TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 191, 1974

INTERPOLATION BETWEEN H^p SPACES: THE REAL METHOD

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ABSTRACT. The interpolation spaces in the Lions-Peetre method between H^p spaces, $0 < p < \infty$, are calculated.

0. **Introduction.** The intermediate spaces between H^1 and L^∞, and hence between H^{p_0} and H^{p_1}, $1 \leq p_i < \infty$, in the real method, have been calculated in [2]. In this note we calculate the intermediate spaces between H^p and L^∞ in the real method, for $p < 1$. The method used in [2] fails hopelessly in this case, and more sophisticated ideas (developed in [1]) have to be employed. We prove

$$(1) \qquad (H^{p_0}, L^\infty)_{\theta,q} = H^{p,q} \quad \text{where } \frac{1}{p} = \frac{1-\theta}{p_0}, \ 0 < \theta < 1, \ 0 < q \leq \infty,$$

where $H^{p,q}$ is defined as follows:

$$f \in H^{p,q} \quad \text{iff} \quad \sup_{0<t} t^{-n} |\phi_t * f| = f^+ \in L^{p,q},$$

with ϕ a sufficiently regular function, and $\int \phi \neq 0$.

The interesting case is of course $p = q$. There is however no added difficulty in considering the general case. For $p > 1$, $H^{p,q} = L^{p,q}$ and so, we get the result of [2]. Using reiteration we get of course

$$(2) \qquad (H^{p_0,q_0}, H^{p_1 q_1})_{\theta,q} = H^{p,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_1}, \ 0 < \theta < 1, \ 0 < q \leq \infty.$$

It is interesting to note that when $p_0 < 1 < p_1$ we cannot pass to the dual spaces. The dual of H^{p_0}, $p_0 < 1$, is a certain Hölder space, and it has been shown by Stein and Zygmund in [4], that the interpolation spaces between Hölder and Lebesgue spaces are not Lebesgue spaces (as we would get for certain values of θ if we take formally the dual of (2)). The reason we cannot pass to the dual is of course that H^{p_0,q_0}, $p_0 < 1$, is not a Banach space.

We shall use freely in this note, results from interpolation theory and from the Fefferman-Stein theory of H^p spaces. The reader can consult [2] for a brief outline of the relevant results of interpolation theory, and [1] for those of H^p spaces.

Received by the editors July 2, 1973.

AMS (MOS) subject classifications (1970). Primary 42A40, 46E99.

I. Interpolation of H^p and L^∞. In this section we first set down the basic decomposition of a H^p function into "good" and "bad" parts. Our main tool is the characterization of H^p as a space of distributions on R^n given in [1], which we now review. Fix a smooth function ψ on R^n satisfying

$$|||\psi|||_N = \sum_{|\alpha|\le N} \int_{R^n} (1+|x|)^N \left|\frac{\partial^\alpha}{\partial x^\alpha}\psi(x)\right| dx < \infty$$

for some large N, and $\int_{R^n}\psi(x)dx = 1$. For a distribution f on R^n, set $f^+(x) = \sup_{t>0}|\psi_t * f(x)|$ where $\psi_t(y) = t^{-n}\psi(y/t)$, and say that $f \in H^p$ if $f^+ \in L^p$. It is shown in [1] that the H^p classes so defined do not depend on the choice of ψ, and are isomorphic to the usual H^p classes. Moreover, the "grand" maximal function

$$f^*(x) = \sup_{|||\phi|||_N \le 1} \sup_{|x-y|<10t} |\phi_t * f(y)|$$

belongs to L^p if $f \in H^p$, and we have the inequality $\|f^*\|_p \le C\|f\|_{H^p}$. Finally, the Schwartz class S is dense in H^p, $0<p<\infty$.

Now fix $p_0 < p < \infty$.

Lemma A. *Let $f \in S$ and $\alpha > 0$ be given. Then f may be written as the sum of two functions g and b which satisfy*

$$\|g\|_\infty \le C\alpha, \qquad \|b\|^{p_0}_{H^{p_0}} \le C \int_{\{f^*(x)>\alpha\}} (f^*(x))^{p_0}\, dx.$$

Proof. (Compare with the proof of Lemma 11 in [1].) Set $\Omega = \{f^*(x) > \alpha\}$. The proof of the Whitney extension theorem [3] exhibits a collection $\{Q_j\}$ of cubes and a family $\{\phi_j\}$ of smooth functions on R^n, with the properties

(1) Ω is the disjoint union of the $\{Q_j\}$.

(1′) $\chi_\Omega = \Sigma_j\phi_j$ and each $\phi_j \ge 0$.

(2) distance $(R^n - \Omega, Q_j) \sim$ diameter $(Q_j) \equiv d_j$. Let x_j be the center of Q_j and y_j a point in $R^n - \Omega$ satisfying $|y_j - x_j| \le 10 d_j$. Thus $f^*(y_j) \le \alpha$.

(2′) ϕ_j is supported in the cube Q_j expanded by the factor $6/5$, say. Also $\phi_j(x) \ge c > 0$ for $x \in Q_j$.

(2″) $\|\partial^\alpha\phi_j/\partial x^\alpha\|_\infty \le C_\alpha d_j^{-|\alpha|}$ for each multi-index α.

Denote by Q_j^* the cube Q_j expanded by a factor of 2. Now $f = f\cdot\chi_{R^n-\Omega} + \Sigma_j f\cdot\phi_j$. We shall define $g = f\cdot\chi_{R^n-\Omega} + \Sigma_j P_j\cdot\phi_j$, where $P_j(x)$ is the unique polynomial of degree $\le N$ (large, to be picked later) satisfying

$$\int_{R^n} (x-x_j)^\alpha P_j(x)\phi_j(x)\, dx = \int_{R^n} (x-x_j)^\alpha f(x)\phi_j(x)\, dx, \quad \text{for } |\alpha| \le N.$$

First of all, we claim that $\|P_j\|_{L^\infty(Q_j^*)} \le C\alpha$. To prove this, we may first translate and dilate R^n so that

$$\begin{cases} x_j = \text{center } (Q_j) = 0 \quad \text{and} \\ d_j = \text{diameter } (Q_j) = 1. \end{cases}$$

Next, let $\pi_1, \cdots, \pi_2$ be an orthonormal base for the Hilbert space of polynomials of degree $\le N$ with norm

$$\|P\|^2 = \int_{R^n} |P(x)|^2 \phi_j(x)\, dx.$$

An elementary argument shows that the coefficients of the π_l are bounded above by a "constant" depending only on N and n. Therefore $\Phi^l(x) = \pi_l(y_j - x)\phi_j(y_j - x)$ satisfies $|||\Phi^l|||_N \le C$ with C depending only on N, n, so that

$$\left| \int_{R^n} f(x)\pi_l(x)\phi_j(x)\, dx \right| = |\Phi^l * f(y_j)| \le Cf^*(y_j) \le C\alpha.$$

On the other hand,

$$P_j = \sum_{l=1}^{L} \left(\int_{R^n} f(x)\pi_l(x)\phi_j(x)\, dx \right) \pi_l,$$

which implies that $\|P_j\|_{L^\infty(Q_j^*)} \le C\alpha$, as claimed.

Now for the "good" function g we have

$$\begin{aligned} |g(x)| &\le |f(x)\chi_{R^n - \Omega}(x)| + \sum_j |P_j(x)|\phi_j(x) \\ &\le \alpha\chi_{R^n - \Omega} + \sum_j C\alpha\phi_j(x) \le C\alpha\chi_{R^n - \Omega} + C\alpha\chi_\Omega = C\alpha, \end{aligned}$$

i.e. $\|g\|_\infty \le C\alpha$.

It remains to determine the H^{p0} "norm" of the "bad" function $b = f - g = \Sigma_j (f(x) - P_j(x))\phi_j(x) \equiv \Sigma_j b_j(x)$. To do so, we fix ψ as above, and undertake to study $b_j^+(x)$, i.e. to estimate

$$(1) \qquad \left| t^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right) (f(y) - P_j(y))\phi_j(y)\, dy \right|.$$

We can take ψ supported in $|z| < 1$.

We can assume $x_j = 0$.

Case 1. $x \in Q_j^*$ and $t \le d_j$. Then for $\Phi(z) = \psi(z)\phi_j(x - tz)$ we may check that $\|\partial^\gamma \Phi / \partial x^\gamma\|_\infty \le C_\gamma$ and since Φ is supported in $|z| \le 1$, $|||\Phi|||_N \le C$ which implies

$$\left| t^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right) f(y)\phi_j(y)\,dy \right| = |\Phi_t * f(x)| \le Cf^*(x).$$

Since

$$\left| t^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right) P_j(y)\phi_j(y)\,dy \right|$$
$$\le \|P_j\|_\infty \left\| t^{-n}\psi\left(\frac{x-y}{t}\right)\phi_j(y) \right\|_{L^1(dy)} \le C\alpha \le Cf^*(x),$$

we have

$$\left| t^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right)(f(y) - P_j(y))\phi_j(y)\,dy \right| \le Cf^*(x).$$

Case 2. $x \in Q_j^*$ and $t > d_j$. Then for $\Phi(z) = \psi(d_j z/t)\phi_j(x - d_j z)$ we have again $|||\Phi|||_N \le C$ by calculations similar to the ones we did not do in Case 1. So

$$\left| t^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right) f(y)\phi_j(y)\,dy \right| \le \left| d_j^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right) f(y)\phi_j(y)\,dy \right|$$
$$= |\Phi_{d_j} * f(x)| \le Cf^*(x),$$

and since

$$\left| d_j^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right) P_j(y)\phi_j(y)\,dy \right|$$
$$\le \|P_j\|_{L^\infty(Q_j^*)} \left\| d_j^{-n}\psi\left(\frac{x-y}{t}\right)\phi_j(y) \right\|_{L^1(dy)} \le C\alpha \le Cf^*(x),$$

we have again

$$\left| t^{-n} \int_{R^n} \psi\left(\frac{x-y}{t}\right)(f(y) - P_j(y))\phi_j(y)\,dy \right| \le Cf^*(x),$$

From Cases 1 and 2 we see that $b_j^+(x) \le Cf^*(x)$ for $x \in Q_j^*$.

Case 3. $x \notin Q_j^*$. We consider only the case $t > \frac{1}{2}|x| > d_j$, since otherwise the integrand in (1) vanishes identically. Regarding x and t as fixed, and letting y vary, we may use Taylor's formula to write

$$\psi\left(\frac{x-y}{t}\right) = [\text{Polynomial of degree} \le N \text{ in } y] + R(y),$$

where the remainder term $R(y)$ satisfies the estimates $|\partial^\gamma R(y)/\partial y^\gamma| \le Cd_j^{-|\gamma|}(d_j/|x|)^{N+1}$. So

$$\begin{aligned}&\left|t^{-n}\int_{R^n}\psi\left(\frac{x-y}{t}\right)(f(y)-P_j(y))\phi_j(y)\,dy\right|\\ &\quad=\left|t^{-n}\int_{R^n}[\text{Polynomial of degree}\le N\text{ in }y]\,(f(y)-P_j(y))\phi_j(y)\,dy\right.\\ &\qquad\qquad\left.+\,t^{-n}\int_{R^n}R(y)(f(y)-P_j(y))\phi_j(y)\,dy\right|\\ &\quad\equiv|A+B|.\end{aligned}$$

Now $A=0$, by virtue of our choice of P_j. To estimate B, we set $\Phi(z)=R(y_j-d_jz)\phi_j(y_j-d_jz)$. The function $\Phi(z)$ is supported in $\{|z|\le 20\}$, and our estimates for the derivatives of $R(y)$ and $\phi_j(y)$ show that $|\partial^\gamma\Phi(z)/\partial z^\gamma|\le C_\gamma(d_j/|x|)^{N+1}$, which implies $|||\Phi|||_N\le C(d_j/|x|)^{N+1}$. Therefore,

$$\begin{aligned}\left|t^{-n}\int_{R^n}R(y)f(y)\phi_j(y)\,dy\right|&\le\left|d_j^{-n}\int_{R^n}R(y)f(y)\phi_j(y)\,dy\right|\\ &=|\Phi_{d_j}*f(y_j)|\le C(d_j/|x|)^{N+1}f^*(y_j)\le C\alpha(d_j/|x|)^{N+1}.\end{aligned}$$

On the other hand, since $\|P_j\|_{L^\infty(Q_j^*)}\le C\alpha$, we again have, trivially,

$$\left|t^{-n}\int_{R^n}R(y)P_j(y)\phi_j(y)\,dy\right|\le C\alpha\left(\frac{d_j}{|x|}\right)^{N+1},$$

so that

$$|B|=\left|t^{-n}\int_{R^n}R(y)(f(y)-P_j(y))\phi_j(y)\,dy\right|\le C\alpha\left(\frac{d_j}{|x|}\right)^{N+1}.$$

Now from Cases 1–3, we know that

$$\begin{aligned}b_j^+(x)&\le Cf^*(x)\quad\text{if }x\in Q_j^*,\\ &\le C\alpha(d_j/|x-x_j|)^{N+1}\quad\text{if }x\notin Q_j^*,\end{aligned}$$

Consequently, for $p_0\le 1$,

$$\int_{R^n}(b_j^+(x))^{p_0}\,dx\le C\int_{Q_j^*}(f^*(x))^{p_0}\,dx+C\alpha^{p_0}\int_{R^n-Q_j^*}\left(\frac{d_j}{|x-x_j|}\right)^{(N+1)p_0}dx.$$

If N is picked so large that $(N+1)p_0>n$, then the last integral on the right is $C\alpha^{p_0}|Q_j|$, which is already dominated by the first integral on the right. Thus

$$\int_{R^n} (b_j^+(x))^{p_0}\, dx \le C \int_{Q_j^*} (f^*(x))^{p_0}\, dx.$$

Now it is easy to piece our estimates for b_j^+ together into an estimate for b^+. For, $b = \Sigma_j b_j$, so $b^+ \le \Sigma_j b_j^+$, so that $(b^+)^{p_0} \le \Sigma_j (b_j^+)^{p_0}$ (recall that $p_0 \le 1$), which implies

$$\int_{R^n} (b^+(x))^{p_0}\, dx \le \sum_j \int_{R^n} (b_j^+(x))^{p_0}\, dx$$

$$\le C \sum_j \int_{Q_j^*} (f^*(x))^{p_0}\, dx = C \int_{R^n} \left(\sum_j \chi_{Q_j^*}(x)\right) (f^*(x))^{p_0}\, dx.$$

The geometry of the Whitney cubes is such that $\Sigma_j \chi_{Q_j^*}(x) \le C\chi_\Omega(x)$, so that at last,

$$\int_{R^n} (b^+(x))^{p_0}\, dx \le C \int_\Omega (f^*(x))^{p_0}\, dx = C \int_{\{f^* > \alpha\}} (f^*(x))^{p_0}\, dx.$$

Thus $\|b\|_{H^{p_0}}^{p_0} \le C\int_{\{f^* > \alpha\}} (f^*(x))^{p_0} dx$, as claimed. The proof of Lemma A is complete. Q.E.D.

We can now prove the theorem announced:

Theorem 1. *For* $0 < p_0 < 1$, $0 < \theta < 1$, $0 < q \le \infty$

$$(H^{p_0}, L^\infty)_{\theta,q} = H^{p,q} \quad \textit{where} \quad 1/p = (1-\theta)/p_0.$$

Proof. Let $f \in H^{p,q}$. Denote by $\tilde{f}^*$ the nonincreasing rearrangement of f^*. Fix $t > 0$, and take in Lemma A, $\alpha = \tilde{f}^*(t^{p_0})$. We then have

$$K(t, f; H^{p_0}, L^\infty) \le \|b_t\|_{H^{p_0}} + t\|g_t\|_{L^\infty}.$$

$$\|b_t\|_{H^{p_0}} \le C \left(\int_{\{f^*(x) > \tilde{f}^*(t^{p_0})\}} (f^*(x))^{p_0}\, dx \right)^{1/p_0} \le C\left(\int_0^{t^{p_0}} (\tilde{f}^*(s))^{p_0}\, ds \right)^{1/p_0},$$

so that

$$\int_0^\infty (t^{-\theta}\|b_t\|_{H^{p_0}})^q\, \frac{dt}{t} \le C \int_0^\infty t^{-\theta q}\left(\int_0^{t^{p_0}} (\tilde{f}^*(s))^{p_0}\, ds \right)^{q/p_0} \frac{dt}{t}$$

$$= C \int_0^\infty t^{-\theta q/p_0}\left(\int_0^t (\tilde{f}^*(s))^{p_0}\, ds \right)^{q/p_0} \frac{dt}{t}.$$

By Hardy's inequality (if $q \ge p_0$) or by a modification of it (for $q < p_0$, see [2])

$$\int_0^\infty (t^{-\theta}\|b_t\|_{H^{p_0}})^q \frac{dt}{t} \leq C \int_0 t^{q(1-\theta)/p_0}(\bar{f}^*(t))^q \frac{dt}{t} = C \cdot \|f^*\|^q_{L^{p,q}}.$$

Further

$$\int_0^\infty (t^{(1-\theta)}\|g_t\|_{L^\infty})^q \frac{dt}{t} \leq C \int_0^\infty (t^{(1-\theta)}\bar{f}^*(t^{p_0}))^q \frac{dt}{t}$$

$$\leq C \cdot \int_0^\infty (t^{1/p}\bar{f}^*(t))^q \frac{dt}{t} = C\|f^*\|_{L^{p,q}},$$

so that $(\int_0^\infty (t^{-\theta}K(t,f))^q dt/t)^{1/q} \leq C\|f^*\|_{L^{p,q}}$. We have shown

$$H^{p,q} \subset (H^{p_0}, L^\infty)_{\theta,q}.$$

The inverse inclusion is trivial:

Consider the sublinear operator $T: f \to f^+$. We have $T: L^\infty \to L^\infty$ and $T: H^{p_0} \to L^{p_0}$. Therefore $T: (H^{p_0}, L^\infty)_{\theta,q} \to (L^{p_0}, L^\infty)_{\theta,q} = L^{p,q}$. That is $f \in (H^{p_0}, L^\infty)_{\theta,q}$ implies $f^+ \in L^{p,q}$ and $f \in H^{p,q}$. The proof is complete.

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