Interpolation of Calderón and Ovčinnikov Pairs (*).

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Summary. – *In this paper we study those Banach pairs* (A_0, A_1) *for which all interpolation is described by Peetre's* K*-method of interpolation. Special emphasis is given to duality and to the case when* (A_0, A_1) *is a pair of* K*-spaces.*

1. – Introduction.

Our aim in this paper will be to study two extreme cases occurring in interpolation theory. Let $\bar{A} = (A_0, A_1)$ be a (compatible) pair of Banach spaces. For $a \in A_0 + A_1$ let $K(t, a, \bar{A})$ be the K-functional. Consider the following two statements:

$$\text{(K)} \qquad K(t, b, \bar{A}) \leqslant K(t, a, \bar{A})$$

and

$$\text{(B)}_\lambda \qquad b = Ta \text{ for some linear operator } T\colon \bar{A} \to \bar{A} \text{ with operator norm less than } \lambda.$$

It is wellknown (and easy to prove) that $(\mathrm{B})_1$ implies (K). If conversely (K) implies $(\mathrm{B})_\lambda$, for some $\lambda < \infty$, we say that $\bar{A}$ is a Calderón pair.

The assumption that $\bar{A}$ is a Calderón pair is a rather strong condition. In fact if $\bar{A}$ is Calderón then every interpolation space A with respect to $\bar{A}$ is a K-space. This means that for some interpolation space E with respect to $\bar{l}_\infty = \big(l_\infty, l_\infty((2^{-\nu})_{\nu\in\mathbf{Z}})\big)$

$$\big\|\big(K(2^\nu, a, \bar{A})\big)_{\nu\in\mathbf{Z}}\big\|_E$$

is an equivalent norm on A. The space E may be described using extremal interpolation functors (cf. th. 4.1). This result due to BRUDNYĬ-KRUGLJAK [5] is proved in sect. 4.2. Similarily one may in most cases describe A in terms of J-spaces. As J- and K- are dual functors this suggest duality results for Calderón pairs. Indeed let $\bar{A}$ be a reflexive pair and denote the dual pair by $\bar{A}^*$. Then $\bar{A}$ is Calderón iff $\bar{A}^*$ is Calderón. See th. 4.11.

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We now specialize the situation somewhat. Let $\bar{A}$ be a mutually closed Banach pair and let E_0, E_1 be two Banach interpolation spaces with respect to $\bar{l}_\infty$. Assume further that E_0, E_1 satisfy the assumptions of the classical equivalence theorem (see [17]) and consider the pair $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ of K-spaces. Then CWIKEL [8] and DMITRIEV-OVČINNIKOV [13] showed that the pair $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ is a Calderón pair whenever (E_0, E_1) is Calderón. In sect. 4.4 we prove a converse. Namely, for almost all pairs $\bar{A}$ holds that $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ is a Calderón pair iff (E_0, E_1) is a Calderón pair. This extends some partial results due to OVČINNIKOV [19].

Let us now sum up the contents of this paper briefly. In section 2 we introduce our terminology. In particular we define J- and K-spaces. In section 3 we define extremal interpolation functors and relate them to J- and K-spaces. Sect. 4 is devoted to Calderón pairs. In sect. 5 we examine OVČINNIKOV pairs which are in a sense the opposite to (relative) Calderón pairs. We describe them using extremal interpolation functors and derive some duality results.

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CONVENTIONS. – The relation $X \subseteq Y$, where X and Y are topological vector spaces, means that we have a continuous imbedding. The equivalence notion $A \approx B$, where A and B are quasi-norms, means that $c_1 A \leqslant B \leqslant c_2 A$ for some positive constants c_1, c_2. Two quasi-normed vector spaces are considered equal if their quasi-norms are equivalent. We use the notation $(\alpha_\nu)_\nu$ for any scalar or vector valued sequence with $\boldsymbol{Z}$ as index set. Further all sums without summation index are taken over all of $\boldsymbol{Z}$. As usual C will denote any immaterial, positive constant.

2. – Preliminaries and notation.

Let A_0 and A_1 be two quasi-Banach spaces. We say that $\bar{A} = (A_0, A_1)$ is a quasi-Banach pair if both A_0 and A_1 are continuously embedded in some Hausdorff topological vector space $\mathcal{H}$. We let

$$\Delta(\bar{A}) = A_0 \cap A_1 \quad \text{ and } \Sigma(\bar{A}) = A_0 + A_1 .$$

A quasi-Banach space A is called an intermediate space with respect to $\bar{A}$ iff $\Delta(\bar{A}) \subseteq A \subseteq \Sigma(\bar{A})$, continuous imbeddings. Let $\bar{A}$ and $\bar{B}$ be two quasi-Banach pairs. Let T be a linear operator mapping $\Sigma(\bar{A})$ into $\Sigma(\bar{B})$. We write $T\colon \bar{A} \to \bar{B}$ if the restrictions $T|_{A_i}$ are bounded linear mappings from A_i into B_i. We define the operator quasi-norm of T by $\|T\|_{\bar{A},\bar{B}} = \max_{i=1,0} (\|T|_{A_i}\|_{A_i,B_i})$, where $\|\cdot\|_{A_i,B_i}$ is the usual operator quasi-norm. The vector space of all $T: \bar{A} \to \bar{B}$ with $\|T\|_{\bar{A},\bar{B}} < \infty$ is denoted by $\mathfrak{L}(\bar{A}, \bar{B})$.

Let A and B be two intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. A and B are said to be relative interpolation spaces between $\bar{A}$ and $\bar{B}$ if whenever $T: \bar{A} \to \bar{B}$ it follows that $\|Ta\|_B \leqslant c\|T\|_{\bar{A},\bar{B}}\|a\|_A$, $a \in A$. If $\bar{A} = \bar{B}$ and $A = B$ we simply say that A is an interpolation space with respect to $\bar{A}$.

Let A be an intermediate space with respect to $\bar{A}$. We denote by A^0, the closure of $\Delta(\bar{A})$ in A. In particular, $\Sigma(\bar{A})^0$ denotes the closure of $\Delta(\bar{A})$ in $\Sigma(\bar{A})$.

Let $t > 0$. The K-functional is defined for $a \in \Sigma(\bar{A})$ by

$$K(t, a, \bar{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1})$$

and similarily the J-functional for $a \in \Delta(\bar{A})$ by

$$J(t, a, \bar{A}) = \max (\|a\|_{A_0}, t\|a\|_{A_1}) .$$

Let $\bar{A}$ be a pair of p-normed quasi-Banach spaces. Take $a \in \Sigma(\bar{A})^0$ and write $a = \sum_\nu a_\nu$ where $a_\nu \in \Delta(\bar{A})$ and the series converges in $\Sigma(\bar{A})$. The series $\sum_\nu a_\nu$ is called a representation of a if in addition

$$\Big(\sum_\nu (\min(1, 2^{-\nu}) J(2^\nu, a_\nu, \bar{A}))^p\Big)^{1/p}$$

is finite. From lemma 5.1 in [10] follows that every $a \in \Sigma(\bar{A})^0$ has a representation. Further any representation of a is p-absolutely convergent in $\Sigma(\bar{A})$.

A sequence $\omega = (\omega_\nu)_{\nu \in \mathbf{Z}}$ is called a weight sequence if each ω_ν is positive. Let $0 < p \leqslant \infty$. The space $l_p(\omega) = l_p(\omega_\nu)$ is defined to consist of all sequences $\alpha = (\alpha_\nu)_\nu$ such that $(\alpha_\nu\omega_\nu)_\nu \in l_p$, i.e. $\big(\sum_\nu |\alpha_\nu\omega_\nu|^p\big)^{1/p}$ is finite. We define $c_0(\omega)$ in a similar way. Let $\bar{l}_p$ and $\bar{c}_0$ denote the pairs $(l_p, l_p(2^{-\nu}))$ and $(c_0, c_0(2^{-\nu}))$ respectively. We denote by e_μ the sequence $(\delta_{\nu,\mu})_\nu$, $\mu \in \mathbf{Z}$.

Let $\mathfrak{T}$ denote the set of all positive functions φ on $\mathbb{R}_+$ such that both $\varphi(t)$ and $t\varphi(1/t)$ are nondecreasing. We let $\mathfrak{T}_0$ denote the subset of $\mathfrak{T}$ consisting of all φ with $\min(1, 1/t)\varphi(t) \to 0$ as $t \to 0, \infty$. Observe that for $\varphi \in \mathfrak{T}$ then $(\varphi(2^\nu))_\nu \in \Sigma(\bar{l}_\infty)$. Similary $(\varphi(2^\nu))_\nu \in \Sigma(\bar{c}_0)$, whenever $\varphi \in \mathfrak{T}_0$. On $\mathfrak{T}$ we define an involution by $\varphi^*(t) = 1/\varphi(1/t)$.

We next define J- and K-spaces. Let E be an interpolation space with respect to $\bar{l}_\infty$. The K-space $\bar{A}_{E:K}$ consist of all $a \in \Sigma(\bar{A})$ with $(K(2^\nu, a, \bar{A}))_\nu \in E$. We put $\|a\|_{\bar{A}_{E:K}} = \|(K(2^\nu, a, \bar{A}))_\nu\|_E$. Assume that both A_0 and A_1 are p-normed vector spaces and let F be any interpolation space with respect to $\bar{l}_p$: The J-space $\bar{A}_{F:J}$ is defined to consist of all $a \in \Sigma(\bar{A})^0$ such that there is a representation $a = \sum_\nu a_\nu$ with $(J(2^\nu, a_\nu, \bar{A}))_\nu \in F$. We put

$$\|a\|_{\bar{A}_{F:J}} = \inf_{a=\sum_\nu a_\nu} \|(J(2^\nu, a_\nu, \bar{A}))_\nu\|_F .$$

Take $\varphi \in \mathfrak{P}$. If $E = l_\infty(1/\varphi(2^\nu))$ and $F = l_p(1/\varphi(2^\nu))$ we write $\bar{A}_{\varphi,\infty:K}$ respectively $\bar{A}_{\varphi,p:J}$. If $\varphi(t) = t^\theta$, $0 \leqslant \theta \leqslant 1$, we get the spaces $\bar{A}_{\theta,\infty:K}$ and $\bar{A}_{\theta,p:J}$. If $0 < \theta < 1$, we may by the classical equivalence theorem omit the indices J and K (see [4], p. 44).

A quasi-Banach pair $\bar{A}$ is called mutually closed iff $A_i = \bar{A}_{i,\infty:K}$, $i = 0, 1$. $\bar{A}$ is named a regular pair if $\Delta(\bar{A})$ is dense in both A_0 and A_1. For a regular Banach pair $\bar{A}$ one may form the dual pair $\bar{A}^* = (A_0^*, A_1^*)$ (see [4], p. 32). The duality for the sequence spaces occuring in this paper is with respect to the form

$$\langle \alpha, \beta \rangle = \sum_\nu \alpha_\nu \overline{\beta_{-\nu}} .$$

Let $\bar{A}$ be a mutually closed Banach pair and take $a \in \Sigma(\bar{A})^0$. One may then find a representation $a = \sum_\nu a_\nu$ such that for all $t > 0$ holds

$$\sum_\nu \min(1, t/2^\nu) J(2^\nu, a_\nu, \bar{A}) \leqslant 18 K(t, a, \bar{A}) . \tag{2.1}$$

This is a consequence of th. 4 in [5]. See also [9], th. 4, [17], th. 3.2.

Let $\bar{A}$ be a Banach pair and take $a \in \Sigma(\bar{A})$. Pick linear functionals Λ_ν on $\Sigma(\bar{A})$ such that $\Lambda_\nu(a) = K(2^\nu, a, \bar{A})$ and for all $b \in \Sigma(\bar{A})$ holds $|\Lambda_\nu(b)| \leqslant K(2^\nu, b, \bar{A})$. The (first) fundamental operator

$$T^a \colon \bar{A} \to \bar{l}_\infty$$

is defined by

$$T^a(b) = (\Lambda_\nu(b))_\nu .$$

Clearly T is a norm one linear operator with $T^a(a) = (K(2^\nu, a, \bar{A}))_\nu$.

Take $a \in \Sigma(\bar{A})^0$ and let $a = \sum_\nu a_\nu$ be a representation of a. The (second) fundamental operator

$$T_{(a_\nu)_\nu} \colon \bar{l}_1 \to \bar{A}$$

is defined by

$$T_{(a_\nu)_\nu}(\gamma) = \sum_{a_\nu \neq 0} \gamma_\nu \frac{a_\nu}{J(2^\nu, a_\nu, \bar{A})} .$$

It follows that $T_{(a_\nu)_\nu}\big((J(2^\nu, a_\nu, \bar{A}))_\nu\big) = a$. These operators made their first explicit appearance in Cwikel's paper [8]. See also [6] and [13].

For results concerning real interpolation spaces we refer to [4], [5], [6], [7], [8], [9], [10], [13], [15], [17], [22].

3. – Extremal interpolation functors.

In a now classical paper, ARONSZAJN and GAGLIARDO [1] introduced extremal interpolation functors. Let us recall their constructions. We take two fixed Banach pairs $\bar{A}$ and $\bar{B}$. Let A and B be two intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. Assume further that A is p-normed. We define the orbit functor $\mathrm{Orb}_{\bar{A}}(A, \cdot)$ as follows. Let $\bar{X}$ be a Banach pair. Then $x \in \mathrm{Orb}_{\bar{A}}(A, \bar{X})$ iff we may write $x = \sum_{i\geq 0} T_i a_i$ (convergence in $\Sigma(\bar{X})$), where $T_i \in \mathfrak{L}(\bar{A}, \bar{X})$, $a_i \in A$ and the sum $\sum_{i\geq 0} (\|T_i\|_{\bar{A},\bar{X}} \|a_i\|_A)^p$ is convergent. Put

$$\|x\|_{\mathrm{Orb}_{\bar{A}}(A,\bar{X})} = \inf\Big(\sum_{i\geq 0} (\|T_i\|_{\bar{A},\bar{X}} \|a_i\|_A)^p\Big)^{1/p}$$

where the infimum is taken over all admissible representations of x. We now turn to define the coorbit functor $\mathrm{Corb}_{\bar{B}}(\cdot, B)$. The space $\mathrm{Corb}_{\bar{B}}(\bar{X}, B)$ consists of all $x \in \Sigma(\bar{X})$ such that $Tx \in B$ for any $T \in \mathfrak{L}(\bar{X}, \bar{B})$. We put

$$\|x\|_{\mathrm{Corb}_{\bar{B}}(\bar{X},B)} = \sup_{\|T\|_{\bar{X},\bar{B}} \leq 1} \|Tx\|_B .$$

The choice of the term « extremal interpolation functors » alludes to the following simple results. Let $\mathcal{F}$ be any interpolation functor with $\mathcal{F}(\bar{A}) \supseteq A$. Then it follows that

$$\mathrm{Orb}_{\bar{A}}(A, \bar{X}) \subseteq \mathcal{F}(\bar{X})$$

for any Banach pair $\bar{X}$ such that $\mathcal{F}(\bar{X})$ is p-normed. Similary if $\mathcal{F}(\bar{B}) \subseteq B$ we may infer that

$$\mathcal{F}(\bar{X}) \subseteq \mathrm{Corb}_{\bar{B}}(\bar{X}, B) .$$

For proofs we refer to [1] or [4], pp. 29-33. The extension to the quasi-Banach case is straightforward.

Two more interpolation methods are of importance in our investigation. Take $a \in \Sigma(\bar{A})$. The space $O_{\bar{A}}(a, \bar{X})$, the orbit of a in $\bar{X}$, consists of all $x \in \Sigma(\bar{X})$ that may be written as $x = Ta$, where $T \in \mathfrak{L}(\bar{A}, \bar{X})$. Put

$$\|x\|_{O_{\bar{A}}(a,\bar{X})} = \inf\{\|T\|_{\bar{A},\bar{X}} : Ta = x\} .$$

This functor has been studied extensively by OVČINNIKOV [18], [19]. Let $\bar{B}$ be a regular pair and take $b^* \in \Sigma(\bar{B}^*)$. Consider the following norm on $\Delta(\bar{X})$. For $x \in \in \Delta(\bar{X})$ put

$$\|x\|_{\mathrm{Co}_{\bar{B}}(b^*,\bar{X})} = \sup_{\|T\|_{\bar{X},\bar{B}}\leq 1} |\langle Tx, b^*\rangle| .$$

We let $\mathrm{Co}_{\bar{B}}(b^*, \bar{X})$, the coorbit of b^* in $\bar{X}$, be the completion of $\Delta(\bar{X})$ in this norm.

The following proposition relates our various functors to each other.

PROPOSITION 3.1. – $O_{\bar{A}}(a, \bar{X}) = \mathrm{Orb}_{\bar{A}}(O_{\bar{A}}(a, \bar{A}), \bar{X})$ and

$$\mathrm{Co}_{\bar{B}}(b^*, \bar{X}) = \mathrm{Corb}_{\bar{B}}(\bar{X}, \mathrm{Co}_{\bar{B}}(b^*, \bar{B}))^0 .$$

The proof of these simple facts is left for the reader.

It turns out that most interpolation methods occuring in analysis can be described in terms of extremal interpolation functors. Let us recall the most important examples.

EXAMPLE 3.2. – Let E and F be two interpolation spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ respectively. Then holds for any Banach pair $\bar{X}$

$$\bar{X}_{E:K} = \mathrm{Corb}_{\bar{l}_\infty}(\bar{X}, E)$$

and

$$\bar{X}_{F:J} = \mathrm{Orb}_{\bar{l}_1}(F, \bar{X}) .$$

These results are due to BRUDNYĬ-KRUGLJAK [5], th. 2 and JANSON [15], th. 14. The proof depends mainly on a clever use of the two fundamental operators. In this connection we also refer to [13].

EXAMPLE 3.3. – As an example of a coorbit functor let us investigate the functor $\mathrm{Co}_{\bar{l}_1}(\varphi, \cdot)$, where $\varphi = (\varphi(2^\nu))_\nu$, $\varphi \in \mathfrak{T}$. We claim that for any Banach pair $\bar{X}$ holds

$$\mathrm{Co}_{\bar{l}_1}(\varphi, \bar{X}) = \mathrm{Corb}_{\bar{l}_1}\big(\bar{X}, l_1(\varphi(2^{-\nu}))\big)^0 .$$

Indeed take $x \in \Delta(\bar{X})$ and let $T \in \mathfrak{L}(\bar{X}, \bar{l}_1)$ be any norm one operator. Then $Tx = = (\langle x, x_\nu^*\rangle)_\nu$ for some sequence $x_\nu^* \in \Sigma(\bar{X})^*$. We have

$$\langle Tx, \varphi\rangle = \sum_\nu \varphi(2^{-\nu})\langle x, x_\nu^*\rangle$$

and consequently holds

$$|\langle Tx, \varphi\rangle| \leq \sum_\nu \varphi(2^{-\nu})|\langle x, x_\nu^*\rangle| = \|Tx\|_{l_1(\varphi(2^{-\nu}))} .$$

This shows the inclusion $\supseteq$. If we compose T with diagonal operators of norm one it follows that

$$\|Tx\|_{l_1(\varphi(2^{-\nu}))} \leqslant \sup_{\|S\|_{\bar{X},\bar{l}_1} \leqslant 1} |\langle Sx, \varphi \rangle| .$$

Consequently $\mathrm{Co}_{\bar{l}_1}(\varphi, \bar{X}) \subseteq \mathrm{Corb}_{\bar{l}_1}\big(\bar{X}, l_1(\varphi(2^{-\nu}))\big)^0$.

The functor $\mathrm{Corb}_{l_1}\big(\cdot, l_1(\varphi(2^{-\nu}))\big)$ plays a central role in OVČINNIKOV's work [18], [19], and is there denoted by $\varphi_u^*(\cdot)$. See also JANSON [15], th. 5.

We remark that explicit descriptions of the functors $O_{\bar{l}_\infty}(\varphi, \cdot)$, $\varphi \in \mathfrak{T}$, may be found in JANSON [15], chap. 4. By th. 4 of [15], $O_{\bar{l}_\infty}(\varphi, \cdot)$ equals the $\varphi_l(\cdot)$ functor of OVČINNIKOV [18]. If $\varphi \in \mathfrak{T}_0$ we may consider the orbit functor $O_{\bar{c}_0}(\varphi, \cdot)$. From th. 5 of [15] follows that this functor coincides with the $\pm$ method of interpolation introduced in [14].

From the characterization of K- and J-interpolation as extremal interpolation methods we may infer that for any interpolation functor $\mathcal{F}$ and for any Banach pair $\bar{A}$ holds

$$\bar{A}_{\mathcal{F}(\bar{l}_1):J} \subseteq \mathcal{F}(\bar{A}) \subseteq \bar{A}_{\mathcal{F}(\bar{l}_\infty):K} . \tag{3.1}$$

In particular holds that

$$\bar{B}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_1):J} \underset{\text{(i)}}{\subseteq} \mathrm{Orb}_{\bar{A}}(A, \bar{B}) \underset{\text{(ii)}}{\subseteq} \bar{B}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_\infty):K} \tag{3.2}$$

and

$$\bar{A}_{\mathrm{Corb}_{\bar{B}}(\bar{l}_1,B):J} \underset{\text{(iii)}}{\subseteq} \mathrm{Corb}_{\bar{B}}(\bar{A}, B) \underset{\text{(iv)}}{\subseteq} \bar{A}_{\mathrm{Corb}_{\bar{B}}(\bar{l}_\infty,B):K} \tag{3.3}$$

whenever A and B are intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. A natural question is to find conditions on $\bar{A}$ and $\bar{B}$ such that equality holds in (3.2) and (3.3). As we will see equality in (ii) and (iii) characterizes relative Calderón pairs. Further whenever $\bar{A}$ and $\bar{B}$ are of type (O), see chap. 5 below, there is equality in (i) and (iv).

For later reference we state here also the following inclusions. For $a \in \Sigma(\bar{A})$ holds with $\varphi(t) = K(t, a, \bar{A})$

$$\bar{B}_{\varphi,1:J} \subseteq O_{\bar{A}}(a, \bar{B}) \subseteq \bar{B}_{\varphi,\infty:K} . \tag{3.4}$$

Let $\bar{B}$ be a regular pair. Take $b^* \in \Sigma(\bar{B}^*)$ and put $\varphi(t) = K^*(t, b^*, \bar{B}^*)$. Then similary

$$\bar{A}_{\varphi,1:J} \subseteq \mathrm{Co}_{\bar{B}}(b^*, \bar{A}) \subseteq \bar{A}^0_{\varphi,\infty:K} . \tag{3.5}$$

Proofs of these formulas will be given below. See also [7], [19].

4. – Calderön pairs.

4.1. *Definitions.*

Let $\bar{A}$ and $\bar{B}$ be two quasi-Banach pairs. Let $a \in \Sigma(\bar{A})$ and $b \in \Sigma(\bar{B})$ be two elements such that for all $t > 0$ holds

$$K(t, b, \bar{B}) \leqslant K(t, a, \bar{A}) . \tag{4.1}$$

If b is any norm one element in $O_{\bar{A}}(a, \bar{B})$, clearly (4.1) is satisfied. If conversely (4.1) implies that $b = Ta$ for some $T: \bar{A} \to \bar{B}$ we say that the orbit $O_{\bar{A}}(a, \bar{B})$ is described by the K-method. One may then choose T such that the norm of T is bounded by a constant not depending on b. This is a easy consequence of the open mapping theorem. Thus the orbit is described by the K-method iff

$$O_{\bar{A}}(a, \bar{B}) = \bar{B}_{\varphi, \infty: K} \tag{4.2}$$

where $\varphi(t) = K(t, a, \bar{A})$.

Let $1 \leqslant \lambda < \infty$. We say that $\bar{A}$ and $\bar{B}$ are of type $\lambda - (C)$ iff whenever (4.1) holds and $\lambda' > \lambda$ one may find $T: \bar{A} \to \bar{B}$, of norm less than λ', such that $Ta = b$. If for some $\lambda < \infty$, $\bar{A}$ and $\bar{B}$ are of type $\lambda - (C)$ we simply say that $\bar{A}$ and $\bar{B}$ are of type (C). Alternatively we say that $\bar{A}$ and $\bar{B}$ are relative Calderón. If $\bar{A} = \bar{B}$ we call $\bar{A}$ a Calderón pair.

The known concrete Calderón pairs in the literature fall into two classes. The first concern weighted l_p spaces. Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. Put $\bar{A} = \big(l_{p_0}(\omega_0), l_{p_1}(\omega_1)\big)$ and $\bar{B} = \big(l_{q_0}(\sigma_0), l_{q_1}(\sigma_1)\big)$, where $\omega_i, \sigma_i, i = 0, 1$, are weight sequences. Then if $1 \leqslant p_i \leqslant q_i \leqslant \infty$, $i = 0, 1$, $\bar{A}$ and $\bar{B}$ are of type (C) by a theorem of DMITRIEV [11], cor. 1. The diagonal case, $p_i = q_i$, $i = 0, 1$, may be found in [7] and [25]. See also [23], [24]. If however $p_0 > q_0$ or $p_1 > q_1$, $\bar{A}$ and $\bar{B}$ are not relative Calderón. See [19], th. 4. The other type of Calderón pairs are those who are covered by th. 4.15 and th. 4.17 below. See also [8], th. 1, [13], th. 2 and [19], th. 7.

Let $\bar{A}$ be a Banach pair. Then $\bar{A}$ and $\bar{l}_\infty$ are of type (C). Similarly $\bar{l}_1$ and $\bar{A}$ are of type $18 - (C)$ whenever $\bar{A}$ is mutually closed (see [10], sect. 4).

4.2. *Calderón pairs and extremal interpolation functors.*

We now extend (4.2) to general orbit functors.

THEOREM 4.1. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs and A any intermediate space with respect to $\bar{A}$. If $\bar{A}$ and $\bar{B}$ are of type (C) then

$$\mathrm{Orb}_{\bar{A}}(A, \bar{B}) = \bar{B}_{\mathrm{Orb}_{\bar{A}}(A, \bar{l}_\infty): K} . \tag{4.3}$$

PROOF. – Choose p such that A is a p-normed. Take $b \in \bar{B}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_\infty):K}$, i.e. $\big(K(2^\nu, b, \bar{B})\big)_\nu \in \mathrm{Orb}_{\bar{A}}(A, \bar{l}_\infty)$. By the definition of norm of $\big(K(2^\nu, b, \bar{B})\big)_\nu$ in $\mathrm{Orb}_{\bar{A}}(A, \bar{l}_\infty)$ we may write

$$\big(K(2^\nu, b, \bar{B})\big)_\nu = \sum_{i\geqslant 0} T_i a_i$$

where $T_i \in \mathfrak{L}(\bar{A}, \bar{l}_\infty)$, $a_i \in A$ and the series

$$\Big(\sum_{i\geqslant 0} (\|T_i\|_{\bar{A},\bar{l}_\infty} \|a_i\|_A)^p\Big)^{1/p}$$

converges. Thus

$$K(2^\nu, b, \bar{B}) \leqslant \sum_{i\geqslant 0} \|T_i\|_{\bar{A},\bar{l}_\infty} K(2^\nu, a_i, \bar{A}) .$$

The series converges since $A \subseteq \Sigma(\bar{A})$. From the BRUDNYĬ-KRUGLJAK theorem on K-divisibility [5], th. 4 (see also [9], th. 1, [17], th. 3.1) now follows that there exists $b_i \in \Sigma(\bar{B})$ such that $b = \sum_{i\geqslant 0} b_i$ and for all $t > 0$ holds

$$K(t, b_i, \bar{B}) \leqslant 28 \|T_i\|_{\bar{A},\bar{l}_\infty} K(t, a_i, \bar{A}) . \tag{4.4}$$

As $\bar{A}$ and $\bar{B}$ are of type $\lambda - (C)$, for some $\lambda < \infty$, (4.4) implies that $b_i = S_i a_i$ where $S_i \in \mathfrak{L}(\bar{A}, \bar{B})$ and $\|S_i\|_{\bar{A},\bar{B}} \leqslant 29\lambda \|T_i\|_{\bar{A},\bar{l}_\infty}$.

Consequently $b = \sum_{i\geqslant 0} b_i = \sum_{i\geqslant 0} S_i a_i$ where $\big(\sum_{i\geqslant 0} (\|S_i\|_{\bar{A},\bar{B}} \|a_i\|_A)^p\big)^{1/p}$ is finite, i.e. $b \in \mathrm{Orb}_{\bar{A}}(A, \bar{B})$. The converse inclusion follows from (3.2).

Let A and B be two intermediate spaces between $\bar{A}$ and $\bar{B}$. They are called relative K-monotone if whenever $a \in A$, $b \in \Sigma(\bar{B})$ and (4.1) holds it follows that $\|b\|_B \leqslant c\|a\|_A$. If this holds with $A = B$, $\bar{A} = \bar{B}$ we say that A is a K-monotone. Our next results are related to cor. 3 and cor. 4 in [5]. See also [12].

COROLLARY 4.2. – Let A and B be two K-monotone p-normed intermediate spaces with respect to $\bar{A}$ and $\bar{B}$. Then

$$B \supseteq \bar{B}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_\infty):K}$$

and

$$A \subseteq \bar{A}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_\infty):K} .$$

PROOF. – We argue as in th. 4.1 up to (4.4). The estimate (4.4) implies that $\|b_i\|_B \leqslant c\|T_i\|_{\bar{A},\bar{l}_\infty} \|a_i\|_A$ and consequently

$$\|b\|_B \leqslant \Big(\sum_{i\geqslant 0} \|b_i\|_B^p\Big)^{1/p} \leqslant c \Big(\sum_{i\geqslant 0} (\|T_i\|_{\bar{A},\bar{l}_\infty} \|a_i\|_A)^p\Big)^{1/p} .$$

Thus $\bar{B}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_\infty):K} \subseteq B$.

As $A \subseteq \operatorname{Orb}_{\bar{A}}(A, \bar{A})$ it follows from (3.2) that $A \subseteq \bar{A}_{\operatorname{Orb}_{\bar{A}}(A,\bar{l}_\infty):K}$.

COROLLARY 4.3. – Let $\bar{A}$ be a Banach pair. Then holds for any K-monotone intermediate space A with respect to $\bar{A}$ that

$$A = \bar{A}_{\operatorname{Orb}_{\bar{A}}(A,\bar{l}_\infty):K}\,.$$

In particular if $\bar{A}$ is Calderón then every interpolation space with respect to $\bar{A}$ is a K-space.

PROOF. – Apply cor. 4.2 with $A = B$, $\bar{A} = \bar{B}$.

Let $\bar{A}$ be a Banach pair. In the fundamental paper [5], BRUDNYĬ and KRUGLJAK proved that if $\bar{A}$ is a Calderón pair and A is an interpolation space with respect to $\bar{A}$ then for some interpolation space E with respect to $\bar{l}_\infty$ holds $A = \bar{A}_{E:K}$. See [5], cor. 4. The point we wish to emphasize here is the use of extremal interpolation functors, as it provides us with an explicit description of space E. Further our approach connects this deep result of BRUDNYÎ-KRUGLJAK with the elementary formula (4.2).

We now prove a dual version of th. 4.1.

THEOREM 4.4. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that $\bar{A}$ is mutually closed and that $\bar{A}$ and $\bar{B}$ are of type (C). Then holds for every intermediate space B with respect to $\bar{B}$

$$\bar{A}_{\operatorname{Corb}_{\bar{B}}(\bar{l}_1,B):J} = \operatorname{Corb}_{\bar{B}}(\bar{A}, B) \cap \Sigma(\bar{A})^0\,.$$

PROOF. – Take $a \in \operatorname{Corb}_{\bar{B}}(\bar{A}, B) \cap \Sigma(\bar{A})^0$. Choose $a_\nu \in \Delta(\bar{A})$ such that $a = \sum_\nu a_\nu$ and such that for all $t > 0$ holds

$$K\big(t, (J(2^\nu, a_\nu, \bar{A}))_\nu, \bar{l}_1\big) \leqslant 18K(t, a, \bar{A})\,. \tag{4.5}$$

See (2.1). We claim that $\big(J(2^\nu, a_\nu, \bar{A})\big)_\nu \in \operatorname{Corb}_{\bar{B}}(\bar{l}_1, B)$.

Indeed take $T \in \mathfrak{L}(\bar{l}_1, \bar{B})$ of norm one. Put $b = T\big((J(2^\nu, a_\nu, \bar{A}))_\nu\big)$. Then $b = \sum_\nu b_\nu$ where $b_\nu = J(2^\nu, a_\nu, \bar{A})\,Te_\nu$. Clearly $b_\nu \in \Delta(\bar{B})$ and

$$J(2^\nu, b_\nu, \bar{B}) \leqslant J(2^\nu, a_\nu, \bar{A})\,. \tag{4.6}$$

From (4.5) and (4.6) we infer that

$$K(t, b, \bar{B}) \leqslant 18K(t, a, \bar{A})\,.$$

Consequently we may find $S\colon \bar{A} \to \bar{B}$ with $\|S\|_{\bar{A},\bar{B}} \leqslant c$ and $Sa = b$. Now

$$\|T\big((J(2^\nu, a_\nu, \bar{A}))_\nu\big)\|_B = \|b\|_B = \|Sa\|_B \leqslant c \sup_{\|V\|_{\bar{A},\bar{B}} \leqslant 1} \|Va\|_B = c\|a\|_{\mathrm{Corb}_{\bar{B}}(\bar{A},B)}$$

i.e. $(J(2^\nu, a_\nu, \bar{A}))_\nu \in \mathrm{Corb}_{\bar{B}}(\bar{l}_1, B)$.

The converse inclusion is a consequence of (3.3).

COROLLARY 4.5. – Let $\bar{A}$ be a Banach pair of type (C). For any interpolation space A with respect to $\bar{A}$ then holds

$$A \cap \Sigma(\bar{A})^0 = \bar{A}_{\mathrm{Corb}_{\bar{A}}(\bar{l}_1, A):J}\,.$$

PROOF. – By lemma 3 in [7] every pair of type (C) is mutually closed. As $A = \mathrm{Corb}_{\bar{A}}(\bar{A}, A)$ the cor. now follows from th. 4.4.

We also have the following result, dual to (4.2).

PROPOSITION 4.6. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs with $\bar{A}$ mutually closed, $\bar{B}$ regular and assume in addition that they are of type (C). Let $b^* \in \Sigma(\bar{B}^*)$. Then holds

$$\bar{A}_{\varphi,1:J} = \mathrm{Co}_{\bar{B}}(b^*, \bar{A})$$

with $\varphi(t) = K^*(t, b^*, \bar{B}^*)$.

PROOF. – By prop. 3.1 and th. 4.4 we only need to show that $\mathrm{Co}_{\bar{B}}(b^*, \bar{l}_1) = l_1(1/\varphi(2^\nu))$. Let $T\colon \bar{l}_1 \to \bar{B}$ be any norm one operator. Observe that T has norm not exceeding one iff $J(2^\nu, Te_\nu, \bar{B}) \leqslant 1$, $\nu \in \mathbf{Z}$. Consider any sequence $\alpha \in \Delta(\bar{l}_1)$ of finite support. Then

$$|\langle T\alpha, b^*\rangle| = \Big|\sum_\nu \alpha_\nu \langle Te_\nu, b^*\rangle\Big| \leqslant \sum_\nu |\alpha_\nu| K(2^{-\nu}, b^*, \bar{B}^*) J(2^\nu, Te_\nu, \bar{B}) \leqslant \sum_\nu |\alpha_\nu| 1/\varphi(2^\nu)\,.$$

This shows the inclusion $\supseteq$. But the argument may be reversed. Indeed let $\varepsilon > 0$ and choose $b_\nu \in \Delta(\bar{B})$ such that

$$\langle b_\nu, b^*\rangle \geqslant (1-\varepsilon) K(2^{-\nu}, b^*, \bar{B}^*)$$

and $J(2^\nu, b_\nu, \bar{B}) \leqslant 1$. Define $T\colon \bar{l}_1 \to \bar{B}$ by $T(e_\nu) = b_\nu$. It follows that

$$(1-\varepsilon) \sum_\nu |\alpha_\nu|/\varphi(2^\nu) \leqslant \sup_{\|S\|_{\bar{l}_1,\bar{B}} \leqslant 1} |\langle S\alpha, b^*\rangle|$$

and the inclusion $\subseteq$ follows.

Similary (4.2) follows from th. 4.1 by choosing $A = O_{\bar{A}}(a, \bar{A})$.

We now give some examples of applications of th. 4.1 and 4.4.

EXAMPLE 4.7. – i) Put $\bar{A} = \bar{l}_1$. Let $\bar{B}$ be any Banach pair such that $\bar{l}_1$ and $\bar{B}$ are of type (C). This is the case whenever $\bar{B}$ is mutually closed (combine (2.1) with th. 4.4 of [10]). Let F be any interpolation space with respect to $\bar{l}_1$. From th. 4.1 and ex. 3.2 now follows that

$$\bar{B}_{F:J} = \bar{B}_{(\bar{l}_\infty)_{F:J}:K} .$$

See also [5], th. 5, [17], th. 3.17.

ii) Let $\bar{A}$ be any mutually closed Banach pair and put $\bar{B} = \bar{l}_\infty$. By th. 4.1 in [10], $\bar{A}$ and $\bar{l}_\infty$ are of type (C). Using ex. 3.2 and th. 4.4 we conclude that

$$\bar{A}_{E:K} \cap \Sigma(\bar{A})^0 = \bar{A}_{(\bar{l}_1)_{E:K}:J} .$$

Here E is any interpolation space with respect to $\bar{l}_\infty$. See also [5], th. 6, [10], th. 4.6, [17], th. 3.19.

Let $\omega = (\omega_\nu)_\nu$ be a weight sequence. The space $\mathcal{F}L^p(\omega)$, $1 \leq p \leq \infty$, is defined to consist of all sequences $(\gamma_\nu)_\nu$ such that $\sum_\nu e^{i\nu x}\gamma_\nu\omega_\nu \in L^p[0, 2\pi]$. Similary we define $\mathcal{F}\mathcal{M}(\omega)$, where $\mathcal{M}$ denotes the space of all bounded measures on $[0, 2\pi]$.

Put $\overline{\mathcal{F}L^p} = (\mathcal{F}L^p(2^{\nu\theta}), \mathcal{F}L^p(2^{-\nu(1-\theta)}))$ and $\mathcal{F}L^p = \mathcal{F}L^p(1)$. We denote by $[\ ]_\theta$ and $[\ ]^\theta$ Calderón's two methods of interpolation. See [4], chap. 4.

PROPOSITION 4.8. – i) Let $\bar{B}$ be a Banach pair such that $\overline{\mathcal{F}L^1}$ and $\bar{B}$ are of type (C). If $0 < \theta < 1$ then

$$[\bar{B}]_\theta = \bar{B}^0_{\theta,\infty}$$

and

$$[\bar{B}]^\theta = \bar{B}_{\theta,\infty} .$$

ii) Let $\bar{B}$ be a mutually closed Banach pair such that $\bar{B}$ and $\overline{\mathcal{F}L^\infty}$ are of type (C). Then

$$[\bar{B}]_\theta = \bar{B}_{\theta,1}$$

where $0 < \theta < 1$.

PROOF. – i) By th. 22 of [15] holds $[\bar{B}]_\theta = \mathrm{Orb}_{\overline{\mathcal{F}L^1}}(\mathcal{F}L^1, \bar{B})$ and $[\bar{B}]^\theta = \mathrm{Orb}_{\overline{\mathcal{F}L^1}}(\mathcal{F}\mathcal{M}, \bar{B})$. From th. 4.1 we now infer that

$$[\bar{B}]_\theta = \bar{B}_{[\bar{l}_\infty]_\theta:K} = \bar{B}_{c_0(2^{-\nu\theta}):K} = \bar{B}^0_{\theta,\infty}$$

and

$$[\bar{B}]_\theta = \bar{B}_{[\bar{l}_\infty]^\theta:K} = \bar{B}_{l_\infty(2^{-\nu\theta}):K} = \bar{B}_{\theta,\infty} .$$

ii) From th. 24 of [15] follows

$$[\bar{B}]_\theta = \mathrm{Corb}_{\overline{\mathcal{F}L^\infty}}(\bar{B}, \mathcal{F}L^\infty)^0 .$$

Now ii) follows from th. 4.4 as $[\bar{l}_1]_\theta = l_1(2^{-\nu\theta})$.

EXAMPLE 4.9. – i) Let $\bar{A}$ be a Banach pair such that $\bar{l}_\infty$ and $\bar{A}$ are of type (C). Th. 4.1 implies that for any interpolation space E with respect to $\bar{l}_\infty$ holds

$$\mathrm{Orb}_{\bar{l}_\infty}(E, \bar{A}) = \bar{A}_{E:K} .$$

In particular we have for every $\varphi \in \mathcal{P}$

$$O_{\bar{l}_\infty}(\varphi, \bar{A}) = \bar{A}_{\varphi,\infty:K} .$$

See [10], th. 4.2, [2], lemma 6.

ii) Let $\bar{A}$ be a mutually closed Banach pair. Assume further that $\bar{A}$ and $\bar{l}_1$ are relative Calderón (see [10], th. 4.5). Then holds

$$\bar{A}_{F:J} = \mathrm{Corb}_{\bar{l}_1}(\bar{A}, F) \cap \Sigma(\bar{A})^0$$

whenever F is an interpolation space with respect to $\bar{l}_1$. Just apply th. 4.4. Further whenever $\varphi \in \mathcal{P}$ holds

$$\bar{A}_{\varphi,1:J} = \mathrm{Co}_{\bar{l}_1}(\varphi^*, \bar{A}) .$$

This example will be of some interest later in sect. 5.3.

4.3. *Duality.*

This section is devoted to duality theorems for relative Calderón pairs.
The key theorem of this section is

THEOREM 4.10. – Let $\bar{A}$ and $\bar{B}$ be two regular Banach pairs. Assume further that $\bar{B}^*$ is regular. Take $b_0 \in \Sigma(\bar{B})$. Then holds isometrically

$$\mathrm{Co}_{\bar{B}^*}(b_0, \bar{A})^* = O_{\bar{B}}(b_0, \bar{A}^*) .$$

PROOF. – Take $a^* \in O_{\bar{B}}(b_0, \bar{A}^*)$ and write $a^* = Tb_0$ where $T \in \mathfrak{L}(\bar{B}, \bar{A}^*)$. Then

$T^*|_{\bar{A}}\colon \bar{A} \to \bar{B}^*$. For $a \in \Delta(\bar{A})$ we now have

$$\begin{aligned}|\langle a, a^*\rangle| &= |\langle b_0, T^*|_{\bar{A}}\, a\rangle| \leqslant \\ &\leqslant \|T^*|_{\bar{A}}\|_{\bar{A},\bar{B}^*}\|a\|_{\mathrm{Co}_{\bar{B}^*}(b_0,\bar{A})} \leqslant \\ &\leqslant \|T\|_{\bar{B},\bar{A}^*}\|a\|_{\mathrm{Co}_{\bar{B}^*}(b_0,\bar{A})}\,.\end{aligned}$$

Consequently $a^* \in \mathrm{Co}_{\bar{B}^*}(b_0, \bar{A})^*$.

Conversely let a^* be any norm one, linear functional on $\mathrm{Co}_{\bar{B}^*}(b_0, \bar{A})$. For $a \in \Delta(\bar{A})$ we then have

$$(4.7)\qquad |\langle a, a^*\rangle| \leqslant \sup_{\|T\|_{\bar{A},\bar{B}^*}\leqslant 1} |\langle b_0, Ta\rangle| = \sup_{\|T^*\|_{\bar{B}^{**},\bar{A}^*}\leqslant 1} |\langle a, T^* b_0\rangle| \leqslant \sup_{\|S\|_{\bar{B},\bar{A}^*}\leqslant 1} |\langle a, Sb_0\rangle|\,.$$

Consider the set $E = \{Sb_0\colon \|S\|_{\bar{B},\bar{A}^*} \leqslant 1\}$ of $\Sigma(\bar{A}^*)$. We claim that E is $\sigma(\Sigma(\bar{A}^*), \Delta(\bar{A}))$-closed. Indeed the unit ball U of $\mathfrak{L}(\bar{B}, \bar{A}^*)$ is compact in the topology generated by all seminorms of the form $|\langle a, Sb\rangle|$ where $a \in \Delta(\bar{A})$ and $b \in \Sigma(\bar{B})$. Thus E is the image of the compact set U under the continuous map $U \ni S \to Sb_0 \in E$ Hence E is closed.

We claim that $a^* \in E$. If not then we may find, using Hahn-Banach a continuous linear functional that strictly separates the closed convex set E from the set $\{a^*\}$ (see [16], p. 244). Hence for some $a \in \Delta(A)$

$$\sup_{\|S\|_{\bar{B},\bar{A}^*}\leqslant 1} \langle a, Sb_0\rangle < \langle a, a^*\rangle\,.$$

By (4.7) this is a contradiction, hence $\|a^*\|_{O_{\bar{B}}(b_0,\bar{A}^*)} \leqslant 1$. The proof is complete.

In [15] one may find further duality results for extremal interpolation functors. In particular th. 2 of [15] is closely related to our th. 4.10. Infact our proof was partly extracted from the proof of th. 2 in [15].

We may now pass to one of our main results in this section.

THEOREM 4.11. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that $\bar{A}$ is regular and mutually closed and that both $\bar{B}$ and $\bar{B}^*$ are regular. Then if $\bar{A}$ and $\bar{B}^*$ are of type (C) it follows that $\bar{B}$ and $\bar{A}^*$ are of type (C).

PROOF. – Take $b \in \Sigma(\bar{B}) \subseteq \Sigma(\bar{B}^{**})$. By prop. 4.6 we infer that

$$\mathrm{Co}_{\bar{B}^*}(b, \bar{A}) = \bar{A}_{\varphi^*, 1:J}$$

where $\varphi(t) = K(t, b, \bar{B})$. If we take duals we may conclude that

$$(4.8)\qquad \bar{A}^*_{\varphi,\infty:K} = O_{\bar{B}}(b, \bar{A}^*)\,.$$

Here we used our th. 4.10 and th. 3.2 in [10]. Further the equality (4.8) holds uniformly in b. From (4.2) we conclude that $\bar{B}$ and $\bar{A}^*$ are of type (C).

We note some corollaries.

COROLLARY 4.12. – Let $\bar{B}$ be a Banach pair such that both $\bar{B}$ and $\bar{B}^*$ are regular. Then if $\bar{B}^*$ is of type (C) it follows that $\bar{B}$ and $\bar{B}^{**}$ are of type (C). In particular, if $\bar{B}$ is a reflexive pair $\bar{B}$ is of type (C) iff $\bar{B}^*$ is of type (C).

COROLLARY 4.13. – Let the assumptions of th. 4.11 be fulfilled. Take $b \in \Sigma(\bar{B})$. Then the orbit

$$O_{\bar{B}^{**}}(b, \bar{A}^*)$$

is described by the K-method.

PROOF. – The cor. will follow once we have proven that

$$O_{\bar{B}^{**}}(b, \bar{A}^*) = O_{\bar{B}}(b, \bar{A}^*) . \tag{4.9}$$

To show this it suffices to show that every mapping $T: \bar{B} \to \bar{A}^*$ may be extended to $\bar{B}^{**}$. As is easily seen $(T^*|_A)^*$ is such an extension.

REMARK. – The arguments leading to (4.9) were extracted from the proof of th. 6 in [15]. See also [14], th. 3.3.

Our next corollary is related to ex. 4.9.

COROLLARY 4.14. – Let $\bar{A}$ be a regular, mutually closed Banach pair. Then if $\bar{A}$ and $\bar{l}_1$ are of type (C) it follows that $\bar{c}_0$ and $\bar{A}^*$ are of type (C).

PROOF. – Apply th. 4.11 with $\bar{B} = \bar{c}_0$.

Let $\bar{A}$ be a regular Banach pair. Take $\varphi \in \mathfrak{I}_0$. From ex. 3.3 and th. 4.10 it now follows that

$$\Big(\mathrm{Corb}_{\bar{l}_1}\big(\bar{A}, l_1(\varphi(2^{-\nu}))\big)^0\Big)^* = O_{\bar{c}_0}(\varphi, \bar{A}^*) .$$

This relation has previously been proven by both JANSON and OVČINNIKOV. See [15], th. 12 and [20].

4.4. *Calderón pairs of K-spaces.*

Let $\bar{A}$ be a Banach pair. In [8] CWIKEL proved that the pair $(\bar{A}_{\theta_0,p_0}, \bar{A}_{\theta_1,p_1})$ is of type (C) provided $0 < \theta_0, \theta_1 < 1$ and $1 \leqslant p_0, p_1 \leqslant \infty$. This result was later extended by DMITRIEV and OVČINNIKOV in [13]. Our next theorem further refines these results. Throughout this section let $\bar{E}$ and $\bar{F}$ denote two Banach pairs which are pairs of interpolation spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ respectively.

THEOREM 4.15. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that the pair $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$ is mutually closed. Assume that for all $t > 0$ holds

$$K(t, b, \bar{B}_{F_0:J}, \bar{B}_{F_1:J}) \leqslant K(t, a, \bar{A}_{E_0:K}, \bar{A}_{E_1:K})$$

where $b \in \Sigma(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})^0$. Then if $\bar{E}$ and $\bar{F}$ are of type $\lambda - (C)$ it follows that $b = Ta$ for some linear operator

$$T\colon (\bar{A}_{E_0:K}, \bar{A}_{E_1:K}) \to (\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$$

of norm less that $c\lambda$. (Thus $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ and $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$ are «almost» of type $c\lambda - (C)$.)

PROOF. – By lemma 3.16 of [17] we may pick $b_\nu \in \Delta(\bar{B})$ such that $b = \sum_\nu b_\nu$ and for all $t > 0$ holds

$$K\big(t, (J(2^\nu, b_\nu, \bar{B}))_\nu, \bar{F}\big) \leqslant cK(t, b, \bar{B}_{F_0:J}, \bar{B}_{F_1:J}) .$$

Similary by th. 3.6 of [17] (see also [5], th. 6.1) holds

$$K\big(t, (K(2^\nu, a, \bar{A}))_\nu, \bar{E}\big) \approx K(t, a, \bar{A}_{E_0:K}, \bar{A}_{E_1:K}) .$$

Our assumptions now imply that

$$K\big(t, (J(2^\nu, b_\nu, \bar{B}))_\nu, \bar{F}\big) \leqslant c_1 K\big(t, (K(2^\nu, a, \bar{A}))_\nu, \bar{E}\big) .$$

Consequently we may find $U\colon \bar{E} \to \bar{F}$, of norm less than $2c_1\lambda$, satisfying

$$(J(2^\nu, b_\nu, \bar{B}))_\nu = U\big((K(2^\nu, a, \bar{A}))_\nu\big) .$$

Notice that we further have

$$T^a\colon (\bar{A}_{E_0:K}, \bar{A}_{E_1:K}) \to (E_0, E_1)$$

and

$$T_{(b_\nu)_\nu}\colon (F_0, F_1) \to (\bar{B}_{F_0:J}, \bar{B}_{F_1:J}) .$$

Thus the operator $T = T_{(b_\nu)_\nu} U T^a$ has the desired properties. The proof is complete.

REMARK. – The pair $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$ is mutually closed whenever both $\bar{B}$ and $((\bar{l}_\infty)_{F_0:J}, (\bar{l}_\infty)_{F_1:J})$ are mutually closed. Indeed $\bar{B}_{F_i:J} = \bar{B}_{(\bar{l}_\infty)_{F_i:J}:K}$, $i = 0, 1$. If we apply the reiteration theorem for K-spaces (see [5], th. 6.1, [17], cor. 3.9) we find that $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})_{i,\infty:K} = \bar{B}_{D_i:K}$ where $D_i = ((\bar{l}_\infty)_{F_0:J}, (\bar{l}_\infty)_{F_1:J})_{i,\infty:K}$. By assumptions holds $D_i = (\bar{l}_\infty)_{F_i:J}$ and thus we infer that $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})_{i,\infty:K} = \bar{B}_{F_i:J}$, $i = 0, 1$.

REMARK. – Assume that $\bar{F}$ is a regular pair. Then th. 4.15 implies that $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ and $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$ are of type (C). (See [17]).

REMARK 4.16. – Assume that for $i = 0, 1$, holds $F_i = (\bar{l}_\infty)_{F_i:J}$. Then if $\bar{E}$ and $\bar{F}$ are of type (C) it follows that $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ and $(\bar{B}_{F_0:K}, \bar{B}_{F_1:K})$ are of type (C). Indeed our assumptions implies that $\bar{B}_{F_i:K} = \bar{B}_{F_i:J}$, $i = 0, 1$ (see [17], lemma 2.8). Take $b \in \Sigma(\bar{B})^0$ and let $b = \sum_\nu b_\nu$ be the representation provided by the fundamental lemma ([4], p. 33). Then

$$K\big(t, \big(J(2^\nu, b_\nu, \bar{B})\big)_\nu, \bar{F}\big) \leqslant cK(t, b, \bar{B}_{F_0:K}, \bar{B}_{F_1:K}) .$$

The rest is as in the proof of th. 4.15.

As we remarked above this result is an extension of th. 2 in [13]. If one in addition to the assumptions in remark 4.16 assumes that for $i = 0, 1$, holds $E_i = (\bar{l}_1)_{E_i:K}$ one gets DMITRIEV-OVČINNIKOV's theorem ([13], th. 2).

In order to obtain a partial converse of th. 4.15 we need to introduce some further terminology.

A Banach pair $\bar{A}$ is called K-surjective iff for every $\varphi \in \mathfrak{T}$ one may find $a \in \Sigma(\bar{A})$ such that for some positive constants c_1 and c_2 holds

$$c_1 K(t, a, \bar{A}) \leqslant \varphi(t) \leqslant c_2 K(t, a, \bar{A}) . \tag{4.10}$$

We further require that c_1 and c_2 may be chosen independently of φ. $\bar{A}$ is called K_0-surjective iff for every $\varphi \in \mathfrak{T}_0$ one may find $a \in \Sigma(\bar{A})^0$ satisfying (4.10).

We note that $\bar{l}_\infty$ is K-surjective. Examples of K_0-surjective pairs are $\bar{l}_p$, $1 \leqslant p \leqslant \infty$ and $(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$. See also [19], lemma 1.

The main result of this section is the following converse to th. 4.15.

THEOREM 4.17. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that $\bar{A}$ is a mutually closed K_0-surjective pair and that $\bar{B}$ is K-surjective. Then if $(\bar{A}_{F_0:J}, \bar{A}_{F_1:J})$ and $(\bar{B}_{E_0:K}, \bar{B}_{E_1:K})$ are of type (C) it follows that $\bar{F}$ and $\bar{E}$ are of type (C).

PROOF. – Assume that for all $t > 0$ holds

$$K(t, b, \bar{E}) \leqslant K(t, a, \bar{F}) .$$

Choose $x_b \in \Sigma(\bar{B})$ such that

$$K(t, b, \bar{l}_\infty) \approx K(t, x_b, \bar{B}) .$$

As $\bar{B}$ and $\bar{l}_\infty$ are of type (C) we may find $S_1 \in \mathfrak{L}(\bar{B}, \bar{l}_\infty)$ with $b = S_1 x_b$. Interpolating we find S_1: $(\bar{B}_{E_0:K}, \bar{B}_{E_1:K}) \to (E_0, E_1)$. From [17], th. 3.6 (see also [5], th. 6. a) we infer that

$$K(t, b, \bar{E}) \approx K(t, x_b, \bar{B}_{E_0:K}, \bar{B}_{E_1:K}) .$$

Pick $x_a \in \Sigma(\bar{A})$ satisfying

$$K(t, a, \bar{l}_1) \approx K(t, x_a, \bar{A}) .$$

As $\bar{l}_1$ and $\bar{A}$ are relative Calderón we may construct $S_2\colon \bar{l}_1 \to \bar{A}$ satisfying $S_2 a = x_a$. Further $S_2\colon (F_0, F_1) \to (\bar{A}_{F_0:J}, \bar{A}_{F_1:J})$. We claim that

$$K(t, a, \bar{F}) \approx K(t, x_a, \bar{A}_{F_0:J}, \bar{A}_{F_1:J}) .$$

Indeed noticing that $\bar{A}_{F_i:J} = \bar{A}_{D_i:K}$, where $D_i = (\bar{l}_\infty)_{F_i:J}$, $i = 0, 1$ (see ex. 4.7) we infer that

$$\begin{aligned} K(t, x_a, \bar{A}_{F_0:J}, \bar{A}_{F_1:J}) &\approx K(t, x_a, \bar{A}_{D_0:K}, \bar{A}_{D_1:K}) \approx \\ &\approx K\big(t, (K(2^\nu, x_a, \bar{A}))_\nu, \bar{D}\big) \approx \\ &\approx K\big(t, (K(2^\nu, a, \bar{l}_1))_\nu, \bar{D}\big) \approx \\ &\approx K\big(t, a, (\bar{l}_1)_{D_0:K}, (\bar{l}_1)_{D_1:K}\big) \approx \\ &\approx K\big(t, a, (\bar{l}_1)_{F_0:J}, (\bar{l}_1)_{F_1:J}\big) \approx K(t, a, \bar{F}) . \end{aligned}$$

We now have

$$K(t, x_b, \bar{B}_{E_0:K}, \bar{B}_{E_1:K}) \leqslant cK(t, x_a, \bar{A}_{F_0:J}, \bar{A}_{F_1:J}) .$$

Choose S: $(\bar{A}_{F_0:J}, \bar{A}_{F_1:J}) \to (\bar{B}_{E_0:K}, \bar{B}_{E_1:K})$ with $Sx_a = x_b$. Put $T = S_1 S S_2$. It follows that $T\colon \bar{F} \to \bar{E}$ and $Ta = b$. As all estimates are uniform we conclude that $\bar{F}$ and $\bar{E}$ are of type (C).

REMARK. – If the pair $\bar{E}$ is regular it suffices to assume that $\bar{B}$ is K_0-surjective. If we combine remark 4.16 and th. 4.17 we get the following corollary.

COROLLARY 4.18. – Let $\bar{A}$ be a mutually closed, K_0-surjective Banach pair. Assume further that for $i = 0, 1$, holds $E_i = (\bar{l}_\infty)_{E_i:J}$. Then $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ is Calderón iff (E_0, E_1) is Calderón.

In [19] OVČINNIKOV constructed a pair of sequence spaces $\bar{E}$ which are not of type (C). In fact

$$E_0 = l_1(2^{-\nu\theta}) \cap l_\infty(2^{-\nu\theta}|\nu|) \quad \text{and} \quad E_1 = l_1(2^{-\nu\theta}) + l_\infty(2^{-\nu\theta}|\nu|) ,$$

where $0 < \theta < 1$, will do. As this $\bar{E}$ satisfies the assumptions of cor. 4.18 it follows that $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ is never of type (C) provided $\bar{A}$ is mutually closed and K_0-surjective. This is th. 7 of [19]. Let us remark that this was part of the motivation for our th. 4.17.

5. – Ovčinnikov pairs.

5.1. *Definitions.*

Let T be a linear operator from $\bar{l}_\infty$ into $\bar{l}_1$. By a fundamental theorem of OVČINNIKOV [18], th. 1, we infer that

$$T\colon l_\infty(1/\varphi(2^\nu)) \to l_1(1/\varphi(2^\nu))$$

for any $\varphi \in \mathfrak{T}$, or equivalently

$$T\colon (\bar{l}_\infty)_{\varphi,\infty:K} \to (\bar{l}_1)_{\varphi,1:J}\,.$$

Further $\|T\|_{l_\infty(1/\varphi(2^\nu)),l_1(1/\varphi(2^\nu))} \leqslant 2K_G\|T\|_{\bar{l}_\infty,\bar{l}_1}$, where K_G is the Grothendieck constant.

The purpose of this chapter is to generalize Ovcinnikov's theorem to other pairs besides $\bar{l}_\infty$ and $\bar{l}_1$. As in [22] we make the following definitions. Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. We say that $\bar{A}$ and $\bar{B}$ are of type (O) if whenever $T\colon \bar{A} \to \bar{B}$ it follows that

$$T\colon \bar{A}_{\varphi,\infty:K} \to \bar{B}_{\varphi,1:J} \tag{5.1}$$

for any $\varphi \in \mathfrak{T}$. We further require that for some $\lambda < \infty$ holds

$$\|T\|_{\bar{A}_{\varphi,\infty:K},\,\bar{B}_{\varphi,1:J}} \leqslant \lambda \|T\|_{\bar{A},\bar{B}}\,. \tag{5.2}$$

Alternatively we say that $\bar{A}$ and $\bar{B}$ are relative Ovcinnikov. $\bar{A}$ and $\bar{B}$ are of type $(O)_0$ if $T \in \mathfrak{L}(\bar{A}, \bar{B})$ implies that

$$T\colon \bar{A}^0_{\varphi,\infty:K} \to \bar{B}_{\varphi,1:J}\,, \tag{5.3}$$

whenever $\varphi \in \mathfrak{T}_0$, and for some $\lambda < \infty$ holds

$$\|T\|_{\bar{A}^0_{\varphi,\infty:K},\,\bar{B}_{\varphi,1:J}} \leqslant \lambda \|T\|_{\bar{A},\bar{B}}\,. \tag{5.4}$$

By Ovčinnikov's theorem $\bar{l}_\infty$ and $\bar{l}_1$ are of type (O) (with $\lambda = 2K_G$). More examples may be found in PEETRE [22].

5.2. *Nuclearity.*

Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. We say that $T \in \mathfrak{L}(\bar{A}, \bar{B})$ is a nuclear operator from $\bar{A}$ into $\bar{B}$ if there exist $(b_\nu)_\nu \in \Delta(\bar{B})$ and $(a_\nu^*) \in \Sigma(\bar{A})^*$ with

$$\sum_\nu \max\left(\|a_\nu^*\|_{A_0^*}\|b_\nu\|_{B_0}, \ \|a_\nu^*\|_{A_1^*}\|b_\nu\|_{B_1}\right) < \infty \tag{5.5}$$

such that $Ta = \sum_\nu \langle a, a_\nu^* \rangle b_\nu$ for any $a \in \Sigma(\bar{A})$. We write $T: \bar{A} \xrightarrow{n} \bar{B}$. We define the nuclear norm of T, denoted by $\|T\|_{\bar{A},\bar{B}}^n$, as the infimum of all expressions appearing in (5.5). It is clear that if $T: \bar{A} \xrightarrow{n} \bar{B}$ then each map $T: A_i \to B_i$, $i = 0, 1$, is nuclear in the usual sense. The converse is in general not true. One instance when it is true is when $\bar{B} = \bar{l}_1$. Indeed if $T: A_i \to l_1(2^{-\nu i})$, $i = 0, 1$, are nuclear we may write $T(\cdot) = \sum_\nu \langle \cdot, a_\nu^* \rangle e_\nu$ where $\sum_\nu 2^{-\nu i}\|a_\nu^*\|_{A_i^*} < \infty$, $i = 0, 1$. Consequently $T: \bar{A} \xrightarrow{n} \bar{l}_1$.

Let $\bar{A}$ be a regular pair. We observe that if $T: \bar{A} \xrightarrow{n} \bar{B}$ then there exist two pairs of weights $\bar{\omega} = (\omega^0, \omega^1)$ and $\bar{\sigma} = (\sigma^0, \sigma^1)$ so that we have a commutative diagram of the form

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\quad T \quad} & \bar{B} \\ \Big\downarrow{\scriptstyle S_1} & & \Big\uparrow{\scriptstyle S_2} \\ (l_\infty(1/\omega^0), l_\infty(1/\omega^1)) & \xrightarrow{S_\alpha} & (l_1(\sigma^0), l_1(\sigma^1)) \, . \end{array} \tag{5.6}$$

Here S_1 and S_2 are norm one linear operators. S_α is a multiplier transform defined by a sequence $\alpha = (\alpha_\nu)_\nu$ such that

$$\sum_\nu |\alpha_\nu| \max\left(\omega_\nu^0\sigma_\nu^0, \ \omega_\nu^1\sigma_\nu^1\right) < \infty \, . \tag{5.7}$$

Further the infimum of (5.7) over all possible factorisations equals $\|T\|_{\bar{A},\bar{B}}^n$.

To see that $T: \bar{A} \xrightarrow{n} \bar{B}$ implies that we have the factorisation (5.6) choose $(b_\nu)_\nu \in \Delta(\bar{B})$ and $(a_\nu^*)_\nu \in \Sigma(\bar{A})^*$ such that (5.5) is fulfilled. Put $E = \{\nu: a_\nu^* \neq 0\}$. For $\nu \in E$ put $\omega_\nu^i = \|a_\nu^*\|_{A_i^*}$, $\sigma_\nu^i = \|b_\nu\|_{B_i}$, $i = 0, 1$. If we take $\alpha_\nu = 1$ for $\nu \in E$, $\alpha_\nu = 0$ for $\nu \notin E$ it is a matter of routine to construct S_1 and S_2 such that (5.6) is satisfied. Conversely (5.6) implies that $T: \bar{A} \xrightarrow{n} \bar{B}$, $\bar{A}$ regular or not.

We may now construct nuclear orbit functors. Take $a \in \Sigma(\bar{A})$. We denote by $O_{\bar{A}}^n(a, \bar{B})$ the space of all $b \in \Sigma(\bar{B})$ that may be written as $b = Ta$ where $T: \bar{A} \xrightarrow{n} \bar{B}$. We put

$$\|b\|_{O_{\bar{A}}^n(a,\bar{B})} = \inf\{\|T\|_{\bar{A},\bar{B}}^n : b = Ta\} \, .$$

We similary define $\mathrm{Orb}_{\bar{A}}^n(A, \cdot)$, $\mathrm{Corb}_{\bar{B}}^n(\cdot, B)$ and $\mathrm{Co}_{\bar{B}}^n(b^*, \cdot)$ where $b^* \in \Sigma(\bar{B}^*)$.

A description of our nuclear orbit functors is provided by the following

PROPOSITION 5.1. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Take $a \in \Sigma(\bar{A})$, $a \neq 0$, and put $\varphi(t) = K(t, a, \bar{A})$. Then holds uniformly in a

$$O^n_{\bar{A}}(a, \bar{B}) = \bar{B}_{\varphi, 1:J} . \tag{5.8}$$

Further for any intermediate space A with respect to $\bar{A}$ holds

$$\mathrm{Orb}^n_{\bar{A}}(A, \bar{B}) = \bar{B}_{\mathrm{Orb}^n_{\bar{A}}(A, \bar{l}_1):J} .$$

PROOF. – Take $b \in \bar{B}_{\varphi, 1:J}$ and choose $b_\nu \in \Delta(\bar{B})$ such that

$$b = \sum_\nu b_\nu \quad \text{and} \quad \sum_\nu \frac{J(2^\nu, b_\nu, \bar{B})}{K(2^\nu, a, \bar{A})} < \infty .$$

Define the multiplier transform $T: \bar{l}_\infty \to \bar{l}_1$ by

$$T(\gamma) = \left(\frac{J(2^\nu, b_\nu, \bar{B})}{K(2^\nu, a, \bar{A})} \gamma_\nu\right)_\nu$$

and put $S = T_{(b_\nu)_\nu} T T^a$. Clearly (5.6) is fulfilled and we conclude that $S: \bar{A} \xrightarrow{n} \bar{B}$. As $b = Sa$ it follows that $b \in O^n_{\bar{A}}(a, \bar{B})$.

Assume now that $b = Ta$ where $T: \bar{A} \xrightarrow{n} \bar{B}$. Write $T(\cdot) = \sum_\nu \langle \cdot, a^*_\nu \rangle b_\nu$ such that (5.5) is fulfilled. Let I_K be the subset of $\boldsymbol{Z}$ defined by the condition: $\nu \in I_K$ iff

$$\|b_\nu\|_{B_0} \leqslant 2^K \|b_\nu\|_{B_1} < 2\|b_\nu\|_{B_0} .$$

The sets I_K, $K \in \boldsymbol{Z}$, constitute a disjoint union of the set $\{\nu: b_\nu \neq 0\}$. Further for $\nu \in I_K$ holds

$$J(2^{-K}, a^*_\nu, \bar{A}^*) J(2^K, b_\nu, \bar{B}) \leqslant 2 \max_{i=0,1} \left(\|a_\nu\|_{A^*_i} \|b_\nu\|_{B_i}\right) .$$

Put $b^K = \sum_{\nu \in I_K} \langle a, a^*_\nu \rangle b_\nu$. As

$$\begin{aligned} J(2^K, b^K, \bar{B}) &\leqslant \sum_{\nu \in I_K} J(2^{-K}, a^*_\nu, \bar{A}^*) J(2^K, b_\nu, \bar{B}) K(2^K, a, \bar{A}) \leqslant \\ &\leqslant 2 \left(\sum_{\nu \in I_K} \max_{i=0,1} \left(\|a^*_\nu\|_{A^*_i} \|b_\nu\|_{B_i}\right)\right) K(2^K, a, \bar{A}) \end{aligned} \tag{5.9}$$

it follows that $b^K \in \Delta(\bar{B})$. Clearly $b = \sum_K b^K$. From our estimates, (5.9), we now infer that

$$\sum_K \frac{J(2^K, b^K, \bar{B})}{K(2^K, a, \bar{A})} \leqslant 2 \sum_K \sum_{\nu \in I_K} \max_{i=0,1} (\|a_\nu^*\|_{A_i^*} \|b_\nu\|_{B_i}) < \infty .$$

Consequently $b \in \bar{B}_{\varphi,1:J}$.

By (3.1) we certainly have $\bar{B}_{\mathrm{Orb}^n_{\bar{A}}(A,\bar{l}_1):J} \subseteq \mathrm{Orb}^n_{\bar{A}}(A, \bar{B})$ so it suffices to show the converse inclusion. Take $b \in \mathrm{Orb}^n_{\bar{A}}(A, \bar{B})$ and write $b = \sum_i T_i a_i$, where $T_i\colon \bar{A} \xrightarrow{n} \bar{B}$, $a_i \in A$ and $\left(\sum_i (\|T_i\|^n_{\bar{A},\bar{B}} \|a_i\|_A)^p\right)^{1/p} < \infty$. From (5.8) follows that $b_i = T_i a_i \in \bar{B}_{K(2^\nu, a_i, \bar{A}),1:J}$. Choose $(b_\nu^i)_\nu \in \Delta(\bar{B})$ such that $b_i = \sum_\nu b_\nu^i$ and

$$\sum_\nu \frac{J(2^\nu, b_\nu^i, \bar{B})}{K(2^\nu, a_i, \bar{A})} \leqslant 3 \|T_i\|^n_{\bar{A},\bar{B}} .$$

It follows that

$$\sum_i \sum_\nu \min(1, 2^{-\nu}) J(2^\nu, b_\nu^i, \bar{B}) \leqslant c\left(\sum_i (\|T_i\|^n_{\bar{A},\bar{B}} \|a_i\|_A)^p\right)^{1/p} .$$

Thus if we put $b_\nu = \sum_i b_\nu^i$ then $b_\nu \in \Delta(\bar{B})$ and $b = \sum_\nu b_\nu$. Further

$$(J(2^\nu, b_\nu, \bar{B}))_\nu \leqslant \sum_i (J(2^\nu, b_\nu^i, \bar{B}))_\nu \tag{5.10}$$

where the series converges in $\Sigma(\bar{l}_1)$. Define multiplier transforms $S_i\colon \bar{l}_\infty \to \bar{l}_1$ by

$$S_i(\gamma) = \left(\frac{J(2^\nu, b_\nu^i, \bar{B})}{K(2^\nu, a_i, \bar{A})} \gamma_\nu\right)_\nu ,$$

and put $V_i = S_i T^{a_i}$. Clearly $V_i\colon \bar{A} \xrightarrow{n} \bar{l}_1$ with $\|V_i\|^n_{\bar{A},\bar{l}_1} \leqslant 3\|T_i\|_{\bar{A},\bar{B}}$. From (5.10) now follows that

$$(J(2^\nu, b_\nu, \bar{B}))_\nu \leqslant \sum_i V_i(a_i)$$

where $\left(\sum_i (\|V_i\|^n_{\bar{A},\bar{l}_1} \|a_i\|_A)^p\right)^{1/p} < \infty$.

Consequently $(J(2^\nu, b_\nu, \bar{B}))_\nu \in \mathrm{Orb}^n_{\bar{A}}(A, \bar{l}_1)$, i.e. $b \in \bar{B}_{\mathrm{Orb}^n_{\bar{A}}(A,\bar{l}_1):J}$. The proof is complete.

REMARK 5.2. – From the proof of (5.8) follows that

$$\|b\|_{\bar{B}_{\varphi,1:J}} \leqslant 2\|b\|_{O^n_{\bar{A}}(a,\bar{B})} \leqslant 2\|b\|_{\bar{B}_{\varphi,1:J}} . \tag{5.11}$$

REMARK 5.3. – BERGH [3], (cf. [4], p. 31) showed that if for some $\nu \in \mathbf{Z}$ holds $J(2^\nu, b, \bar{B}) \leqslant K(2^\nu, a, \bar{A})$ then $b \in O_{\bar{A}}(a, \bar{B})$. Bergh's result apparently is a special case of (5.8). Just write $b = \sum_\nu b_\nu$ where $b_\mu = 0$ if $\mu \neq \nu$ and $b_\nu = b$. CWIKEL in [7] on the other hand proved, if we have $K(2^\nu, b, \bar{B}) \leqslant c_\nu K(2^\nu, a, \bar{A})$ with $\sum_\nu c_\nu < \infty$ then $b \in O_{\bar{A}}^-(a, \bar{B})$. This follows from (5.8) by writing $b = \sum_\nu b_\nu$ where the representation is the one that is provided by the fundamental lemma ([4], p. 45). See also [19], sect. 1. If in (5.8) one puts $\bar{B} = \bar{l}_1$ one gets prop. 4 in [18].

We now turn to nuclear coorbit functors.

PROPOSITION 5.4. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. For every intermediate space B with respect to $\bar{B}$ holds

$$\mathrm{Corb}^n_{\bar{B}}(\bar{A}, B) = \bar{A}_{\mathrm{Corb}^n_{\bar{B}}(\bar{l}_\infty, B):K}.$$

PROOF. – Take $a \in \bar{A}_{\mathrm{Corb}^n_{\bar{B}}(\bar{l}_\infty, B):K}$, i.e. $(K(2^\nu, a, \bar{A}))_\nu \in \mathrm{Corb}^n_{\bar{B}}(\bar{l}_\infty, B)$. Take $T: \bar{A} \xrightarrow{n} \bar{B}$. We claim that $b = Ta \in B$. From (5.8) we infer that $b \in \bar{B}_{K(2^\nu, a, \bar{A}), 1:J}$. Choose $b_\nu \in \Delta(\bar{B})$ such that $b = \sum_\nu b_\nu$ and

$$\sum_\nu \frac{J(2^\nu, b_\nu, \bar{B})}{K(2^\nu, a, \bar{A})} \leqslant 3\|T\|^n_{\bar{A}, \bar{B}}.$$

It follows that $b = T_{(b_\nu)_\nu} S\big((K(2^\nu, a, \bar{A}))_\nu\big)$, where $S: \bar{l}_\infty \to \bar{l}_1$ is a suitable multiplier transform. Consequently

$$\|Ta\|_B = \|T_{(b_\nu)_\nu} S\big((K(2^\nu, a, \bar{A}))_\nu\big)\|_B \leqslant \\ \leqslant 3 \sup_{\|V\|^n_{\bar{l}_\infty, \bar{B}} \leqslant 1} \|V\big((K(2^\nu, a, \bar{A}))_\nu\big)\|_B = 3\|(K(2^\nu, a, \bar{A}))_\nu\|_{\mathrm{Corb}^n_{\bar{B}}(\bar{l}_\infty, B)},$$

i.e. $a \in \mathrm{Corb}^n_{\bar{B}}(\bar{A}, B)$. The converse inclusion follows from (3.1).

PROPOSITION 5.5. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume that $\bar{B}$ is regular and take $b^* \in \Sigma(\bar{B}^*)$. Then holds uniformly in b^*

$$\mathrm{Co}^n_{\bar{B}}(b^*, \bar{A}) = \bar{A}^0_{\varphi, \infty:K}$$

where $\varphi(t) = K^*(t, b^*, \bar{B}^*)$.

PROOF. – Take $a \in \Delta(\bar{A})$ and let $T: \bar{A} \xrightarrow{n} \bar{B}$ be any nuclear map of finite rank. Write $T(\cdot) = \sum_\nu \langle \cdot, a_\nu^* \rangle b_\nu$ where $a_\nu^* \in (A_0^0)^* \cap (A_1^0)^*$, $b_\nu \in \Delta(\bar{B})$ and the series is finite. Choose sets I_K as in the proof of prop. 5.1. Then

$$|\langle Ta, b^* \rangle| = \Big|\sum_K \sum_{\nu \in I_K} \langle a, a_\nu^* \rangle \langle b_\nu, b^* \rangle\Big| \leqslant \\ \leqslant \sum_K \sum_{\nu \in I_K} K(2^K, a, \bar{A}) J(2^{-K}, a_\nu^*, \bar{A}^*) J(2^K, b_\nu, \bar{B}) K(2^{-K}, b^*, \bar{B}^*) \leqslant \\ \leqslant 2 \sup_K \big(K(2^K, a, \bar{A}) K(2^{-K}, b^*, \bar{B}^*)\big) \sum_\nu \max_{i=0,1} \big(\|a_\nu^*\|_{A_i^*} \|b_\nu\|_{B_i}\big).$$

Consequently $\bar{A}^0_{\varphi, \infty:K} \subseteq \mathrm{Co}^n_{\bar{B}}(b^*, \bar{A})$.

To prove the converse inclusion consider rank one operators of the form $T(\cdot) = \langle \cdot, a_\nu^* \rangle b_\nu$ where $J(2^{-\nu}, a_\nu^*, \bar{A}^*) \leqslant 1$ and $J(2^\nu, b_\nu, \bar{B}) \leqslant 1$.

5.3. *Characterizations of type* (O).

In this section we will elucidate the possibility of describing pairs of type (O) using extremal interpolation functors. From our results it will follow that OVČINNIKOV pairs are in a sense opposite to Calderón pairs. The reader is asked to compare the results of this section with (3.4), (3.5) and sect. 4.2. Almost all our results will be given for pairs of type (O) only. Observe that type (O) implies type $(O)_0$.

Let E and F be intermediate spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ respectively. Consider the following condition:

$$\text{if } \alpha = (\alpha_\nu)_\nu \in E \text{ and } \sum_{\alpha_\nu \neq 0} \left|\frac{\beta_\nu}{\alpha_\nu}\right| < \infty \text{ then } \beta = (\beta_\nu)_\nu \in F . \tag{5.12}$$

Clearly (5.12) is fulfilled whenever E and F are relative interpolation spaces between $\bar{l}^\infty$ and $\bar{l}_1$: Just consider the multiplier transform T defined by $T(\gamma) = \big((\beta_\nu/\alpha_\nu)\gamma_\nu\big)_\nu$. Then $T: \bar{l}_\infty \to \bar{l}_1$ and $T\alpha = \beta$. We now have the following extension of the property (5.1). See [22].

PROPOSITION 5.6. – Let E and F be interpolation spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ respectively. Let $\bar{A}$ and $\bar{B}$ be two Banach pairs and assume further that $\bar{A}$ and $\bar{B}$ are of type (O). Take $T \in \mathfrak{L}(\bar{A}, \bar{B})$. Then

$$T: \bar{A}_{E:K} \to \bar{B}_{F:J} \tag{5.13}$$

whenever E and F satisfy (5.12).

PROOF. – Take $a \in \bar{A}_{E:K}$, i.e. $\big(K(2^\nu, a, \bar{A})\big)_\nu \in E$. Put $\varphi(t) = K(t, a, \bar{A})$. Then $a \in \bar{A}_{\varphi,\infty:K}$ and by (5.1) $Ta \in \bar{B}_{\varphi,1:J}$. Write $Ta = \sum_\nu b_\nu$ where $b_\nu \in \Delta(\bar{B})$ satisfies

$$\sum_\nu \frac{J(2^\nu, b_\nu, \bar{B})}{K(2^\nu, a, \bar{A})} < \infty .$$

If we apply (5.12) with $\alpha = \big(K(2^\nu, a, \bar{A})\big)_\nu$ and $\beta = \big(J(2^\nu, b_\nu, \bar{B})\big)_\nu$ it follows that $\big(J(2^\nu, b_\nu, \bar{B})\big)_\nu \in F$, i.e. $Ta \in \bar{B}_{F:J}$.

Let us apply prop. 5.6 with $\bar{A} = \bar{l}_\infty$ and $\bar{B} = \bar{l}_1$. As $E = (\bar{l}_\infty)_{E:K}$ and $F = (\bar{l}_1)_{F:J}$ we see that E and F are relative interpolation spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ whenever (5.12) is fulfilled. We conclude that (5.12) characterizes relative interpolation spaces between $\bar{l}_\infty$ and $\bar{l}_1$.

We now confront type (O) with orbit functors.

PROPOSITION 5.7. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Then $\bar{A}$ and $\bar{B}$ are of type (O) iff the relation

$$O_{\bar{A}}(a, \bar{B}) = \bar{B}_{\varphi,1:J}\,, \tag{5.14}$$

where $\varphi(t) = K(t, a, \bar{A})$, holds uniformly in $a \in \Sigma(\bar{A})$.

PROOF. – Assume that $\bar{A}$ and $\bar{B}$ are of type (O). Take $T \in \mathfrak{L}(\bar{A}, \bar{B})$ and $a \in \Sigma(\bar{A})$. If we apply (5.1) with $\varphi(t) = K(t, a, \bar{A})$ it follows that $Ta \in \bar{B}_{\varphi,1:J}$. Thus $O_{\bar{A}}(a, \bar{B}) \subseteq \bar{B}_{\varphi,1:J}$. The converse inclusion is a consequence of prop. 5.1.

Conversely assume (5.14). Take $T \in \mathfrak{L}(\bar{A}, \bar{B})$ and $\eta \in \mathfrak{T}$. If $a \in \bar{A}_{\eta,\infty:K}$ it follows from (5.14) that $Ta \in \bar{B}_{\varphi,1:J}$. Write $Ta = \sum_\nu b_\nu$ in the usual way and note that

$$\sum_\nu \frac{J(2^\nu, b_\nu, \bar{B})}{\eta(2^\nu)} \leqslant \sup_\nu \left(\frac{K(2^\nu, a, \bar{A})}{\eta(2^\nu)}\right) \sum_\nu \frac{J(2^\nu, b_\nu, \bar{B})}{K(2^\nu, a, \bar{A})}\,.$$

Consequently $Ta \in \bar{B}_{\eta,1:J}$, and we conclude that $\bar{A}$ and $\bar{B}$ are of type (O).

Let us give an application of prop. 5.7. Let $\bar{A}$ be a regular Banach pair such that $\bar{A}^*$ also is regular. Let $\bar{B}$ be a dual Banach pair. We claim that $\bar{A}$ and $\bar{B}$ are of type (O) whenever $\bar{A}^{**}$ and $\bar{B}$ are of type (O). Indeed if $a \in \Sigma(\bar{A})$ then holds by (4.9) $O_{\bar{A}^{**}}(a, \bar{B}) = O_{\bar{A}}(a, \bar{B})$ (isometrically). From (5.14) now follows that

$$O_{\bar{A}}(a, \bar{B}) = \bar{B}_{K(2^\nu, a, \bar{A}^{**}),1:J} = \bar{B}_{K(2^\nu, a, \bar{A}),1:J}\,.$$

The last equality is a consequence of the equality $K(t, a, \bar{A}^{**}) = K(t, a, \bar{A})$. Thus $\bar{A}$ and $\bar{B}$ are of type (O). If we apply this argument with $\bar{A} = \bar{c}_0$, $\bar{B} = \bar{l}_1$, $\bar{A}^{**} = \bar{l}_\infty$, Ovcinnikov's theorem now implies

PROPOSITION 5.8. – $\bar{c}_0$ and $\bar{l}_1$ are of type (O).

REMARK. – Prop. 5.8 is due to JANSON [15]. See also [22].

To describe type $(O)_0$ we use coorbit functors.

PROPOSITION 5.9. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs and assume further that they are regular. Then $\bar{A}$ and $\bar{B}$ are of type $(O)_0$ iff the relation

$$\mathrm{Co}_{\bar{B}}(b^*, \bar{A}) = \bar{A}^0_{\varphi,\infty:K}\,, \tag{5.15}$$

where $\varphi(t) = K^*(t, b^*, \bar{B}^*)$, holds uniformly in $b^* \in \Sigma(\bar{B}^*)$.

PROOF. – Assume that $\bar{A}$ and $\bar{B}$ are of type $(O)_0$. Take $b^* \in \Sigma(\bar{B}^*)$, $T \in \mathfrak{L}(\bar{A}, \bar{B})$ and $a \in \Delta(\bar{A})$. As $a \in \bar{A}^0_{\varphi,\infty:K}$ (5.3) implies that $Ta \in \bar{B}_{\varphi,1:J}$ As $b^* \in \bar{B}^*_{\varphi^*,\infty:K} = (\bar{B}_{\varphi,1:J})^*$ (see [10], th. 3.2) we infer that

$$|\langle Ta, b^*\rangle| \leqslant c\|Ta\|_{\bar{B}_{\varphi,1:J}}\|b^*\|_{\bar{B}^*_{\varphi^*,\infty:K}} \leqslant c\lambda\|T\|_{\bar{A},\bar{B}}\|a\|_{\bar{A}_{\varphi,\infty:K}}\,.$$

Consequently $\bar{A}^0_{\varphi,\infty:K} \subseteq \mathrm{Co}_{\bar{B}}(b^*, \bar{A})$. The converse inclusion follows from prop. 5.5.

Conversely assume (5.15). Take $\eta \in \mathfrak{T}_0$ and $T \in \mathfrak{L}(\bar{A}, \bar{B})$. For $a \in \Delta(\bar{A})$ we now have

$$|\langle Ta, b^*\rangle| \le c\|T\|_{\bar{A},\bar{B}} \sup_\nu \frac{K(2^\nu, a, \bar{A})}{\eta(2^\nu)} \sup_\nu \frac{K(2^\nu, b^*, \bar{B}^*)}{1/\eta(2^{-\nu})} = c\|T\|_{\bar{A},\bar{B}}\|a\|_{\bar{A}_{\eta,\infty:K}}\|b^*\|_{\bar{B}^*_{\eta^*,\infty:K}} .$$

By duality this inequality implies that $Ta \in \bar{B}_{\eta,1:J}$. Thus

$$T\colon \bar{A}^0_{\eta,\infty:K} \to \bar{B}_{\eta,1:J} ,$$

i.e. $\bar{A}$ and $\bar{B}$ are of type $(O)_0$.

REMARK 5.10. – Assume further that $\bar{B}^*$ is a regular Banach pair. By modifying the proof of prop. 5.9 one may show that $\bar{A}$ and $\bar{B}^*$ are of type $(O)_0$ iff the relation $\mathrm{Co}_{\bar{B}^*}(b, \bar{A}) = \bar{A}^0_{\varphi,\infty:K}$, where $\varphi(t) = K^*(t, b, \bar{B})$, holds uniformly in $b \in \Sigma(\bar{B})$.

REMARK 5.11. – Prop. 5.7 may be extended in the following way. If $\bar{A}$ and $\bar{B}$ are of type (O) then

$$\mathrm{Orb}_{\bar{A}}(A, \bar{B}) = \bar{B}_{\mathrm{Orb}_{\bar{A}}(A,\bar{l}_1):J} \tag{5.16}$$

and

$$\mathrm{Corb}_{\bar{B}}(\bar{A}, B) = \bar{A}_{\mathrm{Corb}_{\bar{B}}(\bar{l}_\infty,B):K} . \tag{5.17}$$

Here A and B are intermediate spaces with respect to $\bar{A}$ and $\bar{B}$ respectively. Cf (3.2) and (3.3).

We may now prove the principal result of this section. See also [22].

THEOREM 5.12. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Then $\bar{A}$ and $\bar{B}$ are of type (O) iff for every $a \in \Sigma(\bar{A})$ holds uniformly

$$O_{\bar{A}}(a, \bar{B}) = O^n_{\bar{A}}(a, \bar{B}) = \bar{B}_{\varphi,1:J} , \tag{5.18}$$

where $\varphi(t) = K(t, a, \bar{A})$. Further for any intermediate space A with respect to $\bar{A}$ then holds

$$\mathrm{Orb}_{\bar{A}}(A, \bar{B}) = \mathrm{Orb}^n_{\bar{A}}(A, \bar{B}) = \bar{B}_{\mathrm{Orb}^n_{\bar{A}}(A,\bar{l}_1):J}$$

whenever $\bar{A}$ and $\bar{B}$ are of type (O).

PROOF. – To prove the first part just combine prop. 5.1, (5.8) with prop. 5.7.

The second part will follow from prop. 5.1 once we have proven that $\mathrm{Orb}_{\bar{A}}(A, \bar{B}) \subseteq \mathrm{Orb}^n_{\bar{A}}(A, \bar{B})$. Take $b \in \mathrm{Orb}_{\bar{A}}(A, \bar{B})$ and write $b = \sum_i T_i a_i$, where $T_i \in \mathfrak{L}(\bar{A}, \bar{B})$,

$a_i \in A$ and $\left(\sum_i (\|T_i\|_{\bar{A},\bar{B}}\|a_i\|_A)^p\right)^{1/p} < \infty$. As $T_i a_i \in O_{\bar{A}}(a_i\, \bar{B})$ it follows from (5.18) that we may find $S_i \colon \bar{A} \xrightarrow{n} \bar{B}$ with $S_i a_i = T_i a_i$ and $\|S_i\|^n_{\bar{A},\bar{B}} \leqslant c \|T_i\|_{\bar{A},\bar{B}}$. Consequently $b = \sum_i S_i a_i$ where $\left(\sum_i (\|S_i\|^n_{\bar{A},\bar{B}}\|a_i\|_A)^p\right)^{1/p} < \infty$, i.e. $b \in \mathrm{Orb}^n_{\bar{A}}(A, \bar{B})$. The proof is complete.

Similary if we combine prop. 5.5 with prop. 5.9 we obtain the following characterization of type $(O)_0$.

THEOREM 5.13. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs and assume further that $\bar{A}$ and $\bar{B}$ are regular. Then $\bar{A}$ and $\bar{B}$ are of type $(O)_0$ iff for every $b^* \in \Sigma(\bar{B}^*)$ holds uniformly

$$\mathrm{Co}_{\bar{B}}(b^*, \bar{A}) = \mathrm{Co}^n_{\bar{B}}(b^*, \bar{A}) = \bar{A}^0_{\varphi,\infty:K}\,,$$

where $\varphi(t) = K^*(t, b^*, \bar{B}^*)$.

For general coorbit functors holds:

THEOREM 5.14. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs of type (O). For every intermediate space B with respect to $\bar{B}$ then holds

$$\mathrm{Corb}_{\bar{B}}(\bar{A}, B) = \mathrm{Corb}^n_{\bar{B}}(\bar{A}, B) = \bar{A}_{\mathrm{Corb}^n_{\bar{B}}(\bar{l}_\infty, B):K}\,.$$

Ovčinnikov's theorem has the consequence that if $T\colon \bar{A} \to \bar{B}$ then

$$T\colon O_{\bar{l}_\infty}(\varphi, \bar{A}) \to \mathrm{Corb}_{\bar{l}_1}\left(\bar{B}, l_1(1/\varphi(2^\nu))\right),$$

where $\varphi \in \mathcal{T}$. On the other hand it follows from (3.3) and (3.4) that

$$\bar{B}_{\varphi,1:J} \subseteq \mathrm{Corb}_{\bar{l}_1}\left(\bar{B}, l_1(1/\varphi(2^\nu))\right) \tag{5.19}$$

and

$$O_{\bar{l}_\infty}(\varphi, \bar{A}) \subset \bar{A}_{\varphi,\infty:K}\,. \tag{5.20}$$

Thus $\bar{A}$ and $\bar{B}$ will be of type (O) if there is equality in (5.19) and (5.20). From ex. 4.9 now follows.

PROPOSITION 5.15. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs with $\bar{B}$ mutually closed and regular. If $\bar{l}_\infty$ and $\bar{A}$ are of type (C) and $\bar{B}$ and $\bar{l}_1$ are of type (C) it follows that $\bar{A}$ and $\bar{B}$ are of type (O).

Let $\bar{E}$ and $\bar{F}$ be two pairs of Banach interpolation spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ respectively. From th. 4.15 we infer that $(\bar{B}_{E_0:K}, \bar{B}_{E_1:K})$ and $\bar{l}_1$ are of type (C) whenever $\bar{E}$ and $\bar{l}_1$ are of type (C). If we in addition assumes that for $i = 0, 1$, holds $F_i = (\bar{l}_\infty)_{F_i:J}$, remark 4.16 implies that $\bar{l}_\infty$ and $(\bar{A}_{F_0:J}, \bar{A}_{F_1:J})$ are of type (C) if $\bar{l}_\infty$

and $\bar{F}$ are of type (C). Prop. 5.15 now implies that

$$(\bar{A}_{F_0:J}, \bar{A}_{F_1:J}) \quad \text{and} \quad (\bar{B}_{E_0:K}, \bar{B}_{E_1:K})$$

are of type (O).

Let us give an example. Let $0<\theta_i<1$, $i=0,1,2,3$. Then

$$(\bar{A}_{\theta_0,\infty}, \bar{A}_{\theta_1,\infty}) \quad \text{and} \quad (\bar{B}_{\theta_2,1}, \bar{B}_{\theta_3,1})$$

are of type (O). To prove this just note that $\bar{l}_\infty$ and $(l_\infty(2^{-\theta_0\nu}), l_\infty(2^{-\theta_1\nu}))$ are of type (C). Similary $(l_1(2^{-\theta_2\nu}), l_1(2^{-\theta_3\nu}))$ and $\bar{l}_1$ are relative Calderón. The result now follows from the above.

5.4. *Duality.*

Theorem 5.16. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that $\bar{A}$, $\bar{B}$ and $\bar{B}^*$ are regular pairs. Then $\bar{A}$ and $\bar{B}^*$ are of type $(O)_0$ iff $\bar{B}$ and $\bar{A}^*$ are of type (O).

Proof. – Assume that $\bar{A}$ and $\bar{B}^*$ are of type $(O)_0$ and take $b\in\Sigma(\bar{B})\subseteq\Sigma(\bar{B}^{**})$. From prop. 5.9 (and remark 5.10) follows that

$$\mathrm{Co}_{\bar{B}^*}(b,\bar{A}) = \bar{A}^0_{\varphi,\infty:K} \tag{5.21}$$

where $\varphi(t)=K^*(t,b,\bar{B})$. As $b\in\Sigma(\bar{B})^0$, i.e. $K(t,b,\bar{B})=o(\max(1,t))$ as $t\to 0,\infty$, we infer that

$$\bar{A}^0_{\varphi,\infty:K} = \bar{A}_{c_0(1/\varphi):K}\,.$$

See [4], th. 3.4.2. If we take duals in (5.21) we conclude that

$$O_{\bar{B}}(b,\bar{A}) = \bar{A}^*_{\varphi^*,1:J}\,.$$

Here we used our th. 4.10 and th. 3.1 in [10]. By prop. 5,7, $\bar{B}$ and $\bar{A}^*$ are of type (O).

If conversely $\bar{B}$ and $\bar{A}^*$ are of type (O) it would follow that the two Banach spaces $\mathrm{Co}_{\bar{B}^*}(b,\bar{A})$ and $\bar{A}^0_{\varphi,\infty:K}$ have the same dual. As $\mathrm{Co}_{\bar{B}^*}(b,\bar{A})\subseteq\bar{A}^0_{\varphi,\infty:K}$ they must coincide. Now prop. 5.9 (and remark 5.10) implies that $\bar{A}$ and $\bar{B}^*$ are of type $(O)_0$.

5.5. *K-spaces of type (O).*

Let $\bar{E}$ and $\bar{F}$ be two Banach pairs consisting of interpolation spaces with respect to $\bar{l}_\infty$ and $\bar{l}_1$ respectively. Our first result shows that if $\bar{F}$ and $\bar{E}$ are of type (O) this property is transplanted to certain pairs of J- and K-spaces. Cf. th. 4.15.

THEOREM 5.17. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that the pair $(\bar{A}_{F_0:J}, \bar{A}_{F_1:J})$ is mutually closed and regular. Then if $\bar{F}$ and $\bar{E}$ are of type (O) it follows that $(\bar{A}_{F_0:J}, \bar{A}_{F_0:J})$ and $(\bar{B}_{E_0:K}, \bar{B}_{E_1:K})$ are of type (O).

PROOF. – Take $T: (\bar{A}_{F_0:J}, \bar{A}_{F_1:J}) \to (\bar{B}_{E_0:K}, \bar{B}_{E_1:K})$ and $a \in \Sigma(\bar{A}_{F_0:J}, \bar{A})_{F_1:J}$. Put $b = Ta$. Choose $a_\nu \in \Delta(\bar{A})$ such that $a = \sum_\nu a_\nu$ and $\big(J(2^\nu, a_\nu, \bar{A})\big)_\nu \in \Sigma(\bar{F})$. If we put $S = T^b T T_{(b_\nu)_\nu}$ then $S: \bar{F} \to \bar{E}$ and

$$S\Big(\big(J(2^\nu, a_\nu, \bar{A})\big)_\nu\Big) = \big(K(2^\nu, b, \bar{B})\big)_\nu .$$

As $\bar{F}$ and $\bar{E}$ are of type (O) we infer that

$$T: \bar{A}_{(\bar{F})_{\varphi,\infty:K}:J} \to \bar{B}_{(\bar{E})_{\varphi,1:J}:K} , \tag{5.22}$$

where $\varphi \in \mathfrak{T}$. Using two reiteration theorems for real interpolation spaces, [17], th. 3.11, th. 3.15, (5.22) implies that

$$T: (\bar{A}_{F_0:J}, \bar{A}_{F_1:J})_{\varphi,\infty:K} \to (\bar{B}_{E_0:K}, \bar{B}_{E_1:K})_{\varphi,1,J} .$$

The proof is complete.

The assumption that $(\bar{A}_{F_0:J}, \bar{A}_{F_1:J})$ is mutually closed can sometimes be dispensed with. Cf. remark 4.16.

A converse to th. 5.17 is furnished by the following theorem.

THEOREM 5.18. – Let $\bar{A}$ and $\bar{B}$ be two Banach pairs. Assume further that $\bar{A}$ is K-surjective and that $\bar{B}$ is mutually closed and K_0-surjective. Then if $(\bar{A}_{E_0:K}, \bar{A}_{E_1:K})$ and $(\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$ are of type (O) it follows that $\bar{E}$ and $\bar{F}$ are of type (O).

PROOF. – Take $T \in \mathfrak{L}(\bar{E}, \bar{F})$ and $\varphi \in \mathfrak{T}$. We want to show that

$$T: \bar{E}_{\varphi,\infty:K} \to \bar{F}_{\varphi,1:J} .$$

Take $a \in \Sigma(\bar{E})$ and put $b = Ta$. Choose $x_a \in \Sigma(\bar{A})$ satisfying

$$K(t, x_a, \bar{A}) \approx K(t, a, \bar{l}_\infty) . \tag{5.23}$$

As $\bar{A}$ and $\bar{l}_\infty$ are of type (C) we may construct $T_1: (\bar{A}_{E_0:K}, \bar{A}_{E_1:K}) \to (E_0, E_1)$ with $T_1 x_a = a$.

Pick $x_b \in \Sigma(\bar{B})$ with

$$K(t, x_b, \bar{B}) \approx K(t, b, \bar{l}_1) . \tag{5.14}$$

As $\bar{B}$ is mutually closed we may find $T_2\colon (F_0, F_1) \to (\bar{B}_{F_0:J}, \bar{B}_{F_1:J})$ with $T_2 b = x_b$. Put $S = T_2 T T_1$. Then $Sx_a = x_b$. From our assumptions we infer that

$$S\colon (\bar{A}_{E_0:K}, \bar{A}_{E_1:K})_{\varphi,\infty:K} \to (\bar{B}_{F_0:J}, \bar{B}_{F_1:J})_{\varphi,1:J},$$

where $\varphi \in \mathfrak{T}$. By reiteration (see [5], th. 6, [17], th. 3.6, th. 3.14) this simplifies to

$$S\colon \bar{A}_{E:K} \to \bar{B}_{F:J}$$

where $E = \bar{E}_{\varphi,\infty:K}$ and $F = \bar{F}_{\varphi,1:J}$. Now (5.23) yields

$$\|a\|_E \approx \|a\|_{(\bar{l}_\infty)_{E:K}} \approx \|x_a\|_{\bar{A}_{E:K}}.$$

As $\bar{B}$ is mutually closed (5.24) implies that

$$\|b\|_F \approx \|x_b\|_{\bar{A}_{F:J}}.$$

So altogether we now have

$$\|b\|_{\bar{F}_{\varphi,1:J}} \leq c\|a\|_{\bar{E}_{\varphi,\infty:K}},$$

i.e. $\bar{E}$ and $\bar{F}$ are of type (O). The proof is complete.

REMARK. – Let $\bar{E}$ be a regular pair. It is then only necessary to assume that $\bar{A}$ K_0-surjective.

We now wish to apply th. 5.18 in our analysis of type (O). Let us make the following assumptions. $\bar{A}$ is a K-surjective Banach pair and $\bar{B}$ is a mutually closed, K_0-surjective Banach pair. If $\bar{A}$ and $\bar{B}$ are of type (O) it follows from the proof of th. 5.18 that

$\bar{A}$ and $\bar{l}_1$ are of type (O) and

$\bar{l}_\infty$ and $\bar{B}$ are of type (O).

Take $\varphi \in \mathfrak{T}$. Remark. 5.11, (5.17) applied to $\bar{A}, \bar{l}_1$ and $\bar{l}_1(1/\varphi(2^\nu))$ yields that

$$\operatorname{Corb}_{\bar{l}_1}\big(\bar{A}, \bar{l}_1(1/\varphi(2^\nu))\big) = \bar{A}_{\varphi,\infty:K}. \tag{5.25}$$

Similary (5.16) implies that

$$O_{\bar{l}_\infty}(\varphi, \bar{B}) = \bar{B}_{\varphi,1:J}. \tag{5.26}$$

Now, quite generaly, a Banach pair $\bar{X}$ is called (uniformly) tame if for every $\varphi \in \mathcal{T}$ holds

$$O_{\bar{l}_\infty}(\varphi, \bar{X}) = \mathrm{Corb}_{\bar{l}_1}\left(\bar{X}, l_1(1/\varphi(2^\nu))\right)$$

(uniformly). This notion is due to OVČINNIKOV [18], who showed that $\bar{l}_\infty$ and $\bar{l}_1$ are tame.

Let us now assume that both $\bar{A}$ and $\bar{B}$ are uniformly tame pairs. Then (5.25) implies that

$$O_{\bar{l}_\infty}(\varphi, \bar{A}) = \bar{A}_{\varphi, \infty:K},$$

i.e. $\bar{l}_\infty$ and $\bar{A}$ are of type (C). Similary (5.26) yields

$$\bar{B}_{\varphi,1:J} = \mathrm{Corb}_{\bar{l}_1}\left(\bar{B}, l_1(1/\varphi(2^\nu))\right) = \mathrm{Co}_{\bar{l}_1}(\varphi, \bar{B}).$$

This last equality is fulfilled whenever $\bar{B}$ and $\bar{l}_1$ are of type (C) (see ex. 4.9). If we invoke prop. 5.15 we have now proven

PROPOSITION 5.19. – Let $\bar{A}$ be a uniformly tame, K-surjective Banach pair and let $\bar{B}$ be a uniformly tame, mutually closed, K_0-surjective Banach pair. Then $\bar{A}$ and $\bar{B}$ are of type (O) iff $\bar{l}_\infty$ and $\bar{A}$ are of type (C) and for every $\varphi \in \mathcal{T}$ holds uniformly $\bar{B}_{\varphi,1:J} = \mathrm{Co}_{\bar{l}_1}(\varphi, \bar{B})$.

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