Interpolation of Cosine Operator Functions (*).

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Summary. – *Using basic techniques from the theory of interpolation spaces equivalence theorems are established for the intermediate spaces between a given Banach space A and the domain $D(\Lambda^r)$ of the r-th power of the infinitesimal generator Λ of a strongly continuous cosine operator function C. The results are applied to the study of second order evolution equations including regularity, order reduction and approximation by finite difference methods.*

1. – Introduction.

Given a strongly continuous cosine operator function C on $\boldsymbol{R}^+$ with values in the Banach algebra $\mathfrak{B}(A)$ of bounded linear operators in a Banach space A and infinitesimal generator Λ, we shall be concerned with the investigation of the intermediate spaces between A and the domain $D(\Lambda^r)$, $r \in \boldsymbol{N}$, as well as with the characterization of the domains of fractional powers $(-\Lambda)^\alpha$, $0 < \alpha < r$. Using the K-method we shall give equivalent characterizations of these intermediate spaces by means of moduli of continuity of C including reduction results both in the case of non-optimal approximation and saturation. With respect to the regularity of solutions of initial-value problems for second order evolution equations and the reduction of well-posed second order problems to equivalent first order ones we shall study mapping properties of C and its strong integral S which are the propagators of such problems. As applications, the results of this paper not only provide characterizations of the Besov spaces $B_p^{2\alpha,q}$ in lights of a cosine operator functional calculus instead of the well-known semigroup approach via the Weierstrass singular integral but also can be used in the approximate solution of initial-value problems for second order hyperbolic P.D.E.'s by finite difference techniques with special emphasis on the case of non-smooth initial data.

2. – Basic facts on cosine operator functions.

In this preliminary section we will summarize some basic facts in cosine operator functional calculus which we will strongly need in the subsequent sections such as the da Prato-Giusti-Sova generation theorem, expansions of the r-th Riemann dif-

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ference of a cosine operator function and representations of fractional powers of its infinitesimal generator. For a detailed discussion of general cosine operator theory we refer to G. DA PRATO, E. GIUSTI [9], H. O. FATTORINI [11], [12] and M. SOVA [21].

In what follows A denotes a Banach space over $\boldsymbol{K}$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$) with norm $\|\cdot\|_A$, $\mathcal{C}(A, B)$ the set of all densely defined closed linear operators Λ with domain $D(\Lambda)$ in A and range $R(\Lambda)$ in another Banach space B over $\boldsymbol{K}$ and $\mathcal{B}(A, B)$ the Banach space of bounded linear operators $\Lambda: A \to B$ with norm $\|\Lambda\| = \sup_{a\neq 0} \|\Lambda a\|_B/\|a\|_A$, $\Lambda \in \mathcal{B}(A, B)$ (if $B = A$ we simply write $\mathcal{C}(A)$ resp. $\mathcal{B}(A)$ instead of $\mathcal{C}(A, A)$ resp. $\mathcal{B}(A, A)$). The resolvent set and the spectrum of $\Lambda \in \mathcal{C}(A)$ will be denoted by $\varrho(\Lambda)$ resp. $\sigma(\Lambda)$ while $R(\lambda; \Lambda)$, $\lambda \in \varrho(\Lambda)$, refers to the resolvent $(\lambda I - \Lambda)^{-1}$.

A transformation $C: \overline{\boldsymbol{R}}^+ \to \mathcal{B}(A)$, where $\overline{\boldsymbol{R}}^+ := [0, \infty)$, is called a cosine operator function if $C(0) = I$ and $C(\cdot)$ satisfies d'Alembert's functional equation, i.e.

$$C(t+s) + C(t-s) = 2C(t)C(s), \quad t, s \in \overline{\boldsymbol{R}}^+, \ t > s. \tag{2.1}$$

A cosine operator function C is called strongly continuous or simply a C_0-cosine operator function if $C(\cdot)a$ is continuous on $\overline{\boldsymbol{R}}^+$ for each $a \in A$ $\left(\text{or equivalently, if } s-\lim_{t\to 0+} C(t)a = a \text{ where } s-\lim \text{ means convergence in the norm topology of } A\right)$.

REMARK 1. - It is clear that a C_0-cosine operator function C can be continuously extended to the whole real line $\boldsymbol{R}$ by setting $C(t) = C(-t)$ for $t < 0$. For notational convenience that extension will be still denoted by C.

Every C_0-cosine operator function C is quasi-bounded in the sense that there are nonnegative constants M and ω such that

$$\|C(t)\| \leqslant M \cosh(\omega t), \quad t \in \boldsymbol{R}^+ \tag{2.2}$$

(cf. [21]; Thm. 2.5]). We shall then say that C is of type (M, ω). In particular, if $\omega = 0$ then C is called equibounded.

The operators $C(\cdot)$ commute for different arguments, i.e.

$$C(t_1)C(t_2) = C(t_2)C(t_1), \quad t_1, t_2 \in \overline{\boldsymbol{R}}^+, \tag{2.3}$$

and if C is of type (M, ω) and $t_\nu \in \overline{\boldsymbol{R}}^+$, $\nu = 1, \dots, d$, then

$$\left\| \prod_{\nu=1}^{d} C(t_\nu) \right\| \leqslant M \prod_{\nu=1}^{d} \cosh(\omega t_\nu) \tag{2.4}$$

(cf. [21]; Thms. 2.9, 2.10).

In the sequel we shall frequently use the following simple expansion of the r-th Riemann difference $[C(t) - I]^r$, $r \in \boldsymbol{N}$:

LEMMA 2.1. – Let C be a cosine operator function and $r \in N$. Then there holds

$$[C(t) - I]^r = 2^{-r}\left[2\sum_{j=1}^{r}(-1)^{r-j}\binom{2r}{r-j}C(jt) + (-1)^r\binom{2r}{r}I\right]. \tag{2.5}$$

PROOF. – The proof of (2.5) is easily done via induction on $r \in N$ making repeated use of d'Alembert's functional equation (2.1).

As in the case of semigroups of operators there is the important notion of the infinitesimal generator: Given a cosine operator function C we associate with C a linear operator $\Lambda: D(\Lambda) \subset A \to A$ as the operator given by

$$\Lambda a := 2\, s - \lim_{t\to 0+} t^{-2}[C(t) - I]a\,, \qquad a \in D(\Lambda)\,, \tag{2.6}$$

where $D(\Lambda)$ is the set of all $a \in A$ for which the limit in (2.6) exists. The operator Λ is called the infinitesimal generator of C. Moreover, as we shall see below, the strong integral of C as given by

$$S(t)a = \int_0^t C(s)a\, ds\,, \qquad a \in A \tag{2.7}$$

plays a decisive role in cosine operator functional calculus.

LEMMA 2.2. – Let C be a C_0-cosine operator function with infinitesimal generator Λ. Then there holds for $r \in N$:

(i) $\Lambda^r \in \mathfrak{C}(A)$;

(ii) If $a \in D(\Lambda^r)$, so does $C(t)a$, $t \in \overline{\boldsymbol{R}}^+$, and

$$\frac{d^{2r-1}}{dt^{2r-1}} C(t)a = \Lambda^r S(t)a = S(t)\Lambda^r a\,, \tag{2.8a}$$

$$\frac{d^{2r}}{dt^{2r}} C(t)a = \Lambda^r C(t)a = C(t)\Lambda^r a; \tag{2.8b}$$

(iii) If $a \in A$ then

$$[C(t) - I]^r a = \Lambda^r \int_0^t\int_0^t \ldots \int_0^t \prod_{\nu=1}^{r}(t - s_\nu)C(s_\nu)a\, ds_1\, ds_2 \ldots ds_r\,. \tag{2.9}$$

PROOF. – The assertion (i), (ii) and (iii) are obvious generalizations of the corresponding relations in case $r = 1$ which have been proved in [11; Lemma 5.4], [21; Lemma 2.14].

The $2r$-th Taylor operator B^{2r} of a C_0-cosine operator function C with domain $D(B^{2r}) \subset D(A^{r-1})$ and range $R(B^{2r}) \subset A$ is defined by

$$B^{2r}a = s - \lim_{t\to 0+} B_t^{2r} a$$

where

$$B_t^{2r} a = (2r)!\, t^{-2r}\left[C(t) - \sum_{j=0}^{r-1} \frac{t^{2j}}{(2j)!} A^j\right] a .$$

LEMMA 2.3. - Let C be a C_0-cosine operator function with infinitesimal generator A and let B^{2r}, $r \in \boldsymbol{N}$, be its Taylor operator of order $2r$. Then there holds

(i) If $a \in D(A^r)$ then $s - \lim_{t\to 0+} B_t^{2r} a = A^r a$;

(ii) If $a \in D(A^{r-1})$ and $s \in \boldsymbol{R}^+$ then

$$s - \lim_{t\to 0+} \int_0^s (s-\tau)\, C(\tau) B_t^{2r} a \, d\tau = [C(s) - I] A^{r-1} a . \tag{2.10}$$

PROOF. - In view of Lemma 2.2 (ii) the Taylor theorem for vector-valued functions gives

$$C(t)a - \sum_{j=0}^{r-1} \frac{t^{2j}}{(2j)!} A^j a = \frac{1}{(2r-1)!} \int_0^t (t-s)^{2r-1} C(s) A^r a \, ds$$

(cf. also [19; Thm. 7]). It follows that

$$B_t^{2r} a = 2rt^{-2r} \int_0^t (t-s)^{2r-1} C(S) A^r a \, ds$$

and

$$\|B_t^{2r} a - A^r a\|_A \leqslant 2rt^{-2r} \int_0^t (t-s)^{2r-1} \|[C(s) - I] A^r a\|_A \, ds = o(1)$$

whence $D(A^r) \subset D(B^{2r})$ and $B^{2r} a = A^r a$.

On the other hand, using Lemma 2.2 (iii) we get

$$\int_0^s (s-\tau)\, C(\tau) B_t^{2r} a \, d\tau = 2r(2r-1) t^{-2r} \int_0^t (t-\tau)\, \tau^{2(r-1)} B_\tau^{2(r-1)} [C(s) - I] a \, d\tau$$

which yields

$$\left\| \int_0^s (s-\tau)\, C(\tau) B_t^{2r} a \, d\tau - [C(s) - I] A^{r-1} a \right\|_A =$$

$$= \left\| 2r(2r-1) t^{-2r} \int_0^t (t-\tau)\, \tau^{2(r-1)} [C(s) - I][B_\tau^{2(r-1)} a - A^{r-1} a] \, d\tau \right\|_A$$

thus proving part (ii) of the assertion.

Equations (2.8*a*), (2.8*b*) in Lemma 2.2 (ii) show that C_0-cosine operator functions are intimately connected with the Cauchy problem for second order evolution equations

$$\frac{d^2}{dt^2}a(t) = Aa(t), \quad t \in \mathbf{R}^+, \tag{2.11a}$$

$$a(0) = a^0, \quad \frac{d}{dt}a(0) = a_t^0. \tag{2.11b}$$

Indeed, as has been shown by H. O. FATTORINI (cf. [11]) the initial-value problem (2.11*a*), (2.11*b*) is uniformly well posed in $\mathbf{R}^+$ if and only if the operator A is the infinitesimal generator of a C_0-cosine operator function (here uniform well-posedness means that, if $(a_n)_{\mathbf{N}}$, $a_n \in C^2(\mathbf{R}^+, A)$, $a_n(t) \in D(A)$, $t \in \mathbf{R}^+$, $n \in \mathbf{N}$, is a sequence of solutions of (2.11*a*) such that $\left(d^k/dt^k\right)a_n(0^+) \to 0$ $(n \to \infty)$, $k = 0, 1$, then $(a_n(\cdot))_{\mathbf{N}}$ converges to zero uniformly on compact subsets of $\mathbf{R}^+$).

In this case the operators C and S are the propagators of (2.11*a*), (2.11*b*) in the sense that, if a^0, $a_t^0 \in D(A)$, then the unique solution $a(t)$, $t \in \overline{\mathbf{R}}^+$, of (2.11*a*), (2.11*b*) is given by

$$a(t) = C(t)a^0 + S(t)a_t^0. \tag{2.12}$$

Using this property along with (2.8*a*), (2.8*b*) and the fact that $D(A)$ is dense in A one can derive the following two functional equations for C resp. S which we shall strongly need later on (cf. [11; Lemma 2.3])

$$C(s+t) = C(s)C(t) + AS(s)S(t), \quad s, t \in \mathbf{R}^+ \tag{2.13a}$$

$$S(s+t) = C(s)S(t) + S(s)C(t), \quad s, t \in \mathbf{R}^+. \tag{2.13b}$$

Obviously, cosine operator functions play the same role for second order evolution equations as do semigroups of operators for first order ones. Moreover, with regard to the generation of C_0-cosine operator functions of type (M, ω) there is the following result which is analogous to the Hille-Phillips-Yosida generation theorem for semigroups:

THEOREM 2.4 (da Prato-Giusti-Sova generation theorem). – An operator $A \in \mathcal{C}(A)$ is the infinitesimal generator of a C_0-cosine operator function of type (M, ω) if and only if

(2.14) (i) If $\lambda > \omega^2$ then $\lambda \in \varrho(A)$;

(ii) $s - \lim_{\lambda \to \infty} \lambda R(\lambda; A)a = a$, $a \in D(A)$ $(\lambda > \omega^2)$;

(iii) $\left\| \frac{d^n}{d\lambda^n} \lambda R(\lambda^2; A) \right\| \leqslant \frac{1}{2} M n! [(\lambda + \omega)^{-(n+1)} + (\lambda - \omega)^{-(n+1)}]$, $\quad \lambda > \omega, n \in \mathbf{N}_0$.

PROOF. – Cf. e.g. [9], [21; Thms. 3.1, 3.2].

The coincidence with semigroup theory is stretched by the fact that $\lambda R(\lambda^2; A)$ resp. $R(\lambda^2; A)$, $\lambda > \omega$, can be shown to be the operational Laplace transforms of C resp. S (cf. [11; Lemma 5.6]), i.e.

$$(2.15a)\qquad \lambda R(\lambda^2; A)a = \int_0^\infty \exp(-\lambda t) C(t)a\, dt$$

$$(2.15b)\qquad R(\lambda^2; A)a = \int_0^\infty \exp(-\lambda t) S(t)a\, dt\,. \qquad (\lambda > \omega,\ a \in A)$$

Conversely, assume that $A \in \mathrm{C}(A)$ with $\lambda \in \varrho(A)$ if $\lambda > \omega$, and that $C(\cdot)$ is a $\mathfrak{B}(A)$-valued strongly continuous function satisfying $C(0) = I$ and an inequality of type (2.2). Then if (2.15a) holds true, $C(\cdot)$ is a C_0-cosine operator function and A is its infinitesimal generator (cf. [11; Lemma 5.8]).

It is an immediate consequence of (2.14) (i), (ii), (iii) that, if A is the infinitesimal generator of a C_0-cosine operator function of type (M, ω), then A also generates a C_0-semigroup of operators of type (M, ω^2). However, the converse is not always true, take e.g. $A = \Delta$ in $A = L^p(\mathbf{R}^d)$, $p \neq 2$, $d > 1$ (cf. [17], [18]).

If C is an equibounded C_0-cosine operator function, then we may define fractional powers $(-A)^\alpha$, $\alpha \in \mathbf{R}^+$, of its infinitesimal generator. In the next section we will treat $D((-A)^\alpha)$, $0 < \alpha < r$, $r \in \mathbf{N}$, equipped in the usual way with the graph norm, as intermediate spaces between A and $D(A^r)$. Preparatory we shall now derive a representation of $(-A)^\alpha$ by means of the r-th Riemann difference of C as well as an inverse formula for the operator $(-A)^\alpha$. In the case of semigroups and groups of operators such an approch has been used by H. BERENS, P. L. BUTZER, U. WESTPHAL in [4] and by U. WESTPHAL in [23] and [24].

We introduce $p_{\alpha,n}(t)$, $t \in \mathbf{R}$, $0 < \alpha \leqslant n$, $n \in \mathbf{N}$, as the function whose Fourier transform $\hat{p}_{\alpha,n}(s)$, $s \in \mathbf{R}$ is given by

$$(2.16)\qquad \hat{p}_{\alpha,n}(s) = |s|^{-2\alpha}(\cos s - 1)^n\,.$$

Furthermore, let $q_{\alpha,n}(t)$, $t \in \mathbf{R}$, be defined by

$$q_{\alpha,n}(t) = t^{-1}\int_0^t p_{\alpha,n}(s)\, ds\,.$$

In view of (2.16) the Fourier transform $\hat{q}_{\alpha,n}(s)$, $s \in \mathbf{R}$, turns out to be

$$(2.17)\qquad \hat{q}_{\alpha,n}(s) = \int_s^\infty t^{-2\alpha}(\cos t - 1)^n \frac{dt}{t}\,, \quad 0 < \alpha < n\,.$$

The functions $p_{\alpha,n}(\cdot)$ and $q_{n,\alpha}(\cdot)$ can be shown to be even functions belonging to $L^1(\boldsymbol{R})$ with

$$C_{\alpha,n} := \int_{-\infty}^{+\infty} q_{\alpha,n}(t)\,dt = \int_0^\infty t^{-2\alpha}(\cos t - 1)^n \frac{dt}{t}, \tag{2.18a}$$

$$\int_{-\infty}^{+\infty} p_{\alpha,n}(t)\,dt = \begin{cases} 0 & , \quad 0 < \alpha < n \\ (-2)^{-n}, & \alpha = n \end{cases} \tag{2.18b}$$

(for details and explicit representations of $p_{\alpha,n}(t)$ resp. $q_{\alpha,n}(t)$ see [15] and [24]).

The following Lemma serves as a useful tool in the characterization of fractional powers of the infinitesimal generator A:

LEMMA 2.5. – Let C be an equibounded C_0-cosine operator function with infinitesimal generator A and let $k, n \in \boldsymbol{N}$.

Then there holds:

(i) If $0 < \alpha < \min(k, n)$ and $\varepsilon, \eta > 0$, then for $a \in A$

$$\int_\eta^\infty t^{-2\alpha}[C(t) - I]^k \frac{dt}{t} \int_0^\infty q_{\alpha,n}\left(\frac{s}{\varepsilon}\right) C(s)\, a \frac{ds}{\varepsilon} = \int_\varepsilon^\infty t^{-2\alpha}[C(t) - I]^n \frac{dt}{t} \int_0^\infty q_{\alpha,k}\left(\frac{s}{\eta}\right) C(s)\, a \frac{ds}{\eta}; \tag{2.19}$$

(ii) If $k < n$, $0 < \alpha \leqslant k$ and $\varepsilon, t > 0$, then for $a \in A$

$$[C(t) - I]^k \int_0^\infty q_{\alpha,n}\left(\frac{s}{\varepsilon}\right) C(s)\, a \frac{ds}{\varepsilon} = t^{2\alpha} \int_\varepsilon^\infty \tau^{-2\alpha}[C(\tau) - I]^n \frac{d\tau}{\tau} \int_0^\infty p_{\alpha,k}\left(\frac{s}{t}\right) C(s)\, a \frac{ds}{t}. \tag{2.20}$$

PROOF. – The identity (2 19) has been proved in [15; Lemma 2.2]. Since (2.20) can be shown in an analogous manner, we will only sketch the proof. For notational convenience we set

$$a_{0,n} := (-2)^n \binom{2n}{n}, \qquad a_{j,n} := (-1)^{n-j} 2^{-n+1} \binom{2n}{n-j}, \qquad j = 1, \ldots, n, \ n \in \boldsymbol{N} \tag{2.21}$$

and

$$b_\varepsilon(\tau) := \begin{cases} 0 & , \quad |\tau| < \varepsilon \\ |\tau|^{-2\alpha-1}, & |\tau| \geqslant \varepsilon \,. \end{cases} \quad (\varepsilon > 0)$$

Then, using the expansion of the k-th resp. the n-th Riemann difference of C given by (2.5) the left-hand side of (2.20) can be transformed into

$$\int_{-\infty}^{+\infty} \left[\frac{1}{2\varepsilon} \sum_{j=0}^{k} a_{j,k}\, q_{\alpha,n}\left(\frac{s - jt}{\varepsilon}\right)\right] C(s)\, a\, ds\,, \tag{2.22}$$

while the right-hand side can be written as

$$\int_{-\infty}^{+\infty}\left[\frac{1}{2}t^{2\alpha-1}\sum_{j=0}^{n}a_{j,n}j^{2\alpha}\int_{0}^{\infty}b_{j\varepsilon}(\tau)p_{\alpha,k}\left(\frac{\tau-s}{t}\right)d\tau\right]C(s)a\,ds\,. \tag{2.23}$$

Therefore, in order to show that (2.20) holds true it is sufficient to establish the identity of the bracketed terms in (2.22) and (2.23). For this purpose we may choose $C(t)=\cos(\lambda t)$, $\lambda\in\boldsymbol{R}$, and $A=\boldsymbol{C}$, since the assertion then follows from the uniqueness of the inverse cosine transform.

For our special choice of C (2.22) and (2.23) can be easily computed giving

$$\frac{1}{2}(\cos(\lambda t)-1)^k\hat{q}_{\alpha,n}(\lambda\varepsilon)a \qquad \text{resp.} \tag{2.24}$$

$$\frac{1}{2}t^{2\alpha}\int_{\varepsilon}^{\infty}\tau^{-2\alpha}(\cos(\lambda\tau)-1)^n\frac{d\tau}{\tau}\hat{p}_{\alpha,k}(\lambda t)a\,. \tag{2.25}$$

The identity of (2.24) and (2.25) is immediately verified in view of (2.16) resp. (2.17).

As apparent consequences of the identities (2.19) and (2.20) in the preceding Lemma we have:

LEMMA 2.6. – Under the hypotheses of Lemma 2.5 there holds:

(i) If $r\in\boldsymbol{N}$ and $0<\alpha<r$ then for any $a\in A$ the integral

$$\int_{0}^{\infty}q_{\alpha,r}(t)C(t)a\,dt$$

belongs to $D((-A)^\alpha)$ and for any $\varepsilon>0$

$$(-A)^\alpha\int_{0}^{\infty}q_{\alpha,r}\left(\frac{t}{\varepsilon}\right)C(t)a\frac{dt}{\varepsilon}=\frac{1}{2}\int_{\varepsilon}^{\infty}t^{-2\alpha}[C(t)-I]^r a\frac{dt}{t}\,; \tag{2.26}$$

(ii) Let $r\in\boldsymbol{N}$ and $0<\alpha\leqslant r$. Then for any $a\in A$ and $t\in\boldsymbol{R}^+$ the integral

$$\int_{0}^{\infty}p_{\alpha,r}\left(\frac{s}{t}\right)C(s)a\frac{ds}{t}$$

belongs to $D((-A)^\alpha)$ and

$$[C(t)-I]^r a=2t^{2\alpha}(-A)^\alpha\int_{0}^{\infty}p_{\alpha,r}\left(\frac{s}{t}\right)C(s)a\frac{ds}{t}\,. \tag{2.27}$$

Moreover, if $a \in D((-A)^\alpha)$ then there holds the inversion formula

$$[C(t)-I]^r a = 2t^{2\alpha}\int_0^\infty p_{\alpha,r}\left(\frac{s}{t}\right) C(s)(-A)^\alpha a\,\frac{ds}{t}\,. \tag{2.28}$$

PROOF. – Part (i) of the assertion can be easily established by means of Lemma 2.5 (i) and [15; Lemma 2.1] while part (ii) follows from Lemma 2.5 (ii) and [15; Lemma 2.1] for $\varepsilon \to 0^+$ if we take $k = r$ and $n > r$.

Using the preceding results we finally arrive at the following characterization of fractional powers of the infinitesimal generator A:

THEOREM 2.7. – Suppose that the assumptions of Lemma 2.5 are met. Then an element $a \in A$ belongs to $D((-A)^\alpha)$, $0 < \alpha < r$, $r \in \boldsymbol{N}$, if and only if the strong limit

$$s-\lim_{\varepsilon\to 0+} C_{\alpha,r}^{-1}\int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r a\,\frac{dt}{t} \tag{2.29}$$

exists in which case it is equal to $(-A)^\alpha a$.

If A is reflexive, then condition (2.29) may be replaced by

$$\sup_{\varepsilon>0}\left\|\int_\varepsilon^\infty t^{-2\alpha}[C(t)-I]^r a\,\frac{dt}{t}\right\|_A < \infty\,. \tag{2.30}$$

PROOF. – In the general case the assertion can be easily verified by Lemma 2.6 (i) (cf. [15; Thm. 2.4]). If A is reflexive, let $(\varepsilon_n)_N$ be a sequence of positive real numbers with $\varepsilon_n \to 0$ $(n\to\infty)$ and define

$$a_n := C_{\alpha,r}^{-1}\int_0^\infty q_{\alpha,r}\left(\frac{t}{\varepsilon_n}\right) C(t)\,a\,\frac{dt}{\varepsilon_n}, \qquad n \in \boldsymbol{N}\,.$$

Obviously, $a_n \in D((-A)^\alpha)$ and $s-\lim_{n\to\infty} a_n = a$. Then, if (2.30) holds true it follows from (2.26) that the sequence $((-A)^\alpha a_n)_N$ is uniformly bounded in A. Consequently a belongs to the completion of $D((-A)^\alpha)$ relative to A. But in the reflexive case that completion is equal to $D((-A)^\alpha)$ (cf. [3; pg. 15]) which proves the sufficiency of (2.30). Since the necessity of (2.30) is evident, this concludes the proof of the assertion.

3. – Interpolation of cosine operator functions.

The characterization of interpolation spaces via C_0-semigroups of operators as initiated by J. L. LIONS [16] and systematically treated by J. PEETRE [20] and P. L. BUTZER, H. BERENS [7] will be adapted to the case of C_0-cosine operator functions. In particular, we will identify interpolation spaces by moduli of continuity of cosine operator functions in the framework of the well-known K-method. For background material on the theory of interpolation spaces the reader is referred to the excellent textbooks by J. BERGH, J. LÖFSTRÖM [5] and H. TRIEBEL [22] where also an extensive bibliography is given.

Let $C: \overline{\boldsymbol{R}}^+ \to \mathcal{B}(A)$ be a C_0-cosine operator function with infinitesimal generator Λ. In the sequel $D(\Lambda^r)$, $r \in \boldsymbol{N}$, will always be equipped with the graph norm thus becoming a Banach space itself. Our first result gives an equivalent characterization of the intermediate spaces $(A, D(\Lambda^r))_{\Theta,q}$, $\Theta = \alpha/r$, $0 < \alpha < r$, $1 \leqslant q < \infty$ $(0 \leqslant \alpha \leqslant r,\ q = \infty)$ by means of the r-th order modulus of continuity of C which is given by

$$\omega_r(t^r, a) = \sup_{|s| \leqslant t} \|[C(s) - I]^r a\|_A\,, \qquad a \in A\,. \tag{3.1}$$

REMARK 1. – In (3.1) we have tacitly used the fact that C can be continuously extended to the whole real line $\boldsymbol{R}$ (cf. Remark 1 in Section 2).

Preparatory we will show the following:

LEMMA 3.1. – *Let C be a C_0-cosine operator function of type (M, ω) with infinitesimal generator Λ. For $r \in \boldsymbol{N}$ and $0 < t \leqslant \delta < \infty$ there exist constants $C_\nu = C_\nu(M, \omega, r, \delta)$, $\nu = 1, 2$, such that for $a \in A$*

$$C_1 K(t^{2r}, a; A, D(\Lambda^r)) \leqslant \omega_r(t^r, a) + \min(1, t^{2r})\|a\|_A \leqslant C_2 K(t^{2r}, a; A, D(\Lambda^r))\,. \tag{3.2}$$

If C is equibounded then we may take $\delta = \infty$.

PROOF. – Using (2.4) and the basic equation (2.9) we find that

$$\|[C(s) - I]^r a\|_A \leqslant M \frac{(\cosh(\omega s) - 1)^r}{(\omega s)^{2r}} s^{2r} \|\Lambda^r a\|_A\,, \qquad a \in D(\Lambda^r)\,. \tag{3.3}$$

On the other hand

$$\|[C(s) - I]^r a\|_A \leqslant (M \cosh(\omega s) + 1)^r \|a\|_A\,, \qquad a \in A\,. \tag{3.4}$$

Hence, if we set $a = a_0 + a_1$, where $a_0 \in A$ and $a_1 \in D(\Lambda^r)$ we get

$$\omega_r(t^r, a) \leqslant \sup_{|s| \leqslant t} \|[C(s) - I]^r a_0\|_A + \sup_{|s| \leqslant t} \|[C(s) - I]^r a_1\|_A \leqslant$$
$$\leqslant (M \cosh(\omega t) + 1)^r \|a_0\|_A + M \frac{(\cosh(\omega t) - 1)^r}{(\omega t)^{2r}} t^{2r} \|\Lambda^r a_1\|_A\,.$$

Since $(\cosh(\omega t) - 1)^r = O((\omega t)^{2r})$ and $\min(1, t^{2r})\|a\|_A \leqslant K(t^{2r}, a; A, D(\Lambda^r))$, this gives the second inequality in (3.2). In order to prove the first one, we set for notational convenience $\tilde{a}_{j,r} := a_{0,r}^{-1} a_{j,r}$, $j = 0, 1, \ldots, r$, where $a_{j,r}$ is given by (2.21).

Further we note that by means of d'Alembert's functional equation

$$\prod_{\nu=1}^{r} C(s_\nu) = 2^{-r+1} \sum_{(\sigma_1, \ldots, \sigma_r) \in P_r} C\left(\sum_{\nu=1}^{r} (-1)^{\sigma_\nu} s_\nu\right) \tag{3.5}$$

where $s_\nu \in \bar{R}^+$, $\nu = 1, \ldots, r$ and $P_r := \{(\sigma_1, \ldots, \sigma_r) | \sigma_1 = 0,\ \sigma_\nu \in \{0, 1\},\ \nu = 2, \ldots, r\}$. Then, if $a \in A$ it follows by (2.5) and (3.5) that

$$\begin{aligned} a &= 2^r (t/r)^{-2r} \sum_{j=0}^{r} \tilde{a}_{j,r} \int_0^{t/r} \int_0^{t/r} \ldots \int_0^{t/r} \prod_{\nu=1}^{r} (t/r - s_\nu)\, C(js_\nu)\, a\, ds_1\, ds_2 \ldots ds_r - \\ &\quad - 2^r (t/r)^{-2r} \sum_{j=1}^{r} \tilde{a}_{rj,} \int_0^{t/r} \int_0^{t/r} \ldots \int_0^{t/r} \prod_{\nu=1}^{r} (t/r - s_\nu)\, C(js_\nu)\, a\, ds_1\, ds_2 \ldots ds_r = \\ &= 2 a_{0,r}^{-1} (t/r)^{-2r} \sum_{(\sigma_1, \ldots, \sigma_r) \in P_r} \int_0^{t/r} \int_0^{t/r} \ldots \int_0^{t/r} \prod_{\nu=1}^{r} (t/r - s_\nu) \left[C\left(\sum_{\nu=1}^{r} (-1)^{\sigma_\nu} s_\nu\right) - I\right]^r a\, ds_1\, ds_2 \ldots ds_r - \\ &\quad - 2^r (t/r)^{-2r} \sum_{j=1}^{r} j^{-2r} \tilde{a}_{j,r} \int_0^{jt/r} \int_0^{jt/r} \ldots \int_0^{jt/r} \prod_{\nu=1}^{r} (jt/r - s_\nu)\, C(s_\nu)\, a\, ds_1\, ds_2 \ldots ds_r\,. \end{aligned} \tag{3.6}$$

Denoting by a_0 and a_1 the first resp. the second term on the right-hand side of (3.6), we obviously have $a_0 \in A$ while $a_1 \in D(\Lambda^r)$ which follows easily from Lemma 2.2 (iii). Moreover, applying (2.4) and (2.9) we get

$$\|a_0\|_A \leqslant 2^{-r} \frac{r!}{(2r)!} \omega_r(t^r, a)\,,$$

$$\|a_1\|_A \leqslant 2^r M \sum_{j=1}^{r} \{|\tilde{a}_{j,r}| (\omega j t/r)^{-2r} [\cosh(\omega j t/r) - 1]^r\} \|a\|_A\,,$$

$$\|\Lambda^r a_1\|_A \leqslant 2^r \sum_{j=1}^{r} \{(j/r)^{-2r} |\tilde{a}_{j,r}|\} t^{-2r} \omega_r(t^r, a)\,.$$

This gives the desired inequality

$$K(t^{2r}, a; A, D(\Lambda^r)) \leqslant \|a_0\|_A + \min(1, t^{2r}) \|a_1\|_{D(\Lambda^r)} \leqslant C[\omega_r(t^r, a) + \min(1, t^{2r}) \|a\|_A]\,.$$

As an immediate consequence of the preceding Lemma we have:

THEOREM 3.2. – Under the assumptions of Lemma 3.1 let $A_{\alpha,r;q}$, $0 < \alpha < r$, $1 \leqslant q < \infty$ (resp. $0 \leqslant \alpha \leqslant r$, $q = \infty$), $r \in N$, be the intermediate spaces $(A, D(\Lambda^r))_{\theta,q}$, $\theta = \alpha/r$, and let $0 < \delta < \infty$. Then for $A_{\alpha,r;q}$ there are the following equivalent norms

$$(3.7)\qquad \text{(i)}\quad \left(\int_0^\delta [t^{-\alpha/r}K(t, a; A, D(A^r))]^q \frac{dt}{t}\right)^{1/q},$$

$$\text{(ii)}\quad \|a\|_A + \left(\int_0^\delta [t^{-2\alpha}\omega_r(t^r, a)]^q \frac{dt}{t}\right)^{1/q},$$

$$\text{(iii)}\quad \|a\|_A + \left(\int_0^\delta [[t^{-2\alpha}\|[C(t)-I]^r a\|_A]^q \frac{dt}{t}\right)^{1/q}$$

with the usual modification in case $q=\infty$.

REMARK 1. – Note that $\delta=\infty$ is an admissible value in (3.7) (i), (ii) and (iii) if the C_0-cosine operator function C is supposed to be equibounded.

COROLLARY 3.3. – Under the same assumptions and with the same notations as in Theorem 3.2 there holds

$$(3.8)\qquad C(t)A_{\alpha,r;q} \subset A_{\alpha,r;q}, \qquad t\in \boldsymbol{R}^+,\ 0<\alpha<r,\ 1\leqslant q<\infty\ (0\leqslant\alpha\leqslant r,\ q=\infty);$$

$$(3.9)\qquad S(t)A_{\alpha,r;q} \subset A_{\alpha+1/2,r;q}, \qquad t\in \boldsymbol{R}^+,\ 0<\alpha<r-1/2,\ 1\leqslant q<\infty\ (0\leqslant\alpha\leqslant r-1/2,\ q=\infty).$$

PROOF. – The asserted relations (3.8) and (3.9) follow immediately from (3.7) (iii).

REMARK 2. – Note that (3.8) is nothing else but the interpolation space property of $A_{\alpha,r;q}$.

Next we shall give a characterization of $D((-\Lambda)^\alpha)$, $0<\alpha<r$, $r\in\boldsymbol{N}$, as intermediate spaces of A and $D(\Lambda^r)$, and we shall derive some reduction results both in the case of non-optimal approximation and saturation.

LEMMA 3.4. – Let $\Lambda\in\mathcal{C}(A)$ be the infinitesimal generator of an equibounded C_0-cosine operator function. Then there holds:

(i) If $r\in\boldsymbol{N}$ and $0<\alpha<r$ then

$$(3.10)\qquad A_{\alpha,r;1} \subset D((-\Lambda)^\alpha) \subset A_{\alpha,r;\infty}.$$

(ii) If $r\in\boldsymbol{N}$ and $0<\alpha<\beta\leqslant r$, $1\leqslant q\leqslant\infty$, then

$$(3.11)\qquad (A, D((-\Lambda)^\beta))_{\theta,q} = A_{\alpha,r;q}, \qquad \theta=\alpha/r$$

while the Favard space $(A, D((-\Lambda)^\alpha))_{1,\infty}$ is the space of all $a\in A$ such that (2.30) holds true normed by

$$\|a\|_A + \sup_{\varepsilon>0} \left\| C_{\alpha,r}^{-1} \int_{\varepsilon}^{\infty} t^{-2\alpha}[C(t)-I]^r a \frac{dt}{t} \right\|_A .$$

In particular, $D((-A)^\beta)$, $0<\alpha<\beta\leqslant r$, is a dense subspace of $A_{\alpha,r;q}$, $1\leqslant q<\infty$.

PROOF. – The inclusions (3.10) can be easily verified by means of Lemma 2.6 (ii) resp. Theorem 2.7. As an immediate consequence we have that $D((-A)^\alpha)$, $0<\alpha<r$, is of class $\mathcal{C}(\theta; A, D(A^r))$, $\theta=\alpha/r$ (cf. [5; Def. 3.5.1], [7; Def. 3.2.15]). This also holds true for $\alpha=0$ and $\alpha=r$ as follows directly from the theory of intermediate spaces (cf. [5; pg. 49], [7; pg. 192]). Then (3.11) is readily established applying the reiteration theorem (cf. [5; Thm. 3.5.3], [7; Thm. 3.2.20]). The characterization of the Favard space can be deduced from Lemma 2.6 (i) (cf. [3; Thm. 4.5] in the semigroup case).

Finally, the rest of the assertion is a well-known property of interpolation spaces (cf. [5; Thm. 3.4.2 (*b*)]).

We denote by $A^0_{\alpha,r;\infty}$, $0\leqslant\alpha\leqslant r$, the closure of $D(A^r)$ in $A_{\alpha,r;\infty}$. In view of Corollary 3.3 the restriction of C to $A_{\alpha,r;\infty}$ defines an equibounded cosine operator function. Our next result shows that $A^0_{\alpha,r;\infty}$ is the largest subspace of $A_{\alpha,r;\infty}$ such that the mapping $C(\cdot)a\colon \overline{\boldsymbol{R}}^+\to A_{\alpha,r;\infty}$, $a\in A^0_{\alpha,r;\infty}$, is strongly continuous.

LEMMA 3.5. – An element $a\in A$ belongs to $A^0_{\alpha,r;\infty}$, $0\leqslant\alpha\leqslant r$, $r\in\boldsymbol{N}$ if and only if

$$\lim_{t\to 0+} \|[C(t)-I]a\|_{A_{\alpha,r;\infty}} = 0 . \tag{3.12}$$

PROOF. – For the necessary part of the proof we may assume $a\in D(A^r)$. Then it follows by (2.27) and (3.7) (iii) that

$$\begin{aligned}
\|[C(t)-I]a\|_{A_{\alpha,r;\infty}} &\leqslant C\{\|[C(t)-I]a\|_A + \sup_{s\in R^+}\|s^{-2\alpha}[C(s)-I]^r[C(t)-I]a\|_A\} = \\
&= C\left\{\|[C(t)-I]a\|_A + 2\sup_{s\in R^+}\left\|\int_0^\infty p_{\alpha,r}(\tau)\,C(\tau s)[C(t)-I](-A)^\alpha a\,d\tau\right\|_A\right\}\leqslant \\
&\leqslant C\left\{\|[C(t)-I]a\|_A + 2M\int_0^\infty |p_{\alpha,r}(\tau)|\,d\tau\,\|[C(t)-I](-A)^\alpha a\|_A\right\}
\end{aligned}$$

which immediately gives (3.12).

Conversely, assume that (3.12) holds true and set

$$a_t := (-1)^r 2^{r+1}\int_0^\infty p_{r,r}\left(\frac{s}{t}\right) C(s)\,a\,\frac{ds}{t}, \qquad t\in\boldsymbol{R}^+ .$$

Lemma 2.6 (ii) tells us that $a_t \in D(\Lambda^r)$. Moreover, using (2.18b) we have

$$\|a_t - a\|_{A_{\alpha,r;\infty}} \leqslant 2^{r+1}\int_0^\infty |p_{r,r}(s)|\,\|[C(st) - I]a\|_{A_{\alpha,r;\infty}}\,ds\,.$$

By the dominated convergence theorem the right-hand side of the preceding inequality tends to zero as $t \to 0^+$. Hence, a belongs to the closure of $D(\Lambda^r)$ in $A_{\alpha,r;\infty}$, i.e. $a \in A^0_{\alpha,r;\infty}$.

REMARK 3. – Since $D(\Lambda^r)$ is a closed subspace of $A_{r,r;\infty}$, we have in particular $A^0_{r,r;\infty} = D(\Lambda^r)$.

The following result characterizes the behavior of the r-th Riemann difference $[C(t) - I]^r a$ for $a \in A_{\alpha,r;q}$ resp. $a \in A^0_{\alpha,r;\infty}$ as t approaches zero:

LEMMA 3.6. – Under the assumptions of Lemma 3.4 there holds:

(i) If $a \in A_{\alpha,r;q}$, $0 < \alpha < r$, $1 \leqslant q < \infty$ (resp. $0 \leqslant \alpha \leqslant r$, $q = \infty$), $r \in \boldsymbol{N}$, then

$$\|[C(t) - I]^r a\|_A = O(t^{2\alpha})\,, \qquad (t \to 0^+)\,. \tag{3.13}$$

Conversely, if (3.13) holds true then $a \in A_{\alpha,r;\infty}$

(ii) Let $0 < \alpha < r$, $r \in \boldsymbol{N}$. Then $a \in A^0_{\alpha,r;\infty}$ iff

$$\|[C(t) - I]^r\|_A = o(t^{2\alpha})\,, \qquad (t \to 0^+)\,. \tag{3.14}$$

PROOF. – If $a \in A_{\alpha,r;q}$ then $a \in A_{\alpha,r;\infty}$. But $a \in A_{\alpha,r;\infty}$ if and only if (3.13) holds true as follows directly from (3.7) (iii). If $a \in A^0_{\alpha,r;\infty}$ then $t^{-\alpha/r}K(t, a; A, D(\Lambda^r)) \to 0$ $(t \to 0^+)$ and vice-versa (cf. [5; Thm. 3.4.2 (c)]) which implies assertion (ii) in view of Lemma 3.1.

We shall now derive some special mapping properties of S which will prove to be a useful tool in the reduction of second order evolution equations to first order systems.

LEMMA 3.7. – Let A be a reflexive Banach space and let C be an equibounded C_0-cosine operator function with infinitesimal generator Λ. Then there holds for $t \in \boldsymbol{R}^+$.

$$S(t)A_{1/2,1;\infty} \subset D(\Lambda)\,. \tag{3.15}$$

PROOF. – Let $a \in A_{1/2,1;\infty}$ and $t \in \boldsymbol{R}^+$. Then, using (2.1) it is easy to show that for $0 < s < t$

$$[C(s) - I]S(t)a = \tfrac{1}{2}\int_0^s [C(t+\tau) - C(t)]a\,d\tau - \tfrac{1}{2}\int_{-s}^0 [C(t+\tau) - C(t)]a\,d\tau$$

whence

$$\|[C(s)-I]S(t)a\|_A \leqslant s \sup_{|\tau|\leqslant s} \|[C(t+\tau)-C(t)]a\|_A .$$

But in view of (3.13) we have

$$\|[C(t+\tau)-C(t)]a\|_A = O(|\tau|)\,, \qquad (|\tau|\to 0)$$

and thus

$$\|[C(s)-I]S(t)a\|_A = O(s^2)\,, \qquad (s\to 0^+)\,.$$

Hence, Lemma 3.6 (i) implies that $S(t)a \in A_{1,1;\infty}$. Finally, since A is reflexive, $A_{1,1;\infty} = D(\Lambda)$ which gives the assertion.

In the special case $A = L^p(\Omega)$, $1<p<\infty$, we can show:

LEMMA 3.8. – Let $A = L^p(\Omega)$, $1<p<\infty$, where $\Omega\subset \boldsymbol{R}^d$, $d\in \boldsymbol{N}$, and suppose that C is an equibounded C_0-cosine operator function with infinitesimal generator Λ. Then there holds for $t\in\boldsymbol{R}^+$

$$S(t)A_{\alpha,1;\infty} \subset D\big((-\Lambda)^{\alpha+1/2}\big)\,, \qquad 0\leqslant\alpha\leqslant \tfrac12\,, \tag{3.16}$$

$$(-\Lambda)^{\alpha+1/2}S(t)a = \tilde{C}^{-1}_{\alpha,1}\int_{-\infty}^{+\infty} |s|^{-2\alpha}C(t-s)a\frac{ds}{s}\,, \qquad a\in A^0_{\alpha,1;\infty}\,, \qquad 0\leqslant\alpha<1/2 \tag{3.17}$$

where $\tilde{C}_{\alpha,1} = -2(1+2\alpha)C_{\alpha+1/2,1}$ and the integral in (3.17) has to be understood as the principal value in the sense of Cauchy.

PROOF. – Since for $\alpha=\frac12$ assertion (3.16) follows directly from Lemma 3.7, we may restrict ourselves to the case $a\in A_{\alpha,1;\infty}$, $0\leqslant\alpha<\frac12$.

Now, if $s, t\in\boldsymbol{R}^+$ there holds

$$[C(s)-I]S(t)a = \tfrac12\int_s^{t+s} C(\tau)a\,d\tau + \tfrac12\int_{-s}^{t-s} C(\tau)a\,d\tau - \int_0^t C(\tau)a\,d\tau \tag{3.18}$$

and thus

$$\frac{d}{ds}[C(s)-I]S(t)a = \frac12[C(t+s)-C(t-s)]a\,.$$

Hence, if $\varepsilon>0$ and we formally integrate by parts we get

$$C^{-1}_{\alpha+1/2,1}\int_\varepsilon^\infty s^{-2\alpha-1}[C(s)-I]S(t)a\frac{ds}{s} = \tag{3.19}$$

$$= 2\tilde{C}^{-1}_{\alpha,1}s^{-2\alpha-1}[C(s)-I]S(t)a\Big|_{s=\varepsilon}^{s=\infty} - \tilde{C}^{-1}_{\alpha,1}\int_\varepsilon^\infty s^{-2\alpha}[C(t+s)-C(t-s)]a\frac{ds}{s}\,.$$

The preceding identity (3.18) also implies

$$\|[C(s)-I]S(t)a\|_A \leqslant 2Mt\|a\|_A$$

whence

$$\lim_{s\to\infty} 2s^{-2\alpha-1}|\tilde{C}_{\alpha,1}^{-1}|\|[C(s)-I]S(t)a\|_A = 0\,. \tag{3.20}$$

On the other hand, using (3.18) once more we get

$$[C(\varepsilon)-I]S(t)a = \tfrac{1}{2}\int_t^{t+\varepsilon}[C(\tau)-C(t)]a\,d\tau - \tfrac{1}{2}\int_{t-\varepsilon}^{t}[C(\tau)-C(t)]a\,d\tau$$

and thus

$$\varepsilon^{-2\alpha-1}\|[C(\varepsilon)-I]S(t)a\|_A \leqslant \varepsilon^{-2\alpha}\sup_{|\tau-t|\leqslant\varepsilon}\|[C(\tau)-C(t)]\|_A\,. \tag{3.21}$$

Since $a\in A_{\alpha,1;\infty}$, it follows from (3.13) and (3.21) that

$$\sup_{\varepsilon>0} 2\varepsilon^{-2\alpha-1}|\tilde{C}_{\alpha,1}^{-1}|\|[C(\varepsilon)-I]S(t)a\|_A < \infty\,. \tag{3.22}$$

Moreover, if $a\in A^0_{\alpha,q;\infty}$, then in case $\alpha=0$ the C_0-property of C (note that $A^0_{0,1;\infty}=A$) and in case $0<\alpha<\frac{1}{2}$ relation (3.14) together with (3.21) imply that

$$\lim_{\varepsilon\to 0+} 2\varepsilon^{-2\alpha-1}|\tilde{C}_{\alpha,1}^{-1}|\|[C(\varepsilon)-I]S(t)a\|_A = 0\,. \tag{3.23}$$

As far as the second term on the right-hand side of (3.19) is concerned we have

$$\begin{aligned} -\tilde{C}_{\alpha,1}^{-1}\int_\varepsilon^\infty s^{-2\alpha}[C(t+s)-C(t-s)]a\frac{ds}{s} &= \\ = \tilde{C}_{\alpha,1}^{-1}\int_{|s|\geqslant\varepsilon}|s|^{-2\alpha}C(t-s)\,a\frac{ds}{s} &= \tilde{C}_{\alpha,1}^{-1}\int_{|s|\geqslant\varepsilon}|s|^{-2\alpha}[C(t-s)-C(t)]a\frac{ds}{s}\,. \end{aligned} \tag{3.24}$$

If we set

$$\varphi_t(s;a) := |t-s|^{-2\alpha}[C(s)-C(t)]a\,,\qquad s\in\boldsymbol{R}\,,$$

then $\varphi_t(\cdot\,;a)\in L^\infty(\boldsymbol{R},A)$, since $\|[C(s)-C(t)]a\|_A = O(|s-t|^{2\alpha})$ as $s\to t$. Hence, if we make use of a special L^p-property of the Hilbert transform (cf. [12; Lemma 2.2]) it can be shown in the same way as in [12; Lemma 2.1] that, if $\varepsilon\to 0^+$, the right-hand side of (3.24) converges in $L^q(I,A)$ for any $I:=[-c,+c]$, $c\in\boldsymbol{R}^+$, and any

$q \geqslant 1$ to the Hilbert transform of $\varphi_t(\cdot, a)$ except for a constant factor. Consequently, if $a \in A_{\alpha,1;\infty}$, then taking (3.19), (3.20), (3.22) and (3.24) into account, (2.30) of Theorem 2.7 implies that (3.16) holds true for almost all $t \in \boldsymbol{R}$, while in case $a \in A^0_{\alpha,1;\infty}$ by means of (3.19), (3.20), (3.23) and (3.24) it follows from (2.29) of Theorem 2.7 that for almost all $t \in \boldsymbol{R}^+$

$$(-\Lambda)^{\alpha+1/2}(S(t)a) = \tilde{C}^{-1}_{\alpha,1}\int_{-\infty}^{+\infty} |s|^{-2\alpha}[C(t-s) - C(t)]a\frac{ds}{s} = \tilde{C}^{-1}_{\alpha,1}\int_{-\infty}^{+\infty} |s|^{-2\alpha}C(t-s)a\frac{ds}{s},$$

the integrals being principal values in the sense of Cauchy.

But using (2.13*b*) and the fact that $S(\cdot)$ is odd, it is easily shown (cf. [12; Thm. 2.3]) that (3.16) and (3.17) are even true for all $t \in \boldsymbol{R}$. Furthermore, $(-A)^{\alpha+1/2}S(\cdot)a$, $a \in A^0_{\alpha,1;\infty}$, is a strongly measurable function and hence strongly continuous which also follows from the functional equation (2.13*b*).

REMARK 4. – If A is the infinitesimal generator of a C_0-cosine operator function C of type (M, ω) with $\omega > 0$, then if we set $A_\varrho := A - \varrho^2 I$ for some $\varrho \geqslant \omega$ we have $\lambda R(\lambda^2; A_\varrho) = \lambda R(\lambda^2 + \varrho^2; A)$ and it follows from Theorem 2.4 that A_ϱ generates an equibounded C_0-cosine operator function C_ϱ. Consequently, in this case we may state the preceding results in terms of A_ϱ and C_ϱ instead of A and C.

We conclude this section with reduction theorems both in the case of optimal and non-optimal approximation. In the case of non-optimal approximation we obtain (cf. [7; Thm. 3.4.6, Corollary 3.4.9] in the semigroup case):

THEOREM 3.9. – For given $0 < \alpha < r$, $r \in \boldsymbol{N}$, suppose that $\alpha = k + \beta$ for some $0 \leqslant k \leqslant r-1$, $0 < \beta \leqslant 1$. Then $a \in A_{\alpha,r;q}$, $1 \leqslant q \leqslant \infty$, if and only if and only if $a \in D(A^{k-j})$, $0 \leqslant j \leqslant k$, and $A^{k-j}a \in A_{j+\beta,j+1;q}$ $(0 < \beta < 1)$ resp. $A^{k-j} \in A_{j+1,j+2;q}$ $(\beta = 1)$.

Moreover, the following norms on $A_{\alpha,r;q}$ are equivalent to those given by (3.7) (i)-(iii)

$$\text{(3.25)}\quad \text{(i)}\qquad \|a\|_{D(A^k)} + \left(\int_0^\delta [t^{-2(\beta-1)}\|B_t^{2(j+1)}A^{k-j}a\|_A]^q \frac{dt}{t}\right)^{1/q}, \quad 0 < \beta < 1;$$

$$\text{(ii)}\quad \begin{cases} \|a\|_{D(A^{k-j})} + \left(\int_0^\delta [t^{-2(\beta+j)}\|C(t) - I]^{j+1}A^{k-1}a\|_A^q \frac{dt}{t}\right)^{1/q}, & 0 < \beta < 1 \\ \|a\|_{D(A^{k-j})} + \left(\int_0^\delta [t^{-2(1+j)}\|[C(t) - I]^{j+2}A^{k-j}a\|_A]^q \frac{dt}{t}\right)^{1/q}, & \beta = 1 \end{cases}$$

where again we have to use the usual modification for $q = \infty$ and $\delta = \infty$ is admissible if C is assumed to be equibounded.

PROOF. – As we have stated in the proof of Lemma 3.4, $D(\Lambda^k)$, $0\leqslant k\leqslant r$, belongs to the class $\mathcal{C}(\theta; A, D(\Lambda^r))$, $\theta=\alpha/r$. Hence, if $\alpha=k+r$, $0\leqslant k\leqslant r-1$, $0<\beta<1$, resp. $0\leqslant k\leqslant r-2$, $\beta=1$, we may apply the reiteration theorem to obtain

$$A_{\alpha,r;q}=(D(\Lambda^{k-j}), D(\Lambda^{k+1}))_{(\beta+1)/(j+1),q} \quad \text{resp.}$$
$$A_{\alpha,r;q}=(D(\Lambda^{k-j}), D(\Lambda^{k+2}))_{(j+1)/(j+2);q}\,.$$

Since the transformation $(I-\Lambda)^{k-j}$, $0\leqslant j\leqslant k$, provides an isomorphism between $D(\Lambda^{k-j})$ and A resp. between $D(\Lambda^{k+\nu})$ and $D(\Lambda^{j+\nu})$, $\nu=1,2$, the interpolation theorem (cf. [7; Thm. 3.2.23]) states that the spaces $(D(\Lambda^{k-j}), D(\Lambda^{k+1}))_{(\beta+1)/(j+1),q}$ and $(A, D(\Lambda^{j+1}))_{(\beta+j)/(j+1),q}$ resp. $(D(\Lambda^{k-j}), D(\Lambda^{k+2}))_{(j+1)/(j+2),q}$ and $(A, D(\Lambda^{j+2}))_{(j+1)/(j+2),q}$ are isomorphic. Hence, the assertions of this theorem are readily established.

In the case of saturation we get (cf. [7; Thm. 3.4.10, Corollary 3.4.11] in the semigroup case):

THEOREM 3.10. – An element $a\in A$ belongs to $A_{r,r;\infty}$, $r\in \boldsymbol{N}$, if and only if $a\in D(\Lambda^{r-k})$ and $\Lambda^{r-k}a\in A_{k,k;\infty}$, $1\leqslant k\leqslant r$.

The following norms on $A_{r,r;\infty}$ are equivalent to those given by (3.7) (i)-(iii)

$$\text{(3.26)}\qquad \text{(i)}\quad \|a\|_{D(\Lambda^{r-k})}+\sup_{0<t<\delta}\left(t^{-2k}\left\|\left[C(t)-\sum_{j=0}^{k-1}\frac{t^{2j}}{(2j)!}\Lambda^j\right]\Lambda^{r-k}a\right\|_A\right),$$
$$\text{(ii)}\quad \|a\|_{D(\Lambda^{r-k})}+\sup_{0<t<\delta}\left(t^{-2k}\|[C(t)-I]^k\Lambda^{r-k}a\|_A\right)$$

($\delta=\infty$ admissible in (3.26) (i), (ii) if C is equibounded).

PROOF. – It follows from (2.9) that for $s, t\in \boldsymbol{R}^+$ and $a\in A$

$$\int_0^s(s-\tau)\,C(\tau)[C(t)-I]a\,d\tau=\Lambda\int_0^t(t-\sigma)\,C(\sigma)\int_0^s(s-\tau)\,C(\tau)a\,d\tau\,d\sigma=$$
$$=\Lambda\int_0^s(s-\tau)\,C(\tau)\int_0^t(t-\sigma)\,C(\sigma)a\,d\sigma\,d\tau=\int_0^t(t-\sigma)\,C(\sigma)[C(s)-I]a\,d\sigma\,.$$

Hence, if $a\in A_{r,r;\infty}$ we get

$$\text{(3.27)}\qquad \|[C(t)-I][C(s)-I]^{r-1}a\|_A=$$
$$=\lim_{\tau\to 0+}\left\|2^r\tau^{-2r}\int_0^\tau\int_0^\tau\!\!\dots\!\int_0^\tau\prod_{\nu=1}^{r}(\tau-\tau_\nu)\,C(\tau_\nu)[C(t)-I][C(s)-I]^{r-1}a\,d\tau_1\,d\tau_2\dots d\tau_r\right\|_A=$$
$$=\lim_{\tau\to 0+}\left\|2^r\int_0^t(t-\tau_r)\,C(\tau_r)\int_0^s\!\!\dots\!\int_0^s\prod_{\nu=1}^{r-1}(s-\tau_\nu)\,C(\tau_\nu)\{\tau^{-2r}[C(\tau)-I]^r a\,d\tau_1\,d\tau_2\dots d\tau_r\right\|_A\leqslant$$
$$\leqslant Mt^2s^{2(r-1)}\sup_{\tau\in R^+}\left(\tau^{-2r}\|[C(\tau)-I]^r a\|_A\right).$$

From the preceding inequality we immediately conclude that $a \in A_{r-1,r-1;\infty}$ and $\|[C(t)-I]a\|_{A_{r-1,r-1;\infty}} \to 0$ $(t \to 0^+)$ which shows $a \in A^0_{r-1,r-1;\infty}$ and thus $a \in D(\Lambda^{r-1})$. Then using (2.9) and (3.27) again it follows that

$$\begin{aligned}&\sup_{t\in R^+} \left(t^{-2}\|[C(t)-I]\Lambda^{r-1}a\|\right) = \\ &= \sup_{t\in R^+}\left(t^{-2}\lim_{s\to 0+}\left\|2^{r-1}s^{-2(r-1)}\int_0^s\int_0^s\dots\int_0^s\prod_{\nu=1}^{r-1}(s-\tau_\nu)\,C(\tau_\nu)[C(t)-I]\Lambda^{r-1}a\,d\tau_1\,d\tau_2\dots d\tau_{r-1}\right\|_A\right) = \\ &= \sup_{t\in R^+}\left(t^{-2}\lim_{s\to 0+}\left(2^{r-1}s^{-2(r-1)}\|[C(s)-I]^{r-1}[C(t)-I]a\|_A\right)\right) \leqslant \\ &\qquad\qquad \leqslant 2^{r-1}M\sup_{\tau\in R^+}\left(\tau^{-2r}\|[C(\tau)-I]^r a\|_A\right.\end{aligned}$$

whence $\Lambda^{r-1}a \in A_{1,1;\infty}$. Repeating the previous steps of proof as often as necessary we conclude that $a \in D(\Lambda^{r-k})$ and $\Lambda^{r-k}a \in A_{r-k,r-k;\infty}$. In order to prove the converse we note that by (2.9)

$$\|[C(t)-I]^r a\|_A \leqslant 2^{-(r-k)}Mt^{2(r-k)}\|[C(t)-I]^{r-k}\Lambda^{r-k}a\|_A$$

which immediately gives the assertion.

The equivalence of the norms (3.26) (i), (ii) with those in Theorem 3.2 is then easily established.

4. – Applications.

Let us consider the second order ordinary differential operator $\Lambda = d^2/dx^2$. It is not hard to show that Λ generates an equibounded C_0-cosine operator function C on $A = L^p(\boldsymbol{R})$, $1 \leqslant p < \infty$, given by

$$(4.1)\qquad \big(C(t)a\big)(x) = \tfrac{1}{2}[a(x+t) + a(x-t)]\,, \qquad x, t \in \boldsymbol{R},\ a \in L^p(\boldsymbol{R})\,.$$

Since $D(\Lambda^r)$, $r \in N$, corresponds to the Sobolev space $W^{2r,p}(\boldsymbol{R})$, the interpolation spaces $A_{\alpha,r;q}$, $0 < \alpha < r$, $1 \leqslant q < \infty$ (resp. $0 \leqslant \alpha \leqslant r$, $q = \infty$) can be identified as the Besov spaces $B_p^{2\alpha,q}$ and hence, the norms given in Theorems 3.2, 3.9 and 3.10 define equivalent norms on these spaces. In particular, using the explicit representation (4.1) of C we can easily compute the r-th Riemann difference $[C(t)-I]^r a$

$$\big([C(t)-I]^r a\big)(x) = (\Delta_t^{2r}a)(x) = \sum_{j=-r}^{+r}(-1)^{r-j}\binom{2r}{r-j}a(x+jt)$$

and via (3.7) (ii), (iii) we thus obtain the well-known characterizations of $B_p^{2\alpha,q}$ by means of the $2r$-th central difference Δ_t^{2r}.

Moreover, since $A_{r,r;\infty} = W^{2r,p}(\boldsymbol{R})$, $1 < p < \infty$, $r \in \boldsymbol{N}$, Theorem 3.10 states that the following norms are equivalent on $W^{2r,p}(\boldsymbol{R})$

$$\|a\|_{W^{r-k,p}} + \sup_{0<t<\infty} \left(t^{-2k} \left\| \Delta_t^{2k} \frac{d^{2(r-k)}}{dx^{2(r-k)}} a \right\|_p \right), \qquad 1 \leqslant k \leqslant r .$$

As an example for the application of the reduction results of Theorem 3.9 let us consider the Besov spaces $B_p^{2\alpha,q}$, $1 < \alpha < 2$.

According to (3.25) (i), (ii) for these spaces the following norms are equivalent

$$\|a\|_{W^{1,p}} + \left(\int_0^\infty \left[t^{-2(\alpha-1)} \left\| \Delta_t^2 \frac{d^2}{dx^2} a \right\|_p \right]^q \frac{dt}{t} \right)^{1/q},$$

$$\|a\|_{W^{1,p}} + \left(\int_0^\infty \left[\frac{4!}{2} \left\| t^{-2\alpha} \Delta_t^2 a - t^{-2(\alpha-1)} \frac{d^2}{dx^2} a \right\|_p \right]^q \frac{dt}{t} \right)^{1/q}.$$

In [7; Chap. 4.3.2] equivalent characterizations of $B_p^{2\alpha,q}$ are obtained considering A as the infinitesimal generator of a holomorphic C_0-semigroup T whose explicit representation is given by the Weierstrass singular integral

$$\big(T(t)a\big)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{+\infty} \exp\left(-s^2/4t\right) a(x-s)\, ds , \qquad x \in \boldsymbol{R},\ t \in \boldsymbol{R}^+ .$$

Indeed, the cosine operator function C and the semigroup T are related by the formula

$$\big(T(t)a\big)(x) = \frac{1}{(\pi t)^{1/2}} \int_0^\infty \exp\left(-s^2/4t\right) \big(C(s)a\big)(x)\, ds$$

which is well-known from the theory of transmutation and related P.D.E.'s (cf. [8], [10]).

REMARK 1. – In higher dimensions, i.e. $A = \Delta$ and $A = L^p(\boldsymbol{R}^d)$, $d > 1$, the applicability of our results is somewhat marred by the fact that Δ only generates an equibounded C_0-cosine operator function C if $p = 2$. For example, in case $d = 2$ resp. $d = 3$ we may use the Poisson-Parzeval resp. the Kirchhoff formula as an explicit representation of C in order to characterize the Besov spaces $B_2^{2\alpha,q}$.

As a further application of the results of Section 3 we shall be concerned with the order reduction of the second order Cauchy problem (2.11*a*), (2.11*b*). In practice one often tries to introduce new variables a_1, a_2 according to $a_1 := a$, $a_2 := (d/dt)a$ and thus to reduce (2.11*a*), (2.11*b*) to the first order system

$$\frac{d}{dt}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad t \in \boldsymbol{R}^+ , \tag{4.2a}$$

with initial conditions

$$a_1(0) = a_0\,, \quad a_2(0) = a_t^0\,. \tag{4.2b}$$

However, such a reduction is not always justified, since it may happen that the second order problem (2.11*a*), (2.11*b*) is well-posed while the first order one is not.

Nevertheless, in the special situation of Lemma 3.8 the reduction as outlined above leads to a well-posed first order problem.

More generally we have:

THEOREM 4.1. – Let $\Omega \subset \boldsymbol{R}^d$, $d \in \boldsymbol{N}$, and $A = L^p(\Omega)$, $1 < p < \infty$. Then an operator $\Lambda \in \mathrm{C}(A)$ is the infinitesimal generator of an equibounded C_0-cosine operator function if and only if the operator-valued matrix

$$\Phi_\alpha = \begin{pmatrix} 0 & \tilde{\Lambda}^{1/2-\alpha} \\ \tilde{\Lambda}^{1/2+\alpha} & 0 \end{pmatrix}, \quad 0 \leqslant \alpha \leqslant 1/2 \tag{4.3}$$

where $\tilde{\Lambda}^\gamma := i^{2\gamma}(-\Lambda)^\gamma$, $\gamma \in \boldsymbol{R}^+$, generates an equibounded C_0-group of operators on $A^0_{\alpha,1;\infty} \times A$.

PROOF. – For the necessary part of the assertion assume that $\Lambda \in \mathrm{C}(A)$ is the infinitesimal generator of an equibounded C_0-cosine operator function C and define

$$\mathcal{C}_\alpha(t) = \begin{pmatrix} C(t) & \tilde{\Lambda}^{1/2-\alpha} S(t) \\ \tilde{\Lambda}^{1/2+\alpha} S(t) & C(t) \end{pmatrix}, \quad 0 \leqslant \alpha \leqslant 1/2,\ t \in \boldsymbol{R}\,. \tag{4.4}$$

By means of the basic identities (2.13*a*), (2.13*b*) and Lemma 3.8 it is easily verified that $\mathcal{C}_\alpha$ is a C_0-group of operators on $A^0_{\alpha,1;\infty} \times A$. Moreover, if $\lambda^2 \in \varrho(\Lambda)$ we get

$$R(\lambda, \Phi_\alpha) = \begin{pmatrix} \lambda R(\lambda^2; \Lambda) & \tilde{\Lambda}^{1/2-\alpha} R(\lambda^2; \Lambda) \\ \tilde{\Lambda}^{1/2+\alpha} R(\lambda^2; \Lambda) & \lambda R(\lambda^2; \Lambda) \end{pmatrix}. \tag{4.5}$$

On the other hand, it follows by (2.15*a*) and (2.15*b*) that

$$\int_0^\infty \exp(-\lambda t)\, \mathcal{C}_\alpha(t)\, dt = R(\lambda; \Phi_\alpha)$$

which readily shows that $\mathcal{C}_\alpha$ is generated by Φ_α. It is also immediately clear in view of (4.5) that $\mathcal{C}_\alpha$ is equibounded.

Conversely, if $\mathcal{C}_\alpha$ is an equibounded C_0-group of operators on $A^0_{\alpha,1;\infty} \times A$ with infinitesimal generator Φ_α then

$$\mathrm{C}_\alpha(t) := \tfrac{1}{2}\left(\mathcal{C}_\alpha(t) + \mathcal{C}_\alpha(-t)\right), \quad t \in \boldsymbol{R}\,,$$

defines an equibounded C_0-cosine operator function on $A^0_{\alpha,1;\infty}\times A$ with infinitesimal generator Φ^2_α. But

$$\Phi^2_\alpha = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

which shows that Λ generates an equibounded C_0-cosine operator function on A.

As an immediate consequence we obtain:

COROLLARY 4.2. – The second order Cauchy problem (2.11*a*), (2.11*b*) is uniformly well-posed iff so is the first order problem

$$\text{(4.6a)} \qquad \frac{d}{dt}\,a(t) = \Phi_\alpha a(t)\,, \qquad t\in \boldsymbol{R},\ 0\leqslant\alpha\leqslant 1/2\,,$$

$$\text{(4.6b)} \qquad a(0) = a^0$$

where $a(t) = (a_1(t), a_2(t))'\in A^0_{\alpha,1;\infty}\times A$, $t\in \boldsymbol{R}$, and there is a one-to-one correspondence between the solutions of (2.11*a*), (2.11*b*) and (4.6*a*), (4.6*b*) given by $a_1(t) = \tilde{\Lambda}^{1/2-\alpha}a(t)$ and $a_2(t) = (d/dt)a(t)$.

REMARK 2. – Under slight modifications the assertions of the preceding theorem and its corollary remain valid if Λ is the infinitesimal generator of a C_0-cosine operator function C of type (M, ω) with $\omega > 0$. In this case one has to replace Φ_α in (4.3) resp. (4.6*a*) by

$$\Phi_\alpha = \begin{pmatrix} 0 & \tilde{\Lambda}_\varrho^{1/2-\alpha} + i\varrho I \\ \tilde{\Lambda}_\varrho^{1/2+\alpha} - i\varrho I & 0 \end{pmatrix}$$

where $\Lambda_\varrho := \Lambda - \varrho^2 I$ for some $\varrho > \omega$ (cf. [12; Thm. 6.9]).

As a final example we shall be concerned with the approximate solution of the Cauchy problem for a second order hyperbolic P.D.E. with special emphasis on the case of non-smooth initial data. The analysis presented here is somewhat similar to that one given in [6] for the approximate solution of initial value problems for parabolic and first order hyperbolic equations, but the main difference is that we shall concentrate on two-step fully discrete schemes instead of one-step methods.

Let us consider the initial-value problem

$$\text{(4.7a)} \qquad \frac{d^2}{dt^2}\,a(x,t) = (Pa)(x,t) := \sum_{|\alpha|\leqslant 2m} p_\alpha D^\alpha a(x,t)\,, \qquad x\in \boldsymbol{R}^d,\ t\in \boldsymbol{R}\,,$$

$$\text{(4.7b)} \qquad a(x,0) = a^0(x)\,, \qquad \frac{d}{dt}\,a(x,0) = a^0_t(x)\,, \qquad x\in \boldsymbol{R}^d$$

where $p_\alpha\in \boldsymbol{C}$, $\alpha = (\alpha_1, \dots, \alpha_d)\in \boldsymbol{N}^d$, $|\alpha| = \sum_{j=1}^{d}\alpha_j$, $n\in \boldsymbol{N}$. If we treat that problem in $L^p(\boldsymbol{R}^d)$, then as we have pointed out in Section 2, (4.7*a*), (4.7*b*) is uniformly well-posed in $\boldsymbol{R}$ if and only if the differential operator P is the infinitesimal generator of a C_0-cosine operator function $C: \boldsymbol{R} \to \mathcal{B}(L^p(\boldsymbol{R}^d))$, and in this case C and its strong integral S are the propagators of the given Cauchy problem. Throughout the following we will restrict ourselves to the case $p = 2$. As mentioned before, the reason for this lies in the fact that even in the simple case $P = \Delta$ the Cauchy problem (4.7*a*), (4.7*b*) is not uniformly well-posed in $\boldsymbol{R}$ unless $d = 1$ or $p = 2$. Nevertheless, with regard to an approximate solution of (4.7*a*), (4.7*b*) we are interested in error estimates in the L^∞-norm, and as we shall see below that purpose naturally involves interpolation spaces, namely the Besov spaces $B_2^{d/2+s,q}$, $s \geqslant 0$.

If we denote by

$$\hat{P}(\xi) := \sum_{|\alpha| \leqslant 2m} p_\alpha (i\xi)^\alpha \tag{4.8}$$

the symbol $\hat{P}$ of P, then $Pa = \mathcal{F}^{-1}(\hat{P}\hat{a})$, $a \in S'$, where $\hat{a} := \mathcal{F}a$ denotes the Fourier transform of a and S_q is the space of tempered distributions. The Fourier transformed initial-value problem now reads

$$\frac{d^2}{dt^2}\hat{a}(\xi, t) = \hat{P}(\xi)\, \hat{a}(\xi, t)\,, \qquad \xi \in \boldsymbol{R}^d,\ t \in \boldsymbol{R}\,, \tag{4.9a}$$

$$\hat{a}(\xi, 0) = \hat{a}^0(\xi)\,, \qquad \frac{d}{dt}\hat{a}(\xi, 0) = \hat{a}_t^0(\xi)\,, \qquad \xi \in \boldsymbol{R}^d\,. \tag{4.9b}$$

It is clear that (4.7*a*), (4.7*b*) is uniformly well-posed in $\boldsymbol{R}$ (with respect to $L^2(\boldsymbol{R}^d)$) iff the same holds true for the transformed problem (4.9*a*), (4.9*b*). Equivalently, P is the infinitesimal generator of a C_0-cosine operator function C of type (M, ω) if and only if $\hat{P}$ generates a C_0-cosine operator function $\hat{C}$ of the same type which can formally be represented by

$$\hat{C}(t) = \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!}\hat{P}(\xi)^j\,. \tag{4.10}$$

The correspondence between C and $\hat{C}$ is given by

$$C(t)a = \mathcal{F}^{-1}(\hat{C}(t)\hat{a})\,, \qquad t \in \boldsymbol{R}\,,$$

at least for $a \in \hat{C}_0^\infty$, the space of functions whose Fourier transform $\hat{a}$ belongs to C_0^∞. Moreover, by means of Theorems 2.4 and 4.1 we have the following simple criterion for well-posedness of (4.7*a*), (4.7*b*):

LEMMA 4.3. – The initial-value problem (4.7*a*), (4.7*b*) is uniformly well posed in $\boldsymbol{R}$ if and only if there exists a polynomial $\hat{Q} = \hat{Q}(\xi)$, $\xi \in \boldsymbol{R}^d$, such that $\hat{P}(\xi) = \hat{Q}(\xi)^2 + c^2$ for some $c \in \boldsymbol{C}$, $\operatorname{Re} c \geqslant \omega$, and

$$|\operatorname{Re} \hat{Q}(\xi)| \leqslant \omega\,, \qquad \xi \in \boldsymbol{R}^d\,. \tag{4.11}$$

As a simple example let us consider the one-dimensional wave equation where $P = d^2/dx^2$. Then $\hat{P}(\xi) = -\xi^2$, $Q(\xi) = i\xi$, $c = 0$, and thus $\hat{C}(t) = \cos(t\xi)$ whence $C(t)a = \mathcal{F}^{-1}(\cos(t\xi)\hat{a}) = \frac{1}{2}[a(\cdot + t) + a(\cdot - t)]$ (cf. (4.1)).

After these preliminaries we will now approximate the given initial-value problem (4.7*a*), (4.7*b*) by fully discrete schemes which are governed by some finite difference approximations in the space variables and a two-step finite difference scheme in the time variable. Let us denote by $h \in \boldsymbol{R}^+$ the step size with respect to the discretization in space and let us define $\Delta_k := \{t_j = jk | j \in \boldsymbol{Z}\}$ as a uniform partition of the real line $\boldsymbol{R}$ with step size $k \in \boldsymbol{R}^+$ which we assume to be related to h by $kh^{-r} = \lambda = \text{const}$ for some $r \in \boldsymbol{N}$. We then introduce finite difference operators A_h and B_h according to

$$(4.12)\qquad \begin{cases} A_h a_h := \sum_{|\nu| \leq p} \alpha_\nu(h) a_h(\cdot + \nu h), \\ B_h a_h := \sum_{|\nu| \leq p} \beta_\nu(h) a_h(\cdot + \nu h) \end{cases}$$

where $\nu = (\nu_1, \ldots, \nu_d) \in \boldsymbol{Z}^d$, $|\nu| = \sum_{j=1}^{d} |\nu_j|$, $p \in \boldsymbol{N}$, and $\alpha_\nu(h), \beta_\nu(h)$ are polynomials in h of degree less or equal $2r$.

For given approximations a_h^0 and a_h^1 to a^0 resp. $a^1 = a(k)$ we consider the two-step fully discrete scheme

$$(4.13)\qquad B_h[a_h^{n+1} + a_h^{n-1}] = A_h a_h^n, \quad n \in \boldsymbol{N}.$$

The symbols $\hat{A}_h$ and $\hat{B}_h$ of the difference operators A_h, B_h are given by

$$\hat{A}_h(\xi) = \sum_{|\nu| \leq p} \alpha_\nu(h) \exp(i\langle \xi, \nu h \rangle),$$

$$\hat{B}_h(\xi) = \sum_{|\nu| \leq p} \beta_\nu(h) \exp(i\langle \xi, \nu h \rangle).$$

If we assume

$$\sum_{|\nu| \leq p} \beta_\nu(0) \exp(i\langle \xi, \nu \rangle) \neq 0, \quad \xi \in \boldsymbol{R}^d$$

then $\hat{B}_h(\xi)$ is bounded away from zero for at least sufficiently small $h \in \boldsymbol{R}^+$, and the Fourier transformed difference equation can be written in the form

$$(4.14)\qquad \hat{a}_h^{n+1} - 2\hat{R}_h(\xi)\hat{a}_h^n + \hat{a}_h^{n-1} = 0, \quad n \in \boldsymbol{N}$$

where

$$(4.15)\qquad \hat{R}_h(\xi) := \tfrac{1}{2}\hat{B}_h^{-1}(\xi)\hat{A}_h(\xi).$$

In the sequel we shall essentially take advantage of the fact that the solutions $a(t)$ resp. $\hat{a}(t)$, $t \in \boldsymbol{R}$, of the initial-value problems (4.7*a*), (4.7*b*) resp. (4.9*a*), (4.9*b*) satisfy a difference equation analogous to (4.14). Indeed, in terms of C resp. $\hat{C}$ we have the three-term recurrence formulas

$$a((n+1)k) - 2C(k)a(nk) + a((n-1)k) = 0\,, \qquad n \in \boldsymbol{N}\,, \tag{4.16}$$

$$\hat{a}((n+1)k) - 2\hat{C}(k)\hat{a}(nk) + \hat{a}((n-1)k) = 0\,, \qquad n \in \boldsymbol{N}\,. \tag{4.17}$$

In order to give explicit representations of the solutions $\hat{a}(t)$, $t \in \Delta_k$, of (4.17), resp. $\hat{a}_h^n$, $n \in \boldsymbol{N}$, of (4.14) we introduce the characteristic polynomials

$$\hat{p}_\nu(\lambda; \xi) := \lambda^2 - 2\hat{e}_\nu(\xi)\lambda + 1\,, \qquad \nu = 1, 2\,, \tag{4.18}$$

where $\hat{e}_1(\xi) = \hat{C}(k)$ and $\hat{e}_2(\xi) = \hat{R}_h(\xi)$.

Then if we choose $R \in \boldsymbol{R}^+$ large enough such that the roots $\lambda_{\mu_{\hat{z}}}^{(\nu)}(\xi)$, $\nu, \mu = 1, 2$, of (4.18) are located inside the circle $|\lambda| = R$, the complex line integrals

$$\left\{\begin{aligned} \hat{P}_\nu^0(n; \xi) &= \frac{1}{2\pi i}\oint_{|\lambda|=R} \hat{p}_\nu^{-1}(\lambda; \xi)\,\lambda^n[\lambda - 2\hat{e}_\nu(\xi)]\,d\lambda\,, \qquad n \in \boldsymbol{N}\,,\\ \hat{P}_\nu^1(n; \xi) &= \frac{1}{2\pi i}\oint_{|\lambda|=R} \hat{p}_\nu^{-1}(\lambda; \xi)\,\lambda^n\,d\lambda\,, \qquad n \in \boldsymbol{N}\,, \end{aligned}\right. \tag{4.19}$$

are well defined, and it can be shown (cf. [13]) that the solutions $\hat{a}(t)$, $t \in \Delta_k$, of (4.17) resp. $\hat{a}_h^n$, $n \in \boldsymbol{N}$, of (4.14) are given by

$$\hat{a}(nk) = \hat{P}_1^0(n; \xi)\hat{a}^0 + \hat{P}_1^1(n; \xi)\hat{a}^1 \quad \text{resp.} \tag{4.20}$$

$$\hat{a}_h^n \;\; = \hat{P}_2^0(n; \xi)\hat{a}_h^0 + \hat{P}_2^1(n; \xi)\hat{a}_h^1\,. \tag{4.21}$$

Due to the fact, if $a^0, a^1 \in \hat{C}_0^\infty$ resp. $a_h^0, a_h^1 \in \hat{C}_0^\infty$, then $a(t) = \mathcal{F}^{-1}\hat{a}(t)$, $t \in \Delta_k$, resp. $a_h^n = \mathcal{F}^{-1}\hat{a}_h^n$, $n \in \boldsymbol{N}$, is a solution of (4.16) resp. (4.13), we define the operators $P_\nu^\mu(n)$, $\nu = 1, 2$, $\mu = 0, 1$, given by

$$P_\nu^\mu(n)a := \mathcal{F}^{-1}(\hat{P}_\nu^\mu(n; \xi)\hat{a})\,, \qquad a \in \hat{C}_0^\infty$$

as the solution operators of (4.16) resp. (4.13).

Therefore, we can cast the consistency of (4.7*a*) and (4.13) in terms of the Fourier transformed equations (4.9*a*) and (4.14) (cf. [6, § 3]).

In this sense the difference equation (4.13) and the second order hyperbolic P.D.E. (4.7*a*) are said to be consistent of order $p > 0$ if

$$\hat{R}_h(\xi) - \hat{C}(k) = O(h^p[1 + |\xi|^{r+p}])\,, \qquad (h, \xi \to 0)\,. \tag{4.22}$$

Moreover, the difference scheme (4.13) is said to be stable if there is $h_1 \in \mathbf{R}^+$ such that for all $h \leqslant h_1$ and any $T \in \mathbf{R}^+$ there exists a constant $C = C(T)$ with

$$\|P_2^\mu(n)a\|_{L^2} \leqslant C\|a\|_{L^2}\,, \qquad \mu = 0, 1,\ nk \leqslant T,\ a \in \hat{C}_0^\infty\,. \tag{4.23}$$

Again, it is more convenient to state stability criteria by means of the Fourier transformed equation (4.14):

LEMMA 4.4. – If there exist positive real numbers h_1 and C such that for all $h \leqslant h_1$

$$\sup_{\xi \in R^d} |\hat{R}_h(\xi)| \leqslant 1 + Ck\,, \tag{4.24}$$

then the difference scheme (4.13) is stable.

PROOF. – If (4.24) holds true then we also have

$$\sup_{\xi \in R^d} |\lambda_\mu^{(2)}(\xi)| \leqslant 1 + Ck\,, \qquad \mu = 1, 2\,. \tag{4.25}$$

Using the theorem of residues we can compute $\hat{P}_2^\mu(n;\xi)$, $\mu = 0, 1$, and we thus obtain

$$\hat{P}_2^0(n;\xi) = \sum_{j=0}^{n} (\lambda_1^{(2)}(\xi))^{n-j}(\lambda_2^{(2)}(\xi))^j\,, \qquad \hat{P}_2^1(n;\xi) = \sum_{j=0}^{n-1} (\lambda_1^{(2)}(\xi))^{n-1-j}(\lambda_2^{(2)}(\xi))^j\,.$$

Then (4.23) is an immediate consequence of (4.25) and Parzeval's relation.

We shall now establish a priori estimates for the global discretization error $a_h^n - a(nk)$ in the L^∞-norm for initial data a^0 and a_t^0 belonging to the Besov spaces $B_2^{\nu_0,\infty}$, $\nu_0 = d/2 + s$, resp. $B_2^{\nu_1,\infty}$, $\nu_1 = (d-1)/2 + s$, where $0 < s < r + p$. For the sake of simplicity we shall assume that the initial approximations a_h^0 and $a_{h,t}^0$ are given by the correct initial data and that the starting value a_h^1 for the difference scheme (4.13) corresponds to the exact solution of (4.7*a*), (4.7*b*) at $t = k$, i.e.

$$\begin{aligned} a_h^0 &= a^0\,, \qquad a_{h,t}^0 = a_t^0\,; \\ a_h^1 &= a(k) = C(k)a^0 + S(k)a_t^0\,. \end{aligned} \tag{4.26}$$

REMARK 3. – In practice an approximation a_h^1 to $a(k)$ may be obtained by a one-step method of appropriate order which can be derived from a reduction of (4.7*a*), (4.7*b*) to an equivalent initial-value problem for a first-order system according to Corollary 4.2.

The following error analysis heavily relies upon two basic results on the Besov spaces $B_2^{\nu_0,1}$, $\nu_0 = d/2 + s$, $s \geqslant 0$:

LEMMA 4.5. – Let $\mu \in C^\infty(\boldsymbol{R}^d)$ be a slowly increasing function and suppose that there is a positive constant K such that

$$|\mu(\xi)| \leqslant K[1 + |\xi|^s], \qquad \xi \in \boldsymbol{R}^d,$$

for some $s \geqslant 0$. Then, if $Ta := \mathcal{F}^{-1}(\mu\hat{a})$, $a \in B_2^{\nu_0,1}$, $\nu_0 = d/2 + s$, we have $Ta \in L^\infty(\boldsymbol{R}^d)$, and there exists a positive constant C such that

$$\|Ta\|_{L^\infty} \leqslant CK\|a\|_{B_2^{\nu_0,1}}.$$

PROOF. – See e.g. [6; Corollary 2.2.1].

LEMMA 4.6. – Let P be a bounded linear operator from $B_2^{s_0,1}$, $s_0 \geqslant 0$, into $L^\infty(\boldsymbol{R}^d)$ and assume that there are constants $K_0, K_1 > 0$ such that

$$\|Pa\|_{L^\infty} \leqslant K_0\|a\|_{B_2^{s_0,1}}, \qquad a \in B_2^{s_0,1}$$

$$\|Pa\|_{L^\infty} \leqslant K_1\|a\|_{B_2^{s_1,1}}, \qquad a \in B_2^{s_1,1},\ s_1 > s_0.$$

Then, if $a \in B_2^{s,\infty}$, $s_0 < s < s_1$, and $q := (s - s_0)/(s_1 - s_0)$, there holds

$$\|Pa\|_{L^\infty} \leqslant CK_0^{1-q}K_1^q\|a\|_{B_2^{s,\infty}}.$$

PROOF. – The assertion of this Lemma is a special case of [6; Corollary 2.5.1]. Our main result now states:

THEOREM 4.7. – Let the initial-value problem (4.7*a*), (4.7*b*) be uniformly well-posed in $\boldsymbol{R}$, let the difference scheme (4.13) be consistent with (4.7*a*), (4.7*b*) of order $p > 0$ and assume that the stability condition (4.24) is satisfied.

Moreover, suppose that $a^0 \in B_2^{\nu_0,\infty}$, $\nu_0 = d/2 + s$, $a_t^0 \in B_2^{\nu_1,\infty}$, $\nu_1 = (d-1)/2 + s$, $0 < s < r + p$, and that the starting values a_h^0 and a_h^1 are given by (4.26). Then, if $h \leqslant h_1$ and $a(nk)$, a_h^n, $nk \leqslant T$, are the solutions of (4.7*a*), (4.7*b*) resp. (4.13), there exists a positive constant $C = C(T)$ such that

$$\|a_h^n - a(nk)\|_{L^\infty} \leqslant Ch^{sp/(r+p)}[\|a^0\|_{B_2^{\nu_0,\infty}} + \|a_t^0\|_{B_2^{\nu_1,\infty}}]. \tag{4.27}$$

PROOF. – The outline of proof is as follows: Using Lemma 4.5 we shall first derive two estimates for the global discretization error in the L^∞-norm for starting values a^0, a^1 belonging to $B_2^{d/2,1}$ resp. $B_2^{\nu_2,1}$, $\nu_2 = d/2 + r + p$, and we shall then establish (4.27) by means of the interpolation result of Lemma 4.6 and Corollary 3.3.

Since the global discretization error $a_h^n - a(nk)$ has the representation

$$a_h^n - a(nk) = \mathcal{F}^{-1}([\hat{P}_2^0(n;\xi) - \hat{P}_1^0(n;\xi)]\hat{a}^0) + \mathcal{F}^{-1}([\hat{P}_2^1(n;\xi) - \hat{P}_1^1(n;\xi)]\hat{a}^1), \tag{4.28}$$

we have to provide estimates for $\hat{P}_2^\mu(n;\xi)-\hat{P}_1^\mu(n;\xi)$, $\mu=0,1$, in order to apply Lemma 4.5. It follows from Lemma 4.3 and Lemma 4.4 that $\hat{C}(k)$ and $\hat{R}_h(\xi)$, $h\leqslant h_1$, can be bounded independently of $\xi\in R^d$. Consequently, in (4.19) we may choose $R\in \boldsymbol{R}^+$ independent of h, ξ such that the characteristic polynomials $\hat{p}_\nu(\lambda;\xi)$, $\nu=1,2$, $|\lambda|=R$, are bounded away from zero uniformly in $\xi\in R^d$ whence

$$|\hat{P}_\nu(n;\xi)|\leqslant C\,,\qquad \xi\in \boldsymbol{R}^d,\ h\leqslant h_1,\ \nu=1,2,\ \mu=0,1\,. \tag{4.29}$$

By a similar argument we find that

$$|\hat{P}_2^\mu(n;\xi)-\hat{P}_1^\mu(n;\xi)|\leqslant C|\hat{C}(k)-\hat{R}_h(\xi)|\,,\qquad \xi\in \boldsymbol{R}^d,\ h\leqslant h_1,\ \mu=0,1\,. \tag{4.30}$$

Now, if a^0, $a^1\in B_2^{d/2,1}$, it follows from (4.28), (4.29) and Lemma 4.5 that

$$\|a_h^n-a(nk)\|_{L^\infty}\leqslant C[\|a^0\|_{B_2^{d/2,1}}+\|a^1\|_{B_2^{d/2,1}}]\,. \tag{4.31}$$

On the other hand, suppose that a^0, $a^1\in B_2^{\nu_2,1}$, $\nu_2=d/2+r+p$.

Then, if $h|\xi|<\sigma_1$, $\sigma_1:=\min(1,\sigma_0)$, for some suitably chosen $\sigma_0>0$, we get by (4.30) and (4.22)

$$|\hat{P}_2^\mu(n;\xi)-\hat{P}_1^\mu(n;\xi)|\leqslant Ch^p[1+|\xi|^{r+p}]\,,\qquad \mu=0,1\,, \tag{4.32}$$

while, if $h|\xi|>\sigma_1$, it follows from (4.29) that

$$|\hat{P}_2^\mu(n;\xi)-\hat{P}_1^\mu(n;\xi)|\leqslant C=C(h|\xi|^{r+p})\left(\frac{\sigma_1}{h|\xi|}\right)^{r+p}\sigma_1^{-(r+p)}\leqslant \tag{4.33}$$
$$\leqslant C(h|\xi|)^{r+p}\leqslant Ch^p[1+|\xi|^{r+p}]\,,\qquad \mu=0,1\,.$$

Hence, in view of (4.28), (4.32) and (4.33), Lemma 4.5 implies that

$$\|a_h^n-a(nk)\|_{L^\infty}\leqslant Ch^p[\|a^0\|_{B_2^{\nu_2,1}}+\|a^1\|_{B_2^{\nu_2,1}}]\,. \tag{4.34}$$

Finally, if $a^0\in B_2^{\nu_0,\infty}$, $a_t^0\in B_2^{\nu_1,\infty}$, then Corollary 3.3 gives $a^1\in B_2^{\nu_0,\infty}$, and we get by means of (4.31), (4.34) and Lemma 4.6

$$\|a_h^n-a(nk)\|_{L^\infty}\leqslant Ch^{sp/(r+p)}[\|a^0\|_{B_2^{\nu_0,\infty}}+\|a^1\|_{B_2^{\nu_0,\infty}}]\,. \tag{4.35}$$

But $a^1=C(t)a^0+S(t)a_t^0$ whence

$$\|a^1\|_{B_2^{\nu_0,\infty}}\leqslant C[\|a^0\|_{B_2^{\nu_0,\infty}}+\|a_t^0\|_{B_2^{\nu_1,\infty}}]\,. \tag{4.36}$$

Inserting (4.36) into (4.35) gives the conclusion.

A special type of two-step fully discrete schemes can be obtained in the following way:

Using symmetric difference quotients:

$$D_\nu a_h(x) := (2h)^{-1}[a_h(x + he_\nu) - a_h(x - he_\nu)] , \qquad \nu = 1, \dots, d$$

where $h \in \mathbf{R}^+$ and e_ν denotes the ν-th unit vector in $\mathbf{R}^d$, we approximate the differential operator P in (4.7a) by the difference operator

$$P_h a_h = \sum_{|\alpha| \leqslant 2m} p_\alpha D_h^\alpha , \qquad D_h^\alpha := \prod_{\nu=1}^{d} D_\nu^{\alpha_\nu} , \tag{4.37}$$

whose symbol $\hat{P}_h$ is given by (cf. [6; § 3.2])

$$\hat{P}_h(\xi) = \hat{P}(h^{-1} \sin(h\xi)) , \qquad \sin \xi = (\sin \xi_1, \dots, \sin \xi_d) .$$

Moreover, in view of (4.10) we approximate $\cosh z$, $z \in \mathbf{C}$, by a rational function $r(z) = q^{-1}(z^2)p(z^2)$, where p and q are polynomials in z^2 of order less or equal s such that $q(k^2\hat{P}_h(\xi)) \neq 0$, $\xi \in \mathbf{R}^d$, for at least sufficiently small $h \in R^+$. Then, if we choose $\hat{A}_h(\xi) = p(k^2\hat{P}_h(\xi))$ and $\hat{B}_h(\xi) = q(k^2\hat{P}_h(\xi))$ and set $A_h a = \mathcal{F}^{-1}(\hat{A}_h \hat{a})$, $B_h = \mathcal{F}^{-1}(\hat{B}_h \hat{a})$, $a \in \hat{C}_0^\infty$, we achieve at a two-step fully discrete scheme of type (4.13).

A suitable choice of rational approximations to $\cosh z$ is given by the parameter dependent family of rational functions

$$r(z; \gamma) := q^{-1}(z^2; \gamma)p(z^2; \gamma) , \qquad z \in \mathbf{C},\ \gamma \in \mathbf{R}^+ , \tag{4.38}$$

where

$$p(z^2; \gamma) := \sum_{\nu=0}^{s} \alpha_{2\nu}(\gamma) z^{2\nu} , \qquad s \in \mathbf{N}$$

$$\alpha_{2\nu}(\gamma) := \sum_{\mu=0}^{\nu} \binom{s}{\mu} \frac{(-1)^\mu \gamma^{2\mu}}{(2(\nu - \mu))!} , \qquad \nu = 0, 1, \dots, s$$

$$q(z^2; \gamma) := (1 - \gamma^2 z^2)^s .$$

For second order hyperbolic initial-boundary value problems involving a self-adjoint, positive definite operator P such rational functions have been successfully used in [1] and [14] to design highly efficient two-step fully discrete schemes where in contrast to (4.37) the discretization in the space variables has been performed by Galerkin type methods. For a detailed discussion of the stability and the accuracy of the methods based on (4.38) we refer to [1], [2] and [14].

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