Interpolation in the Time and Frequency Domains

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Abstract— **In this letter, we clarify the connections between two recently proposed and apparently unrelated approaches to bandlimited interpolation by showing that, in a certain sense made precise below, they are the dual of each other. The advantages of recognizing this duality are discussed.**

I. Introduction

The interpolation of signals and images from sets of uniform or nonuniform samples is one of the most often studied problems in signal theory [1]–[4]. In this letter, we formulate an interpolation problem that contains the bandlimited interpolation/extrapolation problems studied in [5]–[8] as special cases. We show that the approach described in [7] and one of those proposed in [5] are the dual of each other.

The term "dual" is used in the following sense. Any signal x that occurs in signal processing can be regarded as a function defined in a certain Abelian group $\mathcal{G}$. Its Fourier transform $\hat{x}$ is a function defined in another group $\hat{\mathcal{G}}$, the dual group[1] of $\mathcal{G}$. The problem addressed in this paper is a special case of the following: Given values of x and $\hat{x}$, determine x.

Our N-dimensional signals can be regarded as vectors $x \in \mathbb{C}^N$ or as functions defined in the additive group of the integers modulo N, $\mathbb{Z}_N$. The dual group of $\mathbb{Z}_N$ is $\mathbb{Z}_N$, and the Fourier transforms of the signals are also N-dimensional ($\hat{x} = Fx$ is the discrete Fourier transform (DFT) of x, and F is the unitary Fourier matrix). The problem solved in this paper is that of determining x or $\hat{x}$ given some x_i and $\hat{x}_i$.

We consider the two solutions given as "dual" because they both grow out of the same principle, and of the same sequence of steps, although applied to functions defined in a group $\mathcal{G}$, in one case, or in its dual $\hat{\mathcal{G}}$, in the other. In one case $\mathcal{G}$ is the group $\mathcal{T}$ underlying the signals themselves (the time domain), whereas $\hat{\mathcal{G}}$ is its dual $\mathcal{F}$ (the frequency domain). In the other, the roles are reversed, and $\mathcal{G} = \mathcal{F}$, whereas $\hat{\mathcal{G}} = \mathcal{T}$. There is no difference between the two approaches, except for this interchange of the underlying groups.

Our conclusions have important practical implications. The Toeplitz equations proposed in [5], which are the basis for the "superfast" methods developed in [6] and [8], have minimum dimension in the frequency domain (the number of nonzero DFT elements). The approach described in [7], on the other hand, leads to a set of equations with minimum dimension in the time domain (the number of unknown data samples). It is natural to ask whether the two sets of equations are related or not; however, the methods used in [5] and [7] are quite distinct, and examination of these works does not immediately yield the answer.

Understanding the roles played by the time and frequency domains in each approach shows i) that they are the dual of each other; ii) that the results in [6] and [9], originally derived for only one of the methods, hold for both; iii) how to obtain similar dual formulations for the image interpolation problem; and iv) under what circumstances one of the methods is preferable to the other.

II. Results

Problem: Let $x \in \mathbb{C}^N$ denote a signal, and $\hat{x} = Fx$ its DFT. Let S_t and S_f be two proper nonempty subsets of $\{0, 1, \cdots, N-1\}$. Given the sets $x(S_t) \equiv \{x_i : i \in S_t\}$ and $\hat{x}(S_f) \equiv \{\hat{x}_i : i \in S_f\}$, find x and $\hat{x}$.

We define two diagonal matrices D^a and D^u, and the vectors v, u, $\hat{a}$ and $\hat{b}$, by

$$D^u_{ii} = \begin{cases} 0, & i \in S_t \\ 1, & i \notin S_t \end{cases} \qquad D^a_{ii} = \begin{cases} 0, & i \in S_f \\ 1, & i \notin S_f \end{cases}$$

$$v_i = \begin{cases} x_i, & i \in S_t \\ 0, & i \notin S_t \end{cases} \qquad \hat{b}_i = \begin{cases} \hat{x}_i, & i \in S_f \\ 0, & i \notin S_f. \end{cases}$$

Note that $x = u + v$ and $\hat{x} = \hat{a} + \hat{b}$, $u = D^u u$, $D^u v = 0$, $\hat{a} = D^a \hat{a}$, and $D^a \hat{b} = 0$.

Proposition 1: The signal $x \in \mathbb{C}^N$ that solves the problem satisfies $x = Bx + b$, and its DFT satisfies $\hat{x} = T\hat{x} + \hat{v}$. The matrices B and T are circulant projections, with $B = F^H D^a F$ and $T = F D^u F^H$. Explicitly, we have

$$B_{ab} = \frac{1}{N} \sum_{p \notin S_f} e^{j \frac{2\pi}{N}(a-b)p}$$

$$T_{ab} = \frac{1}{N} \sum_{p \notin S_t} e^{-j \frac{2\pi}{N}(a-b)p}.$$

Proof: We use the expressions $x = u + v$ and $\hat{x} = \hat{a} + \hat{b}$ defined above. In the context of the problem, the vectors v and $\hat{b}$ are known, and u and $\hat{a}$ are unknown. Now, we have

$$\begin{aligned} x &= F^H \hat{x} = F^H \hat{a} + F^H \hat{b} \\ &= F^H D^a \hat{a} + b = F^H D^a (\hat{x} - \hat{b}) + b \\ &= F^H D^a \hat{x} + b = F^H D^a F x + b. \end{aligned}$$

Setting $B = F^H D^a F$ yields $x = Bx + b$. Similarly

$$\begin{aligned} \hat{x} &= Fx = Fu + Fv \\ &= FD^u u + \hat{v} = FD^u (x - v) + \hat{v} \\ &= FD^u x + \hat{v} = FD^u F^H \hat{x} + \hat{v} \end{aligned}$$

and setting $T = FD^u F^H$ yields $\hat{x} = T\hat{x} + \hat{v}$. □

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[1] The dual group of an Abelian group is the group of its characters under pointwise multiplication. A character is a homomorphism of the group into the unit circle.

Although $\hat{v}$ and b are known, the equations $x = Bx + b$ and $\hat{x} = T\hat{x} + \hat{v}$ cannot be solved for x or $\hat{x}$, because the matrices $I - B$ and $I - T$ are singular. However, this difficulty can be circumvented.

Proposition 2: Let the cardinal of the complement of S_t in $\{0, 1, \cdots, N-1\}$ (the number of unknown time-domain samples) be denoted by n. Let P be the $n \times n$ principal submatrix of B, obtained by deleting from B all rows and columns whose indexes belong to S_t. Denote by $c \in \mathbb{C}^n$ the vector with elements

$$c_i = \sum_{j \in S_t} B_{ij} v_j + b_i \qquad (i \notin S_t).$$

Then, the vector $y \in \mathbb{C}^n$ formed by the n time-domain unknowns $x_i (i \notin S_t)$ satisfies

$$y = Py + c. \tag{1}$$

Proposition 3: Let the cardinal of the complement of S_f in $\{0, 1, \cdots, N-1\}$ (the number of unknown DFT harmonics) be denoted by r. Let Q be the $r \times r$ principal submatrix of T, obtained by deleting from T all rows and columns whose indices belong to S_f. Denote by $d \in \mathbb{C}^r$ the vector with elements

$$d_i = \sum_{j \in S_f} T_{ij} \hat{b}_j + \hat{v}_i \qquad (i \notin S_f).$$

Then, the vector $z \in \mathbb{C}^r$ formed by the r unknown DFT harmonics $\hat{x}_i (i \notin S_f)$ satisfies

$$z = Qz + d. \tag{2}$$

Proof: The N equations $x = Bx + b$ are redundant, since there are only n unknown time-domain samples x_i, and therefore we look for a subset of n equations. Since $x = u + v$, it follows that $u + v = Bu + Bv + b$; that is

$$u_i + v_i = \sum_{j=0}^{N-1} B_{ij} u_j + \sum_{j=0}^{N-1} B_{ij} v_j + b_i.$$

Considering only $i \notin S_t$, and noting that $u_i = 0$ for all $i \in S_t$, whereas $v_i = 0$ for $i \in S_f$, leads to

$$u_i = \sum_{j \notin S_t} B_{ij} u_j + \sum_{j \in S_t} B_{ij} v_j + b_i \qquad (i \notin S_t)$$

which is (1). This settles Proposition 2.

Proposition 3 can be established by duality (reversing the roles of the time and frequency domains) or by considering the N equations $\hat{x} = T\hat{x} + \hat{v}$, and substituting $\hat{x} = \hat{a} + \hat{b}$ to obtain $\hat{a} + \hat{b} = T\hat{a} + T\hat{b} + \hat{v}$, that is

$$\hat{a}_i + \hat{b}_i = \sum_{j=0}^{N-1} T_{ij} \hat{a}_j + \sum_{j=0}^{N-1} T_{ij} \hat{b}_j + \hat{v}_i.$$

Thus, just as in the first case, we have

$$\hat{a}_i = \sum_{j \notin S_f} T_{ij} \hat{a}_j + \sum_{j \in S_f} T_{ij} \hat{b}_j + \hat{v}_i \qquad (i \notin S_f)$$

which is (2). □

The bandlimited interpolation problem is a special case of the problem considered, in which a subset of the DFT harmonics of the signal is assumed to be zero. For lowpass signals with $2M + 1$ nonzero harmonics or, more generally, whenever S_f is contiguous modulo N [4], [7] the inverse of $I - P$ exists and (1) can be solved for y (if, of course, the number of known x_i exceeds the number of unknown $\hat{x}_i$, $2M + 1$).

Proposition 4: The equations given in [7] are a special case of (1), whereas the Toeplitz equations in [5] are a special case of (2).

Proof: For lowpass signals with $2M + 1$ nonzero harmonics $S_f = \{M+1, M+2, \cdots, N-M-1\}$ and $\hat{b} = 0$. The elements of c in (1) become

$$c_i = \sum_{j \in S_t} B_{ij} v_j \qquad (i \notin S_t)$$

and those of P are the restriction of

$$B_{ab} = \frac{1}{N} \sum_{p=-M}^{M} e^{\mathrm{j}\frac{2\pi}{N}(a-b)p}$$

to $a, b \notin S_t$. Thus, (1) reduces to the equation given in [7].

We now consider (2). The elements of d are simply $d_i = \hat{v}_i$ $(i \notin S_f)$, because $b = 0$. Those of Q are

$$Q_{ab} = \frac{1}{N} \sum_{p \notin S_t} e^{-\mathrm{j}\frac{2\pi}{N}(a-b)p}$$

with $|a|, |b| \leq M$. Thus, the elements of $I - Q$ are

$$\frac{1}{N} \sum_{p \in S_t} e^{-\mathrm{j}\frac{2\pi}{N}(a-b)p}$$

which is the (unweighted) Toeplitz matrix from Section IV of [5]. □

Remark 1: The interpolation problem may, of course, be formulated with respect to an arbitrary unitary matrix U other than F. The specific equations that we have obtained apply to two dual special cases of the generalized problem: the "primal problem," in which x is a signal and $\hat{x}$ its transform with respect to $U = F$, and a "dual problem," in which the signal would really be the DFT of x, and $\hat{x}$ its transform with respect to $U = F^H$.

Remark 2: The other discrete group that commonly occurs in signal processing is $\mathbb{Z}$, which has a continuous dual. Thus, no similar approach may yield algebraic linear equations in both time and frequency. There is a generalization for image processing [10], [11], however (the dual of $\mathbb{Z}_N \times \mathbb{Z}_N$ is again $\mathbb{Z}_N \times \mathbb{Z}_N$).

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