Mathematical Methods in the Applied Sciences, Vol. 11, 105–124 (1989)
MOS subject classification: Primary 65 M 15;
Secondary 65 M 60, 35 K 05

Interpolation of Coefficients and Transformation of the Dependent Variable in Finite Element Methods for the Non-linear Heat Equation

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Communicated by W. Wendland

Error estimates are shown for some spatially discrete Galerkin finite element methods for a non-linear heat equation. The approximation schemes studied are based on the introduction of the enthalpy as a new dependent variable, and also on the application of the Kirchhoff transformation and on interpolation of the non-linear coefficients into standard Lagrangian finite element spaces.

1. Introduction

In this paper we study semidiscrete finite element methods with interpolated coefficients for the non-linear heat equation

$$\begin{aligned} c(u)u_t - \nabla\cdot(a(u)\nabla u) &= f(u), && \text{in } \Omega\times(0,T),\\ u &= 0, && \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0) &= v, && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^d$ with $d\leqslant 3$. For the spatial discretization of (1.1) we shall consider standard piecewise polynomial Lagrangian finite element spaces. Thus, we denote by S_h the space of continuous functions on Ω that reduce to polynomials of degree $\leqslant r-1$ on each simplex of a triangulation of Ω. We seek approximate solutions to (1.1) in the subspace S_{0h} consisting of those functions in S_h that satisfy the boundary condition in (1.1). See Section 2 for the precise statements

0170–4214/89/010105–20$10.00

Received 12 October 1987
Revised 18 December 1987

of our assumptions about the initial-boundary value problem (1.1) and the finite element spaces to be studied.

Consider first the standard semidiscrete Galerkin finite element method of finding u_h: $[0, T] \to S_{0h}$ such that

$$(c(u_h)u_{h,t}, \chi) + (a(u_h)\nabla u_h, \nabla\chi) = (f(u_h), \chi) \quad \text{for } \chi \in S_{0h}, 0<t<T,$$
$$u_h(0) = v_h, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the usual inner product in $L_2(\Omega)$ and $v_h \in S_{0h}$ is an approximation of v. In terms of the standard Lagrangian nodal basis $\{\phi_i\}_{i=1}^{N_h}$ of S_{0h} this reads

$$\mathbf{C}(\mathbf{U})\mathbf{U}' + \mathbf{A}(\mathbf{U})\mathbf{U} = \mathbf{F}(\mathbf{U}), \quad \text{for } 0<t<T,$$
$$\mathbf{U}(0) = \mathbf{V},$$

where $\mathbf{U} = (U_i)$ and $\mathbf{V} = (V_i)$ are the vectors of nodal values of u_h and v_h, respectively, and $\mathbf{C}(\mathbf{U})$ is the non-linear mass matrix with entries $\mathbf{C}(\mathbf{U})_{ij} = (c(\Sigma_k U_k\phi_k)\phi_i, \phi_j)$. Similarly, the non-linear stiffness matrix $\mathbf{A}(\mathbf{U})$ is given by $\mathbf{A}(\mathbf{U})_{ij} = (a(\Sigma_k U_k\phi_k)\nabla\phi_i, \nabla\phi_j)$ and the right-hand side $\mathbf{F}(\mathbf{U})$ by $\mathbf{F}(\mathbf{U})_j = (f(\Sigma_k U_k\phi_k), \phi_j)$.

From the point of view of actually computing the solution we note two difficulties: (i) the system is not written in normal form $Y' = f(t, Y)$, and (ii) the above inner products must be computed by numerical quadrature. The first difficulty can be handled by a classical transformation of the dependent variable. With $H(u) = \int_0^u c(s)ds$—the enthalpy—and $G(u) = \int_0^u a(s)ds$—the Kirchhoff transformation—the differential equation in the semidiscrete problem (1.2) can be written

$$(H(u_h)_t, \chi) + (\nabla G(u_h), \nabla\chi) = (f(u_h), \chi).$$

For the numerical quadrature we shall replace the coefficients by their interpolants. Thus, let I_h be the operator which associates with each continuous function g its interpolant $I_h g \in S_h$ defined by $(I_h g)(P) = g(P)$ for each of the nodes P that define the degrees of freedom of S_h. We are then led to consider the following interpolated coefficient finite element method: find u_h: $[0, T] \to S_{0h}$ such that

$$(I_h H(u_h)_t, \chi) + (\nabla I_h G(u_h), \nabla\chi) = (I_h f(u_h), \chi), \quad \text{for } \chi \in S_{0h}, 0<t<T,$$
$$u_h(0) = v_h. \tag{1.3}$$

Let $\{\phi_i\}_{i=1}^{M_h}$ be the nodal basis of S_h. Thus, the indices $1 \leqslant i \leqslant N_h$ refer to the interior nodes and the indices $N_h + 1 \leqslant i \leqslant M_h$ refer to the boundary nodes. To compute u_h from (1.3) one has to solve the system of ordinary differential equations

$$\sum_{i=1}^{N_h} W_i'(\phi_i, \phi_j) + \sum_{i=1}^{N_h} G(U_i)(\nabla\phi_i, \nabla\phi_j) = \sum_{i=1}^{M_h} f(U_i)(\phi_i, \phi_j), \quad j = 1, \ldots, N_h,$$

where U_i are the nodal values of u_h and $W_i = H(U_i)$, subject to the initial conditions

$$W_j(0) = H(V_j), \quad j = 1, \ldots, N_h.$$

Here we use the fact that, assuming the coefficient c to be positive, the enthalpy is a strictly increasing function, so that $U_i = H^{-1}(W_i)$ is uniquely defined. Thus, one actually computes an approximate enthalpy $w_h(t) = \Sigma_{i=1}^{N_h} W_i(t)\phi_i \in S_{0h}$, from which the temperature $u_h(t) = \Sigma_{i=1}^{N_h} H^{-1}(W_i(t))\phi_i \in S_{0h}$ can be retrieved. Clearly, one can compute the standard mass and stiffness matrices (ϕ_i, ϕ_j) and $(\nabla\phi_i, \nabla\phi_j)$ once and for all and then solve this system iteratively by some standard time-stepping procedure. We shall refrain from analysing this aspect of the problem.

In Section 4 below we estimate the L_2 and H^1 norms of the error in the approximate solution u_h given by (1.3). We first show, for $r \geq 3$, $d \leq 3$, the error estimate

$$\|u_h(t)-u(t)\| + \left(\int_0^t \|\nabla(u_h(\tau)-u(\tau))\|^2 d\tau \right)^{1/2} \leq Ch^{r-1}, \quad \text{for } 0 \leq t \leq T,$$

where $\|\cdot\|$ denotes the norm in $L_2(\Omega)$. Note that the mean square average of the gradient of the error is of optimal order, whereas we have only been able to show a suboptimal error estimate pointwise in time. For the special case where $c \equiv 1$ we obtain, for $r \geq 3$, $d \leq 3$, a similar result, where again the L_2 norm of the error is one order less than optimal, pointwise in time, but where now the mean square average of the error is shown to be of optimal order $O(h^r)$. The case $r=2$, $d=1$ is somewhat particular and we obtain for general $c=c(u)$ an $O(h^2)$ error bound, pointwise in time.

The difficulty in this analysis stems from the way the interpolation is carried out under the gradient in the second term on the left-hand side. We therefore consider also the following method, where the coefficient $a(u_h)$ is interpolated directly: find u_h: $[0, T] \to S_{0h}$ such that

$$(I_h H(u_h)_t, \chi) + ((I_h a(u_h))\nabla u_h, \nabla \chi) = (I_h f(u_h), \chi), \quad \text{for } \chi \in S_{0h}, 0 < t < T,$$

$$u_h(0) = v_h. \tag{1.4}$$

In matrix form this reads

$$\sum_{i=1}^{N_h} W_i'(\phi_i, \phi_j) + \sum_{k=1}^{M_h} \sum_{i=1}^{N_h} a(U_k) U_i (\phi_k \nabla \phi_i, \nabla \phi_j) = \sum_{i=1}^{M_h} f(U_i)(\phi_i, \phi_j),$$

$$j = 1, \ldots, N_h,$$

$$W_j(0) = H(V_j), \quad j = 1, \ldots, N_h,$$

and the above remark about solvability applies to this system as well. We analyse this method in Section 3 and find that, provided that the initial approximation v_h is chosen as an elliptic projection of v, the L_2 norm of the error is of optimal order pointwise in time for $r \geq 2$. For the special case, where $c(u) \equiv 1$, we show an optimal order error estimate without this restriction on v_h.

Several authors have considered numerical quadrature in finite element methods. The effect of quadrature in linear parabolic problems was analysed by Raviart.[10] Christie *et al.*[3] coined the term *product approximation* to refer to finite element techniques based on interpolation. Douglas and Dupont[5] studied approximate problems of the type (1.4) with $c(u) \equiv 1$ and $f(u) \equiv 0$. In their work I_h is allowed to be a more general projection. Nie and Thomée[9], again with $c(u) \equiv 1$, considered the middle term in (1.4) in conjunction with the lumped mass method for the first term in a piecewise linear, two-dimensional setting. Khalsa[8] analysed a finite element method with product approximation for a semilinear parabolic problem with a cubic non-linearity in one space dimension.

The present work was inspired by the papers of Čermák and Zlámal[2] and Borshukova and Konovski,[1] in which the method (1.3) was applied to various heat conduction problems with and without phase change. These papers report on numerical computations and contain no error analysis. Our analysis does not allow phase change, i.e. we do not allow $H(u)$ and $G(u)$ to be non-smooth functions of u.

Such a problem was, however, analysed by Elliott.[6] He assumed that $H(u)$ has a jump discontinuity and that $G(u)=u$ and $f=f(x, t)$. For a completely discrete version of (1.3), using a piecewise linear finite element method for the spatial discretization, he obtained an $O(h^{1/2})$ estimate for the mean square average in time of the L_2 norm of the error.

The product approximation for semilinear elliptic problems was analysed by Sanz-Serna and Abia.[11] Their analysis is based on inverse inequalities and a continuation argument, an approach that we have adopted here, too.

2. Notation and preliminaries

In this section we state our general assumptions about the non-linear initial-boundary value problem (1.1) and the finite element methods to be analysed. We also collect some notation and preliminary results.

Let Ω be a bounded polygonal domain in $\mathbb{R}^d$ with $d \leqslant 3$. We shall assume that the coefficients $c(u)$, $a(u)$ and $f(u)$ of (1.1) are smooth functions of $u \in \mathbb{R}$ and that c and a are uniformly positive:

$$c(u) \geqslant c_0 > 0,\ a(u) \geqslant a_0 > 0 \quad \text{for all } u \in \mathbb{R}. \tag{2.1}$$

We further assume that (1.1) has a unique solution u, which is sufficiently smooth for our purposes. Throughout this paper we thus make the somewhat unrealistic—but commonplace—assumption that the solution of problem (1.1) is very smooth, in spite of the polygonal character of the domain Ω.

For the approximation of (1.1) we shall consider standard piecewise polynomial Lagrangian finite element spaces. Thus, we assume that we have a quasi-uniform family $\{\tau_h\}_{h>0}$ of simplicial triangulations of Ω with the parameter h being the maximal diameter of any simplex K in τ_h. Further, for some integer $r \geqslant 2$, we denote by S_h the space of continuous functions that reduce to polynomials of degree $\leqslant r-1$ on each simplex $K \in \tau_h$, and we let $S_{0h} = \{\chi \in S_h \colon \chi|_{\partial\Omega} = 0\}$. Thus, we have $S_h \subset H^1(\Omega)$ and $S_{0h} \subset H_0^1(\Omega)$.

We shall use the notation $(\cdot,\cdot)$ and $\|\cdot\|$ for the inner product and norm of $L_2 = L_2(\Omega)$ and $\|\cdot\|_{m,p}$ for the norms of the Sobolev spaces $W_p^m = W_p^m(\Omega)$. For $p=2$ we write $H^m = H^m(\Omega)$ and $\|\cdot\|_m$. These norms should be interpreted in the piecewise sense, when applied to functions that are only piecewise differentiable with respect to τ_h. Further, we write

$$\|v\|_{L_p(0,t;X)} = \left(\int_0^t \|v(\tau)\|_X^p \, d\tau\right)^{1/p},$$

with the usual modification for $p=\infty$ and where X could be any of the Banach spaces mentioned above.

We define the interpolation operator $I_h \colon C(\bar{\Omega}) \to S_h$ by the condition that $(I_h v)(P) = v(P)$ for any of the nodes P that define the degrees of freedom of S_h. From the theory of finite elements we quote the following error estimate: for $0 \leqslant m \leqslant r$ and $1 \leqslant p \leqslant \infty$ we have

$$\|I_h v - v\|_{m,p} \leqslant C h^{r-m} \|v\|_{r,p}, \tag{2.2}$$

if v belongs to $C(\bar{\Omega})$ and $W_p^r(K)$ for all $K \in \tau_h$, (see, for instance, Reference 4, Theorem 3.1.6).

We shall often need to be able to estimate high order norms of the error in terms of lower order norms. This can be done by an inverse inequality argument, which we state in the following lemma.

Lemma 1. *Let $0 \leqslant l \leqslant m \leqslant r$, $1 \leqslant q \leqslant p \leqslant \infty$. Then for $\chi \in S_h$ and $v \in W_p^r$ we have*

$$\|\chi - v\|_{m,p} \leqslant Ch^{-(m-l)-[(d/q)-(d/p)]}(\|\chi - v\|_{l,q} + h^{r-l}\|v\|_{r,p}). \tag{2.3}$$

Proof. Using (2.2) and an inverse inequality (Reference 4, Theorem 3.2.6) we obtain

$$\begin{aligned}\|\chi - v\|_{m,p} &\leqslant \|\chi - I_h v\|_{m,p} + \|I_h v - v\|_{m,p}\\ &\leqslant Ch^{-(m-l)-[(d/q)-(d/p)]}\|\chi - I_h v\|_{l,q} + \|I_h v - v\|_{m,p}\\ &\leqslant Ch^{-(m-l)-[(d/q)-(d/p)]}(\|\chi - v\|_{l,q} + \|I_h v - v\|_{l,q})\\ &\quad + \|I_h v - v\|_{m,p}\\ &\leqslant Ch^{-(m-l)-[(d/q)-(d/p)]}(\|\chi - v\|_{l,q} + Ch^{r-l}\|v\|_{r,q})\\ &\quad + Ch^{r-m}\|v\|_{r,p},\end{aligned}$$

which proves the lemma.

In our error analysis we shall also use a Ritz projection $\tilde{u}_h = \tilde{u}_h(t) \in S_{0h}$ of the exact solution u of (1.1). For fixed $t \in [0, T]$, we define this to be the solution of the linear problem

$$(a(u(t))\nabla(\tilde{u}_h(t) - u(t)), \nabla\chi) = 0, \quad \text{for all } \chi \in S_{0h}. \tag{2.4}$$

To discuss this definition (and for later reference) we consider the linear elliptic problem

$$\begin{aligned}\nabla\cdot(a(u)\nabla w) &= g, \quad \text{in } \Omega \times (0, T),\\ w &= 0, \quad \text{on } \partial\Omega \times (0, T).\end{aligned}$$

We may define the solution operator $T = T(u(t))$: $L_2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega)$ (not to be mistaken for the length of the time interval) by

$$(a(u)\nabla Tg, \nabla\chi) = (g, \chi), \quad \chi \in H_0^1(\Omega).$$

The corresponding approximate solution operator $T_h = T_h(u(t))$: $L_2(\Omega) \to S_{0h}$ is given by

$$(a(u)\nabla T_h g, \nabla\chi) = (g, \chi), \quad \chi \in S_{0h}. \tag{2.5}$$

In order to be able to perform the duality argument of standard error analysis, we need to assume that Ω is such that

$$\|Tg\|_2 \leqslant C\|g\|, \quad g \in L_2(\Omega), \tag{2.6}$$

so that

$$\|(T_h - T)g\|_1 \leqslant Ch\|g\|, \quad g \in L_2(\Omega), \tag{2.7}$$

It is well known that this holds, for instance, for convex polygonal domains (see Theorem 3.2.1.2 of Reference 7 and its proof).

In our next lemma we collect some error estimates and maximum norm bounds for $\tilde{u}_h$.

Lemma 2. *There is a constant $C=C(u)$ such that for $0\leqslant t\leqslant T$ we have*

$$\|D_t^j(\tilde{u}_h-u)(t)\|_l\leqslant Ch^{r-l},\quad \text{for } j, l=0, 1, \tag{2.8}$$

and

$$\|D_t^j\tilde{u}_h(t)\|_{1,\infty}\leqslant C,\quad \textit{for } j=0, 1. \tag{2.9}$$

Proof. With the Ritz projection operator $R_h=R_h(u(t))$: $H_0^1(\Omega)\rightarrow S_{0h}$ defined by

$$(a(u(t))\nabla(R_hw-w), \nabla\chi)=0,\quad \text{for all } \chi\in S_{0h}, \tag{2.10}$$

we have $\tilde{u}_h=R_hu$. The case $j=0$ of (2.8) now follows from the standard error analysis for linear elliptic problems with variable coefficients. The case $j=1$ follows in a straightforward way after differentiation of (2.4) with respect to time,

$$(a\nabla(\tilde{u}_{h,t}-u_t), \nabla\chi)+(a_t\nabla(\tilde{u}_h-u), \nabla\chi)=0,\quad \text{for all } \chi\in S_{0h}, \tag{2.11}$$

where we have written a for $a(u)$ and a_t for $a(u)_t=a'(u)u_t$. We refer to Reference 13 (Lemmas 2 and 3 in Chapter 5) for the details.

For the proof of (2.9) we shall use the maximum norm stability of R_h,

$$\|R_hw\|_{0,\infty}\leqslant C(\log(1/h))^{\bar{r}}\|w\|_{0,\infty}, \tag{2.12}$$

where $\bar{r}=1$ if $r=2$, $\bar{r}=0$ otherwise, (cf. (5.9)′ in Reference 12). (Although the results of Reference 12 are formulated for a model problem with constant coefficients, the authors remark that their methods work in our more general situation, as well.) Let $\rho=\tilde{u}_h-u$. An application of (2.12) with $w=u(t)-\chi$, $\chi\in S_{0h}$ arbitrary, shows

$$\|\rho(t)\|_{0,\infty}\leqslant Ch^r(\log(1/h))^{\bar{r}}, \tag{2.13}$$

so that, by Lemma 1, $\|\rho(t)\|_{1,\infty}\leqslant C$ and the case $j=0$ of (2.9) follows.

Next we note that (2.11) can be written

$$(a\nabla(\tilde{u}_{h,t}-u_t), \nabla\chi)+\left(a\nabla\left(\frac{a_t}{a}\rho\right), \nabla\chi\right)-\left(a\rho\nabla\left(\frac{a_t}{a}\right), \nabla\chi\right)=0.$$

This means that $\tilde{u}_{h,t}-R_hu_t+R_h((a_t/a)\rho)=\eta$ is an element of S_{0h}, which satisfies

$$(a\nabla\eta, \nabla\chi)=\left(a\rho\nabla\left(\frac{a_t}{a}\right), \nabla\chi\right),\quad \text{for all } \chi\in S_{0h}. \tag{2.14}$$

By the same token as for $\tilde{u}_h=R_hu$, we have $\|R_hu_t(t)\|_{1,\infty}\leqslant C$. By (2.12) and (2.13), we next obtain

$$\left\|R_h\left(\frac{a_t}{a}\rho\right)(t)\right\|_{0,\infty}\leqslant C(\log(1/h))^{\bar{r}}\left\|\frac{a_t}{a}\rho(t)\right\|_{0,\infty}\leqslant Ch^r(\log(1/h))^{2\bar{r}},$$

which implies $\|R_h((a_t/a)\rho)(t)\|_{1,\infty}\leqslant C$ in view of an inverse inequality.

Finally, setting $\chi=\eta$ in (2.14), we obtain

$$\|\eta(t)\|_1\leqslant C\left\|a\rho\nabla\left(\frac{a_t}{a}\right)\right\|\leqslant C\|\rho(t)\|$$

and an inverse inequality and (2.8) show

$$\|\eta(t)\|_{1,\infty} \leqslant Ch^{r-d/2} \leqslant C.$$

Since $\tilde{u}_{h,t} = R_h u_t - R_h((a_t/a)\,\rho) + \eta$, this proves the remaining case $j=1$ of (2.9) and the proof of the lemma is complete.

3. Analysis of the second method

In this section we shall analyse the following interpolated coefficient finite element method for the non-linear heat equation (1.1): find u_h: $[0, T] \to S_{0h}$ such that

$$(I_h H(u_h)_t, \chi) + ((I_h a(u_h)) \nabla u_h, \nabla \chi) = (I_h f(u_h), \chi), \quad \text{for } \chi \in S_{0h},\ 0 < t < T,$$
$$u_h(0) = v_h. \tag{3.1}$$

We shall show an error estimate, which is of optimal order, provided that the initial approximation is chosen as the Ritz projection $\tilde{v}_h = \tilde{u}_h(0)$ of v defined in Section 2.

Theorem 1. *Let u_h and u be the solutions of* (3.1) *and* (1.1), *respectively, and assume that $v_h = \tilde{v}_h = \tilde{u}_h(0)$. Then there are positive numbers $h_0 = h_0(u, T)$ and $C = C(u, T)$ such that, for $h < h_0$, we have*

$$\|u_h(t) - u(t)\| + h\|u_h(t) - u(t)\|_1 \leqslant Ch^r, \quad \text{for } 0 \leqslant t \leqslant T.$$

In the proof of Theorem 1 we shall follow the standard method of splitting the error into two parts,

$$e = u_h - u = (u_h - \tilde{u}_h) + (\tilde{u}_h - u)$$

where $\tilde{u}_h$ is the Ritz projection of u defined in Section 2. In Lemma 2 we found that $\tilde{u}_h - u$ satisfies the desired error estimate and so it remains to estimate $\theta = u_h - \tilde{u}_h$. This will be done in the following lemma. The proof of the theorem will then be completed by means of a continuation argument.

Lemma 3. *In addition to the assumptions of Theorem* 1, *assume that, for some t_1 with $0 < t_1 \leqslant T$, we have*

$$\|e\|_{L_\infty(0,t_1;L_2)} + \|e_t\|_{L_2(0,t_1;L_2)} \leqslant h^{r-1/4}. \tag{3.2}$$

Then it follows that

$$\|e\|_{L_\infty(0,t_1;L_2)} + h\|e\|_{L_\infty(0,t_1;H^1)} + \|e_t\|_{L_2(0,t_1;L_2)} \leqslant Ch^r,$$

where $C = C(u, T)$ does not depend on t_1.

Proof. Let us first note that, by the inverse inequality argument of Lemma 1, and since $r \geqslant 2$ and $d \leqslant 3$, the hypothesis (3.2) implies

$$\|e\|_{L_\infty(0,t_1;W_4^{r-1})} \leqslant C, \quad \|e\|_{L_\infty(0,t_1;W_\infty^{r-2})} \leqslant C$$

and

$$\|e_t\|_{L_2(0,t_1;W_4^{r-1})} \leqslant C, \quad \|e_t\|_{L_2(0,t_1;W_\infty^{r-2})} \leqslant C,$$

so that, in particular, since u is smooth,

$$\|u_h\|_{L_\infty(0,t_1;W_4^{r-1})} \leqslant C, \quad \|u_h\|_{L_\infty(0,t_1;W_\infty^{r-2})} \leqslant C \tag{3.3}$$

and

$$\|u_{h,t}\|_{L_2(0,t_1;W_4^{r-1})}\leqslant C, \quad \|u_{h,t}\|_{L_2(0,t_1;W_\infty^{r-2})}\leqslant C. \tag{3.4}$$

Using these bounds we shall next show that

$$\|f(u_h)\|_{L_\infty(0,t_1;H^r)}\leqslant C, \tag{3.5}$$

$$\|a(u_h)\|_{L_\infty(0,t_1;H^r)}\leqslant C, \tag{3.6}$$

$$\|H(u_h)_t\|_{L_2(0,t_1;H^r)}\leqslant C, \tag{3.7}$$

$$\|a(u_h)_t\|_{L_2(0,t_1;H^r)}\leqslant C. \tag{3.8}$$

For the proof of (3.5) we let $|\alpha|=r$ and apply Hölder's inequality to the formula

$$D^\alpha f(u_h)=\sum_{i=1}^{r}\sum_{\substack{\Sigma_1^i\beta_j=\alpha\\ \beta_j\neq 0}} f^{(i)}(u_h)D^{\beta_1}u_h\ldots D^{\beta_i}u_h, \tag{3.9}$$

which, since $\|u_h\|_{L_\infty(0,t_1;L_\infty)}\leqslant C$ by (3.3), implies

$$\|D^\alpha f(u_h)\|\leqslant C\sum_{\substack{\Sigma_1^i\beta_j=\alpha\\ \beta_j\neq 0}}\|D^{\beta_1}u_h\|_{0,q_1}\ldots\|D^{\beta_i}u_h\|_{0,q_i}, \tag{3.10}$$

where $\Sigma_1^i\, 1/q_j=1/2$. Now note that—elementwise—any rth order derivative of u_h is identically zero and that factors $D^{\beta_j}u_h$ with $|\beta_j|=r-1$ can occur at most twice in any of the products in equation (3.9)—in fact, they occur twice only if $r=2$. Thus, we may take $q_j=2$ or 4 if $|\beta_j|=r-1$ and $q_j=\infty$ otherwise and (3.5) follows in view of the bounds in (3.3).

For (3.7) we have in a similar manner

$$\begin{aligned}D^\alpha H(u_h)_t&=\sum_{\beta_0+\beta=\alpha}D^{\beta_0}u_{h,t}D^\beta c(u_h)\\ &=\sum_{\beta_0+\beta=\alpha}\sum_{i=0}^{|\beta|}D^{\beta_0}u_{h,t}\sum_{\substack{\Sigma_1^i\beta_j=\beta\\ \beta_j\neq 0}}c^{(i)}(u_h)D^{\beta_1}u_h\ldots D^{\beta_i}u_h,\end{aligned}$$

so that

$$\begin{aligned}\|D^\alpha H(u_h)_t\|_{L_2(0,t_1;L_2)}\leqslant C&\sum_{\Sigma_0^i\beta_j=\alpha}\|D^{\beta_0}u_{h,t}\|_{L_2(0,t_1;L_{q_0})}\\ &\times\|D^{\beta_1}u_h\|_{L_\infty(0,t_1;L_{q_1})}\ldots\|D^{\beta_i}u_h\|_{L_\infty(0,t_1;L_{q_i})},\end{aligned} \tag{3.11}$$

where $\Sigma_0^i\, 1/q_j=1/2$ and (3.7) follows by the same argument as above. The bounds (3.6) and (3.8) are proved in the same way as (3.5) and (3.7).

Next we shall bound $\theta=u_h-\tilde{u}_h$. Consider first the case $r>2$. Using (3.1), (2.4) and the weak form of equation (1.1), we have, for $\chi\in S_{0h}$ and $0<t<t_1$,

$$\begin{aligned}&(c(u_h)\theta_t,\chi)+(a(u_h)\nabla\theta,\nabla\chi)=((I_h-I)f(u_h),\chi)+(f(u_h)-f(u),\chi)\\ &-((I_h-I)H(u_h)_t,\chi)-((c(u_h)-c(u))u_t,\chi)-(c(u_h)(\tilde{u}_{h,t}-u_t),\chi)\\ &-(((I_h-I)a(u_h))\nabla u_h,\nabla\chi)-((a(u_h)-a(u))\nabla\tilde{u}_h,\nabla\chi)\\ &=\sum_{i=1}^{5}(R_i,\chi)+\sum_{i=6}^{7}(R_i,\nabla\chi),\end{aligned} \tag{3.12}$$

where I denotes the identity operator and with the obvious definitions of the terms R_i. Taking $\chi=\theta_t$, we obtain

$$(c(u_h)\theta_t, \theta_t)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(a(u_h)\nabla\theta, \nabla\theta)=\sum_{i=1}^{5}(R_i, \theta_t)+\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=6}^{7}(R_i, \nabla\theta)$$
$$-\sum_{i=6}^{7}(R_{i,t}, \nabla\theta)+(R_8, \nabla\theta),$$

where $R_8=1/2a(u_h)_t\nabla\theta$. Integration with respect to t, using (2.1) and the fact that $\theta(0)=0$, yields

$$\int_0^t \|\theta_t\|^2\mathrm{d}\tau+\|\theta(t)\|_1^2\leqslant C\sum_{i=1}^{5}\int_0^t\|R_i\|\,\|\theta_t\|\,\mathrm{d}\tau+C\sum_{i=6}^{7}\|R_i(t)\|\,\|\theta(t)\|_1$$
$$+C\int_0^t(\|R_{6,t}\|+\|R_{7,t}\|+\|R_8\|)\|\theta\|_1\mathrm{d}\tau,$$

so that, after trivial estimates and a simple kick-back argument,

$$\|\theta_t\|^2_{L_2(0,t;L_2)}+\|\theta\|^2_{L_\infty(0,t;H^1)}\leqslant C\sum_{i=1}^{5}\|R_i\|^2_{L_2(0,t;L_2)}+C\sum_{i=6}^{7}\|R_i\|^2_{L_\infty(0,t;L_2)}$$
$$+C\int_0^t(\|R_{6,t}\|+\|R_{7,t}\|+\|R_8\|)\|\theta\|_1\mathrm{d}\tau.$$

Here we have to bound the various terms on the right-hand side. To begin with, using (2.2), (3.5) and (3.7), we have

$$\|R_1\|_{L_2(0,t;L_2)}\leqslant Ch^r\|f(u_h)\|_{L_2(0,t;H^r)}\leqslant Ch^r,$$
$$\|R_3\|_{L_2(0,t;L_2)}\leqslant Ch^r\|H(u_h)_t\|_{L_2(0,t;H^r)}\leqslant Ch^r.$$

Next, since u_h is bounded by (3.3), using (2.8) we obtain

$$\|R_2\|_{L_2(0,t;L_2)}+\|R_4\|_{L_2(0,t;L_2)}\leqslant C\|u_h-u\|_{L_2(0,t;L_2)}$$
$$\leqslant C(\|\tilde{u}_h-u\|_{L_2(0,t;L_2)}+\|\theta\|_{L_2(0,t;L_2)})$$
$$\leqslant Ch^r+C\|\theta\|_{L_2(0,t;H^1)},$$

and

$$\|R_5\|_{L_2(0,t;L_2)}\leqslant C\|\tilde{u}_{h,t}-u_t\|_{L_2(0,t;L_2)}\leqslant Ch^r.$$

To obtain bounds of R_6 and $R_{6,t}$ we note that, by (3.3) and (3.4) and since $r\geqslant 3$, u_h and $u_{h,t}$ are bounded in $L_\infty(0, t; W^1_\infty)$ and $L_2(0, t; W^1_\infty)$, respectively. We find

$$\|R_6\|_{L_\infty(0,t;L_2)}\leqslant\|(I_h-I)a(u_h)\|_{L_\infty(0,t;L_2)}\|\nabla u_h\|_{L_\infty(0,t;L_\infty)}$$
$$\leqslant Ch^r\|a(u_h)\|_{L_\infty(0,t;H^r)}\|u_h\|_{L_\infty(0,t;W^1_\infty)}\leqslant Ch^r,$$

by (3.6), and

$$\begin{aligned}\int_0^t \|R_{6,t}\| \, \|\theta\|_1 \,d\tau &\leqslant \|R_{6,t}\|_{L_2(0,t;L_2)} \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant (\|(I_h-I)a(u_h)_t\|_{L_2(0,t;L_2)} \|\nabla u_h\|_{L_\infty(0,t;L_\infty)} \\ &\quad + \|(I_h-I)a(u_h)\|_{L_\infty(0,t;L_2)} \|\nabla u_{h,t}\|_{L_2(0,t;L_\infty)}) \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant Ch^r(\|a(u_h)_t\|_{L_2(0,t;H^r)} \|u_h\|_{L_\infty(0,t;W^1_\infty)} \\ &\quad + \|a(u_h)\|_{L_\infty(0,t;H^r)} \|u_{h,t}\|_{L_2(0,t;W^1_\infty)}) \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant Ch^r \|\theta\|_{L_2(0,t;H^1)} \leqslant Ch^{2r} + C\|\theta\|^2_{L_2(0,t;H^1)},\end{aligned}$$

by (3.8) and (3.6). To estimate R_7 and $R_{7,t}$ we shall use the bounds of $\tilde{u}_h$ and $\tilde{u}_{h,t}$ in (2.9). First we obtain

$$\begin{aligned}\|R_7\|_{L_\infty(0,t;L_2)} &\leqslant \|a(u_h)-a(u)\|_{L_\infty(0,t;L_2)} \|\nabla \tilde{u}_h\|_{L_\infty(0,t;L_\infty)} \\ &\leqslant C\|u_h-u\|_{L_\infty(0,t;L_2)} \leqslant C(\|\tilde{u}_h-u\|_{L_\infty(0,t;L_2)} + \|\theta\|_{L_\infty(0,t;L_2)}) \\ &\leqslant Ch^r + C\|\theta\|_{L_\infty(0,t;L_2)}\end{aligned}$$

where, for some $\tilde{t} \in [0, t]$,

$$\begin{aligned}\|\theta\|^2_{L_\infty(0,t;L_2)} = \|\theta(\tilde{t})\|^2 &= \int_0^{\tilde{t}} \frac{d}{d\tau}\|\theta\|^2 d\tau \leqslant 2\int_0^t \|\theta\| \, \|\theta_t\| \,d\tau \\ &\leqslant 2\|\theta\|_{L_2(0,t;L_2)} \|\theta_t\|_{L_2(0,t;L_2)},\end{aligned}$$

since $\theta(0)=0$. Thus, for any $\varepsilon > 0$, we can select C such that

$$\|R_7\|^2_{L_\infty(0,t;L_2)} \leqslant Ch^{2r} + C\|\theta\|^2_{L_2(0,t;H^1)} + \varepsilon\|\theta_t\|^2_{L_2(0,t;L_2)}.$$

Similarly,

$$\begin{aligned}\int_0^t \|R_{7,t}\| \, \|\theta\|_1 \,d\tau &\leqslant \|R_{7,t}\|_{L_2(0,t;L_2)} \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant (\|(a(u_h)-a(u))_t\|_{L_2(0,t;L_2)} \|\nabla \tilde{u}_h\|_{L_\infty(0,t;L_\infty)} \\ &\quad + \|a(u_h)-a(u)\|_{L_2(0,t;L_2)} \|\nabla \tilde{u}_{h,t}\|_{L_\infty(0,t;L_\infty)}) \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant C(\|u_h-u\|_{L_2(0,t;L_2)} + \|u_{h,t}-u_t\|_{L_2(0,t;L_2)}) \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant C(\|\theta\|_{L_2(0,t;L_2)} + \|\tilde{u}_h-u\|_{L_2(0,t;L_2)} \\ &\quad + \|\theta_t\|_{L_2(0,t;L_2)} + \|\tilde{u}_{h,t}-u_t\|_{L_2(0,t;L_2)}) \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant Ch^{2r} + C\|\theta\|^2_{L_2(0,t;H^1)} + \varepsilon\|\theta_t\|^2_{L_2(0,t;L_2)}.\end{aligned}$$

Finally, from (3.4) it follows that

$$\begin{aligned}\int_0^t \|R_8\| \, \|\theta\|_1 \,d\tau &\leqslant C\|u_{h,t}\|_{L_2(0,t;L_\infty)} \|\theta\|_{L_\infty(0,t;H^1)} \|\theta\|_{L_2(0,t;H^1)} \\ &\leqslant C\|\theta\|^2_{L_2(0,t;H^1)} + \varepsilon\|\theta\|^2_{L_\infty(0,t;H^1)}.\end{aligned}$$

Summing up, choosing ε appropriately, we now have

$$\|\theta\|^2_{L_\infty(0,t;H^1)} + \|\theta_t\|^2_{L_2(0,t;L_2)} \leqslant Ch^{2r} + C\|\theta\|^2_{L_2(0,t;H^1)}, \quad \text{for} \quad 0 < t < t_1,$$

which, by Grönwall's inequality, shows

$$\|\theta\|^2_{L_\infty(0,t_1;H^1)} + \|\theta_t\|^2_{L_2(0,t_1;L_2)} \leqslant Ch^{2r}.$$

Together with the appropriate estimates of $\tilde{u}_h - u$ from Lemma 2 this proves the case $r > 2$ of the lemma.

If $r = 2$ (and $d = 3$), then (3.3) and (3.4) no longer imply W^1_∞ bounds for u_h and $u_{h,t}$. But, on the other hand, (2.1) now implies that $I_h a(u_h) \geqslant a_0 > 0$ and we shall use the following variant of equation (3.12):

$$\begin{aligned}(c(u_h)\theta_t, \chi) + (I_h a(u_h)\nabla\theta, \nabla\chi) &= ((I_h - I)f(u_h), \chi) + (f(u_h) - f(u), \chi)\\ &- ((I_h - I)H(u_h)_t, \chi) - ((c(u_h) - c(u))u_t, \chi) - (c(u_h)(\tilde{u}_{h,t} - u_t), \chi)\\ &- (((I_h - I)a(u_h))\nabla\tilde{u}_h, \nabla\chi) - ((a(u_h) - a(u))\nabla\tilde{u}_h, \nabla\chi). \end{aligned} \tag{3.13}$$

Thus, all terms involving $a(u_h)$ can now be estimated using the maximum norm bounds of $\tilde{u}_h$ and $\tilde{u}_{h,t}$ in Lemma 2. This completes the proof.

Proof of Theorem 1. Let t_1^* be the largest t_1 such that (3.2) holds. It is obvious that $t_1^* > 0$. If $t_1^* < T$, then by Lemma 3 we can find $h_0 > 0$ such that, for $h < h_0$, we have

$$\|e\|_{L_\infty(0,t_1^*;L_2)} + \|e_t\|_{L_2(0,t_1^*;L_2)} \leqslant Ch^r \leqslant \frac{1}{2}h^{r-1/4},$$

in contradiction to the maximality of t_1^*. Thus $t_1^* = T$ and the proof is complete.

In Theorem 1 we assumed that $v_h = \tilde{v}_h$ in order to be able to prove an $O(h^r)$ error estimate for $\|\theta_t\|_{L_2(0,t;L_2)}$, which was needed because of the non-linearity in the coefficient $c(u)$—recall how the bound (3.7) was used in estimating the term R_3 in the proof of Lemma 3. (Clearly, it is sufficient to choose v_h in such a way that $\|v_h - \tilde{v}_h\|_1 = \|\theta(0)\|_1$ is of superconvergent order $O(h^r)$.) In our next result we shall assume that c does not depend on u—for simplicity we take $c(u) \equiv 1$—and we shall prove an error estimate without any such restriction on v_h.

Theorem 2. *Let $c(u) \equiv 1$ and let u_h and u be the solutions of* (3.1) *and* (1.1), *respectively, and assume that*

$$\|v_h - v\| \leqslant Ch^r. \tag{3.14}$$

Then there are positive numbers $h_0 = h_0(u, T)$ and $C = C(u, T)$ such that, for $h < h_0$, we have

$$\|u_h(t) - u(t)\| + h\|u_h(t) - u(t)\|_1 \leqslant Ch^r, \quad \text{for} \quad 0 \leqslant t \leqslant T.$$

By the above continuation argument the theorem will follow once we have proved the following lemma.

Lemma 4. *In addition to the assumptions of Theorem* 2, *assume that, for some t_1 with $0 < t_1 < T$, we have*

$$\|e\|_{L_\infty(0,t_1;L_2)} \leqslant h^{r-1/4}. \tag{3.15}$$

Then it follows that

$$\|e\|_{L_\infty(0,t_1;L_2)} + h\|e\|_{L_\infty(0,t_1;H^1)} \leqslant Ch^r,$$

where $C = C(u, T)$ *does not depend on* t_1.

Proof. Our assumption (3.15) implies that the bounds in (3.3), (3.5) and (3.6) hold. Considering first the case of $r > 2$, we take $\chi = \theta$ in equation (3.12) to obtain

$$(\theta_t, \theta) + (a(u_h)\nabla\theta, \nabla\theta) = (R_1 + R_2 + R_5, \theta) + (R_6 + R_7, \nabla\theta),$$

now that $R_3 = R_4 = 0$. Hence, by (2.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta\|^2) + \|\theta\|_1^2 \leqslant C \sum_{\substack{j=1 \\ j \neq 3,4}}^{7} \|R_j\|^2 + \|\theta\|_1^2,$$

so that, after integration with respect to t,

$$\begin{aligned}\|\theta(t)\|^2 &\leqslant \|\theta(0)\|^2 + C \sum_{\substack{j=1 \\ j \neq 3,4}}^{7} \|R_j\|^2_{L_2(0,t;L_2)} \\ &\leqslant Ch^{2r} + C\|\theta\|^2_{L_2(0,t;L_2)}, \quad \text{for} \quad 0 < t < t_1,\end{aligned}$$

where we have used the fact that

$$\|\theta(0)\| = \|v_h - \tilde{v}_h\| \leqslant \|v_h - v\| + \|\tilde{v}_h - v\| \leqslant Ch^r,$$

by (3.14) and (2.8), and simple modifications of the bounds of the terms R_i derived in the proof of Lemma 3. Now an application of Grönwall's inequality shows

$$\|\theta(t)\| \leqslant Ch^r, \quad \text{for} \quad 0 \leqslant t \leqslant t_1,$$

and, by an inverse inequality, it hence follows that

$$\|\theta(t)\|_1 \leqslant Ch^{r-1}, \quad \text{for} \quad 0 \leqslant t \leqslant t_1.$$

Together with the appropriate bounds of $\tilde{u}_h - u$ from Lemma 2, this proves the desired result for $r > 2$. The proof for the case $r = 2$ is based on equation (3.13). This completes the proof.

4. Analysis of the first method

We shall now estimate the error in the approximation u_h given by the semidiscrete problem: find u_h: $[0, T] \to S_{0h}$ such that

$$\begin{aligned}(I_h H(u_h)_t, \chi) + (\nabla I_h G(u_h), \nabla\chi) &= (I_h f(u_h), \chi), \quad \text{for} \quad \chi \in S_{0h},\ 0 < t < T, \\ u_h(0) &= v_h.\end{aligned} \tag{4.1}$$

We have the following result.

Theorem 3. *Let* $r \geqslant 3$ *and let* u_h *and* u *be the solutions of* (4.1) *and* (1.1), *respectively, and assume that*

$$\|v_h - v\| + h\|v_h - v\|_1 \leqslant Ch^{r-1}. \tag{4.2}$$

Then there are positive numbers $h_0 = h_0(u, T)$ and $C = C(u, T)$ such that, for $h < h_0$, we have

$$\|u_h(t)-u(t)\| + h\|u_h(t)-u(t)\|_1 + \left(\int_0^t \|u_h(\tau)-u(\tau)\|_1^2\,d\tau\right)^{1/2} \leqslant Ch^{r-1},$$

for $0 \leqslant t \leqslant T$.

Note that the above $L_2(0,T;H^1)$ error bound is of optimal order, whereas the other bounds are less than optimal. The theorem will follow as before from the following lemma.

Lemma 5. *In addition to the assumptions of Theorem 3, assume that, for some t_1 with $0 < t_1 \leqslant T$, we have*

$$\|e\|_{L_\infty(0,t_1;L_2)} \leqslant h^{r-1-1/4}, \tag{4.3}$$

$$\|e\|_{L_2(0,t_1;H^1)} \leqslant h^{r-1-1/4} \tag{4.4}$$

and

$$\|e_t\|_{L_2(0,t_1;L_2)} \leqslant h^{r-2-1/4}. \tag{4.5}$$

Then it follows that, for $h < h_1$,

$$\|e\|_{L_\infty(0,t_1;L_2)} + \|e\|_{L_2(0,t_1;H^1)} \leqslant Ch^{r-1} \tag{4.6}$$

and

$$\|e\|_{L_\infty(0,t_1;H^1)} + \|e_t\|_{L_2(0,t_1;L_2)} \leqslant Ch^{r-2}, \tag{4.7}$$

where $h_1 = h_1(u, T)$ and $C = C(u, T)$ do not depend on t_1.

Proof. In the same way as in our previous proofs the assumption (4.3) can be used to show that (remember that $r \geqslant 3$ and $d \leqslant 3$)

$$h\,\|u_h\|_{L_\infty(0,t_1;W_4^{r-1})} \leqslant C,$$

$$h^{1/4}\,\|u_h\|_{L_\infty(0,t_1;W_6^{r-2})} \leqslant C,$$

$$\|u_h\|_{L_\infty(0,t_1;W_4^{r-2})} \leqslant C$$

and

$$\|u_h\|_{L_\infty(0,t_1;W_\infty^{r-3})} \leqslant C. \tag{4.8}$$

Similarly, (4.4) leads to

$$\|u_h\|_{L_2(0,t_1;W_4^{r-1})} \leqslant C,$$

$$\|u_h\|_{L_2(0,t_1;W_\infty^{r-2})} \leqslant C$$

and (4.5) implies

$$h^2\,\|u_{h,t}\|_{L_2(0,t_1;W_4^{r-1})} \leqslant C,$$

$$h^{5/4}\,\|u_{h,t}\|_{L_2(0,t_1;W_6^{r-2})} \leqslant C,$$

$$h\,\|u_{h,t}\|_{L_2(0,t_1;W_4^{r-2})} \leqslant C,$$

$$h^{3/4}\,\|u_{h,t}\|_{L_2(0,t_1;W_\infty^{r-3})} \leqslant C.$$

From these bounds it follows that

$$\|f(u_h)\|_{L_2(0,t_1;H^r)} + \|G(u_h)\|_{L_2(0,t_1;H^r)} + \|H(u_h)\|_{L_2(0,t_1;H^r)} \leqslant C, \tag{4.9}$$

$$h\|H(u_h)\|_{L_\infty(0,t_1;H^r)} \leqslant C, \tag{4.10}$$

$$h^2\|H(u_h)_t\|_{L_2(0,t_1;H^r)} \leqslant C. \tag{4.11}$$

For example, for $r = 3$—the most complicated case—we have, in view of (3.10) and since $\|u_h\|_{L_\infty(0,t_1;L_\infty)} \leqslant C$ by (4.8),

$$\|f(u_h)\|_{L_2(0,t_1;H^3)} \leqslant C\big(\|u_h\|_{L_\infty(0,t_1;W_4^1)}\|u_h\|_{L_2(0,t_1;W_4^2)} + \|u_h\|^2_{L_\infty(0,t_1;W_4^1)}\|u_h\|_{L_2(0,t_1;W_\infty^1)}\big) \leqslant C$$

and

$$h\|H(u_h)\|_{L_\infty(0,t_1;H^3)} \leqslant Ch\big(\|u_h\|_{L_\infty(0,t_1;W_4^1)}\|u_h\|_{L_\infty(0,t_1;W_4^2)} + \|u_h\|^3_{L_\infty(0,t_1;W_6^1)}\big) \leqslant C.$$

Similarly, in view of (3.11),

$$\begin{aligned} h^2\|H(u_h)_t\|_{L_2(0,t_1;H^3)} &= h^2\|c(u_h)u_{h,t}\|_{L_2(0,t_1;H^3)} \\ &\leqslant Ch^2\big(\|u_{h,t}\|_{L_2(0,t_1;L_\infty)}\big(\|u_h\|_{L_\infty(0,t_1;W_4^1)}\|u_h\|_{L_\infty(0,t_1;W_4^2)} + \|u_h\|^3_{L_\infty(0,t_1;W_6^1)}\big) \\ &\quad + \|u_{h,t}\|_{L_2(0,t_1;W_4^1)}\|u_h\|_{L_\infty(0,t_1;W_4^2)} + \|u_{h,t}\|_{L_2(0,t_1;W_6^1)}\|u_h\|^2_{L_\infty(0,t_1;W_6^1)} \\ &\quad + \|u_{h,t}\|_{L_2(0,t_1;W_4^2)}\|u_h\|_{L_\infty(0,t_1;W_4^1)}\big) \leqslant C. \end{aligned}$$

Similar arguments apply for $r \geqslant 4$.

We are now ready for the proof of the error estimates. With $\theta = u_h - \tilde{u}_h$ as before, we now have

$$\begin{aligned} (c(u_h)\theta_t, \chi) + (a(u_h)\nabla\theta, \nabla\chi) &= ((I_h - I)f(u_h), \chi) + (f(u_h) - f(u), \chi) \\ &\quad - ((I_h - I)H(u_h)_t, \chi) - ((c(u_h) - c(u))u_t, \chi) - (c(u_h)(\tilde{u}_{h,t} - u_t), \chi) \\ &\quad - (\nabla(I_h - I)G(u_h), \nabla\chi) - ((a(u_h) - a(u))\nabla\tilde{u}_h, \nabla\chi) \\ &= \sum_{i=1}^{5}(R_i, \chi) + \sum_{i=6}^{7}(R_i, \nabla\chi), \end{aligned} \tag{4.12}$$

for $\chi \in S_{0h}$ and $0 < t < t_1$. Note that, except for the sixth term on the right-hand side, this equation is the same as (3.12). We begin with the proof of (4.7). To that end we take $\chi = \theta_t$ in (4.12) to obtain

$$\begin{aligned} (c(u_h)\theta_t, \theta_t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(a(u_h)\nabla\theta, \nabla\theta) &= \sum_{i=1}^{5}(R_i, \theta_t) + (R_6, \nabla\theta_t) + \frac{\mathrm{d}}{\mathrm{d}t}(R_7, \nabla\theta) \\ &\quad - (R_{7,t}, \nabla\theta) + (R_8, \nabla\theta), \end{aligned}$$

where $R_8 = 1/2\, a(u_h)_t\nabla\theta$. Arguing similarly to the proof of Lemma 3, but applying an inverse inequality to the sixth term, we have

$$\begin{aligned} \int_0^t \|\theta_t\|^2\,\mathrm{d}\tau + \|\theta(t)\|_1^2 &\leqslant C\|\theta(0)\|_1^2 + C\int_0^t\left(\sum_{i=1}^{5}\|R_i\| + h^{-1}\|R_6\|\right)\|\theta_t\|\,\mathrm{d}\tau \\ &\quad + C\|R_7(t)\|\,\|\theta(t)\|_1 + C\|R_7(0)\|\,\|\theta(0)\|_1 \\ &\quad + C\int_0^t(\|R_{7,t}\| + \|R_8\|)\|\theta\|_1\,\mathrm{d}\tau, \end{aligned}$$

so that, after trivial estimates and a simple kick-back argument,

$$\|\theta_t\|^2_{L_2(0,t;L_2)} + \|\theta\|^2_{L_\infty(0,t;H^1)} \leqslant C\|\theta(0)\|_1^2 + C\sum_{i=1}^{5}\|R_i\|^2_{L_2(0,t;L_2)}$$
$$+ Ch^{-2}\|R_6\|^2_{L_2(0,t;L_2)} + C\|R_7\|^2_{L_\infty(0,t;L_2)}$$
$$+ C\int_0^t \|R_{7,t}\|\,\|\theta\|_1\,d\tau + C\int_0^t \|R_8\|\,\|\theta\|_1\,d\tau.$$

For the first nine terms on the right-hand side we apply the same arguments as in the proof of Lemma 3. In view of (4.2), (4.9) and (4.11) these terms are thus bounded by

$$Ch^{2r-4} + C\|\theta\|^2_{L_2(0,t;H^1)} + \varepsilon\|\theta\|^2_{L_\infty(0,t;H^1)} + \varepsilon\|\theta_t\|^2_{L_2(0,t;L_2)}.$$

For the last term we use $u_{h,t} = \tilde{u}_{h,t} + \theta_t$ and Lemma 2 to obtain

$$\int_0^t \|R_8\|\,\|\theta\|_1\,d\tau \leqslant C\|\tilde{u}_{h,t}\|_{L_\infty(0,t;L_\infty)}\|\theta\|^2_{L_2(0,t;H^1)}$$
$$+ C\|\theta_t\|_{L_2(0,t;L_2)}\|\theta\|_{L_2(0,t;W^1_\infty)}\|\theta\|_{L_\infty(0,t;H^1)}$$
$$\leqslant C\|\theta\|^2_{L_2(0,t;H^1)} + Ch^{1/4}\|\theta_t\|_{L_2(0,t;L_2)}\|\theta\|_{L_\infty(0,t;H^1)}$$
$$\leqslant C\|\theta\|^2_{L_2(0,t;H^1)} + \varepsilon\|\theta\|^2_{L_\infty(0,t;H^1)} + \varepsilon\|\theta_t\|^2_{L_2(0,t;L_2)}$$

for h small. Here we have also used the fact that

$$\|\theta\|_{L_2(0,t;W^1_\infty)} \leqslant Ch^{-d/2}\|\theta\|_{L_2(0,t;H^1)} \leqslant Ch^{-d/2}\cdot Ch^{r-1-1/4} \leqslant Ch^{1/4},$$

by an inverse inequality and (4.4). Thus, altogether we now have

$$\|\theta_t\|^2_{L_2(0,t;L_2)} + \|\theta\|^2_{L_\infty(0,t;H^1)} \leqslant Ch^{2r-4} + \|\theta\|^2_{L_2(0,t;H^1)}, \quad \text{for} \quad 0 < t < t_1,$$

which by Grönwall's inequality and Lemma 2 proves (4.7).

We now turn to the proof of (4.6). Taking $\chi = \theta$ in (4.12) we have

$$\frac{1}{2}\frac{d}{dt}(c(u_h)\theta,\theta) + (a(u_h)\nabla\theta,\nabla\theta) = ((I_h - I)f(u_h),\theta) + (f(u_h) - f(u),\theta)$$
$$-\frac{d}{dt}((I_h - I)H(u_h),\theta) + ((I_h - I)H(u_h),\theta_t) - ((c(u_h) - c(u))u_t,\theta)$$
$$-(c(u_h)(\tilde{u}_{h,t} - u_t),\theta) - (\nabla(I_h - I)G(u_h),\nabla\theta) - ((a(u_h) - a(u))\nabla\tilde{u}_h,\nabla\theta)$$
$$+\frac{1}{2}(c(u_h)_t\theta,\theta)$$
$$= (R_1 + R_2 + R_4 + R_5,\theta) + (R_6 + R_7,\nabla\theta) + \frac{d}{dt}(\tilde{R}_3,\theta) - (\tilde{R}_3,\theta_t) + (R_9,\theta),$$

where $\tilde{R}_3 = -(I_h - I)H(u_h)$ and $R_9 = 1/2c(u_h)_t\theta$ and the other terms are the same as above. By simple estimates and integration with respect to t we obtain

$$\|\theta\|^2_{L_\infty(0,t;L_2)} + \|\theta\|^2_{L_2(0,t;H^1)} \leqslant C\|\theta(0)\|^2 + C\left(\sum_{i=1}^{2} + \sum_{i=4}^{7}\right)\|R_i\|^2_{L_2(0,t;L_2)}$$
$$+ C\|\tilde{R}_3\|^2_{L_\infty(0,t;L_2)} + C\|\tilde{R}_3\|_{L_2(0,t;L_2)}\|\theta_t\|_{L_2(0,t;L_2)} + C\int_0^t \|R_9\|\,\|\theta\|\,d\tau.$$

By (4.2) and in view of (4.8) and (4.9), the first five terms on the right-hand side can be bounded by

$$Ch^{2r-2} + C\|\theta\|^2_{L_2(0,t;L_2)}$$

just as in the proof of Lemma 3. Also, from (4.9) it follows that

$$\|R_6\|_{L_2(0,t;L_2)} \leqslant Ch^{r-1}\|G(u_h)\|_{L_2(0,t;H^r)} \leqslant Ch^{r-1}.$$

For the seventh term we have

$$\begin{aligned}\|R_7\|_{L_2(0,t;L_2)} &\leqslant C\|u_h - u\|_{L_2(0,t;L_2)}\|\tilde{u}_h\|_{L_\infty(0,t;W^1_\infty)}\\ &\leqslant C\big(\|\theta\|_{L_2(0,t;L_2)} + \|\tilde{u}_h - u\|_{L_2(0,t;L_2)}\big)\\ &\leqslant Ch^r + C\|\theta\|_{L_2(0,t;L_2)}.\end{aligned}$$

Further, using (4.10) we have

$$\|\tilde{R}_3\|_{L_\infty(0,t;L_2)} \leqslant Ch^r\|H(u_h)\|_{L_\infty(0,t;H^r)} \leqslant Ch^{r-1}$$

and, by (4.9) and the already proven error bound (4.7),

$$\begin{aligned}\|\tilde{R}_3\|_{L_2(0,t;L_2)}\|\theta_t\|_{L_2(0,t;L_2)} &\leqslant Ch^r\|H(u_h)\|_{L_2(0,t;H^r)}\|\theta_t\|_{L_2(0,t;L_2)}\\ &\leqslant Ch^r\cdot Ch^{r-2} = Ch^{2r-2}.\end{aligned}$$

For the last term we write $u_{h,t} = u_t + e_t$ to obtain

$$\begin{aligned}\int_0^t \|R_9\|\ \|\theta\|\,\mathrm{d}\tau &\leqslant C\|u_t\|_{L_\infty(0,t;L_\infty)}\|\theta\|^2_{L_2(0,t;L_2)}\\ &\quad + C\|e_t\|_{L_2(0,t;L_2)}\|\theta\|_{L_2(0,t;L_\infty)}\|\theta\|_{L_\infty(0,t;L_2)}\\ &\leqslant C\|\theta\|^2_{L_2(0,t;L_2)} + Ch^{1/4}\|\theta\|_{L_2(0,t;H^1)}\|\theta\|_{L_\infty(0,t;L_2)}\\ &\leqslant C\|\theta\|^2_{L_2(0,t;L_2)} + \varepsilon\|\theta\|^2_{L_2(0,t;H^1)} + \varepsilon\|\theta\|^2_{L_\infty(0,t;L_2)}\end{aligned}$$

for h small. Here we also used the fact that

$$\begin{aligned}\|e_t\|_{L_2(0,t;L_2)}\|\theta\|_{L_2(0,t;L_\infty)} &\leqslant Ch^{r-2}\|\theta\|_{L_2(0,t;W^1_4)}\\ &\leqslant Ch^{r-2}\cdot Ch^{-d/4}\|\theta\|_{L_2(0,t;H^1)}\\ &\leqslant Ch^{1/4}\|\theta\|_{L_2(0,t;H^1)},\end{aligned}$$

by (4.7), Sobolev's inequality and an inverse inequality. Summing up, we now have

$$\|\theta\|^2_{L_\infty(0,t;L_2)} + \|\theta\|^2_{L_2(0,t;H^1)} \leqslant Ch^{2r-2} + C\|\theta\|^2_{L_2(0,t;L_2)}, \quad \text{for} \quad 0 < t < t_1,$$

and (4.6) follows, which completes the proof of the lemma.

In the previous theorem we have presented a suboptimal order estimate for the L_2 norm of the error. Naturally, one might ask about the possibility of obtaining an estimate of optimal order of accuracy. We shall give a positive answer in two cases. The first is the case where $c(u) \equiv 1$ and $r \geqslant 3$, $d \leqslant 3$ and the error is measured in the $L_2(0, T; L_2)$ norm. The other is the case of $r=2$, $d=1$, where we obtain a bound in the $L_\infty(0, T; L_2)$ norm for general $c = c(u)$.

Theorem 4. *Let $c(u) \equiv 1$ and $r \geqslant 3$, $d \leqslant 3$ and let u_h and u be the solutions of* (4.1) *and* (1.1), *respectively, and assume that*

$$\|v_h - v\| \leqslant Ch^r. \tag{4.13}$$

Then there are positive numbers $h_0 = h_0(u, T)$ and $C = C(u, T)$ such that, for $h < h_0$, we have

$$h \|u_h(t) - u(t)\| + \left(\int_0^t \|u_h(\tau) - u(\tau)\|^2 \,\mathrm{d}\tau \right)^{1/2} \leqslant Ch^r, \quad \text{for} \quad 0 \leqslant t \leqslant T.$$

The theorem follows as before from the following lemma.

Lemma 6. *In addition to the assumptions of Theorem 4, assume that, for some t_1 with $0 < t_1 \leqslant T$, we have*

$$\|e\|_{L_\infty(0,t_1;L_2)} \leqslant h^{r-1-1/4}, \tag{4.14}$$

$$\|e\|_{L_2(0,t_1;L_2)} \leqslant h^{r-1/4}. \tag{4.15}$$

Then it follows that, for $h < h_1$,

$$\|e\|_{L_\infty(0,t_1;L_2)} \leqslant Ch^{r-1}, \tag{4.16}$$

$$\|e\|_{L_2(0,t_1;L_2)} \leqslant Ch^r, \tag{4.17}$$

where $h_1 = h_1(u, T)$ and $C = C(u, T)$ do not depend on t_1.

Proof. Just as in the proof of Lemma 5 for $r \geqslant 3$, $d \leqslant 3$ we find that the bounds (4.8) and (4.9) follow from the assumptions (4.14) and (4.15). Moreover, the error bound (4.16) was already proved in Lemma 5.

Since $c(u) \equiv 1$, the error equation (4.12) now becomes

$$\begin{aligned}(\theta_t, \chi) + (a(u)\nabla\theta, \nabla\chi) &= ((I_h - I)f(u_h), \chi) + (f(u_h) - f(u), \chi) \\ &\quad - (\tilde{u}_{h,t} - u_t, \chi) - (\nabla(I_h - I)G(u_h), \nabla\chi) - ((a(u_h) - a(u))\nabla u_h, \nabla\chi) \\ &= \sum_{i=1}^{3} (S_i, \chi) + \sum_{i=4}^{5} (S_i, \nabla\chi),\end{aligned} \tag{4.19}$$

for $\chi \in S_{0h}$ and $0 < t < t_1$. Note that we have, for later convenience, replaced $a(u_h)$ in the second term on the left by $a(u)$, which implies a modification of the fifth term on the right.

To prepare for the proof of (4.17) we recall the operators T and T_h defined in Section 2. It is well known that T_h is a self-adjoint, positive semidefinite bounded operator on $L_2(\Omega)$, which is positive definite on S_{0h}. We shall use the equivalence of norms

$$(\chi, T_h\chi)^{1/2} \cong \|T_h\chi\|_1, \quad \chi \in S_{0h}, \tag{4.20}$$

which immediately follows from (2.5) in view of the positivity and boundedness of $a(u)$.

After these preparations we now set $\chi = T_h\theta$ in (4.19). We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\theta, T_h\theta) + \|\theta\|^2 = \sum_{i=1}^{3} (S_i, T_h\theta) + \sum_{i=4}^{5} (S_i, \nabla T_h\theta) + (S_6, \theta),$$

where $S_6 = 1/2\, T_{h,t}\theta$. By integration and simple estimates, using (4.20) and the boundedness of T_h, we obtain

$$\| T_h \theta \|^2_{L_\infty(0,t;H^1)} + \| \theta \|^2_{L_2(0,t;L_2)} \leqslant C \| \theta(0) \|^2 + C \sum_{i=1}^{3} \int_0^t (S_i, T_h \theta) \,\mathrm{d}\tau + C \sum_{i=4}^{5} \int_0^t (S_i, \nabla T_h \theta) \,\mathrm{d}\tau + C \int_0^t (S_6, \theta) \,\mathrm{d}\tau.$$

In view of (4.13) the first term on the right is bounded by Ch^{2r}. Next we have

$$\begin{aligned} \sum_{i=1}^{3} \int_0^t (S_i, T_h \theta) \,\mathrm{d}\tau &\leqslant \sum_{i=1}^{3} \int_0^t (S_i, T_h S_i)^{1/2} (\theta, T_h \theta)^{1/2} \,\mathrm{d}\tau \\ &\leqslant C \sum_{i=1}^{3} \int_0^t \| S_i \| \, \| T_h \theta \|_1 \,\mathrm{d}\tau \\ &\leqslant C \sum_{i=1}^{3} \| S_i \|_{L_2(0,t;L_2)} \| T_h \theta \|_{L_2(0,t;H^1)} \\ &\leqslant Ch^{2r} + \varepsilon \| \theta \|^2_{L_2(0,t;L_2)} + C \| T_h \theta \|^2_{L_2(0,t;H^1)}, \end{aligned}$$

because

$$\begin{aligned} &\| S_1 \|_{L_2(0,t;L_2)} \leqslant Ch^r \| f(u_h) \|_{L_2(0,t;H^r)} \leqslant Ch^r, \\ &\| S_2 \|_{L_2(0,t;L_2)} \leqslant C \| u_h - u \|_{L_2(0,t;L_2)} \leqslant Ch^r + C \| \theta \|_{L_2(0,t;L_2)}, \\ &\| S_3 \|_{L_2(0,t;L_2)} \leqslant Ch^r, \end{aligned}$$

by (2.2), (4.9), (2.8) and (4.18). Further

$$\begin{aligned} \int_0^t (S_4, \nabla T_h \theta) \,\mathrm{d}\tau &= - \int_0^t (\nabla (I_h - I) G(u_h), \nabla T_h \theta) \,\mathrm{d}\tau \\ &= - \int_0^t (\nabla (I_h - I) G(u_h), \nabla (T_h - T) \theta) \,\mathrm{d}\tau \\ &\quad + \int_0^t ((I_h - I) G(u_h), \Delta T \theta) \,\mathrm{d}\tau \\ &\leqslant \int_0^t \| (I_h - I) G(u_h) \|_1 \, \| (T_h - T) \theta \|_1 \,\mathrm{d}\tau \\ &\quad + \int_0^t \| (I_h - I) G(u_h) \| \, \| T \theta \|_2 \,\mathrm{d}\tau \\ &\leqslant Ch^r \int_0^t \| G(u_h) \|_r \, \| \theta \| \,\mathrm{d}\tau \leqslant Ch^r \| G(u_h) \|_{L_2(0,t;H^r)} \| \theta \|_{L_2(0,t;L_2)} \\ &\leqslant Ch^{2r} + \varepsilon \| \theta \|^2_{L_2(0,t;L_2)}, \end{aligned}$$

where we have used (2.2), (2.7), (2.6) and (4.9).

Next we use $u_h = \tilde{u}_h + \theta$ to write

$$\begin{aligned} \int_0^t (S_5, \nabla T_h \theta) \,\mathrm{d}\tau &= \int_0^t ((a(u_h) - a(u)) \nabla \tilde{u}_h, \nabla T_h \theta) \,\mathrm{d}\tau \\ &\quad + \int_0^t ((a(u_h) - a(u)) \nabla \theta, \nabla T_h \theta) \,\mathrm{d}\tau, \end{aligned}$$

where, by (2.9) and (2.8),

$$\int_0^t ((a(u_h)-a(u))\nabla\tilde{u}_h, \nabla T_h\theta)\,d\tau \leqslant C\,\|e\|_{L_2(0,t;L_2)}\|\tilde{u}_h\|_{L_\infty(0,t;W^1_\infty)}\|T_h\theta\|_{L_2(0,t;H^1)}$$

$$\leqslant C(\|\theta\|_{L_2(0,t;L_2)}+\|\tilde{u}_h-u\|_{L_2(0,t;L_2)})\|T_h\theta\|_{L_2(0,t;H^1)}$$

$$\leqslant Ch^{2r}+\varepsilon\|\theta\|^2_{L_2(0,t;L_2)}+C\|T_h\theta\|^2_{L_2(0,t;H^1)},$$

and, by virtue of Lemma 1 and (4.15) and in view of the restrictions on r and d,

$$\int_0^t ((a(u_h)-a(u))\nabla\theta, \nabla T_h\theta)\,d\tau \leqslant C\,\|e\|_{L_2(0,t;L_\infty)}\|\theta\|_{L_2(0,t;H^1)}\|T_h\theta\|_{L_\infty(0,t;H^1)}$$

$$\leqslant Ch^{-(d/2)+r-1/4}\cdot Ch^{-1}\|\theta\|_{L_2(0,t;L_2)}\|T_h\theta\|_{L_\infty(0,t;H^1)}$$

$$\leqslant Ch^{1/4}\|\theta\|_{L_2(0,t;L_2)}\|T_h\theta\|_{L_\infty(0,t;H^1)}$$

$$\leqslant \varepsilon\|\theta\|^2_{L_2(0,t;L_2)}+\varepsilon\|T_h\theta\|^2_{L_\infty(0,t;H^1)},$$

for h small. For the last term we have

$$\int_0^t (S_6,\theta)\,d\tau=\frac{1}{2}\int_0^t(\theta, T_{h,t}\theta)\,d\tau=\frac{1}{2}\int_0^t(a(u)\nabla T_h\theta, \nabla T_{h,t}\theta)\,d\tau$$

$$=-\frac{1}{2}\int_0^t(a(u)_t\nabla T_h\theta, \nabla T_h\theta)\,d\tau$$

$$\leqslant C\|T_h\theta\|^2_{L_2(0,t;H^1)},$$

by (2.5) and by differentiation of (2.5) with respect to time.

Together these estimates show

$$\|T_h\theta\|^2_{L_\infty(0,t;H^1)}+\|\theta\|^2_{L_2(0,t;L_2)}\leqslant Ch^{2r}+C\|T_h\theta\|^2_{L_2(0,t;H^1)},\quad 0\leqslant t\leqslant t_1,$$

and Grönwall's lemma yields

$$\|T_h\theta\|^2_{L_\infty(0,t_1;H^1)}+\|\theta\|^2_{L_2(0,t_1;L_2)}\leqslant Ch^{2r},$$

which proves (4.17). The lemma is proved.

The case $r=2$, $d=1$ is somewhat special, because of the well-known fact that the interpolation operator I_h then coincides with the standard Ritz projection R_h^0: $H_0^1(\Omega)\to S_{0h}$ defined by

$$(\nabla(R_h^0 w-w), \nabla\chi)=0,\quad \chi\in S_{0h},$$

(see, for instance, p. 80 of Reference 13). In the error equation (4.12) we therefore have $(R_6, \nabla\chi)=0$, so that (4.12) is identical to equation (3.12) in the proof of Lemma 3 with the corresponding term removed. Repeating the appropriate steps of the proof of Theorem 1, with the term $(R_6, \nabla\chi)$ removed, we therefore find that the second order convergence of the second method carries over to the present case. We thus have:

Theorem 5. *Let $r=2$, $d=1$ and u_h and u be the solutions of* (4.1) *and* (1.1), *respectively, and assume that $v_h=\tilde{v}_h=\tilde{u}_h(0)$. Then there are positive numbers $h_0=h_0(u, T)$ and $C=C(u, T)$ such that, for $h<h_0$, we have*

$$\|u_h(t)-u(t)\|+h\|u_h(t)-u(t)\|_1\leqslant Ch^2,\quad \text{for } 0\leqslant t\leqslant T.$$

Acknowledgement

Part of this work was done during a visit by the first author (S. L.) to the Mathematical Sciences Institute at Cornell University. The hospitality of the institute is gratefully acknowledged.

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