Euclidean spaces with finite measure, and proved that the small

Lebesgue space and the grand Lebesgue space are associated spaces to each other. In comparison with [7], Theorem 1.7 provides a different description of the predual of grand Lebesgue spaces.

The structure of this paper is organized as follows. In Section 2, we present some basic notation and properties of interpolation and duality. A dual theorem for Morrey spaces on quasi-metric measure spaces and some embedding properties of generalized grand Morrey spaces are also proved. Sections 3 and 4 are then devoted to the proofs of Theorems 1.3 and 1.7, respectively.

Finally, we make some conventions on notation. We denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. The symbol $A \lesssim B$ means $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. If E is a subset of $\mathcal{X}$, we denote by χ_E its *characteristic function* and by $E^{\complement}$ the set $\mathcal{X} \setminus E$. For all $r \in (0,\infty)$ and $x \in \mathcal{X}$, denote by $B(x,r)$ the ball centered at x with side length r, namely, $B(x,r) := \{y \in \mathcal{X}\colon\ d(x,y) < r\}$.

2. Preliminaries

In this section, we recall some basic notions on the interpolation methods and duality. Some basic properties of generalized grand Morrey spaces are also presented.

First we recall that, for all $p \in (0,\infty]$ and $\lambda \in [0,1)$, the classical *Morrey space* $L^{p,\lambda}(\mathcal{X})$ is defined as the set of all measurable functions f on $\mathcal{X}$ such that

$$\|f\|_{L^{p,\lambda}(\mathcal{X})} := \sup_B \left(\frac{1}{[\mu(B)]^\lambda}\int_B |f(x)|^p \,\mathrm{d}\mu(x)\right)^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathcal{X}$. Obviously, $L^{p,0}(\mathcal{X})$ coincides with the Lebesgue space $L^p(\mathcal{X})$, and $L^{\infty,\lambda}(\mathcal{X}) = L^\infty(\mathcal{X})$.

Next we recall some basic knowledge on the interpolation methods used in this paper.

Definition 2.1. Let X_0, X_1 be a couple of quasi-Banach spaces, which are continuously embedding into a large Hausdorff topological vector space Y. The space $X_0 + X_1$ is defined by

$$X_0 + X_1 := \{y \in Y\colon \text{ there exists } y_i \in X_i,\ i \in \{0,1\} \text{ such that } y = y_0 + y_1\}$$

and its norm is given by

$$\|y\|_{X_0+X_1} := \inf\{\|y_0\|_{X_0} + \|y_1\|_{X_1}\colon\ y = y_0 + y_1, \quad y_i \in X_i,\ i \in \{0,1\}\}.$$

A quasi-Banach space X is called an *intermediate space* with respect to $X_0 + X_1$ if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ with continuous embeddings. If X is an intermediate space with respect to $X_0 + X_1$, let X° be the closure of $X_0 \cap X_1$ in X, and let the *Gagliardo closure* of X with respect to $X_0 + X_1$, denoted by $X^\sim$, be defined as follows: $a \in X^\sim$ if and only if there exists a sequence $\{a_i\}_{i\in\mathbb{N}}$ such that $a_i \to a$ in $X_0 + X_1$ and $\|a_i\|_X \leqslant \lambda < \infty$. Moreover, $\|a\|_{X^\sim} := \inf\{\lambda\}$.

Definition 2.2. Let (X_0, X_1) be a pair of quasi-Banach spaces and $\theta \in (0,1)$.

(i) (The Gagliardo-Peetre method.) We say $a \in \langle X_0, X_1\rangle_\theta$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$ and, for any bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$, $\sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-\theta)a_i}$ converges in X_j, $j \in \{0,1\}$. Moreover, for $j \in \{0,1\}$,

$$\left\| \sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i \right\|_{X_j} \leqslant C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for some nonnegative constant C, independent of $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and $\{a_i\}_{i\in\mathbb{Z}}$. Let

$$\|a\|_{\langle X_0, X_1\rangle_\theta} := \inf\{C\}.$$

(ii) (The $\pm$ method.) We say $a \in \langle X_0, X_1, \theta\rangle$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$ and, for any finite subset $F \in \mathbb{Z}$ and bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$, and $j \in \{0,1\}$,

$$\left\| \sum_{i\in\mathbb{F}} \varepsilon_i 2^{i(j-\theta)} a_i \right\|_{X_j} \leqslant C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for some constant C, independent of F, $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and $\{a_i\}_{i\in\mathbb{Z}}$. Let $\|a\|_{\langle X_0, X_1, \theta\rangle} := \inf\{C\}$.

An important tool we need is the following Calderón product.

Definition 2.3. A quasi-Banach space X of complex-valued measurable functions is called a *quasi-Banach lattice* if, for any $f \in X$ and a function g satisfying $|g| \leqslant |f|$, we have $g \in X$ and $\|g\|_X \leqslant \|f\|_X$.

Given two quasi-Banach lattices X_0 and X_1 and $\theta \in (0,1)$, their Calderón product $X_0^{1-\theta} X_1^\theta$ is defined by

$$X_0^{1-\theta} X_1^\theta := \{f \text{ is a complex-valued measurable function: there exist } f^0 \in X_0,\ f^1 \in X_1 \text{ such that } |f| \leqslant |f^0|^{1-\theta} |f^1|^\theta\}$$

and its norm is given by $\|f\|_{X_0^{1-\theta} X_1^\theta} := \inf\{\|f^0\|_{X_0}^{1-\theta} \|f^1\|_{X_1}^\theta\}$, where the infimum is taken over all $f^0 \in X_0$ and $f^1 \in X_1$ such that $|f| \leqslant |f^0|^{1-\theta} |f^1|^\theta$.

The following interpolation results on classical Morrey spaces can be found in [24], Theorem 1.2 (see also [39]).

Theorem 2.4. *Let $\theta \in (0,1)$, $\lambda \in [0,1)$, $p, p_0, p_1 \in (0,\infty]$ be such that $(1-\theta)/p_0 + \theta/p_1 = 1/p$. Then*

$$\langle L^{p_0,\lambda}(\mathcal{X}), L^{p_1,\lambda}(\mathcal{X})\rangle_\theta = ([L^{p_0,\lambda}(\mathcal{X})]^{1-\theta}[L^{p_1,\lambda}(\mathcal{X})]^{\theta})^\circ = L^{p,\lambda}(\mathcal{X})^\circ$$

and

$$\langle L^{p_0,\lambda}(\mathcal{X}), L^{p_1,\lambda}(\mathcal{X}), \theta\rangle = [L^{p_0,\lambda}(\mathcal{X})]^{1-\theta}[L^{p_1,\lambda}(\mathcal{X})]^{\theta} = L^{p,\lambda}(\mathcal{X}).$$

We also formulate here some results on the predual of Morrey spaces on $\mathcal{X}$.

Definition 2.5. Let $p \in (1,\infty)$ and $\lambda \in [0,1)$. A measurable function b on $\mathcal{X}$ is called a (p',λ)-*block* if b is supported on a ball B and satisfies

$$\left(\int_B |b(x)|^{p'} \, \mathrm{d}\mu(x)\right)^{1/p'} [\mu(B)]^{\lambda/p} \leqslant 1.$$

Definition 2.6. Let $p \in (1,\infty)$ and $\lambda \in [0,1)$. The space $B^{p',\lambda}(\mathcal{X})$ is defined to be the collection of all measurable functions f which can be represented as $f = \sum\limits_{i\in\mathbb{Z}} t_i b_i$ almost everywhere, where $\{t_i\}_i \in l^1$ and $\{b_i\}_i$ is a sequence of (p',λ)-blocks. Moreover, let

$$\|f\|_{B^{p',\lambda}(\mathcal{X})} := \inf\left\{\|\{t_i\}_i\|_{l^1} \colon f = \sum_{i\in\mathbb{Z}} t_i b_i\right\},$$

where the infimum is taken over all possible decompositions of f.

Then we have the following dual theorem on Morrey spaces, whose proof is somehow standard (see [40] for Morrey spaces on $\mathbb{R}^n$ and some of its subsets, or [35] for Morrey spaces on $\mathbb{R}^n$ with nondoubling measures). For the sake of completeness, we present its proof here.

Theorem 2.7. *Let $p \in (1,\infty)$ and $\lambda \in [0,1)$. Then the dual space of $B^{p',\lambda}(\mathcal{X})$ is the Morrey space $L^{p,\lambda}(\mathcal{X})$. Precisely, for any $g \in L^{p,\lambda}(\mathcal{X})$, $\int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)$ induces a bounded linear functional on $B^{p',\lambda}(\mathcal{X})$; conversely, for any $L \in (B^{p',\lambda}(\mathcal{X}))^*$, there exists a $g \in L^{p,\lambda}(\mathcal{X})$ such that $L(f) = \int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)$ for all $f \in B^{p',\lambda}(\mathcal{X})$.*

Proof. We only give the proof for the case $\lambda > 0$, since the proof for the case $\lambda = 0$ is similar and easier. Let $g \in L^{p,\lambda}(\mathcal{X})$ and $f \in B^{p',\lambda}(\mathcal{X})$. Then for any $\varepsilon \in (0,\infty)$, there exist $\{t_i\}_i \in l^1$ and a sequence of (p',λ)-blocks $\{a_i\}_i$, supported on balls $\{B_i\}_i$, such that $f = \sum_i t_i a_i$ almost everywhere and $\|\{t_i\}_i\|_{l^1} \leqslant \|f\|_{B^{p',\lambda}(\mathcal{X})} + \varepsilon$. Then by the Hölder inequality we see that

$$\begin{aligned}
\int_{\mathcal{X}} |f(x)g(x)| \, \mathrm{d}\mu(x) &\leqslant \sum_i |t_i| \int_{B_i} |a_i(x)||g(x)| \, \mathrm{d}\mu(x) \\
&\leqslant \sum_i |t_i| \bigg(\int_{B_i} |a_i(x)|^{p'} \, \mathrm{d}\mu(x)\bigg)^{1/p'} \bigg(\int_{B_i} |g(x)|^p \, \mathrm{d}\mu(x)\bigg)^{1/p} \\
&\leqslant \sum_i |t_i| [\mu(B_i)]^{-\lambda/p} \bigg(\int_{B_i} |g(x)|^p \, \mathrm{d}\mu(x)\bigg)^{1/p} \\
&\leqslant \|\{t_i\}_i\|_{l^1} \|g\|_{L^{p,\lambda}(\mathcal{X})} \leqslant (\|f\|_{B^{p',\lambda}(\mathcal{X})} + \varepsilon) \|g\|_{L^{p,\lambda}(\mathcal{X})}.
\end{aligned}$$

Letting $\varepsilon \to 0$, we then conclude that $\int_{\mathcal{X}} f(x)g(x) \, \mathrm{d}\mu(x)$ defines a bounded linear functional on $B^{p',\lambda}(\mathcal{X})$ with operator norm not greater than $\|g\|_{L^{p,\lambda}(\mathcal{X})}$.

Conversely, let L be a bounded linear functional on $B^{p',\lambda}(\mathcal{X})$. For any fixed ball B_0, let $B_j := 2^j B_0$ for all $j \in \mathbb{N}$ and let $L^{p'}(B_j)$ be the set of all $L^{p'}(\mathcal{X})$-functions supported on B_j. Notice that each function in $L^{p'}(B_j)$ can be regarded as a (p', λ)-block supported on B_j modulo a positive constant. Thus, the linear functional $g \mapsto L(g)$ is bounded on $L^{p'}(B_j)$. Then, by the duality between $L^{p'}(B_j)$ and $L^p(B_j)$, there exists $f_j \in L^p(B_j)$ such that $L(g) = \int_{B_j} f(x)g(x) \, \mathrm{d}\mu(x)$ for all $g \in L^{p'}(B_j)$. Letting $j \to \infty$ and using the uniqueness of each f_j, we can find a function f in $L^p_{\mathrm{loc}}(\mathcal{X})$ such that f equals f_j almost everywhere on B_j. It remains to show $f \in L^{p,\lambda}(\mathcal{X})$. Indeed, for any ball B in $\mathcal{X}$, define

$$g_B(x) := \chi_B(x) \mathrm{sgn}(f(x)) |f(x)|^{p-1}, \quad x \in \mathcal{X}.$$

Then

$$[\mu(B)]^{-\lambda/p} g_B \Big/ \bigg(\int_B |g_B(x)|^{p'} \, \mathrm{d}\mu(x)\bigg)^{1/p'}$$

is a (p', λ)-block supported on B. Moreover, noticing that

$$\frac{1}{[\mu(B)]^\lambda} \int_B |f(x)|^p \, \mathrm{d}\mu(x) = \frac{1}{[\mu(B)]^\lambda} \int_B f(x) g_B(x) \, \mathrm{d}\mu(x) = \frac{1}{[\mu(B)]^\lambda} L(g_B),$$

we know that

$$\begin{aligned}
\frac{1}{[\mu(B)]^\lambda} \int_B |f(x)|^p \, \mathrm{d}\mu(x) &\leqslant \frac{1}{[\mu(B)]^\lambda} \|L\| \|g_B\|_{B^{p',\lambda}(\mathcal{X})} \\
&\leqslant \|L\| \frac{1}{[\mu(B)]^\lambda} [\mu(B)]^{\lambda/p} \bigg(\int_B |g_B(x)|^{p'} \, \mathrm{d}\mu(x)\bigg)^{1/p'} \\
&= \|L\| \bigg(\frac{1}{[\mu(B)]^\lambda} \int_B |f(x)|^p \, \mathrm{d}\mu(x)\bigg)^{1/p'}.
\end{aligned}$$

This implies that $f \in L^{p,\lambda}(\mathcal{X})$ and $\|f\|_{L^{p,\lambda}(\mathcal{X})}$ is not greater than the operator norm $\|L\|$. Thus, we complete the proof of Theorem 2.7. □

Finally, we have the following embedding property of generalized grand Morrey spaces, which is a direct consequence of the Hölder inequality and the fact that $\mu(\mathcal{X}) < \infty$.

Proposition 2.8. *Let $p \in (1, \infty)$, $\lambda \in [0, 1)$, $\varepsilon \in (0, p-1)$ and $\varphi \in \mathcal{W}$. Then*

$$L^{p,\lambda}(\mathcal{X}) \subset L_{\varphi}^{p),\lambda}(\mathcal{X}) \subset L^{p-\varepsilon,\lambda}(\mathcal{X}).$$

These embeddings are where the name "generalized grand Morrey space" comes from.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. To this end, we first need to calculate the Calderón product of two generalized grand Morrey spaces.

Lemma 3.1. *Let $\theta \in (0, 1)$, $\lambda \in [0, 1)$, $p, p_0, p_1 \in (1, \infty]$ be such that $(1-\theta)/p_0 + \theta/p_1 = 1/p$, and $\varphi, \varphi_0, \varphi_1 \in \mathcal{W}$ such that φ is equivalent to $\varphi_0^{1-\theta}\varphi_1^{\theta}$. Then*

$$[L_{\varphi_0}^{p_0),\lambda}(\mathcal{X})]^{1-\theta}[L_{\varphi_1}^{p_1),\lambda}(\mathcal{X})]^{\theta} \subset L_{\varphi}^{p),\lambda}(\mathcal{X}).$$

To prove Lemma 3.1 we need some auxiliary functions.

Definition 3.2. Let θ, p, p_0, p_1 be as in Lemma 3.1. For all $\varepsilon \in [0, p-1]$ and $s, t \in [0, \infty]$, define

$$\begin{aligned}
\widetilde{p_i}(\varepsilon) &:= p_i - \frac{\varepsilon}{p-1}(p_i - 1), \quad i \in \{0, 1\}, \\
H(s, t) &:= \frac{1}{(1-\theta)/s + \theta/t}, \\
h(\varepsilon) &:= H(\widetilde{p_0}(\varepsilon), \widetilde{p_1}(\varepsilon)), \\
p_i(\varepsilon) &:= \widetilde{p_i}(h^{-1}(p-\varepsilon)), \quad i \in \{0, 1\}.
\end{aligned}$$

From Definition 3.2, we can deduce the following properties of $p_0(\varepsilon)$ and $p_1(\varepsilon)$.

Proposition 3.3. *Let all notation be as in Definition 3.2. Then $p_i(\varepsilon)$ is continuous, strictly decreasing and satisfies the following conditions:*

$$p_i(0) = p_i, \quad p_i(p-1) = 1, \tag{3.1}$$

$$\frac{1-\theta}{p_0(\varepsilon)} + \frac{\theta}{p_1(\varepsilon)} = \frac{1}{p-\varepsilon}, \tag{3.2}$$

$$\lim_{\varepsilon \to 0^+} \frac{p_i - p_i(\varepsilon)}{\varepsilon} = \frac{p_0 p_1 (p_i - 1)}{p(p + p_0 p_1 - p_0 - p_1)}. \tag{3.3}$$

Proof. Obviously, both $\widetilde{p_0}(\varepsilon)$ and $\widetilde{p_1}(\varepsilon)$ are continuous and strictly decreasing. Therefore, the function h is also continuous and strictly decreasing, so does h^{-1}. This implies that $p_0(\varepsilon)$ and $p_1(\varepsilon)$ are continuous and strictly decreasing, due to their definitions. On the other hand, (3.1) and (3.2) are easy to check, and hence we only need to prove (3.3).

Notice that $h(\varepsilon)$ is continuous, strictly decreasing and $h(0) = p$, $h(p-1) = 1$. Then for any $\varepsilon \in (0, p-1)$ there exists a unique $\varepsilon' \in (0, p-1)$ such that $p - \varepsilon = h(\varepsilon')$. Thus, by Definition 3.2, we see that

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \frac{p_i - p_i(\varepsilon)}{\varepsilon} &= \frac{p_i - 1}{p - 1} \lim_{\varepsilon' \to 0^+} \frac{\varepsilon'}{p - h(\varepsilon')} \\
&= \frac{p_i - 1}{p - 1} \lim_{\varepsilon' \to 0^+} \frac{\varepsilon'(p-1)[p_0 p_1 (p - 1 - \varepsilon') + p\varepsilon']}{p\varepsilon'[(p-1)(p + p_0 p_1 - p_0 - p_1) - (p_0 - 1)(p_1 - 1)\varepsilon']} \\
&= \frac{p_0 p_1 (p_i - 1)}{p(p + p_0 p_1 - p_0 - p_1)},
\end{aligned}$$

as desired. □

Proposition 3.3 leads to the following useful corollary.

Corollary 3.4. *There exist positive constants K_1, K_2 such that for all $\varepsilon \in [0, p-1]$, $K_1 \varepsilon \leqslant p_i - p_i(\varepsilon) \leqslant K_2 \varepsilon$ and, if $\varphi \in \mathcal{W}$, then $\varphi(p_i - p_i(\cdot)) \in \mathcal{W}$ and is equivalent to φ.*

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let $f \in [L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta}$. Then, by Definition 2.3, we know that there exist $f_0 \in L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})$ and $f_1 \in L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})$ such that $|f(x)| \leqslant |f_0(x)|^{1-\theta}|f_1(x)|^{\theta}$ for almost every $x \in \mathcal{X}$, and

$$\|f_0\|^{1-\theta}_{L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})} \|f_1\|^{\theta}_{L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})} \lesssim \|f\|_{[L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta}}.$$

By virtue of $\varphi \approx \varphi_0^{1-\theta}\varphi_1^{\theta}$ and Corollary 3.4, we see that

$$\begin{aligned}\|f\|_{L^{p),\lambda}_{\varphi}(\mathcal{X})} &= \sup_{\varepsilon\in(0,p-1)} \varphi(\varepsilon)\|f\|_{L^{p-\varepsilon,\lambda}(\mathcal{X})}\\ &\approx \sup_{\varepsilon\in(0,p-1)} \varphi_0^{1-\theta}(\varepsilon)\varphi_1^{\theta}(\varepsilon)\|f\|_{L^{p-\varepsilon,\lambda}(\mathcal{X})}\\ &\approx \sup_{\varepsilon\in(0,p-1)} \varphi_0^{1-\theta}(p_0-p_0(\varepsilon))\varphi_1^{\theta}(p_1-p_1(\varepsilon))\|f\|_{L^{p-\varepsilon,\lambda}(\mathcal{X})}.\end{aligned}$$

Applying Proposition 3.3 and the Hölder inequality, we find that

$$\|f\|_{L^{p-\varepsilon,\lambda}(\mathcal{X})} \leqslant \|f_0\|^{1-\theta}_{L^{p_0(\varepsilon),\lambda}(\mathcal{X})}\|f_1\|^{\theta}_{L^{p_1(\varepsilon),\lambda}(\mathcal{X})}.$$

Hence, letting $\varepsilon_i := p_i - p_i(\varepsilon)$, we further see that

$$\begin{aligned}\|f\|_{L^{p),\lambda}_{\varphi}(\mathcal{X})} &\lesssim \sup_{\varepsilon\in(0,p-1)} \varphi_0^{1-\theta}(p_0-p_0(\varepsilon))\varphi_1^{\theta}(p_1-p_1(\varepsilon))\|f_0\|^{1-\theta}_{L^{p_0(\varepsilon),\lambda}(\mathcal{X})}\|f_1\|^{\theta}_{L^{p_1(\varepsilon),\lambda}(\mathcal{X})}\\ &\lesssim \sup_{\varepsilon\in(0,p-1)} \varphi_0^{1-\theta}(p_0-p_0(\varepsilon))\|f_0\|^{1-\theta}_{L^{p_0(\varepsilon),\lambda}(\mathcal{X})}\\ &\quad\times \sup_{\varepsilon\in(0,p-1)} \varphi_1^{\theta}(p_1-p_1(\varepsilon))\|f_1\|^{\theta}_{L^{p_1(\varepsilon),\lambda}(\mathcal{X})}\\ &\lesssim \sup_{\varepsilon_0\in(0,p_0-1)} (\varphi_0(\varepsilon_0)\|f\|_{M^{p_0-\varepsilon_0}_{\lambda}(\mathcal{X})})^{1-\theta}\\ &\quad\times \sup_{\varepsilon_1\in(0,p_1-1)} (\varphi_1(\varepsilon_1)\|f\|_{M^{p_1-\varepsilon_1}_{u_1}(\mathcal{X})})^{\theta}\\ &\approx \|f_0\|^{1-\theta}_{L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})}\|f_1\|^{\theta}_{L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})} \lesssim \|f\|_{[L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta}}.\end{aligned}$$

This implies that $f \in L^{p),\lambda}_{\varphi}(\mathcal{X})$ which completes the proof of Lemma 3.1. □

We also prove that the converse embedding is true.

Lemma 3.5. *Let $\theta \in (0,1)$, $\lambda \in [0,1)$ and let $p, p_0, p_1 \in (1,\infty]$ be such that $(1-\theta)/p_0 + \theta/p_1 = 1/p$. Assume further that φ, φ_0, $\varphi_1 \in \mathcal{W}$ so that φ, φ_0^{p/p_0} and φ_1^{p/p_1} are equivalent. Then*

$$[L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta} \supset L^{p),\lambda}_{\varphi}(\mathcal{X}).$$

To prove Lemma 3.5, we need the following notation.

Definition 3.6. Let f be a measurable function on $\mathcal{X}$, $p \in (0,\infty]$ and $V \in (0,\infty)$. Define

$$S(f,p,V) := \sup_{\mu(B)=V}\left(\int_B |f(x)|^p \,\mathrm{d}\mu(x)\right)^{1/p},$$

where the supremum is taken over all balls B in $\mathcal{X}$ with measure V. As a special case, $S(f,p,V_0) = 0$ if there is no ball with measure V_0 in $\mathcal{X}$.

Remark 3.7. Obviously, for all $k\in(0,\infty)$,

$$S(f,p,V)=S(f^{1/k},kp,V)^k. \tag{3.4}$$

It is also easy to see that

$$\|f\|_{L^{p,\lambda}(\mathcal{X})}=\sup_{V\in(0,\infty)}S(f,p,V)V^{-\lambda/p}.$$

Now we prove Lemma 3.5.

Proof of Lemma 3.5. Let $f\in L_{\varphi}^{p),\lambda}(\mathcal{X})$. Define $f_i:=|f|^{p/p_i}$, $i\in\{0,1\}$. Then by Remark 3.7, we see that

$$\begin{aligned}\|f_i\|_{L_{\varphi_i}^{p_i),\lambda}(\mathcal{X})}&=\sup_{\varepsilon_i\in(0,p_i-1)}\varphi_i(\varepsilon)\|f_i\|_{L^{p_i-\varepsilon_i,\lambda}(\mathcal{X})}\\&=\sup_{\varepsilon_i\in(0,p_i-1)}\sup_{V\in(0,\infty)}\varphi_i(\varepsilon_i)S(f_i,p_i-\varepsilon_i,V)V^{-\lambda/(p_i-\varepsilon_i)}\\&=\Big[\sup_{\varepsilon_i\in(0,p_i-1)}\sup_{V\in(0,\infty)}\varphi(\varepsilon_i)S(f_i,p_i-\varepsilon_i,V)^{p_i/p}V^{-\lambda p_i/(p(p_i-\varepsilon_i))}\Big]^{p/p_i}.\end{aligned}$$

By Proposition 3.3, we know that $p_i(\varepsilon)$ is strictly decreasing, continuous and satisfies $p_i(0)=p_i$ and $p_i(p-1)=1$. Then for any $\varepsilon_i\in(0,p_i-1)$ there exists a unique $\varepsilon\in(0,p-1)$ such that

$$p_i-p_i(\varepsilon)=\varepsilon_i.$$

Thus, we further have

$$\|f_i\|_{L_{\varphi_i}^{p_i),\lambda}(\mathcal{X})}\lesssim\Big[\sup_{\varepsilon\in(0,p-1)}\sup_{V\in(0,\infty)}\varphi(p_i-p_i(\varepsilon))S(f_i,p_i(\varepsilon),V)^{p_i/p}V^{-\lambda p_i/(p_i(\varepsilon)p)}\Big]^{p/p_i}.$$

Then, by (3.4) and $\varphi\in\mathcal{W}$, we find that

$$\begin{aligned}\|f_i\|_{L_{\varphi_i}^{p_i),\lambda}(\mathcal{X})}&\lesssim\Big[\sup_{\varepsilon\in(0,p-1)}\sup_{V\in(0,\infty)}\varphi\Big(p-\frac{p}{p_i}p_i(\varepsilon)\Big)S\Big(f,\frac{p}{p_i}p_i(\varepsilon),V\Big)V^{-\lambda p_i/(p_i(\varepsilon)p)}\Big]^{p/p_i}\\&\approx\Big[\sup_{\varepsilon\in(0,p-1)}\sup_{V\in(0,\infty)}\varphi(\varepsilon)S(f,p-\varepsilon,V)V^{-\lambda/(p-\varepsilon)}\Big]^{p/p_i}\\&\approx\|f\|_{L_{\varphi}^{p),\lambda}(\mathcal{X})}^{p/p_i}.\end{aligned}$$

In addition, notice that

$$|f|=|f|^{p(1-\theta)/p_0}|f|^{p\theta/p_1}=|f_0|^{1-\theta}|f_1|^{\theta}.$$

Therefore,

$$\|f\|_{[L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta}} \leqslant \|f_0\|^{1-\theta}_{L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})}\|f_1\|^{\theta}_{L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})} \lesssim \|f\|_{L^{p),\lambda}_{\varphi}(\mathcal{X})}.$$

This implies $f \in [L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta}$ and completes the proof of Lemma 3.5. □

Combining Lemmas 3.1 and 3.5, we obtain the following result.

Theorem 3.8. *Let $\theta \in (0,1)$, $\lambda \in [0,1)$ and $p, p_0, p_1 \in (1,\infty]$ be such that $(1-\theta)/p_0 + \theta/p_1 = 1/p$. Assume further that φ, φ_0, $\varphi_1 \in \mathcal{W}$ so that φ, φ_0^{p/p_0} and φ_1^{p/p_1} are equivalent. Then*

$$[L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta} = L^{p),\lambda}_{\varphi}(\mathcal{X}).$$

To prove Theorem 1.3, the following general result obtained by Nilsson in [29], Theorem 2.1, is another important tool.

Theorem 3.9. *Let X_0 and X_1 be two quasi-Banach lattices of type $\mathfrak{C}$. Then*

$$\langle X_0, X_1\rangle_\theta = (X_0^{1-\theta}X_1^{\theta})^{\circ}$$

and

$$X_0^{1-\theta}X_1^{\theta} \subset \langle X_0, X_1, \theta\rangle \subset (X_0^{1-\theta}X_1^{\theta})^{\sim}.$$

Notice that, for all $\delta \in (0,\min(1,p)]$, the $1/\delta$-convexification $(L^{p),\lambda}_{\varphi}(\mathcal{X}))^{(1/\delta)}$ of the generalized grand Morrey space is a Banach space, namely, the generalized grand Morrey space $L^{p),\lambda}_{\varphi}(\mathcal{X})$ is of type $\mathfrak{C}$.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. By Theorems 3.8 and 3.9, we see that

$$\langle L^{p_0),\lambda}_{\varphi_0}(\mathcal{X}), L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})\rangle_\theta = ([L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta})^{\circ} = (L^{p),\lambda}_{\varphi}(\mathcal{X}))^{\circ}$$

and

$$L^{p),\lambda}_{\varphi}(\mathcal{X}) = [L^{p_0),\lambda}_{\varphi_0}(\mathcal{X})]^{1-\theta}[L^{p_1),\lambda}_{\varphi_1}(\mathcal{X})]^{\theta} \subset \langle L^{p_0),\lambda}_{\varphi_0}(\mathcal{X}), L^{p_1),\lambda}_{\varphi_1}(\mathcal{X}), \theta\rangle \subset (L^{p),\lambda}_{\varphi}(\mathcal{X}))^{\sim}.$$

Thus, to complete the proof, we only need to prove that $L^{p),\lambda}_{\varphi}(\mathcal{X}) \supset (L^{p),\lambda}_{\varphi}(\mathcal{X}))^{\sim}$.

To this end, let $f \in (L_\varphi^{p),\lambda}(\mathcal{X}))^\sim$. Then there exists a sequence $\{f_i\}_{i\in\mathbb{N}} \subset L_\varphi^{p),\lambda}(\mathcal{X})$ such that

$$\lim_{i\to\infty} \|f_i - f\|_{L_{\varphi_0}^{p_0),\lambda}(\mathcal{X})+L_{\varphi_1}^{p_1),\lambda}(\mathcal{X})} = 0 \tag{3.5}$$

and

$$\|f_i\|_{L_\varphi^{p),\lambda}(\mathcal{X})} \lesssim \|f\|_{(L_\varphi^{p),\lambda}(\mathcal{X}))^\sim}, \quad i \in \mathbb{N}.$$

Thus, there exist sequences $\{f_i^0\}_{i\in\mathbb{N}} \subset L_{\varphi_0}^{p_0),\lambda}(\mathcal{X})$ and $\{f_i^1\}_{i\in\mathbb{N}} \subset L_{\varphi_1}^{p_1),\lambda}(\mathcal{X})$ such that, for almost every $x \in \mathcal{X}$, $f_i(x) - f(x) = f_i^0(x) + f_i^1(x)$ and

$$\|f_i^0\|_{L_{\varphi_0}^{p_0),\lambda}(\mathcal{X})} + \|f_i^1\|_{L_{\varphi_1}^{p_1),\lambda}(\mathcal{X})} \lesssim \|f_i - f\|_{L_{\varphi_0}^{p_0),\lambda}(\mathcal{X})+L_{\varphi_1}^{p_1),\lambda}(\mathcal{X})} \to 0$$

as $i \to \infty$.

Next, we claim that $\{f_i^0\}_{i\in\mathbb{N}}$ converges to 0 in measure on $\mathcal{X}$. Indeed, if $\{f_i^0\}_{i\in\mathbb{N}}$ does not converge to 0 in measure on $\mathcal{X}$, then there exist positive constants c_0, c_1 such that

$$\mu(\{x \in \mathcal{X} \colon\ |f_i^0(x)| > c_0\}) > c_1, \quad i \in \mathbb{N},$$

which leads to

$$\begin{aligned}
\|f_i^0\|_{L_{\varphi_0}^{p_0),\lambda}(\mathcal{X})} &\geqslant \sup_{\varepsilon\in(0,p-1)} \varphi(\varepsilon)\Big(\frac{1}{[\mu(\mathcal{X})]^\lambda}\int_{\mathcal{X}} |f_i^0(x)|^{p-\varepsilon}\, \mathrm{d}\mu(x)\Big)^{1/(p-\varepsilon)} \\
&\geqslant \sup_{\varepsilon\in(0,p-1)} \varphi(\varepsilon)([\mu(\mathcal{X})]^{-\lambda} c_1 {c_0}^{p-\varepsilon})^{1/(p-\varepsilon)} \\
&\geqslant \sup_{\varepsilon\in(0,p-1)} \varphi(\varepsilon)[\mu(\mathcal{X})]^{-\lambda/(p-\varepsilon)} c_0 {c_1}^{1/(p-\varepsilon)} \geqslant C
\end{aligned}$$

for some positive constant C independent of i. This contradicts (3.5). Hence, the above claim is correct. Following a similar argument, we know that $\{f_i^1\}_{i\in\mathbb{N}}$ converges to 0 in measure as well.

Therefore, there exist a subsequence of $\{f_i^0\}_{i\in\mathbb{N}}$, denoted by $\{f_{i_k}^0\}_{k\in\mathbb{N}}$, and a subsequence of $\{f_i^1\}_{i\in\mathbb{N}}$, denoted by $\{f_{i_k}^1\}_{k\in\mathbb{N}}$, such that both $\{f_{i_k}^0\}_{k\in\mathbb{N}}$ and $\{f_{i_k}^1\}_{k\in\mathbb{N}}$ converge to 0 almost everywhere. Hence, $f_{i_k} = f + f_{i_k}^0 + f_{i_k}^1$ converges to f almost everywhere. From this and the Fatou lemma, we deduce that, for any ball B,

$$\begin{aligned}
&\varphi(\varepsilon)\bigg(\frac{1}{[\mu(B)]^\lambda}\int_B |f(x)|^{p-\varepsilon}\, \mathrm{d}\mu(x)\bigg)^{1/(p-\varepsilon)} \\
&\quad\leqslant \varliminf_{k\to\infty} \varphi(\varepsilon)\bigg(\frac{1}{[\mu(B)]^\lambda}\int_B |f_{i_k}(x)|^{p-\varepsilon}\, \mathrm{d}\mu(x)\bigg)^{1/(p-\varepsilon)} \\
&\quad\leqslant \varliminf_{k\to\infty} \|f_{i_k}\|_{L_\varphi^{p),\lambda}(\mathcal{X})} \lesssim \|f\|_{(L_\varphi^{p),\lambda}(\mathcal{X}))^\sim}.
\end{aligned}$$

This implies that $f \in L_\varphi^{p),\lambda}(\mathcal{X})$ and $L_\varphi^{p),\lambda}(\mathcal{X}) \supset (L_\varphi^{p),\lambda}(\mathcal{X}))^\sim$, which completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.7

First we prove that

$$L_{\varphi}^{p),\lambda}(\mathcal{X}) \subset (B_{\varphi}^{p',\lambda}(\mathcal{X}))^{*}. \tag{4.1}$$

To see this, let $f \in L_{\varphi}^{p),\lambda}(\mathcal{X})$ and $g \in B_{\varphi}^{p',\lambda}(\mathcal{X})$. Then, for any $\delta \in (0,\infty)$, there exist (p',λ,φ)-blocks $\{b_i\}_i$, supported on balls $\{B_i\}_i$, and $\{t_i\}_i \subset l^1$ such that $g = \sum\limits_i t_i b_i$ almost everywhere and $\sum\limits_i |t_i| \leqslant \|g\|_{B_{\varphi}^{p',\lambda}(\mathcal{X})} + \delta$. By the definition of (p',λ,φ)-blocks, we can further pick $\{\varepsilon_i\}_i$ such that for all i,

$$[\varphi(\varepsilon_i)]^{-1}\biggl(\int_{B_i} |b_i(x)|^{(p-\varepsilon_i)'}\,\mathrm{d}\mu(x)\biggr)^{1/(p-\varepsilon_i)'} [\mu(B_i)]^{\lambda/(p-\varepsilon_i)} \leqslant 1+\delta. \tag{4.2}$$

By Theorem 2.7, it is easy to see that

$$\begin{aligned}\biggl|\int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)\biggr| &= \biggl|\int_{\mathcal{X}} f(x)\sum_i t_i b_i(x)\,\mathrm{d}\mu(x)\biggr| \\ &\leqslant \sum_i \biggl|\int_{B_i} f(x)t_i b_i(x)\,\mathrm{d}\mu(x)\biggr| = \sum_i \biggl|\int_{B_i} f(x)t_i b_i(x)\frac{\varphi(\varepsilon_i)(1+\delta)}{\varphi(\varepsilon_i)(1+\delta)}\,\mathrm{d}\mu(x)\biggr| \\ &\leqslant \sum_i \varphi(\varepsilon_i)(1+\delta)\|f\|_{L_{\lambda}^{p-\varepsilon_i}(\mathcal{X})}\biggl\|\frac{t_i b_i}{\varphi(\varepsilon_i)(1+\delta)}\biggr\|_{B^{(p-\varepsilon_i)',\lambda}(\mathcal{X})}.\end{aligned}$$

Notice that by the choice of $\{\varepsilon_i\}_i$ and by (4.2), $\varphi(\varepsilon_i)b_i/(1+\delta)$ are $((p-\varepsilon_i)',\lambda)$-blocks as in Definition 2.5. Then $t_i\varphi(\varepsilon_i)b_i/(1+\delta)$ is a block-decomposition of itself in the sense of Definition 2.6, which implies

$$\biggl\|\frac{t_i b_i}{\varphi(\varepsilon_i)(1+\delta)}\biggr\|_{B^{(p-\varepsilon_i)',\lambda}(\mathcal{X})} \leqslant |t_i|.$$

Therefore,

$$\begin{aligned}\biggl|\int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)\biggr| &\leqslant (1+\delta)\sum_i \varphi(\varepsilon_i)\|f\|_{L_{\lambda}^{p-\varepsilon_i}(\mathcal{X})}|t_i| \\ &\leqslant (1+\delta)\sum_i \|f\|_{L_{\varphi}^{p),\lambda}(\mathcal{X})}|t_i| \\ &\leqslant (1+\delta)\|f\|_{L_{\varphi}^{p),\lambda}(\mathcal{X})}(\|g\|_{B_{\varphi}^{p',\lambda}(\mathcal{X})}+\delta).\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\biggl|\int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)\biggr| \leqslant \|f\|_{L_{\varphi}^{p),\lambda}(\mathcal{X})}\|g\|_{B_{\varphi}^{p',\lambda}(\mathcal{X})}.$$

This implies that any $f \in L_\varphi^{p),\lambda}(\mathcal{X})$ defines a bounded linear functional on $B_\varphi^{p',\lambda}(\mathcal{X})$ via $\int_{\mathcal{X}} f(x)g(x)\,\mathrm{d}\mu(x)$, namely, (4.1) holds.

Next we show the converse embedding $(B_\varphi^{p',\lambda}(\mathcal{X}))^* \subset L_\varphi^{p),\lambda}(\mathcal{X})$. To see this, we notice that, for any $\varepsilon \in (0, p-1)$ and $f \in B^{(p-\varepsilon)',\lambda}(\mathcal{X})$,

$$\|f\|_{B_\varphi^{p',\lambda}(\mathcal{X})} \leqslant \frac{\|f\|_{B^{(p-\varepsilon)',\lambda}(\mathcal{X})}}{\varphi(\varepsilon)}.$$

Hence, $B_\varphi^{p',\lambda}(\mathcal{X}) \supset B^{(p-\varepsilon)',\lambda}(\mathcal{X})$ and then by Theorem 2.7,

$$(B_\varphi^{p',\lambda}(\mathcal{X}))^* \subset (B^{(p-\varepsilon)',\lambda}(\mathcal{X}))^* = L_\lambda^{p-\varepsilon}(\mathcal{X}). \tag{4.3}$$

This means that the elements of $(B_\varphi^{p',\lambda}(\mathcal{X}))^*$ can be seen as functions in the set $\bigcap_{\varepsilon\in(0,p-1)} L_\lambda^{p-\varepsilon}(\mathcal{X})$. Thus, to complete the proof, we only need to prove, if $f \in L_\lambda^{p-\varepsilon}(\mathcal{X}) \setminus L_\varphi^{p),\lambda}(\mathcal{X})$ for some $\varepsilon \in (0, p-1)$, then $f \notin (B_\varphi^{p',\lambda}(\mathcal{X}))^*$, which immediately gives us

$$(B_\varphi^{p',\lambda}(\mathcal{X}))^* \supset L_\varphi^{p),\lambda}(\mathcal{X}).$$

Indeed, by (4.3), we assume that $f \in L^{p-\varepsilon,\lambda}(\mathcal{X})$ for all $\varepsilon \in (0, p-1)$. Since $f \in L_\lambda^{p-\varepsilon}(\mathcal{X}) \setminus L_\varphi^{p),\lambda}(\mathcal{X})$, we know that

$$\limsup_{\varepsilon\to 0^+} \varphi(\varepsilon)\|f\|_{L^{p-\varepsilon,\lambda}(\mathcal{X})} = \infty.$$

Therefore, there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, p-1)$ converging to 0 as $k \to \infty$ such that

$$\|f\|_{L_\lambda^{p-\varepsilon_k}(\mathcal{X})} \varphi(\varepsilon_k) > k$$

for k large enough.

Since $f \in L_\lambda^{p-\varepsilon_k}(\mathcal{X})$, by Theorem 2.7 we know that $f \in (B^{(p-\varepsilon_k)',\lambda}(\mathcal{X}))^*$. Hence there exist elements $\{g_k\}_{k\in\mathbb{N}}$ in $B^{(p-\varepsilon_k)',\lambda}(\mathcal{X})$ such that

$$\|g_k\|_{B^{(p-\varepsilon_k)',\lambda}(\mathcal{X})} = 1$$

and

$$\int_{\mathcal{X}} f(x)g_k(x)\,\mathrm{d}\mu(x) \geqslant \frac{1}{2}\|f\|_{L_\lambda^{p-\varepsilon_k}(\mathcal{X})} > \frac{k}{2}[\varphi(\varepsilon_k)]^{-1}.$$

Let $\widetilde{g}_k := \varphi(\varepsilon_k)g_k$ for all $k \in \mathbb{N}$. Then $\|\widetilde{g}_k\|_{B_\varphi^{p',\lambda}(\mathcal{X})} \leqslant 1$ and

$$\int_{\mathcal{X}} f(x)\widetilde{g}_k(x)\,\mathrm{d}\mu(x) > \frac{k}{2},$$

which tends to ∞ as $k \to \infty$. Hence, $f \notin (B_\varphi^{p',\lambda}(\mathcal{X}))^*$. This implies $L_\varphi^{p),\lambda}(\mathcal{X}) = (B_\varphi^{p',\lambda}(\mathcal{X}))^*$ and completes the proof of Theorem 1.7. $\square$

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