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INTERPOLATION OF MARKOFF TRANSFORMATIONS ON THE FRICKE SURFACE

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Abstract. By the Fricke surfaces, we mean the cubic surfaces defined by the equation $p^2+q^2+r^2-pqr-k=0$ in the Euclidean 3-space with the coordinates (p,q,r) parametrized by constant k. When $k=0$, it is naturally isomorphic to the moduli of once-punctured tori. It was Markoff who found the transformations, called Markoff transformations, acting on the Fricke surface. The transformation is typically given by $(p,q,r)\mapsto(r,q,rq-p)$ acting on $\boldsymbol{R}^3$ that keeps the surface invariant. In this paper we propose a way of interpolating the action of Markoff transformation. As a result, we show that one portion of the Fricke surface with $k=4$ admits a $\mathrm{GL}(2,\boldsymbol{R})$-action extending the Markoff transformations.

Introduction. By the Fricke surface, we mean the space defined by the equation

$$p^2+q^2+r^2-pqr-k=0 \tag{0.1}$$

in the Euclidean 3-space $\boldsymbol{R}^3$ with the coordinates (p,q,r). It has attracted interest innumerably often for over a century. When $k=0$, it is naturally isomorphic to the moduli of once-punctured tori, first considered by Fricke and Klein [2] and often called the Fricke moduli. We refer to the papers [5] and [6] for the natural isomorphism.

It was Markoff who found the transformations, called Markoff transformations, acting on the Fricke surface in relation with the theory of quadratic forms. The transformation, typically given by

$$T:(p,q,r)\mapsto(r,q,rq-p)$$

acting on $\boldsymbol{R}^3$, keeps the surface invariant. As we know that quite a few contributions were made and are still in progress, in this paper we propose a way of interpolating the action of Markoff transformation. As a result, we show that the space $\{(p,q,r)\ ;\ p^2+q^2+r^2-pqr-4=0, p>2, q>2, r>2\}$ admits a $\mathrm{GL}(2,\boldsymbol{R})$-action extending the Markoff transformations.

To give a more precise statement, we first consider the n-times composition of the transformation T:

$$T^n:(p,q,r)\mapsto(b_{n-1}(q)r-b_{n-2}(q)p,\, q,\, b_n(q)r-b_{n-1}(q)p)\,,$$

where $b_n(q)$ is an n-th Chebyshev polynomial. By defining the functions $b_t(q)$ with a continuous parameter t, which interpolate the sequence of Chebyshev polynomials, we define a

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transformation

$$T_t : (p, q, r) \mapsto (b_{t-1}(q)r - b_{t-2}(q)p, q, b_t(q)r - b_{t-1}(q)p)$$

in Section 2. It gives rise to a one-parameter group and its action on the Fricke space is also given in Section 2. By using the symmetry amongst the letters p, q and r, we define two similar one-parameter groups and thus have a group G generated by these one-parameter groups.

Next, in Section 3, we compute the algebra generated by the infinitesimal automorphisms of the one-parameter groups above. It is shown that only when $k = 4$ is the algebra finite dimensional and isomorphic to the Lie algebra $\mathrm{sl}(2, \boldsymbol{R})$. A specific role of the case $k = 4$ is also clarified in [3], where the Markoff transformations are investigated as a dynamical system on the surface.

In Section 4, we treat the case $k = 4$ and, by introducing an affine coordinate system on the surface, we show explicitly that the group $\mathrm{GL}(2, \boldsymbol{R})$ includes the group G and acts on the space $\{(p, q, r)\ ;\ p^2 + q^2 + r^2 - pqr - 4 = 0, p > 2, q > 2, r > 2\}$.

1. Fricke surfaces. We define a function on $\boldsymbol{R}^3$ with coordinates (p, q, r) by

$$\varphi(p, q, r) = \varphi_{(k)}(p, q, r) = p^2 + q^2 + r^2 - pqr - k\,,$$

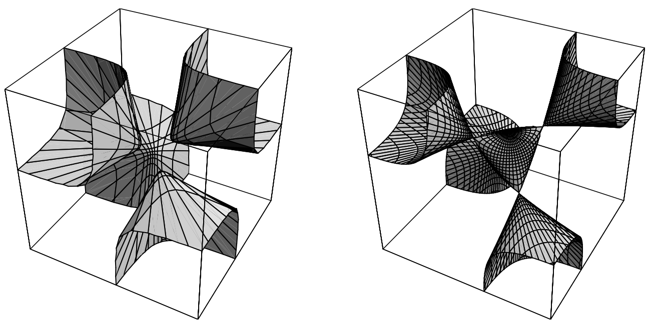
where k is a real parameter and define a surface by

$$V_k = \{(p, q, r) \in \boldsymbol{R}^3\ ;\ \varphi_{(k)} = 0\}\,,$$

called simply the Fricke surface with parameter k. We refer to the book by Fricke and Klein [2] and the articles [5] and [6].

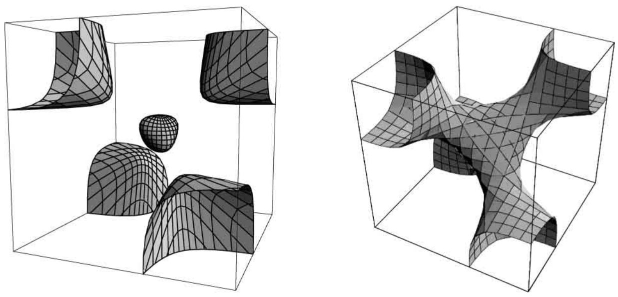
The shape of the surface depends on the parameter. To have an intuitive image, we first present four pictures; refer to Goldman [3] for further pictures.

Figure 1(a), where $k = 0$, consists of four portions (and the origin) and each is asymptotic to the hyperplanes $\{p = \pm 2\}$, $\{q = \pm 2\}$ and $\{r = \pm 2\}$ at infinity; the portion in the first



(a) (b)

FIGURE 1. Fricke surfaces with (a) $k = 0$ and (b) $k = 4$.



(a) (b)

FIGURE 2. Fricke surfaces with (a) $k = 2$ and (b) $k = 8$.

octant is asymptotic to the hyperplanes $\{p = 2\}$, $\{q = 2\}$ and $\{r = 2\}$ and so on. Figure 1(b), where $k = 4$, consists of five portions and they are touching each other at the points $(2, 2, 2)$, $(-2, -2, 2)$, $(2, -2, -2)$ and $(-2, 2, -2)$.

In the case $k < 0$, the surface looks like that with $k = 0$ with the origin excluded. In the case $0 < k < 4$, it looks like Figure 2(a) where $k = 2$. In the case $k > 4$, the surface has only one component and is similar to Figure 2(b) where $k = 8$.

As is seen from the defining equation or from the figures, the surface has an apparent symmetry $K \times \mathcal{S}_3$, where $\mathcal{S}_3$ is the group of permutations of the coordinates p, q and r and $K \cong (\mathbf{Z}_2)^{\times 2}$ is the Klein four-group generated by sign changes $(p, q, r) \to (p, -q, -r)$ and $(p, q, r) \to (-p, q, -r)$; it intertwines the four non-compact components when $k \leq 4$.

To have an extrinsic view more closely, we compute the second fundamental form of the surface. We set

$$\varphi_p = 2p - qr\,, \quad \varphi_q = 2q - pr\,, \quad \varphi_r = 2r - pq\,. \tag{1.1}$$

If we regard the surface as a covering of the pq-plane, the third coordinate r is a function of (p, q) as long as $\varphi_r \neq 0$. Then the first derivatives of r relative to (p, q) are

$$r_p = -\frac{\varphi_p}{\varphi_r}\,, \quad r_q = -\frac{\varphi_q}{\varphi_r}\,.$$

Then the second derivatives are

$$\begin{aligned} r_{pp} &= -2(\varphi_r^2 + q\varphi_p\varphi_r + \varphi_p^2)/\varphi_r^3 = -2(q^2 - 4)(q^2 - k)/\varphi_r^3\,, \\ r_{pq} &= -(-r\varphi_r^2 + p\varphi_p\varphi_r + q\varphi_q\varphi_r + 2\varphi_p\varphi_q)/\varphi_r^3 \\ &= \{r(p^2 - 4)(q^2 - 4) - 2(2r + pq)(4 - k)\}/\varphi_r^3\,, \\ r_{qq} &= -2(\varphi_r^2 + p\varphi_q\varphi_r + \varphi_q^2)/\varphi_r^3 = -2(p^2 - 4)(p^2 - k)/\varphi_r^3\,. \end{aligned}$$

From this formula, a computation when $k = 0$ shows that

$$\varphi_r^4(r_{pp}r_{qq} - r_{pq}^2) = -(p^2-4)(q^2-4)(r^2-4) + 32(p^2+q^2+r^2) - 64\,,$$

which implies that the surface cannot be convex; this is not obvious as seen from Figure 1. When $k = 4$, the determinant simplifies to

$$r_{pp}r_{qq} - r_{pq}^2 = -(r^2-4)/((p^2-4)(q^2-4))\,.$$

This expression and the similar expressions obtained by changing the role of p, q and r imply that the surface is *concave* where $|p| > 2$, $|q| > 2$ and $|r| > 2$, *convex* where $|p| < 2$, $|q| < 2$ and $|r| < 2$ and *degenerate* along six lines $p = \pm 2$, $q = \pm 2$ and $r = \pm 2$.

2. One-parameter groups acting on the surface V_k.

2.1. Definition of one-parameter transformations. Any Fricke surface has a set of simple automorphisms, called the Markoff transformations, which are defined as

$$\begin{aligned} T &: (p,q,r) \mapsto (r,q,rq-p)\,,\\ R &: (p,q,r) \mapsto (p,r,pr-q)\,,\\ S &: (p,q,r) \mapsto (q,qr-p,r)\,. \end{aligned}$$

They satisfy the relation $S = T^{-1}RT$. Let $G_{\mathbf{Z}}$ be the group generated by T, R and S. The function $\varphi_{(k)}$ turns out to be invariant under $G_{\mathbf{Z}}$ irrespective of the value of k; thus, it is a group of automorphisms of the surface V_k. Note that $G_{\mathbf{Z}}$ does not commute with the Klein four-group K and that it does not generally preserve the connected components of V_k; however, the component in the box $p \geq 2$, $q \geq 2$ and $r \geq 2$ (when it exists) is invariant and so is the component in the cube $|p| \leq 2$, $|q| \leq 2$ and $|r| \leq 2$ (when it exists). Refer to [5].

It is well known and easy to show that the correspondence

$$T \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

gives an isomorphism between $G_{\mathbf{Z}}$ and $\mathrm{PSL}_2(\mathbf{Z})$. We refer to [1, 3, 4]. We are curious about a possible continuous group of automorphisms of V_k having $G_{\mathbf{Z}}$ as a subgroup. We start by recalling one of the Chebyshev polynomials which we denote by $b_n(q)$. They are determined by the difference equation

$$\begin{pmatrix} b_n \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n-1} \\ b_{n-2} \end{pmatrix}$$

with the initial conditions $b_0(q) = 1$ and $b_1(q) = q$. This polynomial coincides with $S_n(q)$ in [7] and is equal to the hypergeometric polynomial $(n+1)F(n+2, -n, 3/2; (2-q)/4)$. Induction on n leads to the following lemma.

LEMMA 2.1. *Let $T^n = T(T^{n-1})$ denote the n-times composition of T. Then it is given by*

$$T^n : (p,q,r) \mapsto (b_{n-1}(q)r - b_{n-2}(q)p,\, q,\, b_n(q)r - b_{n-1}(q)p)\,.$$

Similarly, for R and S,

$$R^n : (p, q, r) \mapsto (p, b_{n-1}(p)r - b_{n-2}(p)q, b_n(p)r - b_{n-1}(p)q)\,,$$
$$S^n : (p, q, r) \mapsto (b_{n-1}(r)q - b_{n-2}(r)p, b_n(r)q - b_{n-1}(r)p, r)\,.$$

We next define the function $b_t(q)$ for a continuous parameter t so that it satisfies the equation

$$\begin{pmatrix} b_t \\ b_{t-1} \end{pmatrix} = \begin{pmatrix} q & -1 \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and that it coincides with $b_n(q)$ when $t = n$ is an integer. Such a function is uniquely determined and an explicit expression is given by

$$b_t(q) = \frac{1}{\sqrt{q^2-4}}(\sigma_+^{t+1} - \sigma_-^{t+1})\,, \tag{2.1}$$

where

$$\sigma_+ = \frac{q + \sqrt{q^2-4}}{2} \quad \text{and} \quad \sigma_- = \frac{q - \sqrt{q^2-4}}{2}\,.$$

When $q^2-4 < 0$, we interpret $\sqrt{q^2-4}$ as $i\sqrt{4-q^2}$. In terms of the hypergeometric function, we have $b_t(q) = (t+1)F(t+2, -t, 3/2; (2-q)/4)$. It has the properties given in the following lemma that can be verified by the use of (2.1).

LEMMA 2.2. *We have the following properties*:
(1) $b_{t+1} + b_{t-1} = qb_t$;
(2) $b_{s+t} = b_s b_t - b_{s-1}b_{t-1}$;
(3) $(b_{t-1})^2 - b_t b_{t-2} = 1$;
(4) $(b_t)^2 + (b_{t-1})^2 - qb_t b_{t-1} = 1$;
(5) $b_t|_{q=2} = t+1$.

Using the function b_t, we define a one-dimensional continuous group by the action

$$T_t : (p, q, r) \mapsto (b_{t-1}(q)r - b_{t-2}(q)p, q, b_t(q)r - b_{t-1}(q)p)\,.$$

Property (2) of Lemma 2.2 implies that T_t form a one-parameter family of automorphisms, i.e., $T_{t+s} = T_t \circ T_s$ and Property (3) assures that T_t preserve the surface V_k for any fixed k. Similarly, we define R_t and S_t by

$$R_t : (p, q, r) \mapsto (p, b_{t-1}(p)r - b_{t-2}(p)q, b_t(p)r - b_{t-1}(p)q)\,,$$
$$S_t : (p, q, r) \mapsto (b_{t-1}(r)q - b_{t-2}(r)p, b_t(r)q - b_{t-1}(r)p, r)\,.$$

They have similar properties as those of T_t. Furthermore, by Property (1) of Lemma 2.2, we see that $T^{-1}R_t T = S_t$.

In the following, we denote by G the group generated by T_t and R_t (so, also by S_t).

Now, an important remark is in order. The function $b_t(q)$ is defined for $q > -2$ and is real-valued; it is singular at $q = -2$. Hence, we need to restrict our consideration of the automorphisms above to the part of the surface lying in the set $\{(p, q, r) \in \boldsymbol{R}^3\,;\ p > -2,\ q >$

$-2,\ r > -2\}$; this part will be denoted by V. We will see in the following section that the surface V is invariant under these automorphisms.

2.2. Decomposition of $\boldsymbol{V}$ into $\boldsymbol{G}$-orbits. In order to see the action of G on V, we study the section of V by a plane parallel to any one of the coordinate planes. Such a section is generally a quadratic curve and the one-parameter subgroups $\{T_t\}$, $\{R_t\}$ and $\{S_t\}$ preserve the coordinate function $q,\ p$ and r, respectively. To have an explicit description, we consider T_t in some detail. It defines a motion on the quadratic curve for general fixed value of q.

First, we take care of the case $q = 2$. In this case, $(r-p)^2 = k-4$. Since we have no such points when $k < 4$, assume $k \geq 4$. Then, $r = p \pm \sqrt{k-4}$. Since the image of $(p, 2, r)$ is $(\bar{p}, 2, \bar{r}) := (tr - (t-1)p, 2, (t+1)r - tp)$ in view of Property (5) of Lemma 2.2, we must have $\bar{p} - p = t(r-p)$, $\bar{r} - r = t(r-p)$ and $\bar{r} - \bar{p} = r - p$. Hence, if $r \neq p$, which is possible only when $k > 4$, the point $(\bar{p}, 2, \bar{r})$ runs on the two lines defined by $r = p \pm \sqrt{k-4}$ as t varies. When $k = 4$, we have $r = p$ and the point $(p, 2, p)$ is fixed under T_t, i.e., the whole line $(p, 2, p)$ is pointwise fixed.

We next consider the case where $q > 2$ and introduce new coordinates (P, R) on the pr-plane by

$$P = (p-r)\sqrt{q+2}/2 \quad \text{and} \quad R = (p+r)\sqrt{q-2}/2\,.$$

Then $(p, q, r) \in V$ if and only if (P, R) is on the hyperbola

$$P^2 - R^2 = k - q^2\,.$$

We assume further that $q^2 > k$ for the moment and introduce a parameter ρ on the hyperbola by

$$P = \sqrt{q^2-k}\sinh\rho \quad \text{and} \quad R = \sqrt{q^2-k}\cosh\rho\,.$$

Let $(\bar{p}, q, \bar{r})$ be the image of (p, q, r) under T_t and $\bar{P}$ and $\bar{R}$ the corresponding values of P and R. We then define the value α depending on t by

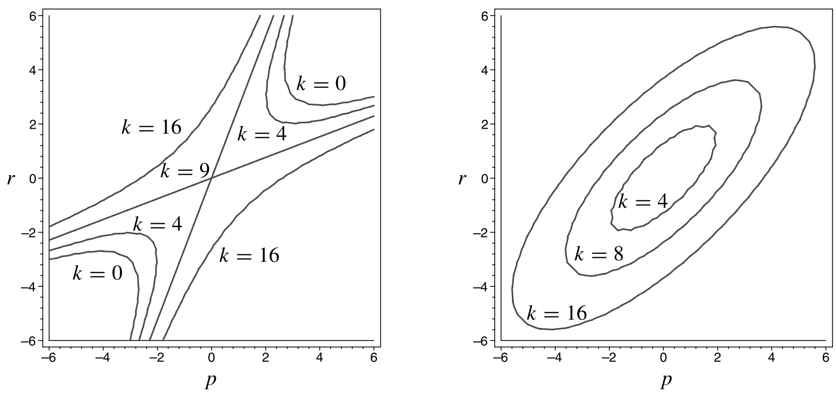
$$\cosh\alpha = \frac{qb_{t-1}(q) - 2b_{t-2}(q)}{2} \quad \text{and} \quad \sinh\alpha = -\frac{1}{2}\sqrt{q^2-4}\,b_{t-1}(q)\,.$$

This can be done, because Property (4) of Lemma 2.2 assures $(\cosh\alpha)^2 - (\sinh\alpha)^2 = 1$. By the definition of b_t, we have $\cosh\alpha = (\sigma_+^t + \sigma_-^t)/2$ and $\sinh\alpha = (-\sigma_+^t + \sigma_-^t)/2$. Then, we can check that

$$\bar{P} = P\cosh\alpha + R\sinh\alpha \quad \text{and} \quad \bar{R} = P\sinh\alpha + R\cosh\alpha\,.$$

That is, relative to the parameter ρ, the motion by T_t is the translation by the amount of α. When t tends to infinity, the value of α also tends to infinity in both sides. Indeed, we can see that $\alpha = t(\log(q - \sqrt{q^2-4})/2)$.

When $q^2 < k$, we only need to replace the role of P with that of R. When $q^2 = k > 4$, the hyperbola reduces to two lines that are written as $r = \sigma_\pm p$. We then see that $p_t = p(\sigma_+)^t$ on the line $r = \sigma_+ p$ and $p_t = p(\sigma_-)^t$ on the line $r = \sigma_- p$.



(a) (b)

FIGURE 3. Section of V by the plane (a) $q = 3$ and (b) $q = 3/2$.

If $q^2 < 4$, we define P and R by

$$P = (p - r)\sqrt{2 + q}/2 \quad \text{and} \quad R = (p + r)\sqrt{2 - q}/2\,;$$

we have $P^2 + R^2 = k - q^2$. Hence, the case where $q^2 > k$ does not occur and the case where $q^2 = k$ leads to $(p, q, r) = (0, q, 0)$, which is a fixed point. So, we need to consider the case where $q^2 < k$. Then (P, R) lies on a circle. By introducing ρ by

$$P = \sqrt{k - q^2}\sin\rho \quad \text{and} \quad R = \sqrt{k - q^2}\cos\rho$$

and α by

$$\cos\alpha = \frac{qb_{t-1}(q) - 2b_{t-2}(q)}{2} \quad \text{and} \quad \sin\alpha = \frac{1}{2}\sqrt{4 - q^2}b_{t-1}(q)\,,$$

we see that the motion under T_t is a rotation on the circle by angle α.

The curves drawn in Figure 3(a) are sections of the surface V by planes parallel to the pr-plane when $q = 3$; the parameter k takes the values 0, 4, 9 and 16. In Figure 3(b), the curves are sections when $q = 3/2$; the parameter k takes the values 4, 8 and 16.

The consideration above shows that the global behavior of the transformation changes depending on the value of k. Referring to the notation (1.1), we set $\varphi_{pt} = 2p_t - qr_t$, $\varphi_{qt} = 2q - p_t r_t$ and $\varphi_{rt} = 2r_t - qp_t$. Then we see that $(\varphi_{pt})^2 = (q^2 - 4)(r_t^2 - 4) + 4(k - 4)$, $(\varphi_{qt})^2 = (r_t^2 - 4)(p_t^2 - 4) + 4(k - 4)$ and $(\varphi_{rt})^2 = (p_t^2 - 4)(q^2 - 4) + 4(k - 4)$. Hence, if $k < 4$, then $p_t^2 - 4$, $q^2 - 4$ and $r_t^2 - 4$ have the same sign, which means that the cube $|p| < 2$, $|q| < 2$, $|r| < 2$ and the box $p > 2$, $q > 2$, $r > 2$ are invariant under T_t. If $k > 4$, then $q^2 > k > 4$, which means that such an isolation as in $k < 4$ is not possible. Indeed, on the curve with $q^2 > k$, the values of p and r are unbounded in both directions and, although the transformation T_t is defined on this curve, the transformation R_t (resp. S_t) becomes undefined when the value p (resp. r) is less than or equal to -2.

We summarize the behavior of the action of G in the case where $k \leq 4$ as follows.

PROPOSITION 2.3. *We have the following cases.*

- Case $k < 0$. *The surface is included in the domain $p^2 > 4$, $q^2 > 4$ and $r^2 > 4$ and the argument for the hyperbola case applies. In particular, the action of the group G on the surface lying in the box $p \geq 2$, $q \geq 2$, $r \geq 2$ is transitive.*
- Case $0 \leq k < 4$. *Points on the surface belong to the domain $p^2 > 4$, $q^2 > 4$, $r^2 > 4$ or to the cube $|p| < 2$, $|q| < 2$, $|r| < 2$. The point $(0, 0, 0)$, when $k = 0$, is isolated and fixed by G. The actions of the group G on the surface lying in the box $p > 2$, $q > 2$, $r > 2$ and that on the surface lying in the cube $|p| < 2$, $|q| < 2$, $|r| < 2$ are transitive.*
- Case $k = 4$. *The surface includes the lines $\{(a, a, 2)\}$, $\{(a, -a, -2)\}$, $\{(a, 2, a)\}$, $\{(a, -2, -a)\}$, $\{(2, a, a)\}$ and$\{(-2, a, -a)\}$. The three lines $\{(a, a, 2)\}$, $\{(a, 2, a)\}$ and $\{(2, a, a)\}$ are pointwise fixed by S_t, T_t and R_t, respectively. In particular, the point $(2, 2, 2)$ is fixed by the group G. The part of the surface lying in the box $p \geq 2$, $q \geq 2$, $r \geq 2$, with $(2, 2, 2)$ deleted, is one orbit of the group G.*

2.3. Invariant area form. We study some local properties of transformations in G. We first remark that the Jacobian of every transformation regarded as a transformation of $\boldsymbol{R}^3(p, q, r)$ is always equal to 1 by Property (2) of Lemma 2.2. We next consider the area form

$$\omega = -\frac{dp \wedge dq}{\varphi_r}$$

defined on the set where $\varphi_r \neq 0$. Owing to the identity $\varphi_p\, dp + \varphi_q\, dq + \varphi_r\, dr = 0$, it is equal to $-dq \wedge dr/\varphi_p$ and $-dr \wedge dp/\varphi_q$ where they are defined. Thus, we can regard ω as an area form away from the set $\{\varphi_p = \varphi_q = \varphi_r = 0\}$.

PROPOSITION 2.4. *The form ω is invariant under the action of G.*

PROOF. We set $T_t(p, q, r) = (p_t, q, r_t)$, where $p_t = b_{t-1}(q)r - b_{t-2}(q)p$ and $r_t = b_t(q)r - b_{t-1}(q)p$. Then, $dp_t \wedge dq = (\partial p_t/\partial p) \wedge dq = (b_{t-1}(q)r_p - b_{t-2}(q))dp \wedge dq$. On the other hand,

$$\begin{aligned} 2r_t - p_t q &= 2b_t(q)r - 2b_{t-1}(q)p - b_{t-1}(q)rq + b_{t-2}(q)pq \\ &= 2r(qb_{t-1} - b_{t-2}) - (2p + qr)b_{t-1} + pqb_{t-2} \\ &= -(2p - qr)b_{t-1} - (2r - pq)b_{t-2} \\ &= -\varphi_p b_{t-1} - \varphi_r b_{t-2}\,. \end{aligned}$$

Since $r_p = -\varphi_p/\varphi_r$, we have $dp_t \wedge dq/(2r_t - p_t q) = dp \wedge dq/(2r - pq)$. □

When $k = 0$, this form ω is known to be the Weil-Petersson Kähler form; we refer to Wolpert [8]. For general values of k, the form determines a Poisson structure; we refer to, for example, [4].

REMARK 2.5. The set $\{\varphi_p = \varphi_q = \varphi_r = 0\}$ consists only of one point $(0, 0, 0)$ when $k = 0$ and of four points $\{(2, 2, 2), (2, -2, -2), (-2, 2, -2), (-2, -2, 2)\}$ when $k = 4$.

Otherwise, it is empty. Since the action of G is transitive on the part lying in the box $p \geq 2$, $q \geq 2$, $r \geq 2$ when $k \leq 4$ as well as in the cube $|p| < 2$, $|q| < 2$, $|r| < 2$ when $k < 4$, the 2-form invariant under G is unique up to a constant.

3. Infinitesimal automorphisms. In this section, we compute the infinitesimal generators of the transformations T_t, R_t and S_t, hoping to unveil the structure of the group G.

We define two vector fields ∂_p and ∂_q by

$$\partial_p = \frac{\partial}{\partial p} - \frac{\varphi_p}{\varphi_r}\frac{\partial}{\partial r} \quad \text{and} \quad \partial_q = \frac{\partial}{\partial q} - \frac{\varphi_q}{\varphi_r}\frac{\partial}{\partial r}\,;$$

both are defined where $\varphi_r \neq 0$. The operators ∂_p and ∂_q are derivations relative to p and q, respectively, of functions on the surface by regarding the variable r as a function of (p, q). Hence, $[\partial_p, \partial_q] = 0$.

The infinitesimal generator of the one-parameter group $\{T_t\}$ is the tangent vector X of the curve

$$c : t \mapsto T_t(p, q, r) = (b_{t-1}(q)r - b_{t-2}(q)p\,, q\,, b_t(q)r - b_{t-1}(q)p) = (p_t, q, r_t)\,,$$

for any fixed (p, q, r). We set

$$\lambda(p, q, r) = \left.\frac{\partial p_t}{\partial t}\right|_{t=0} \quad \text{and} \quad \nu(p, q, r) = \left.\frac{\partial r_t}{\partial t}\right|_{t=0}.$$

Then we have

$$X = \lambda\frac{\partial}{\partial p} + \nu\frac{\partial}{\partial r} = \lambda\partial_p\,.$$

Here we have used the identity $\varphi_p\lambda + \varphi_r\nu = 0$, which follows as the point $T_t(p, q, r)$ is lying on the surface V. Similarly, for $\{R_t\}$, by exchanging p and q, we have

$$Y = \mu\partial_q\,,$$

where $\mu = \mu(p, q, r) = \lambda(q, p, r)$. Our interest here is to see how large the algebra of vector fields generated by X and Y is. (We do not need to worry about $\{S_t\}$ since its infinitesimal generator is included in this algebra because of $S_t = T^{-1}R_tT$.) By a computation, using $[\partial_p, \partial_q] = 0$, we have

$$H := [X, Y] = \lambda\mu_p \cdot \partial_q - \mu\lambda_q \cdot \partial_p\,,$$

and

$$[H, X] = (2\lambda\mu_p\lambda_q + \lambda\mu\lambda_{pq} - \mu\lambda_q\lambda_p)\partial_p - \lambda(\lambda_p\mu_p + \lambda\mu_{pp})\partial_q\,,$$

and $-[H, Y]$ is equal to the right-hand side of the above with the exchange $p \leftrightarrow q$ and $\lambda \leftrightarrow \mu$.

We set

$$f(q) = \left.\frac{\partial b_{t-1}(q)}{\partial t}\right|_{t=0} \quad \text{and} \quad g(q) = \left.\frac{\partial b_t(q)}{\partial t}\right|_{t=0}.$$

By the recurrence relation (1) of Lemma 2.2, we see that $\partial b_{t-2}(q)/\partial t|_{t=0} = g(q)$ and by Property (3) of Lemma 2.2, we have $g(q) = qf(q)/2$. Hence, we get

$$\lambda = \frac{1}{2} f(q)\varphi_r \quad \text{and} \quad \mu = \frac{1}{2} h(p)\varphi_r \,,$$

where h is the function f with variable p in place of q. We set

$$\kappa = k - 4 \,.$$

Then, $\varphi_r^2 = (p^2-4)(q^2-4) + 4\kappa$, $\partial_p \varphi_r = p(q^2-4)/\varphi_r$ and $\partial_q \varphi_r = q(p^2-4)/\varphi_r$. By the definition of $b_t(q)$, we see that

$$f(q) = 2\log((q + \sqrt{q^2-4})/2)/\sqrt{q^2-4} \,.$$

(When $q^2 < 4$, assume that $\sqrt{q^2-4} = i\sqrt{4-q^2}$; in other words, define θ by $\cos\theta = q/2$ and $\sin\theta = \sqrt{4-q^2}/2$ where $0 < \theta < \pi$ and set $f(q) = \theta/\sin\theta$.) Hence,

$$(q^2-4)f_q + qf = 2 \quad \text{and} \quad (q^2-4)f_{qq} + 3qf_q + z = 0 \,.$$

From these identities, we get

$$\lambda_p = p(q^2-4)f(q)/2\varphi_r \,, \quad \lambda_q = (p^2-4+2\kappa f_q)/\varphi_r \,,$$
$$\mu_p = (q^2-4+2\kappa h_p)/\varphi_r \,, \quad \mu_q = q(p^2-4)h(p)/2\varphi_r \,.$$

Hence,

$$2H = -h(p)(p^2-4+2\kappa f_q)\partial_p + f(q)(q^2-4+2\kappa h_p)\partial_q \,. \tag{3.1}$$

A straightforward calculation of $[H, X]$ using the formulas above leads to

$$[H, X] = -\frac{1}{2}\kappa\varphi_r f(q)^2 h_{pp}\partial_q + f(q)\varphi_r(1+\kappa h_p f_q)\partial_p \,. \tag{3.2}$$

Similarly,

$$[H, Y] = \frac{1}{2}\kappa\varphi_r h(p)^2 f_{qq}\partial_p - h(p)\varphi_r(1+\kappa h_p f_q)\partial_q \,.$$

The last two formulas reveal that the algebra generated by X and Y shows a distinctive character depending on whether $\kappa = 0$ or not.

PROPOSITION 3.1. *The algebra generated by X and Y is infinite dimensional unless $\kappa = 0$. When $\kappa = 0$, the algebra is isomorphic to* $\mathrm{sl}(2, \boldsymbol{R})$.

PROOF. The latter statement is easy to see because, when $\kappa = 0$,

$$H = [X, Y] \,, \quad [H, X] = 2X \,, \quad [H, Y] = -2Y \,.$$

To prove the former statement, we pay attention to the vector fields $(\mathrm{Ad}_X)^{k+1}(Y) = (\mathrm{Ad}_X)^k(H)$. We set $(\mathrm{Ad}_X)^k(H) = A_k\partial_q + B_k\partial_p$. Then it is easy to see $A_{k+1} = (1/2)f(q)\varphi_r(A_k)_p$. Since $A_1 = (1/2)\kappa f(q)^2\varphi_r h_{pp}$, we have

$$A_k = \kappa f(q)(f(q)/2)^k D^k(h_p) \,, \quad \text{where } D = \varphi_r\partial_p \,.$$

We note that $h_p = (2 - ph(p))/(p^2 - 4)$ and it satisfies the differential equation $(p^2-4)h_{ppp}+5ph_{pp}+4h_p = 0$. If X and Y generate a finite-dimensional algebra, then we must have a linear relation over $\boldsymbol{R}$ amongst the coefficients $\{A_1, A_2, \dots, A_k\}$ for some k. Let

$$c_k A_k + c_{k-1}A_{k-1} + \cdots + c_1 A_1 = 0$$

be one of linear relations with $c_k \neq 0$. If $\kappa \neq 0$, then $h(p)$ satisfies a differential equation

$$c_k(f(q)/2)^k D^k(h_p) + \cdots + c_1(f(q)/2)D(h_p) = 0$$

for any value of q. We pay attention to the highest-order term,

$$c_k(f(q)\varphi_r/2)^k(\partial_p)^k(h_p)\,,$$

that is actually dependent on q. Its growth order relative to p and q is easily seen to be $O(((\log q)^k \log p)/p^2)$. This means that such a relation cannot be non-trivial. □

REMARK 3.2. We interpolated the iteration T^n of the Markoff transformation T by T_t, by regarding the Chebyshev polynomial b_n as a special case of the hypergeometric function b_t. Then we found that the Lie algebra generated by X and Y is isomorphic to $\mathrm{sl}(2, \boldsymbol{R})$ if and only if $k = 4$. Note that the properties we used for b_t were only (1) and (3) of Lemma 2.2. Here we pose the following problem.

PROBLEM. Find another interpolation T_t of the Markoff transformations T^n so that the Lie algebra generated by infinitesimal generators of T_t and R_t is isomorphic to $\mathrm{sl}(2, \boldsymbol{R})$ when $k \neq 4$.

4. The case where $k = 4$.

4.1. Linearization of the action of $\boldsymbol{G}$. In this section we describe the action of G explicitly on the part of the surface V_4 lying in the box $p \geq 2$, $q \geq 2$, $r \geq 2$; this part will be denoted by S.

A key idea is to consider the map $\phi : \boldsymbol{R}^3(x, y, z) \mapsto \boldsymbol{R}^3(p, q, r)$ defined by

$$p = 2\cosh x\,, \quad q = 2\cosh y\,, \quad r = 2\cosh z\,.$$

Since, when $k = 4$,

$$\begin{aligned}\varphi(\phi(x, y, z)) &= 4((\cosh x)^2 + (\cosh y)^2 + (\cosh z)^2) - 8\cosh x\cosh y\cosh z - 4\\ &= -e^{-x-y-z}(1-e^{x+y+z})(1-e^{-x+y+z})(1-e^{x-y+z})(1-e^{x+y-z})\,,\end{aligned}$$

the map restricted to the plane

$$\mathcal{X} : x + y + z = 0$$

has its image on the surface S. Thus, we have a map

$$\phi : \mathcal{X} \ni (x, y, z) \mapsto (2\cosh x, 2\cosh y, 2\cosh z) \in S\,,$$

which is two-to-one except for the origin.

The action of T_t can lift to the plane $\mathcal{X}$ as follows. Let (x_t, y, z_t) denote the point corresponding to $T_t(p, q, r) = (p_t, q, r_t)$ as in the following diagram.

$$\begin{tikzcd}
(x, y, z) \arrow[r, "\tilde{T}_t"] \arrow[d, "\phi"] & (x_t, y, z_t) \arrow[d, "\phi"] \\
(p, q, r) \arrow[r, "T_t"] & (p_t, q, r_t)
\end{tikzcd}$$

Since $\sqrt{(2\cosh y)^2 - 4} = 2|\sin hy|$ by referring to (2.1), we see that $b_t(q) = (e^{(t+1)y} - e^{-(t+1)y})/(e^y - e^{-y})$. Therefore, $p_t = b_{t-1}(q)r - b_{t-2}(q)p$ implies

$$e^{x_t} + e^{-x_t} = b_{t-1}(q)(e^{x+y} + e^{-x-y}) - b_{t-2}(q)(e^x + e^{-x}),$$

from which we get the identity $x_t = \pm(x + ty)$. Relative to r_t, we have $z_t = \pm(z - ty)$. Namely, the affine transformation

$$\tilde{T}_t : (x, y, z) \mapsto (x + ty, y, z - ty)$$

in the plane $\mathcal{X}$ covers the transformation T_t. Similarly, we can see that the actions of R_t and S_t lift to

$$(x, y, z) \mapsto (x, y + tx, z - tx) \quad \text{and} \quad (x, y, z) \mapsto (x + tz, y - tz, z),$$

respectively. Therefore, we have seen the following proposition.

PROPOSITION 4.1. *The action of G on S lifts to the linear action on $\mathcal{X}$; this action coincides with the linear action of* $\mathrm{SL}(2, \boldsymbol{R})$.

REMARK 4.2. We can extend the action to that of $\mathrm{GL}(2, \boldsymbol{R})$. In fact, for any linear transformation, say g, on $\mathcal{X}$, we get a transformation on S via the map ϕ, since g transforms $(-x, -y, -z)$ to $-g(x, y, z)$. The action of the one-parameter subgroup $\left(\begin{smallmatrix} s & 0 \\ 0 & s \end{smallmatrix}\right)$ on S is given as follows. In the space $\mathcal{X}$, the action is written as

$$(x, y, z) \mapsto (sx, sy, -sx - sy).$$

The (p, q, r)-coordinates of its projection are, by definition,

$$p_s = e^{sx} + e^{sx}, \quad q_s = e^{sy} + e^{-sy}, \quad r_s = e^{s(x+y)} + e^{-s(x+y)}.$$

Then it is not difficult to see

$$p_s = c_s(p), \quad q_s = c_s(q), \quad r_s = c_s(r),$$

where c_s is a function

$$c_s(p) = \left(\frac{p + \sqrt{p^2 - 4}}{2}\right)^s + \left(\frac{p - \sqrt{p^2 - 4}}{2}\right)^s,$$

which is a continuous extension of the Chebyshev polynomial denoted by $C_n(p)$ in [7].

REMARK 4.3. The action on the part where $|p| \leq 2$, $|q| \leq 2$ and $|r| \leq 2$ is not defined globally, because the transformation, say T_t, is singular at $q = -2$. However, the description of the action outside the union of three line segments $p = -2$, $q = -2$ and $r = -2$ is similarly given. It is enough to consider the map $(x, y, z) \mapsto (2\cos x, 2\cos y, 2\cos z)$, which is defined on the set $\{(x, y, z) \in (\boldsymbol{R}/2\pi\boldsymbol{Z})^3 ; x + y + z \equiv 0 \pmod{2\pi}\}$.

4.2. Invariant 1-form. The 2-form ω on S simplifies relative to the coordinates (x, y) on $\mathcal{X}$: it is equal to $-dx \wedge dy$. We define $\theta = (-x\,dy + y\,dx)/2$. Then, obviously, $d\theta = \phi^*\omega$ and it is easy to see that the form θ is invariant under the action of $\mathrm{SL}(2, \boldsymbol{R})$. Any integral curve of $\theta = 0$ is nothing but a line through the origin. If we express it by the equation $x/a = y/b = z/c$ where $a + b + c = 0$, then its push-down on the space S is written as the curve of the form

$$\left(\frac{p + \sqrt{p^2 - 4}}{2}\right)^{1/a} = \left(\frac{q + \sqrt{q^2 - 4}}{2}\right)^{1/b} = \left(\frac{r + \sqrt{r^2 - 4}}{2}\right)^{1/c}.$$

Such a curve starting from the point $(2, 2, 2)$ lifts to a half line starting from the origin. We call such a curve a ray. The half line on S defined by the equation $p = 2$, $q = 2$ or $r = 2$ is one of the rays. The set of rays is parametrized by a circle and the surface S is foliated by the rays.

REMARK 4.4. When $k \neq 4$, there exists no 1-form τ invariant under the action of G so that $d\tau = \omega$.

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REFERENCES

[1] B. H. BOWDITCH, Markoff triples and quasifuchsian groups, Proc. London Math. Soc. 77(1998), 697–736.

[2] R. FRICKE AND F. KLEIN, Vorlesungen über die theorie der automorphen functionen, Band 1 and 2, B. G. Teubner, Leibzig und Berlin, 1897 and 1912.

[3] W. GOLDMAN, The modular group action on real SL(2)-characters of a one-holed torus, Geom. Topol. 7(2003), 443–486.

[4] K. IWASAKI, An area-preserving action of the modular group on cubic surfaces and the Painlevé VI equation, Commun. Math. Phys. 242(2003), 185–219.

[5] L. KEEN, On fundamental domains and the Teichmüller modular group, Contributions to Analysis (eds. L. Ahlfors, I. Kra, B. Maskit and L. Nirenberg), pp.185–194, Academic Press, New York, 1974.

[6] L. KEEN, H. E. RAUCH AND A. T. VASQUEZ, Moduli of punctured tori and the accessory parameter of Lame's equation, Trans. Amer. Math. Soc. 255(1979), 201–230.

[7] TH. J. RIVLIN, Chebyshev Polynomials, Second edition, Pure Appl. Math. (N.Y.), John Wiley & Sons, Inc., New York, 1990.

[8] S. WOLPERT, On the Kähler form of the moduli space of once punctured tori, Comment. Math. Helv. 58 (1983), 246–256.

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