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# INTERPOLATION INEQUALITIES IN BESOV SPACES

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*Dedicated to Professor Takaaki Nishida on the occasion of his sixtieth birthday*

Abstract. In this paper we present an interpolation inequality in the homogeneous Besov spaces on $\mathbb{R}^n$, which reduces to a number of well-known inequalities in special cases.

## 1. Introduction

There are several types of interpolation inequalities in the Sobolev and Besov spaces on $\mathbb{R}^n$; see for instance [1]–[15] and references therein. We prove the following theorem (see below for notation).

**Theorem 1.** *Let* $\lambda, \mu, p, q, r, \theta$ *satisfy* $\lambda, \mu \in \mathbb{R},\ 1 \le p, q \le r \le \infty,\ 0 < \theta < 1$,

$$\lambda > \frac{n}{p} - \frac{n}{r}, \tag{1.1}$$

$$\mu < \frac{n}{q} - \frac{n}{r}, \tag{1.2}$$

$$\theta\left(\lambda - \frac{n}{p} + \frac{n}{r}\right) + (1-\theta)\left(\mu - \frac{n}{q} + \frac{n}{r}\right) = 0. \tag{1.3}$$

*Then there exists a constant* $C > 0$ *such that*

$$\|f; \dot{B}^0_{r,1}\| \le C\|f; \dot{B}^\lambda_{p,\infty}\|^\theta \|f; \dot{B}^\mu_{q,\infty}\|^{1-\theta} \tag{1.4}$$

*for all* $f \in \dot{B}^\lambda_{p,\infty} \cap \dot{B}^\mu_{q,\infty}$.

By the embeddings $\dot{B}^0_{r,1} \hookrightarrow L^r$ and $\dot{H}^\rho_r \hookrightarrow \dot{B}^\rho_{r,\infty}$ with $1 \le r \le \infty,\ \rho \in \mathbb{R}$, we have the following corollary.

**Corollary 2.** *Let* $\lambda, \mu, p, q, r, \theta$ *be as above. Then there exists a constant* $C > 0$ *such that*

$$\|f; L^r\| \le C\|f; \dot{H}^\lambda_p\|^\theta \|f; \dot{H}^\mu_q\|^{1-\theta} \tag{1.5}$$

*for all* $f \in \dot{H}^\lambda_p \cap \dot{H}^\mu_q$.

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The results above generalize various previously known interpolation inequalities. In [9], Miyakawa proved (1.5) in the special cases: (a) $r = \infty,\ \lambda = \mu$; (b) $r = \infty,\ p = q,\ \mu = 0$. In [5], Escobedo and Vega proved (1.5) in the special case: (c) $r = \infty,\ p, q > 1,\ 0 < \lambda,\ \mu < n$. Corollary 2 ensures that (1.5) holds when $\mu \leq 0$. In [11], (1.5) is proved in the special case: (d) $r = \infty,\ p = q = 2$.

There are some available interpolation inequalities not covered by the results above. Complex interpolation yields (1.5) with $1 < p, q, r < \infty,\ 1/r = (1-\theta)/p + \theta/q,\ (1-\theta)\lambda + \theta\mu = 0$ (see also [7]), and therefore (1.1) and (1.2) amount to additional restrictions. The known complex interpolation formulas for Besov spaces do not cover (1.4), however. A possible dependence of the third index on the interpolation inequalities is a novelty of (1.4) (see also [8, 12]).

We know a few more results which seem to be related to (1.4) and (1.5). In [6], Gérard, Meyer, and Oru proved

$$\|f; L^r\| \leq C\|f; \dot{H}^\lambda_p\|^{p/r}\|f; \dot{B}^{-\alpha}_{\infty,\infty}\|^{1-p/r}, \tag{1.6}$$

where $1 < p < r < \infty,\ \alpha(r/p - 1) = \lambda > 0$, in particular, $p = 2,\ r = 6,\ \lambda = 1,\ \alpha = 1/2$. In [4], Cohen, Dahmen, Daubechies, and De Vore proved

$$\|f; L^2\| \leq C\|f; BV\|^{1/2}\|f; B^{-1}_{\infty,\infty}\|^{1/2}, \tag{1.7}$$

where $BV$ denotes the space of functions vanishing at infinity in the weak sense and satisfying the estimate

$$\sup_{y \in \mathbb{R}^n} |y|^{-1} \int |f(x+y) - f(x)| dx \leq C.$$

We prove the theorem in the next section. The proof depends on the standard technique from the Littlewood–Paley theory (see [1, 2, 3, 6, 8, 11, 15] for instance) and therefore the theorem holds for the homogeneous Besov spaces on the Heisenberg group $\mathbb{H}^n$ with necessary modifications (see [1]).

We finally introduce the notation. For any $r$ with $1 \leq r \leq \infty,\ L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $\rho \in \mathbb{R}$ and any $r$ with $1 \leq r \leq \infty$, $\dot{H}^\rho_r$ denotes the homogeneous Sobolev space defined as the space of classes of distributions $f$ modulo polynomials such that $(-\Delta)^{\rho/2} f \in L^r$, where $\Delta$ is the Laplacian in $\mathbb{R}^n$. For any $\rho \in \mathbb{R}^n$ and any $r, m$ with $1 \leq r, m \leq \infty$, $\dot{B}^\rho_{r,m}$ denotes the homogeneous Besov space defined as the space of classes of distributions $f$ modulo polynomials such that $\{2^{\rho j}\|\varphi_j * f; L^r\|\} \in l^m(\mathbb{Z})$, where $*$ denotes the convolution in $\mathbb{R}^n$ and the Fourier transformed functions $\{\hat{\varphi}_j\} \subset C_0^\infty$ satisfy $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n \backslash \{0\},\ 0 \leq \hat{\varphi}_j \leq 1, \operatorname{supp} \hat{\varphi}_j \subset \{\xi; 2^{j-1} \leq |\xi| \leq 2^{j+1}\},\ \hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi)$. We refer to [2, 3, 7, 15] for general information on homogeneous Besov and Triebel–Lizorkin spaces.

## 2. Proof of the theorem

We may assume that $\|f; \dot{B}^\lambda_{p,\infty}\| \neq 0$ and $\|f; \dot{B}^\mu_{q,\infty}\| \neq 0$. From the support properties of $\hat{\varphi}_j$ it follows that

$$\|f; \dot{B}^0_{r,1}\| = \sum_{j\in\mathbb{Z}} \|\varphi_j * f; L^r\| \leq \sum_{j\in\mathbb{Z}} \sum_{k=j-1}^{j+1} \|\varphi_k * \varphi_j * f; L^r\|. \tag{2.1}$$

By the Young inequality, we have

$$\begin{aligned}\|\varphi_k * \varphi_j * f; L^r\| &\le \|\varphi_k; L^m\| \|\varphi_j * f; L^s\| \\ &= 2^{nk(1-1/m)} \|\varphi_0; L^m\| \|\varphi_j * f; L^s\|,\end{aligned} \tag{2.2}$$

where $1 + 1/r = 1/m + 1/s$. We apply (2.2) with $s = p, q$ to (2.1) to obtain

$$\begin{aligned}\|f; \dot{B}^0_{r,1}\| &\le C \sum_{j \ge l} 2^{(n/p-n/r-\lambda)j} \cdot 2^{\lambda j} \|\varphi_j * f; L^p\| \\ &\quad + C \sum_{j<l} 2^{(n/q-n/r-\mu)j} \cdot 2^{\mu j} \|\varphi_j * f; L^q\| \\ &\le C \sum_{j \ge l} 2^{(n/p-n/r-\lambda)j} \|f; \dot{B}^\lambda_{p,\infty}\| + C \sum_{j<l} 2^{(n/q-n/r-\mu)j} \|f; \dot{B}^\mu_{q,\infty}\| \\ &\le C(2^{(n/p-n/r-\lambda)l} \|f; \dot{B}^\lambda_{p,\infty}\| + 2^{(n/q-n/r-\mu)l} \|f; \dot{B}^\mu_{q,\infty}\|) \\ &= C(2^{(n/p-n/r-\lambda)l} a^{1-\theta} + 2^{(n/q-n/r-\mu)l} a^{-\theta}) \|f; \dot{B}^\lambda_{p,\infty}\|^\theta \|f; \dot{B}^\mu_{q,\infty}\|^{1-\theta},\end{aligned}$$

where $a = \|f; \dot{B}^\lambda_{p,\infty}\| / \|f; \dot{B}^\mu_{q,\infty}\|$.

Let $\sigma = (\lambda - n/p + n/r) - (\mu - n/q + n/r) > 0$ and let $l$ be the largest integer that is less than or equal to $\sigma^{-1} \log_2 a$. Then,

$$2^l \le a^{1/\sigma} \le 2 \cdot 2^l, \quad \theta = -(\mu - n/q + n/r)/\sigma, \quad 1 - \theta = (\lambda - n/p + n/r)/\sigma,$$

and therefore

$$2^{(n/p-n/r-\lambda)l} a^{1-\theta} \le (2a^{-1/\sigma})^{\lambda - n/p + n/r} a^{1-\theta} = 2^{\lambda - n/p + n/r},$$

$$2^{(n/q-n/r-\mu)l} a^{-\theta} \le a^{(n/q-n/r-\mu)/\sigma} a^{-\theta} = 1.$$

This proves the theorem.

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