Annali di Matematica pura ed applicata
(IV), Vol. CLIV (1989), pp. 231-242

Interpolation of Linear Operators in the Köthe Dual Spaces (*).

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Summary. – *In this paper we investigate when the Köthe dual spaces Y' and X' are interpolation spaces with respect to couples of the Köthe dual spaces (Y'_0, Y'_1) and (X'_0, X'_1), respectively, where X and Y are interpolation spaces with respect to given couples (X_0, X_1) and (Y_0, Y_1) of Banach function spaces.*

1. – Introduction.

A pair $\boldsymbol{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are both continuously imbedded in some Hausdorff topological vector space V.

For a Banach couple $\boldsymbol{A} = (A_0, A_1)$ we can form the *sum* $A_0 + A_1$ and the *intersection* $A_0 \cap A_1$. They are both Banach spaces in the natural norms $\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1}: a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\}$ for $a \in A_0 + A_1$ and $\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1})$ for $a \in A_0 \cap A_1$.

A Banach space A is called an *intermediate space* between A_0 and A_1 (or with respect to $\boldsymbol{A}$) if $A_0 \cap A_1 \subset A \subset A_0 + A_1$ with continuous inclusions. For brevity, the closure of $A_0 \cap A_1$ in A will be denoted by A^0. We write $\boldsymbol{A}^0 = (A_0^0, A_1^0)$ for a Banach couple $\boldsymbol{A}$. If $\boldsymbol{A}^0 = \boldsymbol{A}$, $\boldsymbol{A}$ is called a *regular couple* and then the dual spaces A_0^* and A_1^* may be regarded as subspaces of $(A_0 \cap A_1)^*$. So (A_0^*, A_1^*) is a Banach couple which we denote by $\boldsymbol{A}^*$. Since $(A_0 + A_1)^* = A_0^* \cap A_1^*$ and $(A_0 \cap A_1)^* = A_0^* + A_1^*$ isometrically (see [2]), so if A is any intermediate space with respect to $\boldsymbol{A}$, such that $A_0 \cap A_1$ is dense in A, then A^* is an intermediate space with respect to $\boldsymbol{A}^*$.

In the theory of interpolation spaces Banach function spaces are importance. We recall some fundamental notation.

Let (Ω, μ) be a measure space with μ complete and σ-finite. We denote by $L^0 = L^0(\Omega, \mu)$ the space of all equivalence classes of μ-measurable real valued functions defined and finite μ-a.e. on Ω, equipped with the topology of convergence in measure on μ-finite sets.

A linear subspace X of L^0 is called an *ideal* (in L^0) if $|x| \leqslant |y|$ μ-a.e. for $x \in L^0$ and $y \in X$ imply $x \in X$. Note that every ideal X in L^0 with supp $X = \Omega$ (supp X is the smallest measurable set outside of which all functions in X are equal to zero)

(*) Entrata in Redazione il 5 novembre 1987.

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is *super order dense* in L^0, i.e., for every $0 \leqslant x \in L^0$ there exists a sequence $(x_n) \subset X$ such that $0 \leqslant x_n \uparrow x$, μ-a.e. (see [9], Lemma 1, p. 138).

We say that an ideal X in L^0 is a *Banach function space* (on (Ω, μ)) if X is a Banach space with the property $|x| \leqslant |y|$ μ-a.e. for $x, y \in X$ implies $\|x\|_X \leqslant \|y\|_X$. Hence, it follows that if X_0 and X_1 are any two Banach function spaces (on (Ω, μ)) then $\boldsymbol{X} = (X_0, X_1)$ forms a Banach couple.

We say that the norm $\|\cdot\|_X$ of a Banach function space X is *continuous* if $x_n \in X$, $0 \leqslant x_n \downarrow 0$, imply $\|x_n\|_X \to 0$, *semi-continuous* if $0 \leqslant x_n \uparrow x$, $x \in X$, imply $\|x_n\|_X \to \|x\|_X$, *monotone complete* if $0 \leqslant x_n \uparrow x$ and $\sup_{n \geqslant 1} \|x_n\|_X < \infty$, imply $x \in X$. If the norm of X is semi-continuous and monotone complete, then we say that it has the *Fatou property*.

The *Köthe dual space* (or *associate space*) X' of X is defined by

$$X' = \{x' \in L^0 \colon \operatorname{supp} x' \subset \operatorname{supp} X,\ \langle |x|, |x'| \rangle < \infty \text{ for all } x \in X\},$$

where $\langle x, x' \rangle = \int_\Omega x x' \, d\mu$ for $(x, x') \in X \times X'$.

The space X' is a Banach function space on (Ω, μ) with the norm

$$\|x'\|_{X'} = \sup\{|\langle x, x' \rangle| \colon \|x\|_X \leqslant 1\},$$

whence follows the *Hölder inequality*

$$|\langle x, x' \rangle| \leqslant \|x\|_X \|x'\|_{X'}.$$

In the remainder we assume that $\operatorname{supp} X = \Omega$.

We note the useful remark that if X is a Banach function space, then by the super order density of any ideal Y with $\operatorname{supp} Y = \operatorname{supp} X$ in L^0 and Lebesgue dominated convergence theorem we get

$$\|x'\|_{X'} = \sup\{|\langle x, x' \rangle| \colon \|x\|_X \leqslant 1,\ x \in Y\}$$

for all $x' \in X'$.

For a given Banach couple $\boldsymbol{X} = (X_0, X_1)$ of Banach function spaces (X_0', X_1') is a Banach couple which we denote by $\boldsymbol{X}'$.

It is well known that if X is a Banach function space, then $\|x\|_{X''} = \|x\|_X$ for $x \in X$, when the norm $\|\cdot\|_X$ is semi-continuous. Moreover $X = X''$ and $\|x\|_X = \|x\|_{X''}$ if and only if the norm of X has the Fatou property (see [9, 19]). In particular the norm of X' has the Fatou property.

Let $\boldsymbol{A} = (A_0, A_1)$ and $\boldsymbol{B} = (B_0, B_1)$ be two Banach couples. We denote by $\mathfrak{L}(\boldsymbol{A}, \boldsymbol{B})$ the Banach space of all linear operators $T \colon A_0 + A_1 \to B_0 + B_1$ such that the restriction of T to the space A_i is a bounded operator from A_i into B_i, $i = 0, 1$, with the norm

$$\|T\|_{\mathcal{L}(\boldsymbol{A},\boldsymbol{B})} = \max\{\|T\|_{A_0\to B_0}, \|T\|_{A_1\to B_1}\}.$$

We say that intermediate spaces A and B (with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively) are *interpolation spaces* with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$ and we write $(A, B) \in Int(\boldsymbol{A}, \boldsymbol{B})$ if every operator from $\mathcal{L}(\boldsymbol{A}, \boldsymbol{B})$ maps A into B. It is a consequence of the closed graph theorem that, then the restriction of T to A is a bounded operator from A into B and

$$\|T\|_{A\to B} \leqslant C\|T\|_{\mathcal{L}(\boldsymbol{A},\boldsymbol{B})}$$

for some positive constant C independent of $T \in \mathcal{L}(\boldsymbol{A}, \boldsymbol{B})$.

If A coincides with B, then A is called an interpolation space with respect to $\boldsymbol{A}$ and $\boldsymbol{B}$ and we write $A \in Int(\boldsymbol{A}, \boldsymbol{B})$; if, moreover, $A_0 = B_0$ and $A_1 = B_1$, then A is called an interpolation space between A_0 and A_1 (or with respect to $\boldsymbol{A}$), and we write $A \in Int\,\boldsymbol{A}$.

In [2] ARONSZAJN and GAGLIARDO showed that if $\boldsymbol{A}$ is a regular Banach couple such that $A_0 \cap A_1$ is a reflexive space and $A \in Int\,\boldsymbol{A}$ with $A_0 \cap A_1$ dense in A, then A^* is an interpolation space between A_0^* and A_1^*. In this paper we investigate when $(Y', X') \in Int(\boldsymbol{Y}', \boldsymbol{X}')$ if we know that $(X, Y) \in Int(\boldsymbol{X}, \boldsymbol{Y})$, where X and Y are function Banach spaces intermediate with respect to given couples $\boldsymbol{X} = (X_0, X_1)$ and $\boldsymbol{Y} = (Y_0, Y_1)$ of Banach function spaces, respectively.

2. – Regular operators. Basic properties.

Let $X \subset L^0(\Omega_1, \mu_1)$ and $Y \subset L^0(\Omega_2, \mu_2)$ be Banach function spaces. We say that the operator $T: X \to Y$ is *regular* if there exists the operator $T': Y' \to X'$ (which we called the (*order*) *adjoint* of T) such that

$$\langle Tx, y'\rangle = \langle x, T'y'\rangle$$

for all $x \in X$ and $y' \in Y'$. Note that if the operator $T: X \to Y$ is regular, then T is linear, moreover if T is bounded, then by Hölder inequality we get that T' is a bounded operator and $\|T'\|_{Y'\to X'} \leqslant \|T\|_{X\to Y}$.

Now we give a useful theorem which characterizes regular operators. First of all, let X be a Banach function space. Let Γ denote the set of all linear continuous functionals defined on the space X by

$$\Gamma = \{f_{x'}: f_{x'}(x) = \langle x, x'\rangle,\ x' \in X'\}.$$

It is easy to see that Γ is a total linear set of the dual space X^*, so on the space X we can define the Γ-topology (which we denote by $\sigma(X, X')$) generated by the family of semi-norms $\{p_{x'}: x' \in X'\}$, where $p_{x'}(x) = |f_{x'}(x)|$.

THEOREM 2.1. – *Let X and Y be Banach function spaces. An operator $T: X \to Y$ is regular if and only if the following condition holds*

$$(1) \qquad x_n \to x \text{ in } \sigma(X, X'), \quad \text{imply} \quad Tx_n \to Tx \text{ in } \sigma(Y, Y').$$

PROOF. – Assume that an operator $T: X \to Y$ is regular. Then there exists an operator $T': Y' \to X'$ such that $\langle Tx, y'\rangle = \langle x, T'y'\rangle$ for all $x \in X$ and $y' \in Y'$. Let $x_n \to x$ in $\sigma(X, X')$, i.e. $\langle x_n, x'\rangle \to \langle x, x'\rangle$ for all $x' \in X'$. Thus

$$\langle Tx_n, y'\rangle = \langle x_n, T'y'\rangle \to \langle x, T'y'\rangle = \langle Tx, y'\rangle$$

for all $y' \in Y'$, whence $Tx_n \to Tx$ in $\sigma(Y, Y')$.

For the converse suppose that an operator $T: X \to Y$ satisfies the condition (1). Take a sequence (x_n) in X with $|x_n| \leqslant x$, $x \in X$ and $x_n \to 0$ a.e. Then $x_n \to 0$ in $\sigma(X, X')$, by Lebesgue dominated convergence theorem. Now, if we set $f_{y'}(x) = \langle Tx, y'\rangle$ for each $x \in X$, where $y' \in Y'$, we see that $f_{y'}(x_n) \to 0$ for each $y' \in Y'$ (by $Tx_n \to 0$ in $\sigma(Y, Y')$). It follows that $f_{y'}$ is an order continuous linear functional on X for each $y' \in Y'$. Therefore, there exists (exactly one) element $x' \in X'$ such that $f_{y'}(x) = \langle x, x'\rangle$ for all $x \in X$ (see [9]). Now it is enough to observe that if we set $T'y' = x'$, then the map $T': Y' \to X'$ is linear and $f_{y'}(x) = \langle x, T'y'\rangle$. Thus $\langle Tx, y'\rangle = \langle x, T'y'\rangle$ for all $x \in X$ and $y' \in Y'$, so the operator T is regular.

COROLLARY 2.2. – *Let X and Y be Banach function spaces. If the operator $T: X \to Y$ is regular and the norm of Y is semi-continuous, then T is bounded.*

PROOF. – Let $x_n \to x$ in X. Then $x_n \to x$ in $\sigma(X, X')$, by the Hölder inequality. Since the operator $T: X \to Y$ is regular, it follows that for each $y' \in Y'$

$$(2) \qquad |\langle Tx_n, y'\rangle| \to |\langle Tx, y'\rangle| \quad \text{as } n \to \infty,$$

by Theorem 2.1.

Now let $\varepsilon > 0$. By semi-continuity of the norm $\|\cdot\|_Y$, we have

$$\|Tx\|_Y = \sup\{|\langle Tx, y'\rangle| : \|y'\|_{Y'} \leqslant 1\}.$$

Then the inequality

$$(3) \qquad \|Tx\|_Y < |\langle Tx, y_0'\rangle| + \varepsilon/2$$

holds for some $y_0' \in Y'$ with $\|y_0'\|_{Y'} \leqslant 1$. Since there exists an $n_0 \in N$ with

$$|\langle Tx, y_0'\rangle| < |\langle Tx_n, y_0'\rangle| + \varepsilon/2$$

for all $n > n_0$, by (2), so from (3), we get

$$\|Tx\|_Y < |\langle Tx_n, y_0'\rangle| + \varepsilon \leqslant \|Tx_n\|_Y + \varepsilon$$

for all $n > n_0$. Since $\varepsilon > 0$ is arbitrary $\|Tx\|_Y \leqslant \liminf_{n\to\infty} \|Tx_n\|_Y$ holds. Consequently, since T is linear our assertion follows from a well known fact that the linear operator S from a Banach space E into a normed space F is continuous if and only if $x_n \to x$ in E imply $\|Sx\|_F \leqslant \liminf_{n\to\infty} \|Sx_n\|_F$.

REMARK. – If X is a Banach function space, then the mapping $x' \mapsto f_{x'}$, where $f_{x'}(x) = \langle x, x' \rangle$ for $x \in X$ is a linear isometry from the Köthe dual X' of X onto X^* if and only if X has continuous norm (see [9]). Hence, by applying Theorem 2.1, we obtain (cf. [8]) the following

THEOREM 2.3. – *Let X and Y be Banach function spaces. A linear bounded operator from X into Y is regular if and only if X has continuous norm.*

3. – Interpolation in the Köthe dual spaces.

Let X, Y be Banach function spaces. An operator $T: X \to Y$ is *positive* if $T(X^+) \subset Y^+$, where $X^+ = \{x \in X: x \geqslant 0\}$. Note that every additive operator T_0: $X^+ \to Y^+$ has a unique extension to a linear operator $T: X \to Y$. This extension is defined by the formula $T(x) = T_0(x^+) - T_0(x^-)$ (see [1], Theorem 1.7, p. 7). Note also that a positive linear operator $T: X \to Y$ is bounded (see [1], Theorem 12.3, p. 175).

Let $\mathbf{X} = (X_0, X_1)$ and $\mathbf{Y} = (Y_0, Y_1)$ be couples of Banach function spaces, then by $\mathfrak{L}_r(\mathbf{X}, \mathbf{Y})$ we denote the subspace of all operators $T \in \mathfrak{L}(\mathbf{X}, \mathbf{Y})$ such that the restrictions $T|_{X_i} := T_{X_i}$ are regular operators from X_i into Y_i, $i = 0, 1$. By $\mathfrak{L}_+(\mathbf{X}, \mathbf{Y})$, we denote the set of all operators $T \in \mathfrak{L}(\mathbf{X}, \mathbf{Y})$ such that $T_{X_i}: X_i \to Y_i$ $(i = 0, 1)$ are positive operators.

If X and Y are Banach function spaces intermediate with respect to $\mathbf{X}$ and $\mathbf{Y}$, respectively, then we write $(X, Y) \in Int_+(\mathbf{X}, \mathbf{Y})$ if $TX \subset Y$ for all $T \in \mathfrak{L}_+(\mathbf{X}, \mathbf{Y})$.

In this section we are interested in the problem, when $(Y', X') \in Int_+(\mathbf{Y}', \mathbf{X}')$ $(Int(\mathbf{Y}', \mathbf{X}'))$ if we know that $(X, Y) \in Int_+(\mathbf{X}, \mathbf{Y})(Int(\mathbf{X}, \mathbf{Y}))$. To answer this problem we need some propositions.

PROPOSITION 3.1. – *Let $\mathbf{X} = (X_0, X_1)$ and $\mathbf{Y} = (Y_0, Y_1)$ be couples of Banach function spaces. If an operator $T \in \mathfrak{L}_r(\mathbf{X}, \mathbf{Y})$, then there exists an operator $T' \in \mathfrak{L}(\mathbf{Y}', \mathbf{X}')$ such that $T'|_{Y_i'} = T'_{X_i}$, where T'_{X_i} is the adjoint of T_{X_i}, $i = 0, 1$.*

PROOF (cf. [2]). – Let $T \in \mathfrak{L}_r(\mathbf{X}, \mathbf{Y})$, then there exist operators $T'_{X_i}: Y_i' \to X_i'$ such that $\langle T_{X_i} x_i, y_i' \rangle = \langle x_i, T'_{X_i} y_i' \rangle$ for all $x_i \in X_i$ and $y_i' \in Y_i'$, $i = 0, 1$. Hence

$$\langle x, T'_{X_0} y' \rangle = \langle T_{X_0} x, y' \rangle = \langle T_{X_1} x, y' \rangle = \langle x, T'_{X_1} y' \rangle$$

for all $x \in X_0 \cap X_1$ and $y' \in Y_0' \cap Y_1'$. Thus $\langle x, T'_{X_0} y' - T'_{X_1} y' \rangle = 0$ for each $x \in X_0 \cap X_1$. This implies that $\|T'_{X_0} y' - T'_{X_1} y'\|_{(X_0 \cap X_1)'} = 0$ for all $y' \in Y_0' \cap Y_1'$, so

$T'_{X_0} = T'_{X_1}$ on $Y'_0 \cap Y'_1$. Now, if we define the operator $T'\colon Y'_0 + Y'_1 \to X'_0 + X'_1$, by $T'y' = T'_{X_0}y'_0 + T'_{X_1}y'_1$ for $y' = y'_0 + y'_1$ with $y'_i \in Y'_i$, $i = 0, 1$. Then simply $T'y'$ is independent of the choice of the decomposition $y' = y'_0 + y'_1$. Obviously $T'|_{Y'_i} = T'_{X_i}$ $(i = 0, 1)$ and $T' \in \mathfrak{L}(\mathbf{Y}', \mathbf{X}')$.

The following useful Proposition follows immediately from Theorem 3.1 in [12].

PROPOSITION 3.2. – *Let (X_0, X_1) be a couple of Banach function spaces, then $(X_0 + X_1)' = X'_0 \cap X'_1$ and $(X_0 \cap X_1)' = X'_0 + X'_1$ with equality of norms.*

In the remainder let $\mathbf{X} = (X_0, X_1)$ and $\mathbf{Y} = (Y_0, Y_1)$ be couples of Banach function spaces.

THEOREM 3.3. – *Let a couple $\mathbf{Y} = (Y_0, Y_1)$ be such that the norm of $Y'_0 + Y'_1$ is continuous.*

(a) If $Y''_i = Y_i$ $(i = 0, 1)$ and $R \in \mathfrak{L}_+(\mathbf{Y}', \mathbf{X}')$, then there exists an operator $T \in \mathfrak{L}_+(\mathbf{X}, \mathbf{Y})$ such that $T' = R$.

(b) If $X_0 \cap X_1$ is dense in X_i $(i = 0, 1)$, $Y''_0 \cap Y''_1 = Y_0 \cap Y_1$, the norms of Y_i are semi-continuous $(i = 0, 1)$ and $R \in \mathfrak{L}(\mathbf{Y}', \mathbf{X}')$, then there exists an operator $T \in \mathfrak{L}(\mathbf{X}\ \mathbf{Y})$ such that $T' = R$.

PROOF. – (*a*) Let an operator $R \in \mathfrak{L}_+(\mathbf{Y}'\ \mathbf{X}')$. Then $R\colon (Y_0 \cap Y_1)' \to (X_0 \cap X_1)'$, by Proposition 3.2. Since $Y'_0 + Y'_1 = (Y_0 \cap Y_1)'$ has a semi-continuous norm, it follows from Theorem 2.3 that the operator $R\colon (Y_0 \cap Y_1)' \to (X_0 \cap X_1)'$ is regular. Thus there exists a linear and bounded operator $S\colon (X_0 \cap X_1)'' \to (Y_0 \cap Y_1)'' = Y_0 \cap Y_1$ satisfying the equality $\langle Ry', x''\rangle = \langle y', Sx''\rangle$ for all $y' \in (Y_0 \cap Y_1)'$ and $x'' \in (X_0 \cap X_1)''$. Hence it follows that S is a positive operator and

$$\langle Ry', x\rangle = \langle y', Sx\rangle \tag{4}$$

for all $y' \in (Y_0 \cap Y_1)' = Y'_0 + Y'_1$ and $x \in X_0 \cap X_1$, by $X_0 \cap X_1 \subset (X_0 \cap X_1)''$. Thus, by the semi-continuity of the norms in Y_i $(i = 0, 1)$ and Hölder inequality we have

$$\|Sx\|_{Y_0} = \sup\{|\langle y'\ Sx\rangle| : \|y'\|_{Y'_0} \leqslant 1\} = \sup\{|\langle Ry'\ x\rangle| : \|y'\|_{Y'_0} \leqslant 1\} \leqslant$$
$$\leqslant \sup\{\|Ry'\|_{X'_0}\|x\|_{X_0} : \|y'\|_{Y'_0} \leqslant 1\} = \|R\|_{Y'_0 \to X'_0}\|x\|_{X_0}$$

and

$$\|Sx\|_{Y_1} \leqslant \|R\|_{Y'_1 \to X'_1}\|x\|_{X_1}$$

for all $x \in X_0 \cap X_1$.

Now let fix $x \in X_0$ with $x \geqslant 0$. Since $X_0 \cap X_1$ is an ideal, so there exists a sequence (x_n) in $X_0 \cap X_1$ such that $0 \leqslant x_n \uparrow x$. Thus $0 \leqslant Sx_n \uparrow$, by positivity of S. Since

$\|Sx_n\|_{Y_0} \leqslant \|S\|_{X_0\to Y_0}\|x_n\|_{X_0} \leqslant \|S\|_{X_0\to Y_0}\|x\|_{X_0}$, it follows from the Fatou property in Y_0 that

$$S_0 x := \lim_{n\to\infty} Sx_n \in Y_0$$

and

$$\|S_0 x\|_{Y_0} = \lim_{n\to\infty} \|Sx_n\|_{Y_0} \leqslant \|S\|_{X_0\to Y_0} \lim_{n\to\infty} \|x_n\|_{X_0} \leqslant \|S\|_{X_0\to Y_0}\|x\|_{X_0}.$$

We show that the operator $S_0: X_0^+ \to Y_0^+$ is additive. To see this we first show that $S_0 x = \lim_{n\to\infty} Sx_n$, where $x \in X_0^+$, is independent of the choice of the sequence $(x_n) \subset \subset X_0 \cap X_1$ such that $0 \leqslant x_n \uparrow x$. Let $x \in X_0^+$, $(x_n) \subset X_0 \cap X_1$ with $0 \leqslant x_n \uparrow x$. Then $|x_n Ry'| \leqslant |x|\,|Ry'|$, $|y' Sx_n| \leqslant |y'|\,|Sx|$ and $\langle |Ry'|, |x|\rangle \leqslant \|Ry'\|_{X_0'}\|x\|_{X_0} < \infty$, $\langle |y'|\ |Sx|\rangle \leqslant \|y'\|_{Y_0'}\|Sx\|_{Y_0} < \infty$ for all $y' \in Y_0'$, by Hölder inequality. Hence $\langle Ry', x_n\rangle \to \langle Ry', x\rangle$ and $\langle y', Sx_n\rangle \to \langle y', S_0 x\rangle$ as $n \to \infty$, by Lebesgue dominated convergence theorem. Finally if we put $S_0 x = y_0$, then

$$\langle Ry', x\rangle = \langle y', y_0\rangle = \langle y', \lim_{n\to\infty} Sx_n\rangle \tag{5}$$

for all $y' \in Y_0'$, by (4). Now if $0 \leqslant y_n \uparrow x$ with $(y_n) \subset X_0 \cap X_1$, then $\lim_{n\to\infty} Sy_n = \bar{y}_0 \in Y_0$ and $\langle Ry', x\rangle = \langle y', y_0\rangle$, $\langle Ry', x\rangle = \langle y', \bar{y}_0\rangle$, by (5). This implies that $\langle y', y_0 - \bar{y}_0\rangle = 0$ for all $y' \in Y_0'$ and consequently $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sy_n$. Hence, we obtain easily that the operator $S_0: X_0^+ \to Y_0^+$ is additive. Thus, S_0 has a unique extension to a linear operator $\bar{S}_0: X_0 \to Y_0$ defined by the formula $\bar{S}_0(x) = S_0(x^+) - S_0(x^-)$ for $x \in X_0$. Obviously $\bar{S}_0$ is bounded and $\|\bar{S}_0 x\|_{Y_0} \leqslant 2\|S\|_{X_0\to Y_0}\|x\|_{X_0}$. Moreover

$$\begin{aligned} \langle y_0', \bar{S}_0 x\rangle &= \langle y_0', S_0 x^+\rangle - \langle y_0', S_0 x^-\rangle = \\ &= \langle Ry_0', x^+\rangle - \langle Ry_0', x^-\rangle = \langle Ry_0', x^+ - x^-\rangle = \langle Ry_0', x\rangle \end{aligned} \tag{6}$$

for all $x \in X_0$ and $y_0' \in Y_0'$, by (5).

Similarly, we define a linear operator $\bar{S}_1: X_1 \to Y_1$ with $\|\bar{S}_1 x\|_{Y_1} \leqslant 2\|S\|_{X_1\to Y_1}\|x\|_X$ and

$$\langle y_1', \bar{S}_1 x\rangle = \langle Ry_1', x\rangle \tag{7}$$

for all $x \in X_1$ and $y_1' \in Y_1'$. Obviously $\bar{S}_0 x = \bar{S}_1 x$ for $x \in X_0 \cap X_1$. We simply set $Tx = \bar{S}_0 x_0 + \bar{S}_1 x_1$ for $x = x_0 + x_1$, where $x_i \in X_i$, $i = 0, 1$, and show that Tx is independent of the choice of the decomposition $x = x_0 + x_1$. Since $\bar{S}_i: X_i \to Y_i$ $(i = 0, 1)$ are positive operators, so $T \in \mathfrak{L}_+(\mathbf{X}, \mathbf{Y})$. By (6) and (7), we see that $T \in \mathfrak{L}_r(\mathbf{X}, \mathbf{Y})$ and $T'_{X_0} = \bar{S}_0' = R_{Y_0'}$, $T'_{X_1} = \bar{S}_1' = R_{Y_1'}$. From Proposition 3.1, we get that there exists an operator $T' \in \mathfrak{L}(\mathbf{Y}', \mathbf{X}')$, such that $T'y' = T'_{X_0} y_0' + T'_{X_1} y_1'$ for $y' = y_0' + y_1'$ with $y_i' \in Y_i'$ $(i = 0, 1)$. To finish the proof note that $T'y' = R_{Y_0'} y_0' + R_{Y_1'} y_1' = R(y_0' + y_1') = Ry'$ for all $y' \in Y_0' + Y_1'$. Consequently $T' = R$.

(*b*) Modyfing the proof of (*a*), we obtain easily the proof of (*b*).

PROPOSITION 3.4. – *Let X and Y be Banach function spaces intermediate with respect to $\mathbf{X}$ and $\mathbf{Y}$, respectively. If $(X, Y) \in Int(\mathbf{X}, \mathbf{Y})$ and $T \in \mathfrak{L}_r(\mathbf{X}, \mathbf{Y})$, then $T'|_{Y'}$ is a bounded linear operator from Y' into X'.*

PROOF. – Let $x \in X_0 \cap X_1$ and $y' = y'_0 + y'_1$, with $y'_0 \in Y'_0$, $y'_1 \in Y'_1$. Then by the construction of the operator $T' \in \mathfrak{L}(\mathbf{Y}', \mathbf{X}')$ (see Proposition 3.1) we have

$$\langle T_X x, y'\rangle = \langle T_X x, y'_0 + y'_1\rangle = \langle T_X x, y'_0\rangle + \langle T_X x, y'_1\rangle = \langle T_{X_0} x, y'_0\rangle + \langle T_{X_1} x, y'_1\rangle =$$

$$= \langle x, T'_{X_0} y'_0\rangle + \langle x, T'_{X_1} y'_1\rangle = \langle x, T'_{X_0} y'_0 + T'_{X_1} y'_1\rangle = \langle x, T' y'\rangle .$$

Thus $\langle T_X x, y'\rangle = \langle x, T'|_{Y'} y'\rangle$ for all $x \in X_0 \cap X_1$ and $y' \in Y'$, by $Y' \subset Y'_0 + Y'_1$. Since $X_0 \cap X_1$ is an ideal it follows from the Hölder inequality that

$$\|T'y'\|_{X'} = \sup \{|\langle x, T'y'\rangle|: \|x\|_X \leqslant 1, x \in X_0 \cap X_1\} =$$

$$= \sup \{|\langle T_X x, y'\rangle|: \|x\|_X \leqslant 1, x \in X_0 \cap X_1\} \leqslant \|T\|_{X \to Y} \|y'\|_{Y'} .$$

THEOREM 3.5. – *Let a Banach couple $\mathbf{Y} = (Y_0, Y_1)$ be such that the norm of $Y'_0 + Y'_1$ is continuous.*

(*a*) *If $Y''_i = Y_i$ $(i = 0, 1)$ and $(X, Y) \in Int_+(\mathbf{X}, \mathbf{Y})$, then $(Y', X') \in Int_+(\mathbf{Y}', \mathbf{X}')$.*

(*b*) *If $X_0 \cap X_1$ is dense in X_i $(i = 0, 1)$, $Y''_0 \cap Y''_1 = Y_0 \cap Y_1$, the norms of Y_i are semi-continuous $(i = 0, 1)$ and $(X, Y) \in Int(\mathbf{X}, \mathbf{Y})$, then $(Y', X') \in Int(\mathbf{Y}', \mathbf{X}')$.*

The proof is clear by virtue of Theorem 3.3 and Proposition 3.4.

COROLLARY 3.6. – *Let $X''_i = X_i$, $Y''_i = Y_i$ $i = 0, 1$ and let the norms of $X_0 + X_1$ and $Y'_0 + Y'_1$ be continuous.*

(*a*) *If $(X, Y) \in Int(\mathbf{X}, \mathbf{Y})$, then $(X'', Y'') \in Int(\mathbf{X}, \mathbf{Y})$.*

(*b*) *If $X'' = X$ and $Y'' = Y$, then $(X, Y) \in Int(\mathbf{X}, \mathbf{Y})$ if and only if $(Y', X') \in Int(\mathbf{Y}', \mathbf{X}')$.*

PROOF. – Since $X_0 \cap X_1$ is an ideal and the norm of $X_0 + X_1$ is continuous, so $X_0 \cap X_1$ is dense in $X_0 + X_1$. In consequence $X_0 \cap X_1$ is dense in both X_0 and X_1 (see [2]). Similarly we get that $Y'_0 \cap Y'_1$ is dense in Y'_i $(i = 0, 1)$. Thus, Corollary follows easily from Proposition 3.2 and Theorem 3.5 (*b*).

REMARK. – In general $X' \in Int(X'_0, X'_1)$ does not imply $X \in Int(X_0, X_1)$. Namely, let $L^1 = L^1(0, \infty)$, $L^\infty = L^\infty(0, \infty)$. Russu has given an example of a symmetric space X on $(0, \infty)$ such that $X \notin Int(L^1, L^\infty)$ (see [10], Theorem 5.11). Since the norm of a symmetric space X' has the Fatou property, so $X' \in Int(L^\infty, L^1) = Int((L^1)', (L^\infty)')$ (see [10], Theorem 4.9, p. 142).

In the remainder we need the following

PROPOSITION 3.7. – *Let X be a Banach function space in $L^0(\Omega, \mu)$. If $Y \subset X$ is a linear ideal in L^0 with supp $Y =$ supp $X = \Omega$, then $(\overline{Y}^X)' = X'$ with equality of norms.*

PROOF. – Let $E = \overline{Y}^X$. Obviously E is a Banach function space and $X' \subset E'$ with $\|x'\|_{E'} \leqslant \|x'\|_{X'}$ for all $x' \in X'$. Now let $x' \in E'$, then

$$(8) \qquad \langle |y|, |x'| \rangle < \infty$$

for all $y \in E$. Let $x \in X$. Since Y is a linear ideal in L^0, so there exists a sequence $(y_n) \subset Y$ such that $0 \leqslant y_n \uparrow |x|$ μ-a.e. Hence by (8)

$$\int_\Omega |x'| y_n \, d\mu \leqslant \|x'\|_{E'} \|y_n\|_E = \|x'\|_{E'} \|y_n\|_X \leqslant \|x'\|_{E'} \|x\|_X$$

and consequently, by Levy's Lemma $\langle |x|, |x'| \rangle \leqslant \|x'\|_{E'} \|x\|_X < \infty$. Thus $x' \in X'$ and $\|x'\|_{X'} \leqslant \|x'\|_{E'}$ and the proof is finished.

From Theorem 3.5 and Proposition 3.7 we obtain the following

COROLLARY 3.8. – *Let a couple $\mathbf{Y} = (Y_0, Y_1)$ be such that the norm of $Y_0' + Y_1'$ is continuous, $Y_0'' \cap Y_1'' = Y_0 \cap Y_1$ and the norms of Y_i are semi-continuous $(i = 0, 1)$. If $(X, Y) \in Int(\mathbf{X}, \mathbf{Y})$ implies $(X^0, Y^0) \in Int(\mathbf{X}^0, \mathbf{Y}^0)$, then $(Y', X') \in Int(\mathbf{Y}', \mathbf{X}')$.*

The next result is a consequence of Corollary 3.6 and Ogasawara's Theorem (see [1], Theorem 14.22, p. 240) from which it follows that a Banach function space E is reflexive if and only if the norms of E and E' are continuous and $E'' = E$.

COROLLARY 3.9. – *Let X_0 and X_1 be reflexive Banach function spaces. Then $X \in Int(X_0, X_1)$ if and only if $X' \in Int(X_0', X_1')$.*

4. – Interpolation in special Banach function spaces.

In this section we give applications of our results to concrete function Banach spaces. Let $\mathbf{A}$ be a couple of Banach spaces. Denote by L^∞, respectively $L^\infty_{1/s}$, the space of all measurable functions x on $\mathbb{R}_+$ such that $|x(s)|$, respectively $|x(s)|/s$, is essentially bounded. Put $\mathbf{L}^\infty = (L^\infty, L^\infty_{1/s})$. For any Banach function space Φ intermediate with respect to $\mathbf{L}^\infty$, the *real interpolation space* (or *K space*) $\mathbf{A}_\Phi$ is defined to consist of all $a \in A_0 + A_1$ such that $K(\cdot, a; \mathbf{A}) \in \Phi$, with the norm $\|a\|_{\mathbf{A}_\Phi} = \|K(\cdot, a; \mathbf{A})\|_\Phi$, where for $a \in A_0 + A_1$ and $t > 0$

$$K(t, a; \mathbf{A}) = \inf \{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \colon a = a_0 + a_1, \ a_0 \in A_0, \ a_1 \in A_1\}$$

is the K-functional of Peetre.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two Banach couples. We say that $\boldsymbol{A}$ has the *Calderón property* relative to $\boldsymbol{B}$ if the condition

$$K(t, b; \boldsymbol{B}) \leqslant K(t, a; \boldsymbol{A}) \quad \text{for all } t > 0$$

implies the existence of an operator $T \in \mathfrak{L}(\boldsymbol{A}, \boldsymbol{B})$ (whose norm depends only on the couples $\boldsymbol{A}$ and $\boldsymbol{B}$) such that $Ta = b$.

THEOREM 4.1. – *Let $\boldsymbol{X} = (X_0, X_1)$ and $\boldsymbol{Y} = (Y_0, Y_1)$ be Banach function spaces and let $\boldsymbol{Y}$ be such that $Y_0' + Y_1'$ has continuous norm, $Y_0'' \cap Y_1'' = Y_0 \cap Y_1$ and the norms of Y_i are semi-continuous ($i = 0, 1$). If $E \subset \boldsymbol{X}_\Phi$ and $F \supset \boldsymbol{Y}_\Phi$ are Banach function spaces intermediate with respect to $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, then $(F', E') \in Int(\boldsymbol{Y}', \boldsymbol{X}')$. In particular, if $\boldsymbol{X}$ has the Calderón property relative to $\boldsymbol{Y}$ and $(X, Y) \in Int(\boldsymbol{X}, \boldsymbol{Y})$, then $(Y', X') \in Int(\boldsymbol{Y}', \boldsymbol{X}')$.*

PROOF. – First we note that $(E, F) \in Int(\boldsymbol{X}, \boldsymbol{Y})$. Since $K(t, a; A_0^0, A_1^0) = K(t, a; A_0, A_1)$ for all $a \in (A_0 + A_1)^0 = A_0^0 + A_1^0$, so $E^0 \subset (X_0, X_1)_\Phi^0 = (X_0^0, X_1^0)_\Phi^0$ and $F^0 \supset (Y_0, Y_1)_\Phi^0 = (Y_0^0, Y_1^0)_\Phi^0$. Hence $(E^0, F^0) \in Int(\boldsymbol{X}^0, \boldsymbol{Y}^0)$ and consequently $(F', E') \in Int(\boldsymbol{Y}', \boldsymbol{X}')$, by Corollary 3.9.

If $\boldsymbol{X}$ has the Calderón property relative to $\boldsymbol{Y}$ and $(X, Y) \in Int(\boldsymbol{X}, \boldsymbol{Y})$, then there exists a Banach function space $\Phi \in Int\, \boldsymbol{L}^\infty$ such that $X \subset \boldsymbol{X}_\Phi$ and $Y \supset \boldsymbol{Y}_\Phi$ (see [4, 15]). Thus the proof of the theorem is complete.

It is well known (see [5, 7, 16, 18]) that the couple (L^{p_1}, L^{p_0}) has the Calderón property relative to (L^{q_0}, L^{q_1}), where $1 \leqslant p_0 \leqslant q_0 \leqslant \infty$ and $1 \leqslant p_1 \leqslant q_1 \leqslant \infty$. Thus, by Theorem 4.1 we get the following

COROLLARY 4.2. – *Let $\boldsymbol{X} = (L^{p_0}, L^{p_1})$ and $\boldsymbol{Y} = (L^{q_0}, L^{q_0})$, $1 \leqslant p_0 \leqslant q_0 < \infty$, $1 \leqslant p_1 \leqslant q_1 < \infty$, $q_0, q_1 \neq 1$. If $(X, Y) \in Int(\boldsymbol{X}, \boldsymbol{Y})$, then $(Y', X') \in Int\left((L^{q_0'}, L^{q_1'}), (L^{p_0'}, L^{p_1'})\right)$, where $1/p_i + 1/p_i' = 1/q_i + 1/q_i' = 1$ ($i = 0, 1$).*

If a Banach function space $\Phi \in Int\, \boldsymbol{L}^\infty$ satisfies some conditions, then it is possible to show that for any couples of Banach function spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ we have $(Y', X') \in Int(\boldsymbol{Y}', \boldsymbol{X}')$, where $X \subset \boldsymbol{X}_\Phi$ and $Y \supset \boldsymbol{Y}_\Phi$ are Banach function spaces intermediate with respect to $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Namely, let Φ^1 be the dual space to Φ under bilinear form

$$(f, g) = \int_{\mathbb{R}_+} f(t)\, g\left(\frac{1}{t}\right) \frac{dt}{t}.$$

By J_Ψ we denote the *J-method* of interpolation (see [3, 4, 6], for more details). Thus the above assertion follows from the following theorem (see [14])

THEOREM 4.3. – *Assume that* (X_0, X_1) *is a couple of Banach function spaces. If a Banach function space* $\Phi \in Int\, \mathbf{L}^\infty$ *is such that* $\Phi \cap L^\infty \neq L^\infty \cap L^\infty_{1/s}$ *and* $\Phi \cap L^\infty_{1/s} \neq \neq L^\infty \cap L^\infty_{1/s}$, *then* $(X_0, X_1)'_\Phi = J_{\Phi^1}(X'_0, X'_1)$.

Before proving the next result we recall that the *Marcinkiewicz space* M_φ on the interval $I = (0, l)$, $0 < l \leqslant \infty$, with Lebesgue measure, is defined by

$$M_\varphi = \left\{x \in L^0 \colon \|x\|_\varphi = \sup_{0<t<l} \left(\frac{1}{\varphi(t)} \int_0^t x^*(s)\, ds\right) < \infty\right\},$$

where the function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is *quasi-concave* ($\varphi(s) \leqslant \max\{1, s/t\}\varphi(t)$ for all $s, t \in \mathbb{R}_+$) and x^* is the nonincreasing rearrangement of the function x. It is well known (see [10]) that if $\varphi(0_+) = 0$, then the Köthe dual of M_φ is the *Lorentz space* defined by

$$\Lambda_\varphi = \left\{x \in L^0 \colon \|x\|_{\Lambda_\varphi} = \int_0^l x^*(s)\, d\varphi(s) < \infty\right\},$$

moreover $M''_\varphi = M_\varphi$. It is easy to verify that if $l < \infty$ ($l = \infty$), then Λ_φ has continuous norm if and only if $\varphi(0+) = 0$ ($\varphi(0+) = 0$ and $\varphi(\infty) = \infty$). In what follows we assume that $\varphi(0+) = 0$ and $\varphi(\infty) = \infty$ if $l = \infty$. Now we apply the Theorem 4.1 to obtain the following result (cf. [11, 17]).

THEOREM 4.4. – $(\Lambda_\varphi, \Lambda_\psi) \in Int\, ((\Lambda_{\varphi_0}, \Lambda_{\varphi_1}), (\Lambda_{\psi_0}, \Lambda_{\psi_1}))$ *if and only if there exists a constant* $c > 0$ *such that for all* $s, t \in I$

$$\frac{\psi(t)}{\varphi(s)} \leqslant c \max\left\{\frac{\psi_0(t)}{\varphi_0(s)}, \frac{\psi_1(t)}{\varphi_1(s)}\right\}. \tag{9}$$

PROOF. – The inequality (9) is equivalent to the following condition (see [11, 13]): $\psi(t) \leqslant \psi_0(t) f(\psi_1(t)/\psi_0(t))$, $\varphi_0(t) f(\varphi_1(t)/\varphi_0(t)) \leqslant c\varphi(t)$ for some quasi-concave function f, $c > 0$ and all $t \in I$. Hence, applying the reiteration theorem (see [4]), we obtain

$$M_\varphi \subset M_{\varphi_0 f(\varphi_1/\varphi_0)} = (M_{\varphi_0}, M_{\varphi_1})_\Phi$$

and

$$M_\psi \supset M_{\psi_0 f(\psi_1/\psi_0)} = (M_{\psi_0}, M_{\psi_1})_\Phi\,,$$

where $\Phi = L^\infty_{1/f}$. Since $\Lambda_{\psi_0} + \Lambda_{\psi_1} = \Lambda_{\min(\psi_0, \psi_1)}$, we have $M'_{\psi_0} + M'_{\psi_1} = \Lambda_{\min(\psi_0, \psi_1)}$ and so Theorem 4.1 applies. A necessary condition (9) is well known to be a necessary condition for the interpolation of symmetric spaces (see [11]).

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