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INTERPOLATION-BASED $\mathcal{H}_2$-MODEL REDUCTION OF BILINEAR CONTROL SYSTEMS*

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Abstract. In this paper, we will discuss the problem of optimal model order reduction of bilinear control systems with respect to the generalization of the well-known $\mathcal{H}_2$-norm for linear systems. We revisit existing first order necessary conditions for $\mathcal{H}_2$-optimality based on the solutions of generalized Lyapunov equations arising in bilinear system theory and present an iterative algorithm which, upon convergence, will yield a reduced system fulfilling these conditions. While this approach relies on the solution of certain generalized Sylvester equations, we will establish a connection to another method based on generalized rational interpolation. This will lead to another way of computing the $\mathcal{H}_2$-norm of a bilinear system and will extend the pole-residue optimality conditions for linear systems, also allowing for an adaption of the successful iterative rational Krylov algorithm to bilinear systems. By means of several numerical examples, we will then demonstrate that the new techniques outperform the method of balanced truncation for bilinear systems with regard to the relative $\mathcal{H}_2$-error.

Key words. model order reduction, bilinear systems, $\mathcal{H}_2$-optimality, Sylvester equations

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1. Introduction.

The need for efficient numerical treatment of complex dynamical processes often leads to the problem of model order reduction; i.e., the approximation of large-scale systems resulting from, e.g., partial differential equations, by significantly smaller ones. Since model reduction of linear systems has been studied for several years now, there exists a well-established theory including error bounds and structure-preserving properties fulfilled by a reduced-order model. However, although there are still a lot of open and worthwhile problems, recently more and more attention has been paid to nonlinear systems which are inevitably more complicated. As a first step into this direction, the class of bilinear systems has been pointed out to be an interesting interface between fully nonlinear and linear control systems; see [10, 22, 23, 24, 27]. More precisely, these special systems are of the form

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + \displaystyle\sum_{k=1}^{m} N_k x(t) u_k(t) + Bu(t), \\ y(t) = Cx(t), \quad x(0) = x_0, \end{cases} \tag{1.1}$$

with $A, N_k \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $u(t) = [u_1(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. Throughout the rest of this paper, we will assume that we have a zero initial condition $x_0 = 0$. However, if this does not hold true, one can easily embed all our results into the above setting by incorporating x_0 in an enlarged input vector

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of the form $\begin{bmatrix} B & x_0 \end{bmatrix}$. Due to the previous structure, which is obviously closely related to the state space representation of linear systems, many concepts known from linear model order reduction have been shown to possess bilinear analogues. As was already discussed in [10, 22, 23, 24], a variety of biological, physical, and economical phenomena naturally result in bilinear models. Here, models for nuclear fusion, mechanical brakes or biological species can be mentioned as typical examples. Interestingly enough, a completely similar structure is obtained for a certain type of linear stochastic differential equations. Some interesting applications, like, e.g., the Fokker–Planck equation, are discussed in [20]. Coming back to the actual reduction problem, we are formally aiming at the construction of another bilinear system,

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \displaystyle\sum_{k=1}^{m} \hat{N}_k \hat{x}(t) u_k(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t), \quad \hat{x}(0) = 0, \end{cases} \tag{1.2}$$

with $\hat{A}, \hat{N}_k \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times m}$, $\hat{C} \in \mathbb{R}^{p\times\hat{n}}$. Since $\hat{\Sigma}$ should approximate Σ in some sense, we want $\hat{y} \approx y$ for all admissible inputs $u \in L^2[0,\infty[$. Moreover, in order to ensure a significant speed-up in numerical simulations, we demand $\hat{n} \ll n$. There are different ways of achieving this goal. Similar to concepts used in the context of linear systems, there exist SVD-based approaches leading to a reasonable generalization of the method of balanced truncation; see [6, 31]. While these methods have been proven to perform very well, they require the solution of two generalized Lyapunov equations which cause serious memory problems already for medium-sized systems. More precisely, while for the case of linear systems there exist methods that make use of the eigenvalue decomposition of the system matrix A and, consequently, lead to algorithms that find an explicit solution in $\mathcal{O}(n^3)$, this is no longer possible for the bilinear case. This is due to the fact that one usually cannot find a simultaneous diagonalization of the matrices A and $N_1, \dots, N_m$, which would be necessary in order to generalize the ideas of, e.g., the Bartels–Stewart algorithm; see [5]. On the other hand, several interpolation-based ideas have evolved that try to approximate generalized transfer functions by projecting the original model on appropriate Krylov subspaces; see [4, 3, 9, 13, 15, 25, 26]. Despite the fact that a memory efficient implementation is possible, the worse approximation quality compared to the method of balanced truncation make these approaches unfavorable. Moreover, while the choice of optimal interpolation points with respect to a certain norm has been solved for the linear case (see [12, 19]) this is still an open question for bilinear system theory. The goal of this paper now is to reveal an appropriate generalized interpolation framework for bilinear systems that allows us to propose two different iterative algorithms that aim at finding a local $\mathcal{H}_2$-minimum of the so-called error system. For the first one, we will have to study certain generalized Sylvester equations. The second approach extends the iterative rational Krylov algorithm (IRKA/MIRIAm) (see [12, 19]) to the bilinear case. We will now proceed as follows. In the subsequent section, we will give a brief review on optimal $\mathcal{H}_2$-model reduction for linear systems. This will include a recapitulation of first order necessary conditions as well as a discussion of the solution provided by IRKA. In section 3, we will focus on the $\mathcal{H}_2$-norm for bilinear systems, initially introduced in [31]. Here, we present an alternative computation of the norm of the error system which, in section 4, will enable us to derive first order necessary conditions that extend the ones known from the linear case. Finally, we will study several numerical examples which will underscore the superiority of the methods proposed in section 5 and conclude with a short summary.

2. $\mathcal{H}_2$-optimal model reduction for linear systems. Since later on we will extend the concepts from linear $\mathcal{H}_2$-model reduction, we briefly review the existing theory for linear continuous time-invariant systems, i.e.,

$$\Sigma_\ell : \begin{cases} \dot{x}(t) = A_\ell x(t) + B_\ell u(t), \\ y(t) = C_\ell x(t), \quad x(0) = 0, \end{cases} \tag{2.1}$$

with dimensions as defined in (1.1) and transfer function $H_\ell(s) = C_\ell \left(sI_n - A_\ell\right)^{-1} B_\ell$. So far, we did not further specify criteria which allow us to measure the quality of a reduced-order system. Here, we deal with the problem of finding a reduced-order model which approximates the original system as accurately as possible with respect to the $\mathcal{H}_2$-norm. Recall that for linear systems, this norm is defined as

$$||\Sigma_\ell||_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr}\left(H_\ell(-i\omega) H_\ell^T(i\omega)\right) d\omega \right)^{\frac{1}{2}},$$

where tr denotes the trace of a matrix. As is well known, there exist two alternative computations for this norm. The first relies on the solution of the Lyapunov equations corresponding to the system, i.e.,

$$A_\ell P_\ell + P_\ell A_\ell^T + B_\ell B_\ell^T = 0, \quad A_\ell^T Q_\ell + Q_\ell A_\ell + C_\ell^T C_\ell = 0.$$

It can be shown that it holds

$$||\Sigma_\ell||_{\mathcal{H}_2}^2 = \operatorname{tr}\left(C_\ell P_\ell C_\ell^T\right) = \operatorname{tr}\left(B_\ell^T Q_\ell B_\ell\right).$$

Rather recently, in [2], under the assumption that H_ℓ has only simple poles, Antoulas provided a new derivation based on the poles and residues of the transfer function:

$$||\Sigma_\ell||_{\mathcal{H}_2}^2 = \sum_{k=1}^{n} \operatorname{tr}\left(\operatorname{res}\left[H_\ell(-s) H_\ell^T(s), \lambda_k\right]\right),$$

where λ_k denotes the eigenvalues of the system matrix A_ℓ and

$$\operatorname{res}\left[H_\ell(-s) H_\ell^T(s), \lambda_k\right] = \lim_{s \to \lambda_k} H_\ell(-s) H_\ell^T(s)(s - \lambda_k).$$

Based on these expressions, it is possible to derive first order necessary conditions for $\mathcal{H}_2$-optimality, i.e., for locally minimizing the norm of the error system $||\Sigma_\ell - \hat{\Sigma}_\ell||_{\mathcal{H}_2}$; see, e.g., [19, 21, 30]. On the one hand, the Lyapunov-based norm computation leads to the Wilson conditions

$$P_{12}^T Q_{12} + P_{22} Q_{22} = 0, \quad Q_{12}^T B + Q_{22}\hat{B} = 0, \quad \hat{C} P_{22} - C P_{12} = 0, \tag{2.2}$$

where

$$P^{err} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad Q^{err} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix},$$

are the solutions of the Lyapunov equations of the error system

$$A^{err} = \begin{bmatrix} A_\ell & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B_\ell \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C_\ell & -\hat{C} \end{bmatrix}.$$

Equivalently, it is possible to characterize the optimality via interpolation-based conditions. Initially derived in [21] and picked up again in [19, 11, 29], the reduced systems' transfer function has to tangentially interpolate the transfer function of the original system at the mirror images of its own poles i.e., for $1 \leq k \leq \hat{n}$,

$$\tilde{C}_k^T \hat{H}(-\hat{\lambda}_k) = \tilde{C}_k^T H(-\hat{\lambda}_k), \tag{2.3}$$

$$\hat{H}(-\hat{\lambda}_k)\tilde{B}_k = H(-\hat{\lambda}_k)\tilde{B}_k, \tag{2.4}$$

$$\tilde{C}_k^T \hat{H}'(-\hat{\lambda}_k)\tilde{B}_k = \tilde{C}_k^T H'(-\hat{\lambda}_k)\tilde{B}_k, \tag{2.5}$$

where $R\Lambda R^{-1} = \hat{A}$ is the spectral decomposition of $\hat{A}$ with $\Lambda = \operatorname{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_{\hat{n}})$, $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$, and the subscript k denotes the kth column of a matrix. For later purposes, it is important to note that there is another way of writing down the above conditions. For this, we will make use of the Kronecker product notation and some simple properties of the vect () operator:

$$\operatorname{vect}(I_n) = \mathcal{I}_n,\ \operatorname{tr}\left(X^T Y\right) = \operatorname{vect}(X)^T \operatorname{vect}(Y),\ \operatorname{vect}(XYZ) = \left(Z^T \otimes X\right)\operatorname{vect}(Y). \tag{2.6}$$

Note that the right-hand side of (2.3) consists of m columns. Considering now the jth of those, we obtain

$$\begin{aligned}
&\tilde{C}_k^T C_\ell \left(-\hat{\lambda}_k I_n - A_\ell\right)^{-1} B_j \\
&= \begin{pmatrix} \tilde{C}_1^T C_\ell & \ldots & \tilde{C}_{\hat{n}}^T C_\ell \end{pmatrix} \begin{bmatrix} -\hat{\lambda}_1 I_n - A_\ell & & \\ & \ddots & \\ & & -\hat{\lambda}_{\hat{n}} I_n - A_\ell \end{bmatrix}^{-1} (e_k \otimes B_j) \\
&= \operatorname{vect}\left(C_\ell^T \tilde{C}\right)^T \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A_\ell\right)^{-1} \left(e_k e_j^T \otimes B_\ell\right) \mathcal{I}_m \\
&= \mathcal{I}_p^T \left(\tilde{C} \otimes C_\ell\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A_\ell\right)^{-1} \left(e_k e_j^T \otimes B_\ell\right) \mathcal{I}_m.
\end{aligned}$$

Hence, condition (2.3) is the same as requiring

$$\begin{aligned}
&\mathcal{I}_p^T \left(\tilde{C} \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(e_k e_j^T \otimes \hat{B}\right) \mathcal{I}_m \\
&\quad = \mathcal{I}_p^T \left(\tilde{C} \otimes C_\ell\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A_\ell\right)^{-1} \left(e_k e_j^T \otimes B_\ell\right) \mathcal{I}_m,
\end{aligned} \tag{2.7}$$

for $k = 1, \ldots, \hat{n}$ and $j = 1, \ldots, m$. Similarly, we can derive conditions equivalent to (2.4) and (2.5):

$$\begin{aligned}
&\mathcal{I}_p^T \left(e_j e_k^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m \\
&\quad = \mathcal{I}_p^T \left(e_j e_k^T \otimes C_\ell\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A_\ell\right)^{-1} \left(\tilde{B}^T \otimes B_\ell\right) \mathcal{I}_m,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\mathcal{I}_p^T \left(\tilde{C} \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(e_k e_k^T \otimes I_{\hat{n}}\right) \\
&\qquad \times \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m \\
&\quad = \mathcal{I}_p^T \left(\tilde{C} \otimes C_\ell\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A_\ell\right)^{-1} \left(e_k e_k^T \otimes I_n\right) \\
&\qquad \times \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A_\ell\right)^{-1} \left(\tilde{B}^T \otimes B_\ell\right) \mathcal{I}_m.
\end{aligned} \tag{2.9}$$

Based on these conditions, in [19, 11, 29] the authors have proposed iterative rational Krylov algorithms (IRKA/MIRIAm) which, upon convergence, yield a locally $\mathcal{H}_2$-optimal reduced system. Here, the crucial observation is that if we construct the reduced system by the Petrov–Galerkin projection $\mathcal{P} = VW^T$, i.e.,

$$\hat{A} = W^T A_\ell V, \quad \hat{B} = W^T B_\ell, \quad \hat{C} = C_\ell V,$$

with $V = \begin{bmatrix} V_1 & \dots & V_{\hat{n}} \end{bmatrix}$ and $W = \begin{bmatrix} W_1 & \dots & W_{\hat{n}} \end{bmatrix}$ given as

$$V_i = (\sigma_i I_n - A_\ell)^{-1} B_\ell \tilde{B}_i, \tag{2.10}$$

$$W_i = \left(\sigma_i I_n - A_\ell^T\right)^{-1} C_\ell^T \tilde{C}_i, \tag{2.11}$$

we can guarantee that the transfer function of $\hat{\Sigma}_\ell$ tangentially interpolates the values and first derivatives of the original systems' transfer function at the points σ_i. Again, for later purposes it will be important to note that (2.10) and (2.11) can be rewritten by using a vectorized notation:

$$\operatorname{vect}(V) = (\operatorname{diag}(\sigma_1, \dots, \sigma_{\hat{n}}) \otimes I_n - I_{\hat{n}} \otimes A_\ell)^{-1} (\tilde{B}^T \otimes B_\ell)\mathcal{I}_m, \tag{2.12}$$

$$\operatorname{vect}(W) = \left(\operatorname{diag}(\sigma_1, \dots, \sigma_{\hat{n}}) \otimes I_n - I_{\hat{n}} \otimes A_\ell^T\right)^{-1} (\tilde{C}^T \otimes C_\ell^T)\mathcal{I}_p. \tag{2.13}$$

3. $\mathcal{H}_2$-norm for bilinear systems. In this section, we will review a possible generalization of the $\mathcal{H}_2$-norm for bilinear systems introduced in [31].

DEFINITION 3.1. *We define the $\mathcal{H}_2$-norm for bilinear systems as*

$$||\Sigma||_{\mathcal{H}_2}^2 = \operatorname{tr}\left(\sum_{k=1}^{\infty} \int_0^\infty \dots \int_0^\infty \sum_{\ell_1,\dots,\ell_k=1}^{m} g_k^{(\ell_1,\dots,\ell_k)} (g_k^{(\ell_1,\dots,\ell_k)})^T ds_1 \dots ds_k \right),$$

with $g_k^{(\ell_1,\dots,\ell_k)}(s_1,\dots,s_k) = Ce^{As_k} N_{\ell_1} e^{As_{k-1}} N_{\ell_2} \cdots e^{As_1} b_{\ell_k}$.

It has been shown that the above definition makes sense in the case of the existence of certain generalized observability and reachability Gramians associated with bilinear systems. These, in turn, satisfy the generalized Lyapunov equations

$$AP + PA^T + \sum_{k=1}^{m} N_k P N_k^T + BB^T = 0, \tag{3.1}$$

$$A^T Q + QA^T + \sum_{k=1}^{m} N_k^T Q N_k + C^T C = 0, \tag{3.2}$$

and can be computed via the limit of an infinite series of linear Lyapunov equations. Basically, these assumptions are closely related to the notion of stability of Σ. For a more detailed insight, we refer to [31]. Hence, in the following we will always assume that the original system Σ is stable, meaning that the eigenvalues of the system matrix A lie in the open left complex plane and, moreover, the matrices N_k are sufficiently bounded. More precisely, we state the following result on bounded-input-bounded-output (BIBO) stability of bilinear systems, initially obtained in [28].

THEOREM 3.2. *Let a bilinear system Σ be given and assume that A is asymptotically stable, i.e., there exist real scalars $\beta > 0$ and $0 < \alpha \leq -\max_i(\operatorname{Re}(\lambda_i(A)))$, such that*

$$||e^{At}|| \leq \beta e^{-\alpha t}, \quad t \geq 0.$$

Further assume that $||u(t)|| = \sqrt{\sum_{k=1}^m |u_k(t)|^2} \leq M$ *uniformly on* $[0,\infty[$ *with* $M > 0$, *and denote* $\Gamma = \sum_{k=1}^m ||N_k||$. *Then* Σ *is BIBO stable, i.e., the corresponding Volterra series of the solution* $x(t)$ *uniformly converges on the interval* $[0,\infty[$ *if* $\Gamma < \frac{\alpha}{M\beta}$.

Our stability assumption is motivated by the explicit solution formulas for (3.1) and (3.2) and the demand of having positive semidefinite solutions P and Q, respectively:

$$\operatorname{vect}(P) = -\left(A \otimes I_n + I_n \otimes A + \sum_{k=1}^m N_k \otimes N_k\right)^{-1} \operatorname{vect}\left(BB^T\right), \tag{3.3}$$

$$\operatorname{vect}(Q) = -\left(A^T \otimes I_n + I_n \otimes A^T + \sum_{k=1}^m N_k^T \otimes N_k^T\right)^{-1} \operatorname{vect}\left(C^T C\right). \tag{3.4}$$

Similarly to the linear case, the $\mathcal{H}_2$-norm now can be computed with the help of the solutions P and Q; see [31].

PROPOSITION 3.3. *Let* Σ *be a bilinear system. Assume that* A *is asymptotically stable and the reachability Gramian* P *and the observability Gramian* Q *exist. Then it holds that*

$$||\Sigma||_{\mathcal{H}_2}^2 = \operatorname{tr}\left(CPC^T\right) = \operatorname{tr}\left(B^T Q B\right).$$

Since in the subsequent section we want to derive first order necessary conditions for $\mathcal{H}_2$-optimality that extend the interpolation conditions (2.3), (2.4), and (2.5) for linear systems, we propose the following alternative derivation.

THEOREM 3.4. *Let* Σ *be a stable bilinear system. Then it holds that*

$$||\Sigma||_{\mathcal{H}_2}^2 = \mathcal{I}_p^T (C \otimes C) \left(-A \otimes I_n - I_n \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} (B \otimes B)\, \mathcal{I}_m.$$

Proof. For the proof, recall the properties from (2.6), together with the results from Proposition 3.3 and the solution formulas (3.3) and (3.4), respectively,

$$\begin{aligned}
||\Sigma||_{\mathcal{H}_2}^2 &= \operatorname{tr}\left(CPC^T\right) = \operatorname{vect}\left(C^T\right)^T \operatorname{vect}\left(PC^T\right) = \operatorname{vect}\left(C^T\right)^T (C \otimes I) \operatorname{vect}(P) \\
&= \operatorname{vect}\left(C^T\right)^T (C \otimes I) \left(-A \otimes I - I \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} \operatorname{vect}\left(BB^T\right) \\
&= \left(\left(C^T \otimes I\right) \operatorname{vect}\left(C^T\right)\right)^T \left(-A \otimes I - I \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} (B \otimes B)\, \mathcal{I}_m \\
&= \left(\operatorname{vect}\left(C^T C\right)\right)^T \left(-A \otimes I - I \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} (B \otimes B)\, \mathcal{I}_m \\
&= \left(\operatorname{vect}\left(C^T I_p C\right)\right)^T \left(-A \otimes I - I \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} (B \otimes B)\, \mathcal{I}_m \\
&= \left((C^T \otimes C^T)\mathcal{I}_p\right)^T \left(-A \otimes I - I \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} (B \otimes B)\, \mathcal{I}_m \\
&= (\mathcal{I}_p)^T (C \otimes C) \left(-A \otimes I - I \otimes A - \sum_{k=1}^m N_k \otimes N_k\right)^{-1} (B \otimes B)\, \mathcal{I}_m. \qquad \square
\end{aligned}$$

4. $\mathcal{H}_2$-optimality conditions for bilinear systems. Next, we want to discuss necessary conditions for $\mathcal{H}_2$-optimality. As in the linear case, for this we have to consider the norm of the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$, which is defined as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad N_k^{err} = \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.$$

Based on the assertions from Proposition 3.3 in [31], it is shown that the reduced system matrices have to fulfill conditions that extend the Wilson conditions to the bilinear case:

$$\begin{aligned} Q_{12}^T P_{12} + Q_{22} P_{22} = 0, \quad Q_{22}\hat{N}_k P_{22} + Q_{12}^T N_k P_{12} = 0, \\ Q_{12}^T B + Q_{22}\hat{B} = 0, \quad \hat{C}P_{22} - CP_{12} = 0, \end{aligned} \tag{4.1}$$

where

$$P^{err} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad Q^{err} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \tag{4.2}$$

are the solutions of the generalized Lyapunov equations

$$A^{err}P^{err} + P^{err}(A^{err})^T + \sum_{k=1}^m N_k^{err} P^{err} (N_k^{err})^T + B^{err}(B^{err})^T = 0, \tag{4.3}$$

$$(A^{err})^T Q^{err} + Q^{err}A^{err} + \sum_{k=1}^m (N_k^{err})^T Q^{err} N_k^{err} + (C^{err})^T C^{err} = 0. \tag{4.4}$$

Since we are heading for a generalization of the iterative rational Krylov algorithm, next we want to derive necessary conditions based on the computation formula from Theorem 3.4. A simple analysis of the structure of the error system leads to the following expression for the error functional E.

Corollary 4.1. *Let Σ and $\hat{\Sigma}$ be the original and reduced bilinear systems, respectively. Then*

$$\begin{aligned} E^2 &:= \|\Sigma^{err}\|_{\mathcal{H}_2}^2 := \|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2}^2 = (\mathcal{I}_p)^T \left(\begin{bmatrix} C & -\tilde{C} \end{bmatrix} \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix} \right) \\ &\quad \times \left(- \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} - I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} - \sum_{k=1}^m \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix} \right)^{-1} \\ &\quad \times \left(\begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \right) \mathcal{I}_m, \\ &= (\mathcal{I}_p)^T \left(\begin{bmatrix} C & -\tilde{C} \end{bmatrix} \otimes \begin{bmatrix} C & -\tilde{C} \end{bmatrix} \right) \\ &\quad \times \left(- \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} - I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} - \sum_{k=1}^m \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \right)^{-1} \\ &\quad \times \left(\begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \otimes \begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \right) \mathcal{I}_m, \end{aligned}$$

where $R\Lambda R^{-1} = \hat{A}$ is the spectral decomposition of $\hat{A}$ and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$, $\tilde{N}_k = R^T \hat{N}_k^T R^{-T}$.

The above representation is motivated by the demand of having optimization parameters Λ, $\tilde{N}_k$, $\tilde{B}$, and $\tilde{C}$ that can be chosen to minimize $||\Sigma - \hat{\Sigma}||^2_{\mathcal{H}_2}$, at least locally. Before we proceed, let us introduce a specific permutation matrix

$$M = \begin{bmatrix} I_{\hat{n}} \otimes \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix} & I_{\hat{n}} \otimes \begin{bmatrix} \mathbf{0}^T \\ I_{\hat{n}} \end{bmatrix} \end{bmatrix},$$

which will simplify the computation of Kronecker products for certain block matrices. For this, consider one of the block structures appearing in Corollary 4.1 for which we can show that

$$\begin{aligned}
&M^T \left(\tilde{N}_k^T \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix} \right) M \\
&= \begin{bmatrix} I_{\hat{n}} \otimes \begin{bmatrix} I_n & \mathbf{0}^T \end{bmatrix} & I_{\hat{n}} \otimes \begin{bmatrix} \mathbf{0} & I_{\hat{n}} \end{bmatrix} \end{bmatrix} \left(\tilde{N}_k^T \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix} \right) \begin{bmatrix} I_{\hat{n}} \otimes \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix} & I_{\hat{n}} \otimes \begin{bmatrix} \mathbf{0}^T \\ I_{\hat{n}} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} I_{\hat{n}} \otimes \begin{bmatrix} I_n & \mathbf{0}^T \end{bmatrix} & I_{\hat{n}} \otimes \begin{bmatrix} \mathbf{0} & I_{\hat{n}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \tilde{N}_k^T \otimes \begin{bmatrix} N_k \\ \mathbf{0} \end{bmatrix} & \tilde{N}_k^T \otimes \begin{bmatrix} \mathbf{0}^T \\ \hat{N}_k \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{N}_k^T \otimes N_k & 0 \\ 0 & \tilde{N}_k^T \otimes \hat{N}_k \end{bmatrix}.
\end{aligned}$$

For the differentiation with respect to the optimization parameters, we make use of the product rule for Kronecker products (see Lemma A.1 in Appendix A):

$$\begin{aligned}
\frac{\partial E^2}{\partial \tilde{C}_{ij}} &= 2 \cdot (\mathcal{I}_p)^T \left(\begin{bmatrix} 0 & -e_i e_j^T \end{bmatrix} \otimes \begin{bmatrix} C & -\tilde{C} \end{bmatrix} \right) \\
&\quad \times \left(- \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} - I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} - \sum_{k=1}^m \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \right)^{-1} \\
&\quad \times \left(\begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \otimes \begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \right) \mathcal{I}_m \\
&= 2 \cdot (\mathcal{I}_p)^T \left(\begin{bmatrix} 0 & -e_i e_j^T \end{bmatrix} \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix} \right) \\
&\quad \times \left(- \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} - I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} - \sum_{k=1}^m \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix} \right)^{-1} \\
&\quad \times \left(\begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \right) \mathcal{I}_m \\
&= 2 \cdot (\mathcal{I}_p)^T \left(-e_i e_j^T \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix} \right) \left(-\Lambda \otimes \begin{bmatrix} I_n & 0 \\ 0 & I_{\hat{n}} \end{bmatrix} - I_{\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \right. \\
&\quad \left. \times - \sum_{k=1}^m \tilde{N}_k^T \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix} \right)^{-1} \left(\tilde{B}^T \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \right) \mathcal{I}_m \\
&= 2 \cdot (\mathcal{I}_p)^T \left(-e_i e_j^T \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix} \right) \left(M \left(- \begin{bmatrix} \Lambda \otimes I_n & 0 \\ 0 & \Lambda \otimes I_{\hat{n}} \end{bmatrix} \right. \right. \\
&\quad \left. \left. - \begin{bmatrix} I_{\hat{n}} \otimes A & 0 \\ 0 & I_{\hat{n}} \otimes \hat{A} \end{bmatrix} - \sum_{k=1}^m \begin{bmatrix} \tilde{N}_k^T \otimes N_k & 0 \\ 0 & \tilde{N}_k^T \otimes \hat{N}_k \end{bmatrix} \right) M^T \right)^{-1} \left(\tilde{B}^T \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \right) \mathcal{I}_m
\end{aligned}$$

$$= -2 \cdot (\mathcal{I}_p)^T \left(e_i e_j^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m$$
$$+ 2 \cdot (\mathcal{I}_p)^T \left(e_i e_j^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m.$$

Here, the last step is justified by the fact that M is a permutation matrix and, thus, $M^T M = I$ and the identities:

$$\left(-e_i e_j^T \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix}\right) M = \begin{bmatrix} -e_i e_j^T \otimes C & e_i e_j^T \otimes \hat{C} \end{bmatrix}, \; M^T \left(\tilde{B} \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix}\right) = \begin{bmatrix} \tilde{B} \otimes B \\ \tilde{B} \otimes \hat{B} \end{bmatrix}.$$

Setting the resulting expression equal to zero reveals that $\hat{\Sigma}$ has to satisfy

$$\begin{aligned} &(\mathcal{I}_p)^T \left(e_i e_j^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m \\ &= (\mathcal{I}_p)^T \left(e_i e_j^T \otimes \hat{C}\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m. \end{aligned} \tag{4.5}$$

In view of (2.4) in the form of (2.8), we see that this demand naturally extends the interpolation-based condition known from the linear case. For the differentiation with respect to the poles of $\hat{A}$, we use the second part of the Lemma A.1 in order to obtain

$$\begin{aligned} \frac{\partial E^2}{\partial \lambda_i} &= 2 \cdot \mathcal{I}_p^T \left(\begin{bmatrix} C - \tilde{C} \end{bmatrix} \otimes \begin{bmatrix} C & -\tilde{C} \end{bmatrix}\right) \\ &\times \left(\begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} + I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} + \sum_{k=1}^{m} \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix}\right)^{-1} \\ &\times \left(\begin{bmatrix} 0 & 0 \\ 0 & e_i e_i^T \end{bmatrix} \otimes I_{n+\hat{n}}\right) \\ &\times \left(\begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} + I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} + \sum_{k=1}^{m} \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix}\right)^{-1} \\ &\times \left(\begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \otimes \begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix}\right) \mathcal{I}_m \\ &= 2 \cdot \mathcal{I}_p^T \left(\begin{bmatrix} C - \tilde{C} \end{bmatrix} \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix}\right) \\ &\times \left(\begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} + I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \sum_{k=1}^{m} \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix}\right)^{-1} \\ &\times \left(\begin{bmatrix} 0 & 0 \\ 0 & e_i e_i^T \end{bmatrix} \otimes I_{n+\hat{n}}\right) \\ &\times \left(\begin{bmatrix} A & 0 \\ 0 & \Lambda \end{bmatrix} \otimes I_{n+\hat{n}} + I_{n+\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \sum_{k=1}^{m} \begin{bmatrix} N_k & 0 \\ 0 & \tilde{N}_k^T \end{bmatrix} \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix}\right)^{-1} \\ &\times \left(\begin{bmatrix} B \\ \tilde{B}^T \end{bmatrix} \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix}\right) \mathcal{I}_m \end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot \mathcal{I}_p^T \left(-\tilde{C} \otimes \begin{bmatrix} C & -\hat{C} \end{bmatrix}\right) \\
&\quad \times \left(\Lambda \otimes I_{n+\hat{n}} + I_{\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \sum_{k=1}^{m} \tilde{N}^T \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix}\right)^{-1} \left(e_i e_i^T \otimes I_{n+\hat{n}}\right) \\
&\quad \times \left(\Lambda \otimes I_{n+\hat{n}} + I_{\hat{n}} \otimes \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \sum_{k=1}^{m} \tilde{N}_k^T \otimes \begin{bmatrix} N_k & 0 \\ 0 & \hat{N}_k \end{bmatrix}\right)^{-1} \left(\tilde{B}^T \otimes \begin{bmatrix} B \\ \hat{B} \end{bmatrix}\right) \mathcal{I}_m \\
&= -2 \cdot \mathcal{I}_p^T \left(\tilde{C} \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \\
&\quad \times \left(e_i e_i^T \otimes I_n\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m \\
&\quad + 2 \cdot \mathcal{I}_p^T \left(\tilde{C} \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \\
&\quad \times \left(e_i e_i^T \otimes I_{\hat{n}}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m.
\end{aligned}
$$

Once more, we find an interpolation-based condition generalizing (2.5) in the form of (2.9) if we set the last expression equal to zero:

$$
\begin{aligned}
&(\mathcal{I}_p)^T \left(\tilde{C} \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \\
&\qquad \times \left(e_i e_i^T \otimes I_n\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m \\
&\quad = (\mathcal{I}_p)^T \left(\tilde{C} \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \\
&\qquad \times \left(e_i e_i^T \otimes I_{\hat{n}}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m.
\end{aligned} \tag{4.6}
$$

Finally, as a matter of careful analysis, we obtain similar optimality conditions when differentiating with respect to $\tilde{B}$ and $\tilde{N}_k$, respectively,

$$
\begin{aligned}
&\mathcal{I}_p^T \left(\tilde{C} \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \left(e_j e_i^T \otimes B\right) \mathcal{I}_m \\
&\quad = \mathcal{I}_p^T \left(\tilde{C} \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^{m} \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \left(e_j e_i^T \otimes \hat{B}\right) \mathcal{I}_m,
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
&\mathcal{I}_p^T \left(\tilde{C} \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \\
&\qquad \times \left(e_j e_i^T \otimes N\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^{m} \tilde{N}_k^T \otimes N_k\right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m
\end{aligned}
$$

$$
\begin{aligned}
(4.8)\qquad &= \mathcal{I}_p^T \left(\tilde{C} \otimes \hat{C}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^m \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \\
&\quad \times \left(e_j e_i^T \otimes \hat{N}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} - \sum_{k=1}^m \tilde{N}_k^T \otimes \hat{N}_k\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}\right) \mathcal{I}_m.
\end{aligned}
$$

Hence, the previous derivations can be summarized in the following theorem.

THEOREM 4.2. *Let Σ denote a BIBO stable bilinear system. Assume that $\hat{\Sigma}$ is a reduced bilinear system of dimension $\hat{n}$, minimizing the $\mathcal{H}_2$-norm of the error system among all bilinear systems of dimension $\hat{n}$. Then $\hat{\Sigma}$ fulfills* (4.5)–(4.8).

Remark 4.1. At this point one might wonder why it makes sense to denote the above conditions as being of interpolatory nature. Note that if the inverse

$$
\left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k\right)^{-1}
$$

exists, we can rewrite it by means of the Neumann Lemma as an infinite series of the form

$$
\sum_{i=0}^{\infty} \left(\left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A\right)^{-1} \left(\sum_{k=1}^m \tilde{N}_k^T \otimes N_k\right)\right)^i \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A\right)^{-1}.
$$

Now each term of this series in a way corresponds to the Volterra series representation arising for bilinear control systems. More precisely, as has been discussed in [17], one can show that the above conditions mean that the Volterra series representation evaluated at the mirror images of the poles of the reduced system has to coincide with that of the original system. In other words, conditions (4.5)–(4.8) describe interpolation conditions for the Volterra series representation of a bilinear control system.

5. Generalized Sylvester equations and bilinear IRKA. Now that we have specified first order necessary conditions for $\mathcal{H}_2$-optimality, in this section we will propose two algorithms that iteratively construct a reduced-order system which locally minimizes the $\mathcal{H}_2$-error. We will start with a procedure based on certain generalized Sylvester equations which in the linear case reduces to the concept discussed in [29]. For this, let us consider the following two matrix equations:

$$
AX + X\hat{A}^T + \sum_{k=1}^m N_k X \hat{N}_k^T + B\hat{B}^T = 0, \tag{5.1}
$$

$$
A^T Y + Y\hat{A} + \sum_{k=1}^m N_k^T Y \hat{N}_k - C^T \hat{C} = 0. \tag{5.2}
$$

Obviously, the solutions $X, Y \in \mathbb{R}^{n \times \hat{n}}$ can be explicitly computed by vectorizing both sides and making use of the vect ()-operator. However, this requires solving two linear systems of equations:

$$
\left(-I_{\hat{n}} \otimes A - \hat{A} \otimes I_n - \sum_{k=1}^m \hat{N}_k \otimes N_k\right) \operatorname{vect}(X) = \operatorname{vect}\left(B\hat{B}^T\right),
$$

$$
\left(I_{\hat{n}} \otimes A^T + \hat{A}^T \otimes I_n + \sum_{k=1}^m \hat{N}_k^T \otimes N_k^T\right) \operatorname{vect}(Y) = \operatorname{vect}\left(C^T \hat{C}\right).
$$

Throughout the rest of this paper, we will assume that there exist unique solutions satisfying these Sylvester equations. Due to the properties of the eigenvalue computation of Kronecker products, this certainly is satisfied if the eigenvalues of $\hat{A}$ are located in $\mathbb{C}_-$ and the norms of $\hat{N}_k$ are sufficiently small. However, in view of Theorem 3.2 we have already mentioned that this basically characterizes a stable bilinear system. Although, in general, this cannot be ensured by our proposed algorithms, we did not observe unstable reduced-order systems so far. For a similar discussion for the linear case, we refer to [19]. For the sake of completeness, we mention that under appropriate assumptions X and Y can be computed as the limit of an infinite series of linear Sylvester equations.

LEMMA 5.1. *Let* $\mathcal{L}, \Pi : \mathbb{R}^{n\times\hat{n}} \to \mathbb{R}^{n\times\hat{n}}$ *denote two linear operators defined by the bilinear systems* Σ *and* $\hat{\Sigma}$, *with* $\mathcal{L}(X) := AX + X\hat{A}^T$ *and* $\Pi(X) := \sum_{k=1}^m N_k X \hat{N}_k^T$. *If the spectral radius* $\rho(\mathcal{L}^{-1}\Pi) < 1$, *then the solution* X *of the generalized Sylvester equation* (5.1) *is given as* $X = \lim_{j\to\infty} X_j$, *with*

$$AX_1 + X_1\hat{A}^T + B\hat{B}^T = 0, \quad AX_j + X_j\hat{A}^T + \sum_{k=1}^m N_k X_{j-1}\hat{N}_k^T + B\hat{B}^T = 0, \quad j > 1.$$

A dual statement is obviously true for (5.2). Since the statement is a direct consequence of the theory of convergent splittings, we dispense with the proof and instead refer to [14] for an equivalent discussion on bilinear Lyapunov equations.

Remark 5.1. Although the aforementioned splitting at least theoretically yields a possible way of solving the generalized Sylvester equation (5.1), the procedure strongly depends on the size of the spectral radius $\rho(\mathcal{L}^{-1}\Pi)$. Moreover, so far it seems hard to state properties of a bilinear control system that automatically ensure the desired convergence.

Let us now focus on Algorithm 1, which in each step constructs a reduced system $\hat{\Sigma}$ by a Petrov–Galerkin type projection $\mathcal{P} = V(W^TV)^{-1}W^T$, determined by the solutions of the generalized Sylvester equations associated with the preceding system matrices.

Algorithm 1. *Generalized Sylvester iteration*

Input: A, N_k, B, C, $\hat{A}$, $\hat{N}_k$, $\hat{B}$, $\hat{C}$
Output: $\hat{A}^{opt}$, $\hat{N}_k^{opt}$, $\hat{B}^{opt}$, $\hat{C}^{opt}$

1: **while** (not converged) **do**
2: Solve $AX + X\hat{A}^T + \sum_{k=1}^m N_k X\hat{N}_k^T + B\hat{B}^T = 0$.
3: Solve $A^TY + Y\hat{A} + \sum_{k=1}^m N_k^T Y\hat{N}_k - C^T\hat{C} = 0$.
4: $V = \mathrm{orth}(X)$, $W = \mathrm{orth}(Y)$
5: $\hat{A} = \left(W^TV\right)^{-1} W^TAV$, $\hat{N}_k = \left(W^TV\right)^{-1} W^TN_kV$, $\hat{B} = \left(W^TV\right)^{-1} W^TB$, $\hat{C} = CV$
6: **end while**
7: $\hat{A}^{opt} = \hat{A}$, $\hat{N}_k^{opt} = \hat{N}_k$, $\hat{B}^{opt} = \hat{B}$, $\hat{C}^{opt} = \hat{C}$

Remark 5.2. Note that the two main steps of Algorithm 1 consist of finding solutions to generalized Sylvester equation of the form

$$AX + X\hat{A}^T + \sum_{k=1}^m N_k X\hat{N}_k^T + B\hat{B}^T = 0,$$

determined by the large size matrix A from the original system and the small size matrix $\hat{A}$ from the reduced system. Similar to the generalized Lyapunov equations arising for bilinear control systems, solving a matrix equation of this type might still pose a severe challenge. However, one might think of considering the explicit system of linear equations, given by the Kronecker formulation

$$\left(-I_{\hat{n}} \otimes A - \hat{A} \otimes I_n - \sum_{k=1}^{m} \hat{N}_k \otimes N_k\right) \operatorname{vect}(X) = -\operatorname{vect}\left(BB^T\right),$$

which can be solved by means of an iterative Krylov subspace based solver. As a preconditioning technique, one naturally might think of the corresponding simplified Sylvester equation appearing in the linear case (i.e., $N_k = 0$) which can be efficiently solved by means of a Schur decomposition; see [7].

Finally, we are ready to prove one of our two main results.

THEOREM 5.2. *Assume Algorithm* 1 *converges. Then,* $\hat{A}^{opt}$, $\hat{N}_k^{opt}$, $\hat{B}^{opt}$, $\hat{C}^{opt}$ *fulfill the necessary* $\mathcal{H}_2$*-optimality conditions* (4.1).

Proof. Let $\bar{A}$, $\bar{N}_k$, $\bar{B}$, $\bar{C}$ denote the matrices corresponding to the next to last step in the while loop. Due to convergence, $\hat{\Sigma}^{opt}$ is a state space transformation of $\bar{\Sigma}$, i.e. $\exists T \in \mathbb{R}^{\hat{n}\times\hat{n}}$ nonsingular, such that

$$\bar{A} = T^{-1}\hat{A}^{opt}T, \quad \bar{N}_k = T^{-1}\hat{N}_k^{opt}T, \quad \bar{B} = T^{-1}\hat{B}^{opt}, \quad \bar{C} = \hat{C}^{opt}T.$$

Furthermore, according to step 4 of Algorithm 1, we have

$$V^{opt} = X^{opt}F, \quad W^{opt} = Y^{opt}G,$$

with $F, G \in \mathbb{R}^{\hat{n}\times\hat{n}}$ nonsingular. Thus, it holds that

$$\begin{aligned}\left((W^{opt})^T V^{opt}\right)^{-1}(W^{opt})^T &= \left(G^T (Y^{opt})^T X^{opt} F\right)^{-1} G^T (Y^{opt})^T \\ &= F^{-1}\left((Y^{opt})^T X^{opt}\right)^{-1}(Y^{opt})^T.\end{aligned}$$

From step 2, it follows

$$AX^{opt} + X^{opt}\bar{A}^T + \sum_{k=1}^{m} N_k X^{opt}\bar{N}_k^T + B\bar{B}^T = 0.$$

Hence,

$$\begin{aligned}&\underbrace{F^{-1}\left(Y^{opt^T}X^{opt}\right)^{-1}(Y^{opt})^T}_{((W^{opt})^T V^{opt})^{-1}(W^{opt})^T} A \underbrace{X^{opt}F}_{V^{opt}} \\ &+ F^{-1}\bar{A}^T F + \sum_{k=1}^{m} F^{-1}\left((Y^{opt})^T X^{opt}\right)^{-1}(Y^{opt})^T N_k X^{opt}\bar{N}_k^T F \\ &\quad + F^{-1}\left((Y^{opt})^T X^{opt}\right)^{-1}(Y^{opt})^T B\bar{B}^T F = 0,\end{aligned}$$

which implies

$$\begin{aligned}&\hat{A}^{opt} + F^{-1}T^T(\hat{A}^{opt})^T T^{-T}F + \sum_{k=1}^{m}\hat{N}_k^{opt}F^{-1}T^T(\hat{N}_k^{opt})^T T^{-T}F \\ &+ \hat{B}^{opt}(\hat{B}^{opt})^T T^{-T}F = 0.\end{aligned}$$

Finally, we end up with

$$\hat{A}^{opt}F^{-1}T^T + F^{-1}T^T(\hat{A}^{opt})^T + \sum_{k=1}^{m} \hat{N}_k^{opt}F^{-1}T^T(\hat{N}_k^{opt})^T + \hat{B}^{opt}(\hat{B}^{opt})^T = 0.$$

From the last line and the fact that we assumed the reduced system to be stable, the solution of the generalized Lyapunov equation is unique and we conclude that $P_{22} = F^{-1}T^T$, where P_{22} is the lower right block from the partitioning in (4.2). Similarly, we obtain

$$A^TY^{opt} + Y^{opt}\bar{A} + \sum_{k=1}^{m} N_k^TY^{opt}\bar{N}_k - C^T\bar{C} = 0.$$

This leads to

$$\begin{aligned}
&F^T(X^{opt})^TA^TY^{opt}((X^{opt})^TY^{opt})^{-1}F^{-T} + F^T(X^{opt})^TY^{opt}\bar{A}((X^{opt})^TY^{opt})^{-1}F^{-T}\\
&+\sum_{k=1}^{m} F^T(X^{opt})^TN_k^TY^{opt}\bar{N}_k((X^{opt})^TY^{opt})^{-1}F^{-T}\\
&-F^T(X^{opt})^TC^T\bar{C}((X^{opt})^TY^{opt})^{-1}F^{-T} = 0,
\end{aligned}$$

which can be transformed into

$$\begin{aligned}
&(\hat{A}^{opt})^T + F^T(X^{opt})^TY^{opt}T^{-T}\hat{A}^{opt}T((X^{opt})^TY^{opt})^{-1}F^{-T}\\
&\quad+\sum_{k=1}^{m} F^T(X^{opt})^TN^TY^{opt}((X^{opt})^TY^{opt})^{-1}\\
&\quad\quad\times F^{-T}F^T(X^{opt})^TY^{opt}\bar{N}_k((X^{opt})^TY^{opt})^{-1}F^{-T}\\
&\quad-F^T(X^{opt})^TC^T\hat{C}^{opt}T((X^{opt})^TY^{opt})^{-1}F^{-T} = 0.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&(\hat{A}^{opt})^T + F^T(X^{opt})^TY^{opt}T^{-T}\hat{A}^{opt}T((X^{opt})^TY^{opt})^{-1}F^{-T}\\
&+\sum_{k=1}^{m}(\hat{N}_k^{opt})^TF^TX^{opt^T}Y^{opt}T^{-1}\hat{N}_k^{opt}T((X^{opt})^TY^{opt})^{-1}F^{-T}\\
&-(\hat{C}^{opt})^T\hat{C}^{opt}T((X^{opt})^TY^{opt})^{-1}F^{-T} = 0,
\end{aligned}$$

and, subsequently,

$$\begin{aligned}
&-(\hat{A}^{opt})^TF^T(X^{opt})^TY^{opt}T^{-1} - F^T(X^{opt})^TY^{opt}T^{-1}(\hat{A}^{opt})^T\\
&\quad-\sum_{k=1}^{m}(\hat{N}_k^{opt})^TF^T(X^{opt})^TY^{opt}T^{-1}\hat{N}_k^{opt} + (\hat{C}^{opt})^T\hat{C}^{opt} = 0.
\end{aligned}$$

Again, the unique solution of the generalized Lyapunov equation of the reduced system satisfies $Q_{22} = -F^T(X^{opt})^TY^{opt}T^{-1}$, with Q_{22} as defined in (4.1). Moreover, due to symmetry of the solution, it follows that $Q_{22} = -T^{-T}(Y^{opt})^TX^{opt}F$. Finally, we will need the solutions of the generalized Sylvester equations arising in (4.3). However, it holds that

$$AX^{opt} + X^{opt}\bar{A}^T + \sum_{k=1}^{m} N_k X^{opt} \bar{N}_k^T + B\bar{B}^T = 0$$

is equivalent to

$$AX^{opt} + X^{opt}T^T(\hat{A}^{opt})^T T^{-T} + \sum_{k=1}^{m} N_k X^{opt} T^T (\hat{N}_k^{opt})^T T^{-T} + B(\hat{B}^{opt})^T T^{-T} = 0,$$

yielding

$$AX^{opt}T^T + X^{opt}T^T(\hat{A}^{opt})^T + \sum_{k=1}^{m} N_k X^{opt} T^T (\hat{N}_k^{opt})^T + B(\hat{B}^{opt})^T = 0.$$

Here, we make use of the unique solution of the generalized Sylvester equation. Thus, it follows that $P_{12} = X^{opt}T^T$. Since the argumentation for the dual Sylvester equation is completely analogous, we will skip the derivation that leads to $Q_{12} = Y^{opt}T^{-1}$. Let us now show the optimality conditions (4.1):

$$Q_{12}^T P_{12} + Q_{22}P_{22} = T^{-T}(Y^{opt})^T X^{opt} T^T - T^{-T}(Y^{opt})^T X^{opt} F F^{-1} T^T = 0,$$

$$\begin{aligned}
&Q_{22}\hat{N}_k^{opt}P_{22} + Q_{12}^T N_k P_{12}\\
&\quad = -T^{-T}(Y^{opt})^T X^{opt} F \hat{N}_k^{opt} F^{-1} T^T + T^{-T}(Y^{opt})^T N_k X^{opt} T^T\\
&\quad = -T^{-T}(Y^{opt})^T X^{opt} F \left((W^{opt})^T V^{opt}\right)^{-1} (W^{opt})^T N_k V^{opt} F^{-1} T^T\\
&\qquad + T^{-1}(Y^{opt})^T N_k X^{opt} T^T\\
&\quad = -T^{-T}(Y^{opt})^T X F F^{-1} \left((Y^{opt})^T X^{opt}\right)^{-1} (Y^{opt})^T N_k V^{opt} F F^{-1} T^T\\
&\qquad + T^{-T}(Y^{opt})^T N_k X^{opt} T^T = 0,
\end{aligned}$$

$$\begin{aligned}
&Q_{12}^T B + Q_{22}\hat{B}^{opt}\\
&\quad = T^{-T}(Y^{opt})^T B - T^{-T}(Y^{opt})^T X^{opt} F \hat{B}^{opt}\\
&\quad = T^{-T}(Y^{opt})^T B - T^{-T}(Y^{opt})^T X^{opt} F \left((W^{opt})^T V^{opt}\right)^{-1} (W^{opt})^T B\\
&\quad = T^{-T}(Y^{opt})^T B - T^{-T}(Y^{opt})^T X^{opt} F F^{-1} \left((Y^{opt})^T X^{opt}\right)^{-1} (Y^{opt})^T B = 0,
\end{aligned}$$

$$\begin{aligned}
\hat{C}^{opt}P_{22} - CP_{12} &= \hat{C}^{opt}F^{-1}T^T - CX^{opt}T^T = CV^{opt}F^{-1}T^T - CX^{opt}T^T\\
&= CX^{opt}FF^{-1}T^T - CX^{opt}T^T = 0. \qquad \square
\end{aligned}$$

Remark 5.3. Note that Algorithm 1 generalizes a Sylvester equation based algorithm for $\mathcal{H}_2$-optimality (see [18]) and thus does not require diagonalizability of $\hat{A}$.

We will now turn our attention to an interpolation-based approach that can be directly derived from Algorithm 1. For a similar derivation in the linear case, see, e.g., [18]. Again, let $\hat{A} = R\Lambda R^{-1}$ denote the eigenvalue decomposition of the reduced system. As already mentioned before, the explicit solution for (5.1) in vectorized form reads

$$
\begin{aligned}
\operatorname{vect}(X) &= \left(-I_{\hat{n}} \otimes A - \hat{A} \otimes I_n - \sum_{k=1}^{m} \hat{N}_k \otimes N_k\right)^{-1} \operatorname{vect}\left(B\hat{B}^T\right) \\
&= \left(-I_{\hat{n}} \otimes A - \hat{A} \otimes I_n - \sum_{k=1}^{m} \hat{N}_k \otimes N_k\right)^{-1} \left(\hat{B} \otimes B\right)\mathcal{I}_m \\
&= \left[(R \otimes I_n)\left(-I_{\hat{n}} \otimes A - \Lambda \otimes I_n - \sum_{k=1}^{m} R^{-1}\hat{N}_k R \otimes N_k\right)\left(R^{-1} \otimes I_n\right)\right]^{-1} \\
&\quad \times \left(\hat{B} \otimes B\right)\mathcal{I}_m \\
&= (R \otimes I_n) \underbrace{\left(-I_{\hat{n}} \otimes A - \Lambda \otimes I_n - \sum_{k=1}^{m} R^{-1}\hat{N}_k R \otimes N_k\right)^{-1} \left(R^{-1}\hat{B} \otimes B\right)\mathcal{I}_m}_{\operatorname{vect}(V)}.
\end{aligned}
$$

From the last line, we can now conclude that

$$
(R \otimes I_n)^{-1} \operatorname{vect}(X) = \operatorname{vect}(V) \text{ and hence } XR^{-T} = V.
$$

Similarly, starting from (5.2), we obtain

$$
\begin{aligned}
\operatorname{vect}(Y) &= \left(I_{\hat{n}} \otimes A^T + \hat{A}^T \otimes I_n + \sum_{k=1}^{m} \hat{N}_k^T \otimes N_k^T\right)^{-1} \operatorname{vect}\left(C^T\hat{C}\right) \\
&= \left(I_{\hat{n}} \otimes A^T + \hat{A}^T \otimes I_n + \sum_{k=1}^{m} \hat{N}_k^T \otimes N_k^T\right)^{-1} \left(\hat{C}^T \otimes C^T\right)\mathcal{I}_p \\
&= \Bigg[\left(R^{-T} \otimes I_n\right)\Bigg(-I_{\hat{n}} \otimes A^T \\
&\quad - \Lambda \otimes I_n - \sum_{k=1}^{m} R^T\hat{N}_k^T R^{-T} \otimes N_k^T\Bigg)\left(-R^T \otimes I_n\right)\Bigg]^{-1} \left(\hat{C}^T \otimes C^T\right)\mathcal{I}_p \\
&= \left(-R^{-T} \otimes I_n\right)\operatorname{vect}(W).
\end{aligned}
$$

Once again, this leads to

$$
\left(-R^{-T} \otimes I_n\right)^{-1} \operatorname{vect}(Y) = \operatorname{vect}(W) \quad \text{and} \quad Y(-R) = W,
$$

where

$$
\operatorname{vect}(W) := \left(-I_{\hat{n}} \otimes A^T - \Lambda \otimes I_n - \sum_{k=1}^{m} R^T\hat{N}_k^T R^{-T} \otimes N_k^T\right)^{-1} \left(R^T\hat{C}^T \otimes C^T\right)\mathcal{I}_p.
$$

According to the proof of Theorem 5.2, as long as $\operatorname{span}\{X\} \subset V$ and $\operatorname{span}\{Y\} \subset W$, we can ensure that the reduced system satisfies the necessary $\mathcal{H}_2$-optimality conditions.

Hence, we have found an equivalent method which obviously extends IRKA to the bilinear case; see Algorithm 2.

Algorithm 2. *Bilinear* IRKA (BIRKA)

Input: A, N_k, B, C, $\hat{A}$, $\hat{N}_k$, $\hat{B}$, $\hat{C}$
Output: $\hat{A}^{opt}$, $\hat{N}_k^{opt}$, $\hat{B}^{opt}$, $\hat{C}^{opt}$

while (not converged) **do**
  $R\Lambda R^{-1} = \hat{A}$, $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C}R$, $\tilde{N}_k = R^T \hat{N}_k^T R^{-T}$
  $\operatorname{vect}(V) = (-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k)^{-1}(\tilde{B}^T \otimes B)\mathcal{I}_m$
  $\operatorname{vect}(W) = (-\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{k=1}^m \tilde{N}_k \otimes N_k^T)^{-1}(\tilde{C}^T \otimes C^T)\mathcal{I}_p$
  $V = \operatorname{orth}(V)$, $W = \operatorname{orth}(W)$
  $\hat{A} = \left(W^T V\right)^{-1} W^T A V$, $\hat{N}_k = \left(W^T V\right)^{-1} W^T N_k V$, $\hat{B} = \left(W^T V\right)^{-1} W^T B$, $\hat{C} = CV$
end while
$\hat{A}^{opt} = \hat{A}$, $\hat{N}_k^{opt} = \hat{N}_k$, $\hat{B}^{opt} = \hat{B}$, $\hat{C}^{opt} = \hat{C}$

Finally, we point out the equivalence between the optimality conditions (4.1) and (4.5). For this, we need the following projection-based identity.

LEMMA 5.3. *Let $V, W \in \mathbb{R}^{n\times \hat{n}}$ be matrices of full rank $\hat{n}$.*
(a) *Let $z \subset \operatorname{span}\{\operatorname{vect}(V)\}$. Then $\left(I_{\hat{n}} \otimes V(W^TV)^{-1}W^T\right) z = z$.*
(b) *Let $z \subset \operatorname{span}\{\operatorname{vect}(W)\}$. Then $z^T \left(I_{\hat{n}} \otimes V(W^TV)^{-1}W^T\right) = z^T$.*

Proof. By assumption, there exists $x \in \mathbb{R}^{n\cdot\hat{n}}$ such that (s.t.)

$$\begin{aligned}\left(I_{\hat{n}} \otimes V(W^TV)^{-1}W^T\right) z &= \left(I_{\hat{n}} \otimes V(W^TV)^{-1}W^T\right) \operatorname{vect}(V)\, x\\ &= \operatorname{vect}\left(V(W^TV)^{-1}W^TV\right) x = \operatorname{vect}(V)\, x = z.\end{aligned}$$

The proof of the second statement is based on the exact same arguments. ∎

THEOREM 5.4. *Assume Algorithm 2 converges. Then $\hat{A}^{opt}$, $\hat{N}_k^{opt}$, $\hat{B}^{opt}$, $\hat{C}^{opt}$ fulfill the necessary interpolation-based $\mathcal{H}_2$-optimality conditions.*

Proof. Since the only difference in proving conditions (4.5)–(4.8) lies in using statement (b) of Lemma 5.3 and the combination of both (a) and (b), respectively, we will restrict ourselves to showing optimality condition (4.5):

$$\begin{aligned}&\mathcal{I}_p^T \left(e_i e_j^T \otimes \hat{C}^{opt}\right) \left(-\Lambda \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A}^{opt} - \sum_{k=1}^m \tilde{N}_k^T \otimes \hat{N}_k^{opt}\right)^{-1} \left(\tilde{B}^T \otimes \hat{B}^{opt}\right) \mathcal{I}_m\\ &\quad= \mathcal{I}_p^T \left(e_i e_j^T \otimes CV\right)\\ &\qquad\times \left[\left(I_{\hat{n}} \otimes (W^TV)^{-1}W^T\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k\right) (I_{\hat{n}} \otimes V)\right]^{-1}\\ &\qquad\times \left(\tilde{B}^T \otimes (W^TV)^{-1}W^T B\right) \mathcal{I}_m\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{I}_p^T \left(e_i e_j^T \otimes CV\right) \\
&\quad \times \left[\left(I_{\hat{n}} \otimes (W^TV)^{-1}W^T\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right) (I_{\hat{n}} \otimes V) \right]^{-1} \\
&\quad \times \left(I_{\hat{n}} \otimes (W^TV)^{-1}W^T\right) \\
&\quad \times \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right)^{-1} \\
&\quad \times \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m \\
&\overset{(5.3a)}{=} \mathcal{I}_p^T \left(e_i e_j^T \otimes CV\right) \\
&\quad \times \left[\left(I_{\hat{n}} \otimes (W^TV)^{-1}W^T\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right) (I_{\hat{n}} \otimes V) \right]^{-1} \\
&\quad \times \left(I_{\hat{n}} \otimes (W^TV)^{-1}W^T\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right) \\
&\quad \times \left(I_n \otimes V(W^TV)^{-1}W^T\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m \\
&= \mathcal{I}_p^T \left(e_i e_j^T \otimes CV\right) \left(I_n \otimes (W^TV)^{-1}W^T\right) \\
&\quad \times \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m \\
&= \mathcal{I}_p^T \left(e_i e_j^T \otimes C\right) \left(-\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{k=1}^m \tilde{N}_k^T \otimes N_k \right)^{-1} \left(\tilde{B}^T \otimes B\right) \mathcal{I}_m. \qquad \square
\end{aligned}
$$

Remark 5.4. Note that, analogously to the case of solving generalized Sylvester and Lyapunov equations, respectively, it is also possible to construct the matrices appearing in Algorithm 2 as the limit of an infinite series of linear IRKA type computations. For this, in each iteration, one starts with

$$V_i^1 = (-\lambda_i I - A)^{-1} B\tilde{B}_i,$$

and continues with

$$V_i^j = (-\lambda_i I - A)^{-1} \left(\sum_{k=1}^m N_k V^{j-1} (\tilde{N}_k)_i \right).$$

The actual projection matrix V is then given as $V = \sum_{j=1}^{\infty} V^j$. A dual derivation obviously yields the projection matrix W. At this point, the interpolatory interpretation of the proposed algorithm is seen once more. The construction of each V^j corresponds in a way to the tangential interpolation framework appearing for linear dynamical systems with multiple inputs and multiple outputs. Furthermore, similar to the statement in Remark 5.2, another way of constructing the projection matrices is given by the use of an iterative solver which again might be implemented with a natural preconditioner determined by the simplified and underlying linear problem which can be easily tackled by the iterative rational Krylov algorithm. Since the latter method is computationally more attractive than the Schur decomposition based approach discussed in [7], the reformulation of Algorithm 1 into Algorithm 2 might turn out to be profitable for practicable computations and will be a topic of further research.

Remark 5.5. Note that the numerical efficiency of both Algorithm 1 and Algorithm 2 heavily depend on the number of iterations needed until the relative change of the eigenvalues of the system matrix A approaches zero. As has already been shown for the linear case (cf. [19]), the iterative rational Krylov algorithm (IRKA) is a simplified Newton iteration where the Jacobian matrix is neglected. Obviously, this means that there exist a lot of examples where both algorithms will not converge at all. Nevertheless, recently there have been some first convergence results for symmetric state space systems; see [16]. However, at this point it seems very hard to generalize those ideas to the bilinear case.

6. Numerical examples. In this section, we will now study several applications of bilinear control systems and discuss the performance of the approaches proposed above. As we already mentioned, the method of balanced truncation for bilinear systems is connected to generalized controllability and reachability Gramians of the underlying system, respectively. Hence, similar to the linear case, we expect this method to yield reduced models with small relative $\mathcal{H}_2$-error as well and we will thus use it for a comparison with our algorithms. However, due to the theoretical equivalence of Algorithms 1 and 2, we will mainly report the results for the latter case. Nevertheless, we remark that if iterative solvers are included in numerical simulations, there might occur differences with respect to robustness and speed of convergence which will be subject to further studies. However, here we computed the projection matrices V and W by solving the large systems of linear equations explicitly instead of using more sophisticated iterative techniques. Finally, all Lyapunov equations were solved by the method proposed in [14] which allows for solving medium-sized systems. All simulations were generated on an Intel Core i7 CPU 920, 8 MB cache, 12 GB RAM, openSUSE Linux 11.1 (x86_64), MATLAB Version 7.11.0.584 (R2010b) 64-bit (glnxa64).

6.1. An interconnected power system. The first application is a model for two interconnected power systems which can be described by a bilinear system of state dimension 17. The hydro unit as well as the steam unit each can be controlled by two input variations resulting in a system with four inputs and three outputs. Since we are only interested in the reduction process, we refer to [1] where a detailed derivation of the dynamics can be found. We have successively reduced the original model to systems varying from $\hat{n} = 1, \ldots, 16$ state variables. A comparison of the associated relative $\mathcal{H}_2$-norm of the error system between our approaches and the method of balanced truncation is shown in Figure 6.1.

As one can see, except for the cases $\hat{n} = 2$, we always obtain better results with the new technique. The initialization of Algorithms 1 and 2 is done completely at random, using arbitrary reduced order models, interpolations points, and tangential directions, respectively. For both algorithms we use the same initialization and, as shown in Figure 6.1, obtain the exact same results. This underscores the theoretical equivalence and thus justifies concentrating on Algorithm 2. As indicated for system dimensions $\hat{n} = 5, 10, 14$, the algorithm does not always converge in a few steps; see Figure 6.2. On the other hand, we see that the relative $\mathcal{H}_2$-error stagnates very fast. Hence, the stopping criterion, which is chosen to be that the relative change of the norm of the poles of the reduced system becomes smaller than $\sqrt{\epsilon}$, where ϵ denotes machine precision, might be too restrictive. Again, finding appropriate criteria seems to be a reasonable topic of further research.

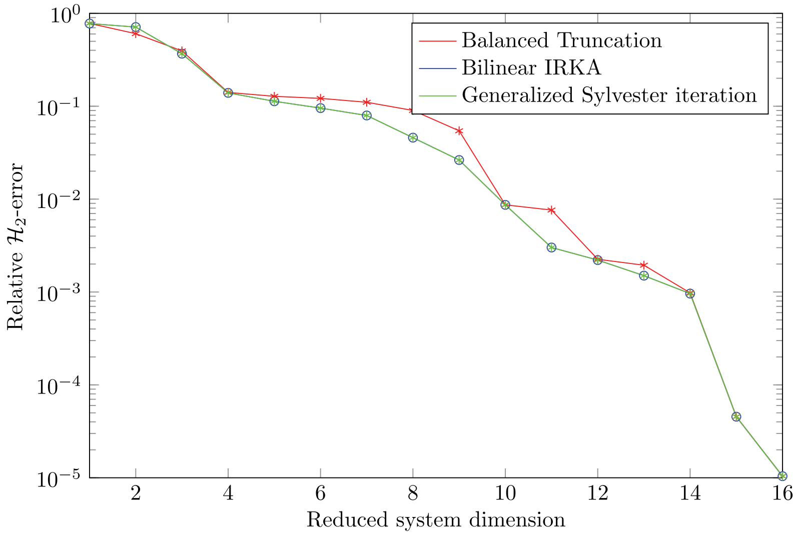

FIG. 6.1. *Power system. Comparison of relative $\mathcal{H}_2$-error between balanced truncation and B-IRKA.*

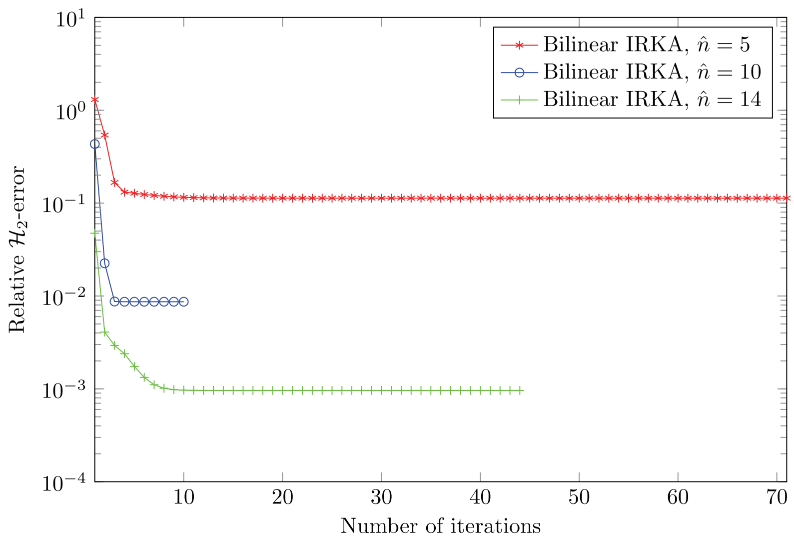

FIG. 6.2. *Power system. Convergence history of the relative $\mathcal{H}_2$-error.*

6.2. Fokker–Planck equation. The second example is an application from stochastic control and was already discussed in [20]. Let us consider a dragged Brownian particle whose one-dimensional motion is described by the stochastic differential equation

$$dX_t = -\nabla V(X_t, t)dt + \sqrt{2\sigma}dW_t,$$

with $\sigma = \frac{2}{3}$ and $V(x, u) = W(x, t) + \Phi(x, u_t) = (x^2 - 1)^2 - xu - x$. As mentioned in [20], we might alternatively consider the underlying probability distribution function

$$\rho(x, t)dx = \mathbf{P}\left[X_t \in (x, x + dx)\right]$$

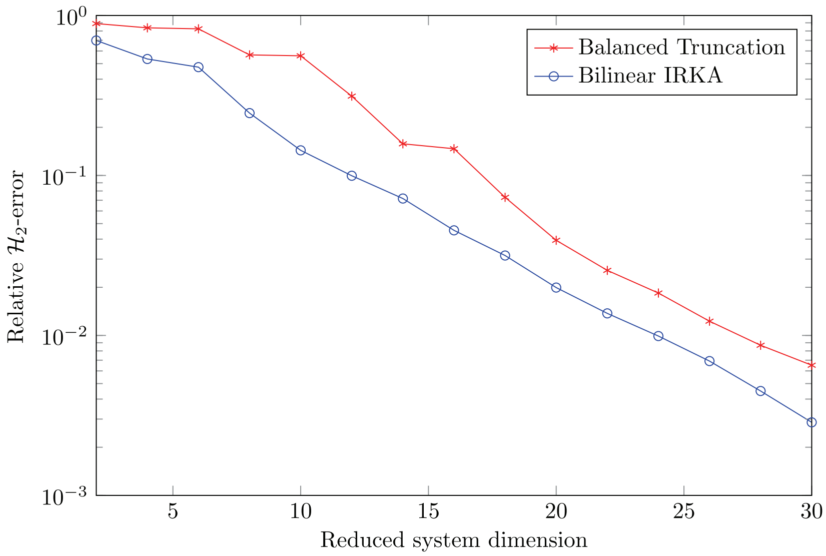

FIG. 6.3. *Fokker–Planck equation. Comparison of relative $\mathcal{H}_2$-error between balanced truncation and B-IRKA.*

which is described by the Fokker–Planck equation,

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \sigma \Delta \rho + \nabla \cdot (\rho \nabla V), && (x,t) \in (a,b) \times (0,T], \\ 0 &= \sigma \nabla \rho + \rho \nabla B, && (x,t) \in \{a,b\} \times [0,T], \\ \rho_0 &= \rho, && (x,t) \in (a,b) \times 0. \end{aligned}$$

After a semidiscretization resulting from a finite difference scheme consisting of 500 nodes in the interval $[-2, 2]$, we obtain a single-input single-output bilinear control system, where we choose the output matrix C to be the discrete characteristic function of the interval $[0.95, 1.05]$. Since we only pointed out the most important parameters of the model, we once more refer to [20] for gaining a more detailed insight into this topic. In Figure 6.3, we again compare the relative $\mathcal{H}_2$-errors between balanced truncation and B-IRKA for varying system dimensions. We observe convergence for all reduced system dimensions and our new method clearly outperforms balanced truncation.

6.3. Viscous Burgers equation. Next, let us consider the viscous Burgers equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}, \quad (x,t) \in (0,1) \times (0,T),$$

subject to initial and boundary conditions

$$v(x,0) = 0, \quad x \in [0,1], \qquad v(0,t) = u(t), \quad v(1,t) = 0, \quad t \geq 0.$$

Introduced in [9], after a spatial semidiscretization of this nonlinear partial differential equation using k nodes in a finite difference scheme, we end up with an ordinary differential equation including a quadratic nonlinearity. As is well known, the Carleman linearization technique (see, e.g., [27]) allows to approximate this system by

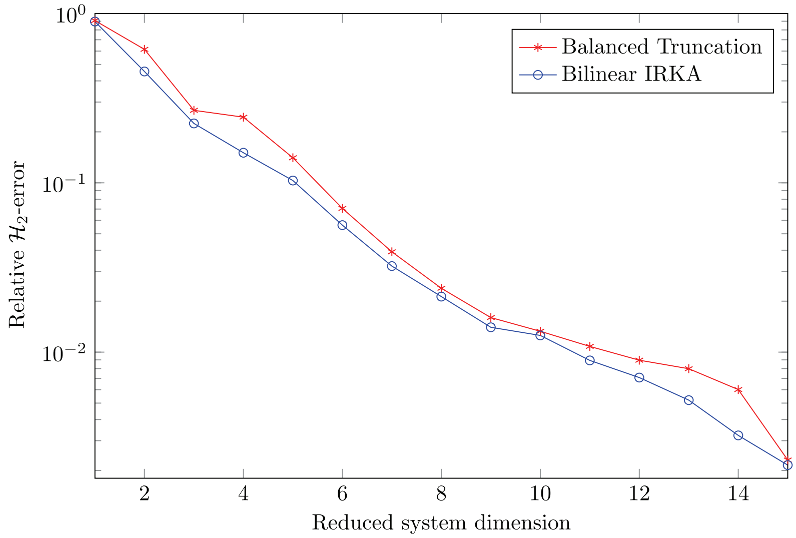

FIG. 6.4. *Burgers equation. Comparison of relative $\mathcal{H}_2$-error between balanced truncation and B-IRKA.*

a bilinearized system of dimension $n = k + k^2$. The simulations are generated with $\nu = 0.1$ and $k = 30$. The measurement vector C is chosen to yield the spatial average value for the quantity v. As shown in Figure 6.4, in all cases the relative $\mathcal{H}_2$-error for the systems constructed by B-IRKA is smaller than that resulting from balanced truncation. Moreover, once more there are no convergence problems at all although we again use random data for the initialization.

6.4. A heat transfer model. Finally, we study another standard bilinear test example resulting from a boundary controlled heat transfer system; see, e.g., [8]. Formally, the dynamics are described by the heat equation subject to Dirichlet and Robin boundary conditions, i.e.,

$$\begin{aligned} x_t &= \Delta x && \text{in } (0,1)\times(0,1),\\ n\cdot\nabla x &= 0.75\cdot u_{1,2,3}(x-1) && \text{on } \Gamma_1,\Gamma_2,\Gamma_3,\\ x &= u_4 && \text{on } \Gamma_4, \end{aligned}$$

where $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 denote the boundaries of Ω. Hence, a spatial discretization using k^2 grid points now yields a bilinear system of dimension $n = k^2$, with four inputs and one output, chosen to be the average temperature on the grid. In order to show that our algorithm also works in large-scale settings, we implement the above system with 10000 grid points. The results for reduced system dimensions $\hat{n} = 2, \ldots, 30$ are given in Figure 6.5 and demonstrate that we can improve the approximation quality with regard to the $\mathcal{H}_2$-norm with a numerically efficient interpolation-based framework. Moreover, in order to show the superiority of the new approach we further plot the results for the reduced systems obtained by IRKA as well as those generated by the new interpolation framework together with some clever, but nonoptimal interpolation points. This means, we use real equidistributed and Chebyshev interpolations points between the smallest and largest real part of the mirror images of the eigenvalues of the system matrix A and stop Algorithm 2 after the first iteration step. However,

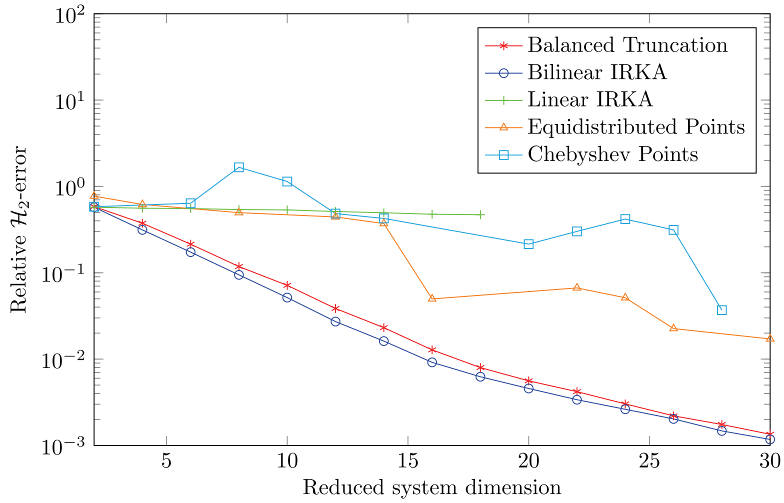

FIG. 6.5. *Heat transfer model. Comparison of relative $\mathcal{H}_2$-error between balanced truncation and B-IRKA.*

the relative $\mathcal{H}_2$-error is computed only when the corresponding reduced systems are stable, leading to positive definite solutions of the Gramians of the error systems. Moreover, as can be seen in Figure 6.5, the linear iterative rational Krylov algorithm converges only for reduced system dimensions up to $\hat{n} = 18$ at all.

Since so far most bilinear reduction methods have been evaluated by means of comparing the relative error for outputs corresponding to typical system inputs, we compute the time response to an input of the form $u_k(t) = \cos(k\pi t),\ k = 1, 2, 3, 4$. The results are plotted in Figure 6.6, where we test the performance for an original

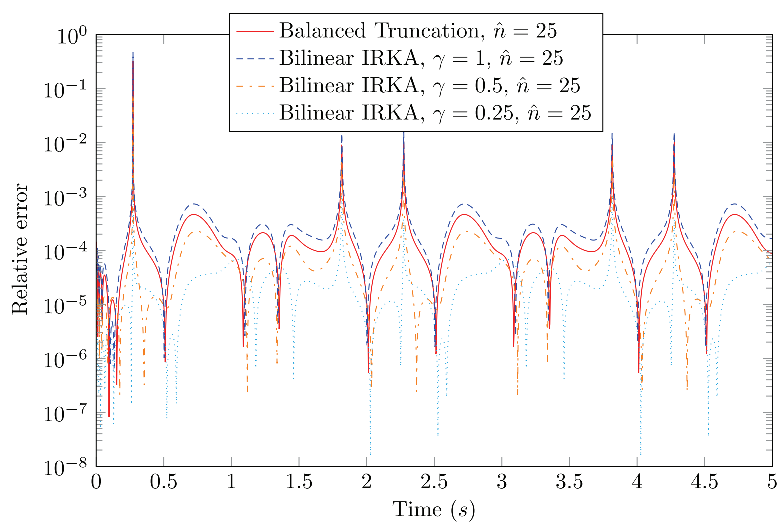

FIG. 6.6. *Heat transfer model. Comparison of relative error to an input of the form $u_k(t) = \cos(k\pi t)$ for a bilinear system of order $n = 2500$ between balanced truncation and B-IRKA for varying scaling factors γ.*

bilinear system of order $n = 2500$ and different scaling values γ. This means, the matrices N_k and B, respectively, are multiplied with γ, while the input signal $u(t)$ is replaced with $\frac{1}{\gamma}u(t)$. Similar experiments are studied in [6]. Interestingly enough, while the convergence results for B-IRKA do not change significantly, the relative error is smaller for smaller values of γ. However, all tested values γ can certainly compete with the approximation quality obtained from balanced truncation.

7. Conclusions. In this paper, we have studied the problem of $\mathcal{H}_2$-model reduction for bilinear systems. Based on an existing generalization of the linear $\mathcal{H}_2$-norm, we have derived first-order necessary conditions for optimality. As has been shown, these can be interpreted as an extension of those obtained for the linear case and lead to a generalization of the iterative rational Krylov algorithm. We have further proposed an equivalent iterative procedure that requires solving certain generalized Sylvester equations. The efficiency of our approaches has been evaluated by several bilinear test examples for which they yield better results than the popular method of balanced truncation. Finally, it was shown that the new method can additionally compete when the approximation quality is measured in terms of the transient response in time domain. As a topic of further and ongoing research, we currently investigate the effect of choosing reasonable initial data in order to improve convergence rates of the algorithms as well as efficient solution techniques for the special generalized Sylvester equations one has to solve in each iteration step.

Appendix A. Product rule for matrices with Kronecker structure.

LEMMA A.1. *Let* $C(x) \in \mathbb{R}^{p\times n}$, $A(y),\ N_k \in \mathbb{R}^{n\times n}$, *and* $B \in \mathbb{R}^{n\times m}$, *with* $x, y \in \mathbb{R}$. *Let*

$$\mathcal{L}(y) = \left(-A(y)\otimes I - I \otimes A(y) - \sum_{k=1}^{m} N_k \otimes N_k\right)$$

and assume that C *and* A *are differentiable with respect to* x *and* y. *Then,*

$$\frac{\partial}{\partial x}\left[(\mathcal{I}_p)^T (C(x)\otimes C(x))\mathcal{L}(y)^{-1}(B\otimes B)\mathcal{I}_m\right]$$
$$= 2\cdot(\mathcal{I}_p)^T \left(\frac{\partial}{\partial x}C(x)\otimes C(x)\right)\mathcal{L}(y)^{-1}(B\otimes B)\mathcal{I}_m$$

and

$$\frac{\partial}{\partial y}\left[(\mathcal{I}_p)^T (C(x)\otimes C(x))\mathcal{L}(y)^{-1}(B\otimes B)\mathcal{I}_m\right]$$
$$= 2\cdot(\mathcal{I}_p)^T \left(C(x)\otimes C(x)\right)\mathcal{L}(y)^{-1}\left(\frac{\partial}{\partial y}A(y)\otimes I\right)\mathcal{L}(y)^{-1}(B\otimes B)\mathcal{I}_m.$$

Proof. For the first part, note that

$$\left(-A(y)\otimes I - I\otimes A(y) - \sum_{k=1}^{m} N_k\otimes N_k\right)^{-1}(B\otimes B)\mathcal{I}_m := \operatorname{vect}\left(P(y)\right)$$

is the solution of the Lyapunov equation

$$A(y)P(y) + P(y)A(y)^T + \sum_{k=1}^{m} N_k P(y) N_k^T + BB^T = 0.$$

Hence, we can conclude that $P(y) = P(y)^T$. Next, using (2.6), we observe that

$$\begin{aligned}
&(\mathcal{I}_p)^T \left(C(x) \otimes \frac{\partial}{\partial x} C(x) \right) \operatorname{vect}(P(y)) = \operatorname{vect}\left(\left(\frac{\partial}{\partial x} C(x)^T C(x) \right) \right)^T \operatorname{vect}(P(y)) \\
&\quad = \operatorname{tr}\left(C(x)^T \frac{\partial}{\partial x} C(x) P(y) \right) = \operatorname{tr}\left(\frac{\partial}{\partial x} C(x) P(y) C(x)^T \right) \\
&\quad = \operatorname{tr}\left(C(x) P(y)^T \frac{\partial}{\partial x} C(x)^T \right) = \operatorname{tr}\left(C(x) P(y) \frac{\partial}{\partial x} C(x)^T \right) \\
&\quad = \operatorname{tr}\left(\left(\frac{\partial}{\partial x} C(x)^T \right) C(x) P(y) \right) = \operatorname{vect}\left(\left(C(x)^T \frac{\partial}{\partial x} C(x) \right) \right)^T \operatorname{vect}(P(y)) \\
&\quad = (\mathcal{I}_p)^T \left(\frac{\partial}{\partial x} C(x) \otimes C(x) \right) \operatorname{vect}(P(y)).
\end{aligned}$$

The last equation implies that we can interchange the derivatives with respect to x. However, the assertion now follows trivially. For the second part, recall that we have $\frac{\partial}{\partial y}\left(A(y)^{-1}\right) = -A(y)^{-1} \frac{\partial A(y)}{\partial y} A(y)^{-1}$. Furthermore, with $Q(x,y)$ we denote the solution of the dual Lyapunov equation

$$A(y)^T Q(x,y) + Q(x,y) A(y) + \sum_{k=1}^{m} N_k^T Q(x,y) N_k + C(x)^T C(x) = 0.$$

Hence, with 2.6, we end up with

$$\begin{aligned}
&(\operatorname{vect}(Q(x,y)))^T \left(I \otimes \frac{\partial}{\partial y} A(y) \right) \operatorname{vect}(P(y)) \\
&\quad = (\operatorname{vect}(Q(x,y)))^T \operatorname{vect}\left(\left(\left(\frac{\partial}{\partial y} A(y) \right) P(y) \right) \right) = \operatorname{tr}\left(Q(x,y)^T \left(\frac{\partial}{\partial y} A(y) \right) P(y) \right) \\
&\quad = \operatorname{tr}\left(P(y)^T \left(\frac{\partial}{\partial y} A(y)^T \right) Q(x,y) \right) = \operatorname{tr}\left(\left(\frac{\partial}{\partial y} A(y)^T \right) Q(x,y)^T P(y) \right) \\
&\quad = \operatorname{tr}\left(\left(Q(x,y) \frac{\partial}{\partial y} A(y) \right)^T P(y) \right) = \left(\operatorname{vect}\left(\left(Q(x,y) \frac{\partial}{\partial y} A(y) \right) \right) \right)^T \operatorname{vect}(P(y)) \\
&\quad = \left(\left(\frac{\partial}{\partial y} A(y)^T \otimes I \right) \operatorname{vect}(Q(x,y)) \right)^T \operatorname{vect}(P(y)) \\
&\quad = (\operatorname{vect}(Q(x,y))^T \left(\frac{\partial}{\partial y} A(y) \otimes I \right) \operatorname{vect}(P(y)).
\end{aligned}$$

Again, the last line proves the second statement. □

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