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Interpolation of operations and Orlicz classes

by

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Dedicated to
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Abstract. This paper deals with the extension of the theorem of Marcinkiewicz concerning interpolation of operations to include Orlicz spaces as intermediate classes. An abstract version of this result is also presented and several applications are discussed, mainly to continuity properties of integral transforms.

Introduction. In this paper we extend the theorem of Marcinkiewicz concerning the interpolation of operations to include Orlicz spaces as intermediate classes. For operations of weak type (p_i, p_i), $i = 0, 1$, this question was considered by A. Zygmund [29] and for operations of weak type (p_i, q_i), $1 \leqslant p_i \leqslant q_i \leqslant \infty$, $i = 0, 1$, it was treated by W. Riordan in his unpublished doctoral dissertation. In fact Theorem 2.3 is essentially contained in Riordan's work, but the proof given here is simpler, it applies to more general situations and the description of the intermediate Orlicz classes is more explicit. Theorems 2.8 and 2.11, which exploit the availability of strong type, are apparently new although a special case is given in [10]. The study of these extensions of the theorem of Marcinkiewicz is of interest because they are not explicitly included in the abstract theory of interpolation.

The abstract formulation of the theorem of Marcinkiewicz is due to A. P. Calderón ([4], [6], [8], [17]) and it makes use of the Lorentz classes $L(p, q)$. The posibility of obtaining an interpolation result which would simultaneously extend known results for $L(p, q)$ and Orlicz spaces has been raised in the Problem 9b) of the June 1975 issue of the Notices of the Amer. Math. Soc., vol. 22, page 199. Theorems 3.17, 3.20, 3.22, 3.25 give an interpolation theorem that contains an answer to that problem. The appropriate setting in this case is a generalization of the spaces $\Lambda_\alpha(X)$ ([22]).

The results alluded to approach have numerous applications. We only consider here some questions discussed in [11] and extensions of the interesting results of [20] and [25] concerning integral transforms. The reader may also be interested in interpolation of operators with change of measure ([23]), for some particular Orlicz classes this is done in [13].

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1. Orlicz classes. In what follows the letters A, B, C, D, E are reserved for *generalized Young's functions*, that is, for functions $A(t)$ defined from $[0,\infty]$ into $[0,\infty]$, $A(0)=0$, such that

(i) A is non-trivial, i.e., $A\not\equiv 0$ or $A(t)\not\equiv\infty$ for $t\in(0,\infty]$;

(ii) $A(t)$ is left continuous.

By the phrase "$A(t)/t$ increases" we refer to those generalized Young's functions such that

(iii) $A(t)/t$ increases in the wide sense.

The inverse of A is defined on $[0,\infty]$ by

$$A^{-1}(t)=\inf\{s: A(s)>t\},\quad \inf\emptyset=\infty.$$

It is easily seen that A^{-1} is a monotone non-decreasing function from $[0,\infty]$ into $[0,\infty]$ which is right continuous and

$$A\big(A^{-1}(t)\big)\leqslant t\leqslant A^{-1}\big(A(t)\big),\quad t\geqslant 0. \tag{1.1}$$

Moreover,

$$A(t)=\sup\{s: A^{-1}(s)<t\},\quad \sup\emptyset=0.$$

If $A(t)/t$ increases, for $t\in[0,\infty)$ let

$$\bar{A}(t)=\sup_{s\geqslant 0}\big(st-A(s)\big). \tag{1.2}$$

$\bar{A}(t)$ is called the *Young's complement* of A. It is readily seen that $\bar{A}(t)/t$ increases, and

$$t\leqslant A^{-1}(t)\bar{A}^{-1}(t)\leqslant 2t,\quad 0\leqslant t<\infty. \tag{1.3}$$

If $A(t)/t$ increases, we define the *regularization* A_0 of A as

$$A_0(t)=\int_0^t\big(A(s)/s\big)\,ds. \tag{1.4}$$

Then A_0 is convex, increasing, zero at zero and non-trivial (A_0 is positive and finite in the same set as A). Moreover,

$$A_0(t)\leqslant A(t)\leqslant A_0(2t). \tag{1.5}$$

Such functions A_0 are called *Young's functions* and (1.4) shows that inequalities involving A are equivalent to inequalities with the Young's function A_0. However, it is often convenient to use A instead of A_0 since

for example $(A\wedge B)(t)=\min\big(A(t),B(t)\big)$ satisfies (i)–(iii) but is not in general convex when A and B are convex.

Let (M,μ) be a positive measure space and let A be a generalized Young's function. Let f be a (complex valued) μ-measurable function defined on M, and set

$$|f|_A=\inf\Big\{K>0: \int_M A\big(|f(x)|/K\big)\,d\mu\leqslant 1\Big\}. \tag{1.6}$$

If $A(t)/t$ increases we let

$$L_A(M,\mu)=L_A=\{f: |f|_{A_0}\leqslant 1\}, \tag{1.7}$$

where $A_0=A$ if A is convex and A_0 is given by (1.4) otherwise. It is known that L_A is a Banach space with $\|f\|_A=|f|_A$, the Orlicz space norm.

In case A is a generalized Young's function and f is a μ-measurable function, let

$$l_A(f)=\inf\Big\{K>0: \int_M A\big(|f(x)|/K\big)\,d\mu\leqslant K\Big\},$$

and

$$L_A(M,\mu)=L_A=\{f: l_A(f)<\infty\}.$$

Then l_A is a metric and if $\lambda\geqslant 1$, then $l_A(\lambda f)\leqslant\lambda l_A(f)$. If A is a Young's function, then l_A and $\|\cdot\|_A$ determine the same uniform topology on L_A ([20], Theorem 9.4).

The concept of regularization was introduced in [9], and Orlicz spaces are studied in detail in [12], [20].

In what follows we assume that μ is non-atomic and $\mu(M)=\infty$. These restrictions exclude sequence spaces and spaces of finite measure, but the reader will have no difficulty in modifying the results that follow to include such instances.

An operation $g=Tf$ defined for f in $L_A(M,\mu)$ and taking values g in $L_B(N,\nu)$ is said to be *bounded* if there is a constant $K>0$ such that

$$\int_N B\big(|Tf(y)|/K\big)\,d\nu\leqslant 1 \tag{1.8}$$

whenever

$$\int_M A\big(|f(x)|\big)\,d\mu\leqslant 1. \tag{1.9}$$

The smallest constant K above is called the *norm* of T and such operators T are said to be of (*strong*) *type* (A,B). When A, B are powers we use the exponents. Thus if $A(t)=t^p$, L_A is L^p and a transformation of type (A,A) is of type (p,p), etc. It is natural to call the function A given by

$$A(t)=0\quad\text{for } 0\leqslant t\leqslant 1,\qquad A(t)=\infty\quad\text{for } t>1$$

a power, sometimes we write $A(t) = t^\infty$. That $L_A = L^\infty$ follows from the definition of Orlicz space.

To define weak type (A, B) we first recall the notation $m(f, \lambda)$ for the *distribution function* of a measurable function f on (M, μ), to wit

$$m(f, \lambda) = \mu(\{x \in M: |f(x)| > \lambda\}), \quad \lambda > 0. \tag{1.10}$$

If T is a mapping of type (A, B), then for each $\lambda > 0$ we have

$$m(Tf, \lambda) \leqslant 1/B(\lambda/K) \tag{1.11}$$

whenever (1.9) holds.

In case (1.11) (but not necessarily (1.10)) holds when (1.9) holds we say that T is of *weak type* (A, B) with norm $\leqslant K$.

If $L_B = L^\infty$ we agree that the notions of weak type and type coincide.

2. Interpolation of operations. Some 20 years ago A. Zygmund supplied the proof of the general form of the theorem of Marcinkiewicz for the Lebesgue classes L^p. Techniques used there are exploited to obtain Orlicz classes as intermediate spaces as well.

An operation $g = Tf$ of a class of functions b on (M, μ) into a class of functions g on (N, ν) is called a *sublinear operation* if it satisfies the following properties:

(i) If $f = f_0 + f_1$ and Tf_i $(i = 0, 1)$ are defined, then Tf is defined;

(ii) $|T(f_0+f_1)| \leqslant |Tf_0| + |Tf_1|$ ν-almost everywhere;

(iii) For any scalar k we have $|T(kf)| = |k||Tf|$ ν-almost everywhere.

We need one more definition. Given the points (α_i, β_i), $i = 0, 1$, $\alpha_0 \neq \alpha_1$, in the region $0 \leqslant \beta_i, \alpha_i < \infty$, we set

$$\varepsilon = \frac{\beta_0 - \beta_1}{\alpha_0 - \alpha_1} \tag{2.1}$$

and

$$\gamma = \frac{\beta_0/\alpha_0 - \beta_1/\alpha_1}{1/\alpha_0 - 1/\alpha_1}. \tag{2.2}$$

Thus the equation of the straight-line passing through the points (α_i, β_i) is given by $y = \varepsilon x + \gamma$. We can now state the interpolation theorem.

(2.3) THEOREM. *Let*

$$0 \leqslant \frac{1}{q_i} = \beta_i \leqslant \frac{1}{p_i} = \alpha_i < \infty, \quad i = 0, 1, \ \alpha_0 \neq \alpha_1, \ \beta_0 \neq \beta_1.$$

Suppose that a sublinear operation $g = Tf$ is simultaneously of weak types (p_i, q_i), $i = 0, 1$, with norms M_0 and M_1, respectively.

Assume that the generalized Young's functions A, B are given by

$$A(t) = \int_0^t a(s)\,ds, \quad B(t) = \int_0^t b(s)\,ds,$$

where a and b are monotone. Further assume that if $l = q_0 \vee q_1 < \infty$ and $m = q_0 \wedge q_1$, then $B(t)/t^l$ decreases, $B(t)/t^m$ increases and

$$\int_0^t B(s)/s^m \frac{ds}{s} \leqslant k_m B(t)/t^m; \tag{2.4}$$

$$\int_t^\infty B(s)/s^l \frac{ds}{s} \leqslant k_l B(t)/t^l. \tag{2.5}$$

If $l = \infty$, we assume there is a $q > m$ such that $B(t)/t^q$ decreases.

If ε and γ are given by (2.1) *and* (2.2) *and if*

$$B^{-1}(t) = A^{-1}(t^\varepsilon)t^\gamma, \tag{2.6}$$

then T is of type (A, B) with norm $K = K(k_m, k_l, M_i, p_i, q_i)$.

Proof. With no loss of generality we may assume that $0 < p_0 < p_1 \leqslant \infty$. It is readily seen that $L_A \subset L^{p_0} + L^{p_1}$. In fact, if for $f \in L_A$ and $u > 0$ we set $f_u = \operatorname{sgn} f \cdot (u \wedge |f|)$ and $f^u = f - f_u$, then our assumptions imply that f_u is in L^{p_1} and f^u in L^{p_0}. Therefore Tf is well defined for f in L_A.

To calculate $I = \int_N B(|Tf(y)|/2K)\,d\nu$ we use the well-known expression

$$I = \int_0^\infty b(\lambda) m(Tf, 2K\lambda)\,d\lambda$$
$$\leqslant \int_0^\infty b(\lambda) m(Tf_u, K\lambda)\,d\lambda + \int_0^\infty b(\lambda) m(Tf^u, K\lambda)\,d\lambda = I_1 + I_2,$$

the last inequality being an immediate consequence of the definition of the distribution function and the fact that T is sublinear.

We consider the case $0 < q_0 < q_1 \leqslant \infty$, thus $m = q_0$, $l = q_1$ and $\varepsilon > 0$.

In the above decomposition of $f = f^u + f_u = f_{u_0} + f_{u_1}$ we choose u as the monotone function of λ such that $u^{-1}(\lambda) = B^{-1}(A^{1/\varepsilon}(\lambda))$. Two cases arise: $q_1 < \infty$ and $q_1 = \infty$. First suppose that $q_1 < \infty$. Since T is of weak type (p_i, q_i) with norm M_i, we have

$$m(Tf_{u_i}, K\lambda) \leqslant (M_i \|f_{u_i}\|_{p_i})^{q_i}/(K\lambda)^{q_i} \tag{2.7}$$
$$= M_i^{q_i} p_i^{q_i/p_i} \left(\int_0^\infty m(f_{u_i}, s) s^{p_i} \frac{ds}{s}\right)^{q_i/p_i} (K\lambda)^{q_i}, \quad i = 0, 1.$$

Since $m(f_{u_1}, s) = m(f_u, s) = m(f, s)$ for $u(\lambda) \leqslant s$ and zero otherwise, it follows that

$$I_1^{p_1/q_1} \leqslant p_1 (M_1/K)^{p_1} \Big\{ \int_0^\infty \frac{b(\lambda)}{\lambda^{q_1}} \Big(\int \chi\big(s/u(\lambda)\big) m(f, s) s^{p_1} \frac{ds}{s} \Big)^{q_1/p_1} d\lambda \Big\}^{p_1/q_1},$$

where χ is the characteristic function of the interval $(0, 1)$. Therefore by Minkowski's integral inequality and the fact that $b(\lambda) \leqslant q_1 B(\lambda)/\lambda$ since $B(\lambda)/\lambda^{q_1}$ decreases we obtain

$$I_1^{p_1/q_1} \leqslant p_1 (M_1/K)^{p_1} \int_0^\infty m(f, s) s^{p_1} \Big\{ \int_0^\infty \chi\big(s/u(\lambda)\big)^{q_1/p_1} \frac{b(\lambda)}{\lambda^{q_1}} d\lambda \Big\}^{p_1/q_1} \frac{ds}{s}$$

$$\leqslant p_1 (M_1/K)^{p_1} q_1^{p_1/q_1} \int_0^\infty m(f, s) s^{p_1} \Big\{ \int_{u^{-1}(s)}^\infty B(\lambda)/\lambda^{q_1} \frac{d\lambda}{\lambda} \Big\}^{p_1/q_1} \frac{ds}{s}.$$

To bound the innermost integral we proceed as follows. By integral conditon (2.5) and relation (2.6) we obtain

$$\int_{u^{-1}(s)}^\infty B(\lambda)/\lambda^{q_1} \frac{d\lambda}{\lambda} \leqslant k_{q_1} B\big(u^{-1}(s)\big)/u^{-1}(s)^{q_1}$$

$$= k_{q_1} B\Big(B^{-1}\big(A(s)^{1/\varepsilon}\big)\Big)/B^{-1}\big(A(s)^{1/\varepsilon}\big)^{q_1}$$

$$\leqslant k_{q_1} A(s)^{1/\varepsilon}/\big(sA(s)^{\gamma/\varepsilon}\big)^{q_1},$$

whence it follows that

$$I_1^{p_1/q_1} \leqslant p_1 (M_1/K)^{p_1} (q_1^{k_{q_1}})^{p_1/q_1} \int_0^\infty m(f, s) s^{p_1} A(s)^{\frac{p_1}{\varepsilon}(1/q_1 - \gamma)} \frac{1}{s^{p_1}} \frac{ds}{s}.$$

But since

$$\frac{p_1}{\varepsilon}(1/q_1 - \gamma) = 1 \quad \text{and} \quad \frac{A(s)}{s} \leqslant \frac{a(s)}{p_0}$$

because $A(s)/s^{p_0}$ increases, it follows that

$$I_1 \leqslant \left(\frac{p_1}{p_0}\right)^{q_1/p_1} (M/K)^{q_1} q_1^{k_{q_1}} \Big(\int_0^\infty m(f, s) a(s) \frac{ds}{s} \Big)^{q_1/p_1}.$$

The same idea will be used to bound I_2, in this argument the letter c will denote a constant which may not be the same in different occurrences. However, the final expression of $c \leqslant (M_0/K)^{q_0} k_{q_0} q_1$. Since $m(f_{u_0}, s) = m(f^u, s) = m(f, u+s)$, from (2.7) it follows that

$$I_2^{p_0/q_0} \leqslant c \Big\{ \int_0^\infty \frac{b(\lambda)}{\lambda^{q_0}} \Big(\int_0^\infty \chi\big(u(\lambda)/s\big) m(f, s) s^{p_0} \frac{ds}{s} \Big)^{q_0/p_0} d\lambda \Big\}^{p_0/q_0},$$

whence by Minkowski's integral inequality we obtain

$$I_2^{p_0/q_0} \leqslant c \int_0^\infty m(f, s) s^{p_0} \Big\{ \int_0^{u^{-1}(s)} B(\lambda)/\lambda^{q_0} \frac{d\lambda}{\lambda} \Big\}^{p_0/q_0} \frac{ds}{s}.$$

By integral condition (2.4) and relation (2.6), it follows that

$$I_2^{p_0/q_0} \leqslant c \int_0^\infty m(f, s) s^{p_0} A(s)^{\frac{p_0}{\varepsilon}(1/q_0 - \gamma)}/s^{p_0} \frac{ds}{s}.$$

Thus since $\frac{p_0}{\varepsilon}(1/q_0 - \gamma) = 1$ and $A(s)/s \leqslant \frac{a(s)}{p_0}$, we finally obtain

$$I_2 \leqslant c \Big\{ \int_0^\infty m(f, s) a(s)\, ds \Big\}^{q_0/p_0}.$$

Let now $K = 2^{1/q_0} M_0 q_1^{1/q_0} k_{q_0}^{1/q_0} \vee 2^{1/q_1} \left(\frac{p_1}{p_0}\right)^{1/p_1} M_1 q_1^{1/q_1} k_{q_1}^{1/q_1}$. It then readily follows that

$$I \leqslant \tfrac{1}{2} \Big\{ \Big(\int_0^\infty m(f, s) a(s)\, ds \Big)^{p_0/q_0} + \Big(\int_0^\infty m(f, s) a(s)\, ds \Big)^{p_1/q_1} \Big\}$$

and consequently $\int_M A\big(|f(x)|\big) d\mu \leqslant 1$ implies that $\int_N B\big(|Tf(y)|/2K\big) d\nu \leqslant 1$ and T is of type (A, B) with norm $\leqslant 2K$.

If $q_1 = \infty$, the notions of weak type and type coincide. First assume $p_1 < \infty$. Since $A(s)/s^{p_1}$ decreases, we have that

$$\|Tf_u\|_\infty \leqslant M_1 |f_u|_{p_1} \leqslant M_1 \Big\{ p_1 \int_0^{u(\gamma)} \frac{s^{p_1}}{A(s)} A(s) m(f, s) \frac{ds}{s} \Big\}^{1/p_1}$$

$$\leqslant \left(\frac{p_1}{p_0}\right)^{1/p_1} M_1 u(\lambda)/A\big(u(\lambda)\big)^{1/p_1} \Big\{ \int_0^\infty a(s) m(f, s)\, ds \Big\}^{1/p_1}$$

$$\leqslant \left(\frac{p_1}{p_0}\right)^{1/p_1} M_1 A^{-1}\big(B(\lambda)^\varepsilon\big)/B(\lambda)^{\varepsilon p_1}.$$

But now $\gamma + \varepsilon/p_1 = 0$. Whence by (2.6) we have

$$\|Tf_u\|_\infty \leqslant \left(\frac{p_1}{p_0}\right)^{1/p_1} M_1 \lambda$$

and $m(Tf_u, K\lambda) = 0$ provided $K \geqslant (p_1/p_0)^{1/p_1} M_1$. We may bound I_2 as before provided there is a q, $q_0 < q < \infty$ such that $B(s)/s^q$ decreases.

This is our assumption. Thus setting $K = (p_1/p_0)^{1/p_1} M_1 \vee M_0 q^{1/q_0} k_{q_0}^{1/q_0}$ it follows that

$$I \leqslant \Big\{\int_0^\infty a(s)\, m(f, s)\, ds\Big\}^{p_0/q_0}.$$

Thus T is of type (A, B) with norm $\leqslant 2K$.

If $p_1 = q_1 = \infty$, then $\gamma = 0$ and $1/\varepsilon = q_0/p_0$. Therefore since

$$\|Tf_u\|_\infty \leqslant M_1 \|f_u\|_\infty = M_1 u(\lambda) = M_1 A^{-1}\big(B(\lambda)^{p_0/q_0}\big),$$

from (2.6) we obtain

$$\|Tf_u\|_\infty \leqslant M_1 \lambda$$

and $m(Tf_u, K\lambda) = 0$ whenever $K \geqslant M_1$. Set $K = M_1 \vee M_0 k_{q_0}^{1/q_0} q^{1/q_0}$. As before it follows that T is of type (A, B) with norm $\leqslant 2K$. We must now consider the case $0 < q_1 < q_0 \leqslant \infty$. Thus $\varepsilon < 0$ and if we set $u^{-1}(\lambda) = B^{-1}\big(A(\lambda)^{1/\varepsilon}\big)$, then $u(\lambda)$ will be monotone decreasing. Whence if as above we put $I \leqslant I_1' + I_2'$ the reader will have no difficulty in computing a bound for I by interchanging the roles of I_1' and I_2' with I_2 and I_1 respectively (see also the proof of Theorem 2.8).

The proof of the theorem is thus complete.

In the next results we use to advantage the assumption of strong type replacing that of weak type. The reader will have no difficulty in verifying that the behaviour of the constant K which gives the norm of the operation in the intermediate spaces improves considerably. In particular it remains bounded as we approach an endpoint where we have of strong type. It is well known that this is not the case in Theorem 2.3.

(2.8) THEOREM. *Let $0 < \beta_i = 1/q_i \leqslant \alpha_i = 1/p_i < \infty$, $i = 0, 1$, $p_0 \neq p_1$, $q_0 \neq q_1$ and let ε, γ be as in Theorem 2.3. Suppose that a sublinear operation $g = Tf$ is of weak type (p_0, q_0) and of type (p_1, q_1), with norms M_0 and M_1 respectively.*

Assume that the generalized Young's functions A, B are given by

$$A(t) = \int_0^t a(s)\, ds, \qquad B(t) = \int_0^t b(s)\, ds$$

with a and b monotone, and that

$$B^{-1}(t) = A^{-1}(t^\varepsilon)\, t^\gamma.$$

If $l = q_0 \vee q_1$ and $m = q_0 \wedge q_1$, then further assume that $B(t)/t^l$ decreases and $B(t)/t^m$ increases. If $q_1 = l$ suppose that

$$\int_0^t \big(B(s)/s^{q_0}\big) \frac{ds}{s} \leqslant k_{q_0} B(t)/t^{q_0}$$

and if $q_1 = m$ suppose that

$$\int_t^\infty \big(B(s)/s^{q_0}\big) \frac{ds}{s} \leqslant k_{q_0} B(t)/t^{q_0}.$$

Then T is of type (A, B) with norm $K = K(k_{q_0}, M_i, p_i, q_i)$.

Proof. We omit the case $q_i = \infty$ since it has already been covered in Theorem 2.3. Out of the several possible cases according to the different values of p_i, q_i, we assume that $0 < p_0 < p_1 < \infty$ and $0 < q_1 < q_0 < \infty$, the consideration of the other cases being similar. Thus $q_0 = l$, $q_1 = m$ and $\varepsilon < 0$ and if we set $u^{-1}(\lambda) = B^{-1}\big(A(\lambda)^{1/\varepsilon}\big)$, then u^{-1} is a monotone decreasing function of λ. Let $f = f_u + f^u$. Then if f is in L_A, $g = Tf$ is defined and

$$\begin{aligned} I &= \int_N B\big(|Tf(y)|/2K\big)\, d\nu \\ &\leqslant \int_0^\infty m(Tf_u, K\lambda)\, b(\lambda)\, d\lambda + \int_0^\infty m(Tf^u, K\lambda)\, b(\lambda)\, d\lambda = I_1 + I_2. \end{aligned}$$

Since $B(\lambda)/\lambda^{q_0}$ decreases, we have that $b(\lambda)\lambda \leqslant q_0 B(\lambda)$ and

$$\begin{aligned} I_1 &= -\int_0^\infty \frac{d}{d\lambda}\Big(q_1 \int_{\lambda K}^\infty m(Tf_u, s)\, s^{q_1} \frac{ds}{s}\Big) \frac{b(\lambda)\lambda}{q_1(\lambda K)^{q_1}}\, d\lambda \\ &\leqslant -\frac{q_0}{q_1 K^{q_1}} \int_0^\infty \frac{d}{d\lambda}\Big(q_1 \int_{\lambda K}^\infty m(Tf_u, s)\, s^{q_1} \frac{ds}{s}\Big) \frac{B(\lambda)}{\lambda^{q_1}}\, d\lambda \\ &= -\frac{q_0}{q_1 K^{q_1}}\, q_1 \int_{\lambda K}^\infty m(Tf_u, s)\, s^{q_1} \frac{ds}{s} \frac{B(\lambda)}{\lambda^{q_1}}\Big|_0^\infty + \\ &\quad + \frac{q_0}{q_1 K^{q_1}} \int_0^\infty q_1 \int_{\lambda K}^\infty m(Tf_u, s)\, s^{q_1} \frac{ds}{s}\, d\big(B(\lambda)/\lambda^{q_1}\big) \\ &= J_1 + J_2. \end{aligned}$$

Let $\varphi(\lambda) = q_1 \int_{\lambda K}^\infty m(Tf_u, s)\, s^{q_1} \frac{ds}{s} \frac{B(\lambda)}{\lambda^{q_1}}$. Since T is of type (p_1, q_1), norm M_1, and $A(s)/s^{p_1}$ decreases, we have that

$$\begin{aligned} (2.9)\qquad \varphi(\lambda) &\leqslant M_1^{q_1} \Big\{p_1 \int_0^{u(\lambda)} m(f, s)\, s^{p_1} \frac{ds}{s}\Big\}^{q_1/p_1} \frac{B(\lambda)}{\lambda^{q_1}} \\ &\leqslant M_1^{q_1} p_1^{q_1/p_1} \frac{u(\lambda)^{q_1}}{A\big(u(\lambda)\big)^{q_1/p_1}} \Big\{\int_0^{u(\lambda)} m(f, s)\, A(s) \frac{ds}{s}\Big\}^{q_1/p_1} \frac{B(\lambda)}{\lambda^{q_1}}. \end{aligned}$$

Whence by the choice of u, the relation $B^{-1}(t) = t^{\gamma}A^{-1}(t^{\varepsilon})$ and the fact that $\varepsilon q_1/p_1+\gamma q_1 = 1$, we have that

$$\varphi(\lambda) \leqslant M_1^{q_1}\left(\frac{p_1}{p_0}\right)^{q_1/p_1}\left\{\int_0^\infty m(f,s)a(s)\,ds\right\}^{q_1/p_1} = L. \tag{2.10}$$

Thus $J_1 \leqslant Lq_0/q_1K^{q_1}$.

Since $d(B(\lambda)/\lambda^{q_1}) \geqslant 0$, we have that

$$\frac{q_1}{q_0}K^{q_1}J_2 = \int_0^\infty q_1\int_{\lambda K}^\infty m(Tf_u,s)s^{q_1}\frac{ds}{s}\,d(B(\lambda)/\lambda^{q_1})$$

$$\leqslant \int_0^\infty M_1^{q_1}|f_u|_{p_1}^{q_1}d(B(\lambda)/\lambda^{q_1})$$

$$= M_1^{q_1}\int_0^\infty\left\{p_1\int_0^{u(\lambda)} m(f,s)s^{p_1}\frac{ds}{s}\right\}^{q_1/p_1}d(B(\lambda)/\lambda^{q_1})$$

$$= M_1^{q_1}\left\{p_1\int_0^{u(\lambda)} m(f,s)s^{p_1}\frac{ds}{s}\right\}^{q_1/p_1}B(\lambda)/\lambda^{q_1}\Big|_0^\infty -$$

$$-M_1^{q_1}p_1^{q_1/p_1}\int_0^\infty (B(\lambda)/\lambda^{q_1})\frac{d}{d\lambda}\left(\int_0^{u(\lambda)} m(f,s)s^{p_1}\frac{ds}{s}\right)^{q_1/p_1}d\lambda = J_3+J_4.$$

As in (2.10) and (2.11) it follows that $J_3 \leqslant L$.

As for J_4 we have

$$J_4/M_1^{q_1}p_1^{q_1/p_1}$$

$$= \int_0^\infty (B(\lambda)/\lambda^{q_1})\frac{q_1}{p_1}\left(\int_0^{u(\lambda)} m(f,s)s^{p_1}\frac{ds}{s}\right)^{q_1/p_1-1}m(f,u(\lambda))u(\lambda)^{p_1-1}du(\lambda)$$

and setting $u(\lambda) = t$ it follows that

$$J_4/M_1^{q_1}p_1^{q_1/p_1-1}q_1$$

$$= \int_0^\infty B(u^{-1}(t))/(u^{-1}(t))^{q_1}\left(\int_0^t m(f,s)s^{p_1}\frac{ds}{s}\right)^{q_1/p_1-1}m(f,t)t^{p_1}\frac{dt}{t}$$

$$\leqslant \int_0^\infty \frac{B(u^{-1}(t))}{(u^{-1}(t))^{q_1}}\left(\frac{t^{p_1}}{A(t)}\right)^{q_1/p_1-1}\left\{\int_0^t m(f,s)A(s)\frac{ds}{s}\right\}^{q_1/p_1-1}m(f,t)t^{p_1}\frac{dt}{t}$$

$$\leqslant \int_0^\infty A(t)\left\{\int_0^t m(f,s)A(s)\frac{ds}{s}\right\}^{q_1/p_1-1}m(f,t)\frac{dt}{t}$$

since

$$\frac{B(u^{-1}(t))}{(u^{-1}(t))^{q_1}}\left(\frac{t^{p_1}}{A(t)}\right)^{q_1/p_1-1}t^{p_1} = A(t)^{1+1/\varepsilon-q_1\gamma/\varepsilon-q_1/p_1}t^{p_1(q_1/p_1-1+1)-q_1} = A(t).$$

Thus $J_4 \leqslant L$.

Therefore combining the above estimates we have

$$J_2 \leqslant \frac{q_0}{q_1}\frac{1}{K^{q_1}}(L+L) = 2Lq_0/q_1K^{q_1},$$

and

$$I_1 \leqslant J_1+J_2 \leqslant 3q_0M_1^{q_1}\left(\frac{p_1}{p_0}\right)^{q_1/p_1}\left(\int_0^\infty m(f,s)a(s)\,ds\right)^{q_1/p_1}q_1K^{q_1}.$$

The term I_2 is taken care of as in Theorem 2.3. Thus for an appropriate constant K we have that $I \leqslant 1$ whenever $\int_M A(|f(x)|)\,d\mu \leqslant 1$, and T is of type (A, B) with norm $\leqslant K$. The reader will have no difficulty in supplying the proof for the remaining cases.

The proof of the theorem is thus complete.

(2.11) THEOREM. *Let $0 \leqslant 1/q_i \leqslant 1/p_i < \infty$, $i = 0, 1$, $q_0 \neq q_1$, and let ε and γ be as in Theorem 2.3. Suppose that a sublinear operation $g = Tf$ is of types (p_i, q_i), $i = 0, 1$, with norms M_0 and M_1 respectively.*

Assume that the generalized Young's functions A, B are given by

$$A(t) = \int_0^t a(s)\,ds, \qquad B(t) = \int_0^t b(s)\,ds$$

with a, b monotone, and that

$$B^{-1}(t) = A^{-1}(t^{\varepsilon})t^{\gamma}.$$

If $l = q_0 \vee q_1 < \infty$ and $m = q_0 \wedge q_1$, then further suppose that $B(t)/t^l$ decreases and $B(t)/t^m$ increases. If $l = \infty$ suppose there is a q, $q > m$, such that $B(t)/t^q$ decreases. Then T is of type (A, B).

The main ideas used to prove the theorem are already contained in Theorem 2.8. The proof is therefore left for the interested reader to verify.

3. The classes $\Lambda(\varphi_X, C)$ and interpolation. In this section we give the abstract formulation of Theorem 2.3. We need to introduce some preliminary material first.

(3.1) *Rearrangement invariant spaces* (r.i. spaces). Let (M, μ) be a totally σ-finite, positive measure space. Let f be measurable and suppose that $m(f, \lambda)$ is finite for each $\lambda > 0$. We can then define the *non-increasing rearrangement* f^* of f as

$$f^*(t) = \inf\{\lambda \geqslant 0 : m(f,\lambda) \leqslant t\}, \quad \inf\emptyset = 0. \tag{3.2}$$

The following properties of f^* are readily verified:

(i) If $|f_1| \leqslant |f_2|$, then $f_1^* < f_2^*$;

(ii) If $t = t_1 + t_2$, then $(f+g)^*(t) \leqslant f^*(t_1) + g^*(t_2)$.

If the function f is locally integrable, then f^* is integrable on every finite interval and we define

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds.$$

The following properties of f^{**} will be used later:

(i) $f^*(t) \leqslant f^{**}(t)$;

(ii) $tf^{**}(t) = \sup\{\int_E |f(x)|\,d\mu\colon\ \mu(E) \leqslant t\}$.

A Banach space X of real valued, Lebesgue measurable functions on a possibly infinite interval $I = (0, l)$ is said to be a *function space* if the following conditions hold:

(i) If $|f| \leqslant |g|$ a.e. and $f \in X$, then $g \in X$ and $\|g\| \leqslant \|f\|$;

(ii) If $\{f_n\}_{n=1}^{\infty} \subset X$ and $\|f_n\| \leqslant M$ and $0 \leqslant f_n \nearrow f$ a.e., then $f \in X$ and $\|f\| \leqslant M$.

A function space X is said to be a *rearrangement invariant space* (r.i. space) if whenever $f \in X$ and f' is any function on I equimeasurable with f, then $f' \in X$ and $\|f'\| = \|f\|$. Examples of r.i. spaces include the Lebesgue L^p-spaces, the Orlicz L_A-spaces and the Lorentz spaces Λ, M and $L(p, q)$. Also if X, Y are r.i. spaces, so is $X+Y$.

The *fundamental function* $\varphi_X(t)$ of an r.i. space is defined as $\varphi_X(t) = \|\chi_{[0,t]}\|$, $0 < t$, where $\chi_{[0,t]}$ is the characteristic function of the interval $[0, t]$. Let X' be the (Banach) space of all measurable functions g on I such that

$$\|g\|_{X'} = \sup\{\left|\int_I f(s)g(s)\,ds\right|\colon\ \|f\| \leqslant 1\} < \infty.$$

Then again X' is a r.i. space and $X'' = X$.

Moreover, the following properties hold

(i) $\varphi_X(t)\varphi_{X'}(t) = t$ (cf. (1.3));

(ii) $\varphi_X(t)$ is a continuous, increasing function, which is absolutely continuous on $[\varepsilon, \infty)$ for each positive ε with $\dfrac{d\varphi_X(t)}{dt} \leqslant \varphi_X(t)/t$ a.e.;

(iii) X has an equivalent r.i. norm $\|\cdot\|_0$ such that the fundamental function $\varphi_{X_0}(t)$ is concave and, moreover, $\varphi_X(t) \leqslant \varphi_{X_0}(t) \leqslant \varphi_X(2t)$, $0 < t < \infty$ (cf. (1.5)).

The following two conditions are convenient to deal with the technical difficulties which arise. We say that $X \in U$ if for some $0 < \alpha < 1$ there are positive constants δ and θ such that

$$\varphi_X(u)/\varphi_X(v) \leqslant \theta(u/v)^{\alpha} \quad \text{for} \quad v/u \geqslant \delta. \tag{3.3}$$

We say that $X \in \mathscr{L}$ if for some $0 < \beta < 1$ there are positive constants γ and θ such that

$$\varphi_X(v)/\varphi_X(u) \leqslant \theta(v/u)^{\beta} \quad \text{if} \quad v/u \leqslant \gamma. \tag{3.4}$$

Since for the Orlicz class L_A we have $\varphi_{L_A}(t) = 1/A^{-1}(1/t)$ ([20], Lemma 2.6), (3.3) essentially reduces to $A^{-1}(t)/t^{\alpha}$ decreases and (3.4) to $A^{-1}(t)/t^{\beta}$ increases.

Spaces X which are in $U \cap \mathscr{L}$ can be renormed so that the fundamental function φ_X satisfies

(iv) $\dfrac{d\varphi_X(t)}{dt} \approx \varphi_X(t)/t$.

Let $\Lambda(X) = \{f\colon f^*(t)$ exists and $\|f\|_{\Lambda(X)} = \int_0^{\infty} f^*(t)\,d\varphi_X(t) < \infty\}$, and

$M(X) = \{f\colon f^{**}(t)$ exists and $\|f\|_{M(X)} = \sup_{t>0}\left(f^{**}(t)\varphi_X(t)\right) < \infty\}$.

It is known that $\Lambda(X) \subset X \subset M(X)$, with continuous embeddings. Since for $X = L^p$ we have $\varphi_X(t) = t^{1/p}$, it follows that $\Lambda(X) = L(p, 1)$ and $M(X) = L(p, \infty)$. The classes $\Lambda(X)$ are described in [22] and [28], for instance.

A sublinear operation T is said to be of *weak type* (X, Y) if T maps $\Lambda(X)$ into $M(Y)$ and there is a constant K such that

$$\sup_{t>0}\{(Tf)^*(t)\varphi_Y(t)\} \leqslant K\|f\|_{\Lambda(X)}. \tag{3.5}$$

The smallest constant K for which (3.3) holds is called the *norm* of T. By Theorem 2.5 of [22] (cf. [6], Theorem 7, and [8], (2.5)) if $Y \in U$, then (3.3) holds if and only if

(3.6) for any measurable set $E \subset M$ we have

$$\sup_{t>0}\{(T\chi_E)^*(t)\varphi_Y(t)\} \leqslant K\varphi_X(\mu(E)),$$

where χ_E is the characteristic function of E and $\mu(E)$ is its measure.

We now introduce the classes $\Lambda(\varphi_X, C)$ which will arise as intermediate spaces when we interpolate operations of weak type (X_i, Y_i), $i = 0, 1$.

For a generalized Young's function C and the measure space $((0, \infty), dt/t)$ let $L_C = L_C((0, \infty), dt/t)$. We then set

$$\Lambda(\varphi_X, C) = \{f\colon f^{**}(t)\varphi_X(t) \text{ is in } L_C\},$$

and we put

$$|f|_{\Lambda(\varphi_X, C)} = |f^{**}\varphi_X|_C.$$

It is readily seen that if f and g are in $\Lambda(\varphi_X, C)$, then so is $f+g$. If $C(t)/t$ increases, then $|f|_{\Lambda(\varphi_X, C)} = \|f^{**}\varphi_X\|_C$ is a norm on $\Lambda(\varphi_X, C)$, and with that norm $\Lambda(\varphi_X, C)$ is a Banach space (this statement is a particular instance of Section 13.4 of [5]). If $C(t)$ is concave, then we may set

$$d(f, g) = \int_0^\infty C\big((f-g)^{**}(t)\varphi_X(t)\big)\frac{dt}{t}.$$

For some functions C, $(\Lambda(\varphi_X, C), d)$ becomes a complete metric space (cf. [8], Section 2). Since we shall not make use of these facts, we omit further comments. The statement of our results will, however, involve functionals closely related to the definition of $|f|_{\Lambda(\varphi_X, C)}$. Thus the following is of interest to us.

(3.7) THEOREM (cf. [6], Theorem 6, [22], Lemma 3.1). *Let C be a Young's function such that $C(t)/t^p$ decreases for some p; then*

$$\int_0^\infty C\big(f^*(t)\varphi_X(t)\big)\frac{dt}{t} \approx \int_0^\infty C\big(f^{**}(t)\varphi_X(t)\big)\frac{dt}{t}, \tag{3.8}$$

provided there is a constant θ (which we may assume $\geqslant 1$) such that

$$\frac{1}{\varphi_{X'}(t)}\int_0^\infty \varphi_{X'}(s)\frac{ds}{s} \leqslant \theta, \qquad t>0, \tag{3.9}$$

and

$$\varphi_{X'}(t)\int_0^\infty \big(1/\varphi_{X'}(s)\big)\frac{ds}{s} \leqslant \theta, \qquad t>0. \tag{3.10}$$

Proof. Since $f^*(t) \leqslant f^{**}(t)$, one of the inequalities is obvious. Let

$$g(t) = f^{**}(t)\varphi_X(t) = \frac{\varphi_X(t)}{t}\int_0^t f^*(s)\,ds.$$

Therefore since $\varphi_X(s)\varphi_{X'}(s) = s$, we have that

$$g(t) = \frac{1}{\varphi_{X'}(t)}\int_0^t f^*(s)\varphi_X(s)\varphi_{X'}(s)\frac{ds}{s},$$

and by Jensen's inequality and (3.7) it follows that

$$C\big(g(t)\big) \leqslant \frac{1}{\theta\varphi_{X'}(t)}\int_0^t C\big(\theta f^*(s)\varphi_X(s)\big)\varphi_{X'}(s)\frac{ds}{s}.$$

But since $C(t)/t^p$ decreases, we have that $C(\theta t) \leqslant \theta^p C(t)$. Thus by interchanging the order of integration we obtain

$$\int_0^\infty C\big(g(t)\big)\frac{dt}{t} \leqslant \theta^{p-1}\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\varphi_{X'}(s)\int_s^\infty \big(1/\varphi_{X'}(t)\big)\frac{dt}{t}\frac{ds}{s}$$

$$\leqslant \theta^{p-1}\theta\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s}.$$

The proof is complete.

To prove a similar result when C is concave we shall make use of the following lemma.

(3.11) LEMMA. *Suppose that f and φ are monotone, positive functions. Further assume that f is decreasing and that φ is increasing and concave. Let C be a concave generalized Young's function. Then for $t>0$ we have*

$$\int_0^t f(s)\varphi(s)\frac{ds}{s} \leqslant \frac{\ln 2}{2}C^{-1}\left(\frac{2}{\ln 2}\int_0^t C\big(f(s)\varphi(s)\big)\frac{ds}{s}\right) \tag{3.12}$$

and

$$\int_t^\infty f(s)\varphi(s)\frac{ds}{s} \leqslant 2\ln 2\, C^{-1}\left(\frac{1}{\ln 2}\int_{t/2}^\infty C\big(f(u)\varphi(u)\big)\frac{du}{u}\right). \tag{3.13}$$

Proof. Let $I(t) = \int_0^t C\big(f(u)\varphi(u)\big)\frac{du}{u}$. Then for $s \leqslant t$ we have

$$I(t) \geqslant \int_{s/2}^s C\big(f(u)\varphi(u)\big)\frac{du}{u} \geqslant C\big(f(s)\varphi(s/2)\big)\ln 2$$

$$\geqslant \frac{\ln 2}{2}C\big(f(s)\varphi(s)\big)$$

since both $C(t)/t$ and $\varphi(t)/t$ decrease. Thus for $s \geqslant t$ we have

$$f(s)\varphi(s) \leqslant C^{-1}\left(\frac{2}{\ln 2}I(t)\right).$$

Let now $D(t)$ be the increasing function given by

$$C(t)D(t) = t.$$

Then

$$\int_0^t f(s)\varphi(s)\frac{ds}{s} = \int_0^t C\big(f(s)\varphi(s)\big)D\big(f(s)\varphi(s)\big)\frac{ds}{s}$$

$$\leqslant D\left(C^{-1}\left(\frac{2}{\ln 2}I(t)\right)\right)I(t) = \frac{\ln 2}{2}C^{-1}\left(\frac{2}{\ln 2}I(t)\right),$$

as we wished to show.

Similarly, let $J(t) = \int_{t/2}^{\infty} C\big(f(u)\varphi(u)\big)\frac{du}{u}$. Then for $s \geqslant t/2$ we have

$$J(t) \geqslant \int_s^{2s} C\big(f(u)\varphi(u)\big)\frac{du}{u} \geqslant C\big(f(2s)\varphi(s)\big)\ln 2$$

and

$$f(2s)\varphi(s) \leqslant C^{-1}\left(\frac{1}{\ln 2}J(t)\right).$$

Thus

$$\int_t^{\infty} f(s)\varphi(s)\frac{ds}{s} = \int_{t/2}^{\infty} f(2s)\varphi(2s)\frac{ds}{s}$$

$$\leqslant 2\int_{t/2}^{\infty} f(2s)\varphi(s)\frac{ds}{s}$$

$$\leqslant 2D\left(C^{-1}\left(\frac{1}{\ln 2}J(t)\right)\right)\int_{t/2}^{\infty} C\big(f(2u)\varphi(u)\big)\frac{du}{u}$$

$$\leqslant 2D\left(C^{-1}\left(\frac{1}{\ln 2}J(t)\right)\right)J(t).$$

Therefore

$$\int_t^{\infty} f(s)\varphi(s)\frac{ds}{s} \leqslant 2\ln 2\, C^{-1}\left(\frac{1}{\ln 2}\int_{t/2}^{\infty} C\big(f(u)\varphi(u)\big)\frac{du}{u}\right).$$

This is the desired conclusion.

(3.14) THEOREM. *Let C be concave. Then* (3.8) *holds provided there is a constant $\delta > 0$ such that*

$$C(st) \leqslant \delta C(s)C(t), \qquad s, t \geqslant 0, \tag{3.15}$$

and for some $\gamma > 0$

$$\int_t^{\infty} C\big(\ln 2/2\varphi_{X'}(s)\big)\frac{ds}{s} \leqslant \gamma/C\big(\varphi_{X'}(t)\big), \qquad t > 0. \tag{3.16}$$

Proof. Let $g(t)$ be as in Theorem 3.7. Then we apply (3.12) with $\varphi(s) = s$ to obtain

$$\int_0^{\infty} C\big(g(t)\big)\frac{dt}{t} \leqslant \int_0^{\infty} C\left(\frac{1}{\varphi_{X'}(t)}\frac{\ln 2}{2}C^{-1}\left(\frac{2}{\ln 2}\int_0^t C\big(f^*(s)s\big)\frac{ds}{s}\right)\right)\frac{dt}{t}$$

$$\leqslant \frac{2\delta^2}{\ln 2}\int_0^{\infty} C\left(\frac{\ln 2}{2\varphi_{X'}(t)}\right)\int_0^t C\big(f^*(s)\varphi_X(s)\big)C\big(\varphi_{X'}(s)\big)\frac{ds}{s}\frac{dt}{t}.$$

Whence inverting the order of integration and by (3.16) it follows that

$$\int_0^{\infty} C\big(g(t)\big)\frac{dt}{t} \leqslant \frac{2}{\ln 2}\delta^2\gamma\int_0^{\infty} C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s}.$$

This proves our result.

We shall now prove the interpolation theorems. The assumption that the spaces in question are in $U \cap \mathscr{L}$ is now made for the remainder of the paper, this simplifies some of the proofs.

(3.17) THEOREM. *Let T be a sublinear operation simultaneously of weak types (X_i, Y_i), with norm K_i, $i = 0, 1$. Let $\varphi_{X_i}, \varphi_{Y_i}$ denote the corresponding fundamental functions. Set $\eta(t) = \varphi_{X_0}(t)/\varphi_{X_1}(t)$, $\xi(t) = \varphi_{Y_0}(t)/\varphi_{Y_1}(t)$ and suppose that η and ξ are strictly monotone increasing assuming values from 0 to ∞. Put $\psi(t) = \eta^{-1}\big(\xi(t)\big)$. Thus ψ is strictly monotone increasing taking values from 0 to ∞. Assume that X is* r.i. *space with fundamental function φ_X such that φ_X/φ_{X_0} decreases and φ_X/φ_{X_1} increases, Let φ_Y, the fundamental function of a* r.i. *space Y, be given by $\varphi_Y(t) = \varphi_X\big(\psi(t)\big)\varphi_{Y_0}(t)/\varphi_{X_0}\big(\psi(t)\big)$. Further assume that*

$$\int_0^t \varphi_{X_0}(s)/\varphi_X(s)\frac{ds}{s} \leqslant \varphi_{X_0}(t)/\varphi_X(t) \tag{3.18}$$

and that

$$\int_t^{\infty} \varphi_Y(s)/\varphi_{Y_0}(s)\frac{ds}{s} \leqslant \varphi_Y(t)/\varphi_{Y_0}(t), \qquad t > 0. \tag{3.19}$$

We also assume that for the positive constant θ,

$$\int_0^t \varphi_{X_1}(s)\frac{ds}{s} \leqslant \theta\varphi_{X_1}(t),$$

actually this condition holds because X_1 is in $U \cap \mathscr{L}$. Furthermore assume that

$$\int_t^{\infty} \varphi_{X_1}(s)\varphi_X(s)\frac{ds}{s} \leqslant \varphi_{X_1}(t)/\varphi_X(t), \qquad t > 0, \tag{3.18'}$$

and

$$\int_0^t \varphi_Y(s)/\varphi_{Y_1}(s)\frac{ds}{s} \leqslant \varphi_Y(t)/\varphi_{Y_1}(t), \quad t > 0. \tag{3.19'}$$

Then, if C is a Young's function such that $C(t)/t^p$ decreases for some $p < \infty$, there is a constant K independent of f but depending on K_i, θ, etc. such that

$$\int_0^\infty C\big((Tf)^*(t)\varphi_Y(t)\big)\frac{dt}{t} \leqslant K\int_0^\infty C\big(f^*(t)\varphi_X(t)\big)\frac{dt}{t}.$$

Proof. For a positive number u consider the decomposition of f given by $f = f_u + f^u$, where $f_u = f$ if $|f| < u$ and zero otherwise. It is known, and easily seen, that

$$(f^u)^*(t) \leqslant \begin{cases} f^*(t) & \text{if} \quad t \leqslant m(f, u), \\ 0 & \text{if} \quad t > m(f, u), \end{cases}$$

and

$$(f_u)^*(t) \leqslant \begin{cases} u & \text{if} \quad t \leqslant m(f, u), \\ f^*(t) & \text{if} \quad t > m(f, u). \end{cases}$$

It is also readily seen that if $\int_0^\infty C\big(f^*(t)\varphi_X(t)\big)\frac{dt}{t} < \infty$, then f_u is in $\Lambda(X_1)$ and f^u is in $\Lambda(X_0)$. Thus Tf if defined. Set $u = f^*\big(\psi(t)\big)$. Since T is of weak type (X_0, Y_0) with norm K_0, it follows that

$$\begin{aligned}(Tf^u)^*(t)\varphi_Y(t) &\leqslant K_0\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\int_0^\infty (f^u)^*(s)\varphi_{X_0}(s)\frac{ds}{s} \\ &\leqslant K_0\int_0^{\psi(t)} f^*(s)\varphi_X(s)\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\frac{\varphi_{X_0}(s)}{\varphi_X(s)}\frac{ds}{s} = K_0 I(t).\end{aligned}$$

By Jensen's inequality and (3.13) it follows that

$$\begin{aligned}\int_0^\infty C\big((Tf^u)^*(t)\varphi_Y(t)\big)\frac{dt}{t} &\leqslant \int_0^\infty\int_0^{\psi(t)} C\big(K_0 f^*(s)\varphi_X(s)\big)\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\frac{\varphi_{X_0}(s)}{\varphi_X(s)}\frac{ds}{s}\frac{dt}{t} \\ &= \int_0^\infty C\big(K_0 f^*(s)\varphi_X(s)\big)\Big\{\int_{\psi^{-1}(s)}^\infty \frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\frac{dt}{t}\Big\}\frac{\varphi_{X_0}(s)}{\varphi_X(s)}\frac{ds}{s} \\ &\leqslant (1 \vee K_0^p)\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s}\end{aligned}$$

on account of (3.19), the definition of φ_Y, and the fact that $C(t)/t^p$ decreases for $p < \infty$. Also since T is of weak type (X_1, Y_1), norm K_1, we have

$$\begin{aligned}(Tf_u)^*(t)\varphi_Y(t) &\leqslant K_1\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\int_0^\infty (f_u)^*(s)\varphi_{X_1}(s)\frac{ds}{s} \\ &\leqslant K_1\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\int_0^{\psi(t)} f^*\big(\psi(t)\big)\varphi_{X_1}(s)\frac{ds}{s} + K_1\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\int_{\psi(t)}^\infty f^*(s)\varphi_X(s)\frac{\varphi_{X_1}(s)}{\varphi_X(s)}\frac{ds}{s} \\ &= K_1\big(I_1(t) + J(t)\big).\end{aligned}$$

By (3.14) we have that

$$I_1(t) \leqslant \theta\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)} f^*\big(\psi(t)\big)\varphi_{X_1}\big(\psi(t)\big) \leqslant \theta f^*\big(\psi(t)\big)\varphi_X\big(\psi(t)\big)$$

ince as it is readily seen also

$$\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)} = \frac{\varphi_X\big(\psi(t)\big)}{\varphi_{X_1}\big(\psi(t)\big)}.$$

Then it follows that

$$I_1(t) \leqslant \frac{2}{\ln 2}\theta I(t).$$

Indeed, since φ_{X_0} increases and $\varphi_{X_0}(s)/s$ decreases, we have

$$\begin{aligned}f^*\big(\psi(t)\big)\varphi_X\big(\psi(t)\big) &= f^*\big(\psi(t)\big)\varphi_{X_0}\big(\psi(t)\big)\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)} \\ &\leqslant \frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\frac{1}{\ln 2}\int_{\psi(t)/2}^{\psi(t)} f^*(s)\varphi_{X_0}(2s)\frac{ds}{s} \\ &\leqslant \frac{2}{\ln 2}\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\int_0^{\psi(t)} f^*(s)\varphi_{X_0}(s)\frac{ds}{s} = \frac{2}{\ln 2}I(t).\end{aligned}$$

Now to bound $J(t)$ we use again Jensen's inequality. By Jensen's inequality, (3.18') and (3.19') we have

$$\begin{aligned}\int_0^\infty C\big(K_1 J(t)\big)\frac{dt}{t} &\leqslant \int_0^\infty\int_{\psi(t)}^\infty C\big(K_1 f^*(s)\varphi_X(s)\big)\frac{\varphi_{X_1}(s)}{\varphi_X(s)}\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\frac{ds}{s}\frac{dt}{t} \\ &= \int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{\varphi_{X_1}(s)}{\varphi_X(s)}\int_0^{\psi^{-1}(s)}\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\frac{dt}{t}\frac{ds}{s} \\ &\leqslant K\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s}.\end{aligned}$$

In view of the above estimates and the sublinearity of T it follows that

$$\int_0^\infty C\big(Tf^*(t)\varphi_Y(t)\big)\frac{dt}{t} = \int_0^\infty C\big((Tf)^*(2t)\varphi_Y(2t)\big)\frac{dt}{t}$$
$$\leqslant \int_0^\infty C\big(2[(Tf_u)^*(t)+(Tf^u)^*(t)]\varphi_Y(t)\big)\frac{dt}{t}$$
$$\leqslant K\Big\{\int_0^\infty C\big((Tf_u)^*(t)\varphi_Y(t)\big)\frac{dt}{t}+\int_0^\infty C\big((Tf^u)^*(t)\varphi_Y(t)\big)\frac{dt}{t}\Big\}.$$

In view of the above inequalities the proof of our theorem is complete.

(3.20) THEOREM. *Let* $T, C, \varphi_{X_i}, \varphi_{Y_i}, \varphi_X, \varphi_Y, \eta, \xi$, *and* ψ *be defined as in Theorem* 3.17, *but assume that now* ξ *decreases. Thus* ψ *also is a monotone decreasing function from* ∞ *to zero. Further assume that* (3.18) *and* (3.18′) *hold and that we also have*

$$(3.19'')\qquad \int_0^t \varphi_Y(s)/\varphi_{Y_0}(s)\frac{ds}{s}\leqslant \varphi_Y(t)/\varphi_{Y_0}(t),\qquad t>0$$

and

$$(3.19''')\qquad \int_t^\infty \varphi_Y(s)/\varphi_{Y_1}(s)\frac{ds}{s}\leqslant \varphi_Y(t)/\varphi_{Y_1}(t),\qquad t>0.$$

Then there is a constant K *such that*

$$\int_0^\infty C\big((Tf)^*(t)\varphi_Y(t)\big)\frac{dt}{t}\leqslant K\int_0^\infty C\big(f^*(t)\varphi_X(t)\big)\frac{dt}{t}.$$

The proof being analogous to that of Theorem 3.17 is omitted.

(3.21) COROLLARY. *If we are either in the situation described in Theorem* 3.17 *or Theorem* 3.20, *then there is a constant* K *such that*

$$|Tf|_{\Lambda(\varphi_Y, C)}\leqslant K|f|_{\Lambda(\varphi_X, C)}.$$

We now prove a similar result for the case when C is a concave generalized Young's function.

(3.22) THEOREM. *Let* $T, \varphi_{X_i}, \varphi_{Y_i}, \varphi_X, \varphi_Y, \eta, \xi$ *and* ψ *be defined as in Theorem* 3.17. *Let* C *be a concave generalized Young's function such that*

$$C(st)\leqslant \delta C(s)C(t),\qquad s,t\geqslant 0,$$

and

$$(3.23)\qquad \int_t^\infty C\left(\frac{\varphi_Y(s)}{\varphi_{Y_0}(s)}\right)\frac{ds}{s}\leqslant \gamma/C\big(\varphi_{Y_0}(t)/\varphi_Y(t)\big),$$

and

$$(3.24)\qquad \int_0^t C\left(\frac{\varphi_Y(s)}{\varphi_{Y_1}(s)}\right)\frac{ds}{s}\leqslant \gamma/C\big(\varphi_{Y_1}(t)/\varphi_Y(t)\big).$$

Then the conclusion of Theorem 3.17 *holds.*

Proof. It will be sufficient to show that

$$\int_0^\infty C\left(\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\int_0^{\psi(t)} f^*(s)\varphi_{X_0}(s)\frac{ds}{s}\right)\frac{dt}{t}+\int_0^\infty C\left(\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\int_{\psi(t)}^\infty f^*(s)\varphi_{X_1}(s)\frac{ds}{s}\right)\frac{dt}{t}$$
$$=I+J\leqslant K\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s},$$

for then the conclusion will follow as in Theorem 3.17.

Let

$$g(t)=\int_0^{\psi(t)} f^*(s)\varphi_{X_0}(s)\frac{ds}{s}.$$

Then by (3.12) it follows that

$$g(t)\leqslant \frac{\ln 2}{2}C^{-1}\left(\frac{2}{\ln 2}\int_0^{\psi(t)} C\big(f^*(s)\varphi_{X_0}(s)\big)\frac{ds}{s}\right)=\frac{\ln 2}{2}C^{-1}\left(\frac{2}{\ln 2}I(t)\right).$$

Thus by (3.15) and (3.23) it follows that

$$I\leqslant \int_0^\infty C\left(\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}g(t)\right)\frac{dt}{t}\leqslant \delta\int_0^\infty C\left(\frac{\ln 2}{2}\,\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\right)\frac{2}{\ln 2}I(t)\frac{dt}{t}$$
$$\leqslant \delta^2\frac{2}{\ln 2}\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)C\left(\frac{\varphi_{X_0}(s)}{\varphi_X(s)}\right)\int_{\psi^{-1}(s)}^\infty C\left(\frac{\ln 2}{2}\,\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\right)\frac{dt}{t}\,\frac{ds}{s}$$
$$\leqslant \delta^2\frac{2}{\ln 2}\gamma\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s}.$$

Let $h(t)=\int_{\psi(t)}^\infty f^*(s)\varphi_{X_1}(s)\frac{ds}{s}$. Then by (3.13) it follows that

$$h(t)\leqslant 2\ln 2\,C^{-1}\left(\frac{1}{\ln 2}\int_{\psi(t)/2}^\infty C\big(f^*(s)\varphi_{X_1}(s)\big)\frac{ds}{s}\right)=2\ln 2\,C^{-1}\left(\frac{1}{\ln 2}J(t)\right).$$

In view of (3.15) and (3.24) we have that

$$\int_0^\infty C\left(\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}h(t)\right)\frac{dt}{t} \leqslant \delta\int_0^\infty C\left(2\ln 2\,\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\right)\frac{1}{\ln 2}\,J(t)\frac{dt}{t}$$

$$\leqslant \frac{\delta^2}{\ln 2}\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\,C\left(\frac{\varphi_{X_1}(s)}{\varphi_X(s)}\right)\int_0^{\psi^{-1}(2s)} C\left(2\ln 2\,\frac{\varphi_Y(t)}{\varphi_{Y_1}(t)}\right)\frac{dt}{t}\,\frac{ds}{s}$$

$$\leqslant \frac{\delta^2\gamma}{\ln 2}\,2\cdot 2\ln 2\int_0^\infty C\big(f^*(s)\varphi_X(s)\big)\frac{ds}{s}$$

since we have $C\big(\varphi_{X_1}(s)/\varphi_X(s)\big) \leqslant C\big(2\varphi_{X_1}(2s)/\varphi_X(2s)\big) \leqslant 2C\big(\varphi_{X_1}(2s)/\varphi_X(2s)\big)$. This completes the proof of the theorem.

To close this section we state the analogue to Theorem 3.20

(3.25) THEOREM. *Let T, φ_{X_i}, φ_{Y_i}, φ_X, φ_Y, η, ξ and ψ be defined as in Theorem 3.20. Let C be a concave generalized Young's function such that $C(st) \leqslant \delta C(s)C(t)$, $s, t \geqslant 0$, and*

$$\int_0^t C\left(\frac{\varphi_Y(t)}{\varphi_{Y_0}(t)}\right)\frac{ds}{s} \leqslant \gamma/C\big(\varphi_{Y_0}(t)/\varphi_Y(t)\big)$$

and

$$\int_t^\infty C\left(\frac{\varphi_Y(s)}{\varphi_{Y_1}(s)}\right)\frac{ds}{s} \leqslant \gamma/C\big(\varphi_{Y_1}(t)/\varphi_Y(t)\big).$$

Then $\displaystyle\int_0^\infty C\big((Tf)^*(t)\varphi_Y(t)\big)\frac{dt}{t} \leqslant K\int_0^\infty C\big((f)^*(t)\varphi_X(t)\big)\frac{dt}{t}$.

The proof is analogous to that of Theorem 3.22 and is therefore omitted.

4. Applications. A particular instance of Theorem 2.8 is the following situation: T is a sublinear operation of weak type $(1,1)$ and of type (p,p) for some $p>1$. If $A(t)/t$ increases, $A(t)/t^p$ decreases and

$$\text{(4.1)}\qquad \int_0^t \frac{A(s)}{s^2}\,dx = O\big(A(t)/t\big),$$

then T is of type (A, A). This is also Theorem 4.8 in [10]. In fact this result is relatively simple. Since $\varepsilon = 1$, $\gamma = 0$, we then have $u(\lambda) = \lambda$ and $B(t) = A(t)$. However, this result cannot be applied to $A(t) = t\wedge t^p$, and consequently to $L_A = L^1 + L^p$, because (4.1) fails to hold. But in its place we have

$$\text{(4.2)}\qquad \int_0^t \frac{A(s)}{s^2}\,ds = O\left(\frac{C(t)}{t}\right),$$

where

$$C(t) = (1+t\log^+ t)\wedge t^p,\qquad A(t)\leqslant \theta C(t),$$

and $L_C = L\log^+ L + L^p$.

The reader will have no difficulty in modifying the proof of Theorem 2.8 using (4.2) instead of (4.1) to obtain the following result.

(4.3) THEOREM. *Let T be a sublinear operation of weak type $(1,1)$ and of type (p,p) for some $p>1$. Then T maps $L\log^+ L + L^p$ continuously into $L^1 + L^p$.*

This is one of the main results in [11], I. In the same vein suppose that T is a sublinear operation of weak type (p_0, q_0) and of type (p_1, q_1), with $p_0 < p_1$ and $q_0 < q_1$. Further assume that if A, B are as in Theorem 2.8, then there are Young's functions $C(t)$ and $D(t)$ such that

$$\int_0^t \frac{B(s)}{s^{q_0+1}}\frac{ds}{s} = O\big(C(t)/t^{q_0}\big),\qquad B(t)\leqslant \theta C(t),$$

and that

$$C\big(t/A^{\varepsilon\gamma}(t)\big)^{p_0/q_0}A^{\varepsilon\gamma p_0}(t) = O\big(D(t)\big),\qquad A(t)\leqslant \theta D(t).$$

Then T maps $L_D + L^{p_1}$ into $L^{q_0} + L^{q_1}$.

If we set $A(t) = t^{p_0}\wedge t^{p_1}$, then the reader will have no difficulty in verifying that T maps $L^{p_0}(\log^+ L)^{p_0/q_0} + L^{p_1}$ continuously into $L^{q_0} + L^{q_1}$. This is one of the main results in [11], II.

We shall return to this remark shortly.

(4.4) *Fractional integration.* Let $0<\alpha<1$ and let

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n(1-\alpha)}}\,dy.$$

We call $I_\alpha f$ the (*Riesz*) *fractional integral* of f of order α. One of the interesting applications of [29], Theorem 6, is precisely the fact that I_α is a linear operation of weak type $\big(1, 1/(1-\alpha)\big)$. In fact this more general result holds.

(4.5) LEMMA. *Let E be a measurable subset of R^n; then $t^{1/p-\alpha}(I_\alpha\chi_E)^{**}(t) \leqslant K|E|^{1/p}$, where $|E|$ = Lebesgue measure of E. Therefore by (3.4) I_α maps $L(p,1)$ continuously into $L(q,\infty)$ for $0<1/q = 1/p-\alpha<1$, $p>1$.*

Proof (due to A. P. Calderón). We shall compute $t(I_\alpha\chi_E)^{**}(t) = \sup\left\{\int_B I_\alpha\chi_E(x)\,dx\right\}$, where $|B|\leqslant t$. As is readily seen

$$I = \int_B I_\alpha\chi_E(x)\,dx = \int \chi_B(x) I_\alpha\chi_E(x)\,dx = \int \chi_E(y) I_\alpha\chi_B(y)\,dy.$$

Given x in R^n among all sets S with $|S| = |E|$, then $I_\alpha\chi_S(x)$ is largest when $S = \{y\colon |x-y|\leqslant k|E|^{1/n}\}$, where k is chosen so that $|S| = |E|$.

Therefore

$$I_\alpha\chi_E(x) \leqslant I_\alpha\chi_S(x) = \int\limits_{|x-y|\leqslant k|E|^{1/n}} \frac{dy}{|x-y|^{n(1-\alpha)}} \leqslant \delta|E|^\alpha,$$

for some constant δ. Thus we have $I \leqslant \delta|B|\,|E|^\alpha$, and interchanging the roles of B and E it also follows that $I \leqslant \delta|E|\,|B|^\alpha$. Thus for any θ, $0 \leqslant \theta \leqslant 1$, we have

$$\begin{aligned} I &\leqslant (\delta|B|\,|E|^\alpha)^\theta(\delta|B|^\alpha|E|)^{1-\theta} \\ &\leqslant \delta|E|^{1+\theta(\alpha-1)}|B|^{\theta(1-\alpha)+\alpha}, \end{aligned}$$

whence it follows that

$$(4.6)\qquad (I_\alpha\chi_E)^{**}(t) \leqslant \delta|E|^{1+\theta(\alpha-1)}t^{(1-\alpha)(\theta-1)}.$$

Given $p \geqslant 1$, let $\theta(1-\alpha) = 1/p-1$. Then (4.6) becomes

$$t^{1/p-\alpha}(I_\alpha\chi_E)^{**}(t) \leqslant \delta|E|^{1/p},$$

and the lemma follows.

Thus I_α is a mapping of weak type (p,q) with $0 < 1/q = 1/p-\alpha < 1$. By Theorem 2.3, then I_α is also a mapping of type (p,q) when $p > 1$. Therefore, by Theorem 2.8, I_α is of type (A,B) with $B^{-1}(t) = A^{-1}(t)t^{-\alpha}$ provided $A(t)/t^p$ decreases and $\int_0^t A(s)/s^2\,ds = O\big(A(t)/t\big)$.

A similar result is given in [18], and interesting applications in [26].

For the particular case of fractional integration, Theorem 2.8 gives a best possible result. Indeed, if I_α maps L_A into L_B with norm K, then $A^{-1}(t)t^{-\alpha} \leqslant \frac{K}{\delta}B^{-1}(t)$, for some constant $\delta > 0$. To see this assume that $\int B\big(|I_\alpha f(y)|/K\big)\,dy \leqslant 1$ whenever $\int A\big(|f(x)|\big)\,dx \leqslant 1$. Let Q be a ball, let $x \in Q$ and let $f(x) = A^{-1}(1/|Q|)\chi_Q(x) = u\chi_Q(x)$. Then

$$I_\alpha f(x) = u\int\limits_Q \frac{dy}{|x-y|^{n(1-\alpha)}} \geqslant u\delta|Q|^\alpha,$$

and since $\int A\big(|f(x)|\big)\,dx \leqslant 1$, we have that

$$\begin{aligned} 1 &\geqslant \int B\big(|I_\alpha f(y)|/K\big)\,dy \geqslant \int\limits_Q B\big(|I_\alpha f(y)|/K\big)\,dy \\ &\geqslant |Q|B(u\delta|Q|^\alpha/K) = |Q|B\big(\delta A^{-1}(1/|Q|)|Q|^\delta/K\big). \end{aligned}$$

Whence multiplying by $1/|Q|$ and taking inverses we obtain the desired conclusion since $t = 1/|Q|$ is arbitrary.

Let us return to the remarks following (4.3). We set there $p_0 = 1$, $q_0 = 1/(1-\alpha)$, $p_1 > 1$ and $q_1 = 1/(1/p_1-\alpha)$. Then I_α maps $L(\log^+ L)^{1-\alpha}+L^{p_1}$ into $L^{q_0}+L^{q_1}$. This is essentially a result of A. Zygmund (see [19]).

The classes $A(\varphi_X, C)$ are suited to make statements about the range of operations more precise and the results obtained extend simultaneously results about Lorentz classes and Orlicz spaces. This is due to the relation

$$\int_0^\infty A\big(f^*(t)\big)\,dt = \int_0^\infty m(f,t)\,a(t)\,dt.$$

In fact if T is a sublinear operation such as in Theorem 2.3 and A, B are given by (2.6) and C is a generalized Young's function such that it is either convex or it is concave and satisfies the appropriate conditions (with φ_Y replaced by B^{-1}, etc.), then

$$\int_0^\infty C\big(B^{-1}(t)(Tf)^*(t)\big)\frac{dt}{t} \leqslant K\int_0^\infty C\big(A^{-1}(t)f^*(t)\big)\frac{dt}{t}.$$

For the particular case of the fractional integral I_α, we have the following result. Let C be a Young's function, let $A(t)/t$ be increasing and $\int_0^t A(s)/s^2\,ds = O\big(A(t)/t\big)$ and $A(t)/t^p$ decreases for some $p_1 > p > 1$. Then

$$(4.7)\qquad \int_0^\infty C\big(s^\alpha A^{-1}(s)(I_\alpha f)^*(s)\big)\frac{ds}{s} \leqslant K\int_0^\infty C\big(A^{-1}(s)f^*(s)\big)\frac{ds}{s}.$$

Of course we also have

$$(4.8)\qquad \int_0^\infty C\big((I_\alpha f)^*(s)/s^\alpha A^{-1}(1/s)\big)\frac{ds}{s} \leqslant K\int_0^\infty C\big(f^*(s)/A^{-1}(1/s)\big)\frac{ds}{s}.$$

Also when C is concave (4.8) will hold provided

$$C(st) \leqslant \delta C(s)C(t), \qquad s,t \geqslant 0$$

and

$$\int_t^\infty C\big(1/sA^{-1}(1/s)\big)\frac{ds}{s} \leqslant \gamma/C\big(tA^{-1}(1/t)\big)$$

and

$$\int_0^t C\big(1/s^{1/p_1}A^{-1}(1/s)\big)\frac{ds}{s} \leqslant \gamma/C\big(t^{1/p_1}A^{-1}(1/t)\big).$$

These conditions can be readily checked for given functions A, C.

We shall now apply our results to integral operators. We begin with a simple case, an extension of Young's convolution theorem.

(4.9) Theorem. *Let f be a μ-measurable function such that f is in $M(X)$, i.e., $\sup_{t>0}\big(\varphi_X(t)f^{**}(t)\big) \leqslant k$. Let $Tg(x) = \int_M f(x-y)g(y)\,d\mu(y)$. Suppose that*

$\varphi_Y(t)/\varphi_{X'}(t)$ *increases*, $\varphi_Y(t)/t\varphi_{X'}(t)$ *decreases. Then for a Young's function* C, T *maps* $\Lambda(\varphi_Y, C)$ *into* $\Lambda(\varphi_Z, C)$, *where* $\varphi_Z(t) = \varphi_Y(t)/\varphi_{X'}(t)$.

Proof. It will suffice to show that for a μ-measurable set E of finite measure we have

$$(4.10) \qquad \varphi_{Z_0}(t)(T\chi_E)^{**}(t) \leqslant \delta\varphi_{Y_0}(\mu(E))$$

where $\varphi_{Z_0}(t) = \varphi_X(t)^{1-\theta}$ and $\varphi_{Y_0}(t) = t/\varphi_X(t)^\theta$ and $0 < \theta < 1$, for then the theorem will follow by interpolation.

Now

$$t(T\chi_E)^{**}(t) = \sup\Big\{\int_B |T\chi_E(y)|\,d\mu:\ \mu(B) \leqslant t\Big\}.$$

But since

$$\begin{aligned}\int_B |T\chi_E(y)|\,d\mu(y) &\leqslant \int\int \chi_B(y)\chi_E(x)|f(x-y)|\,d\mu(x)\,d\mu(y)\\ &\leqslant f^{**}(\mu(E))\mu(E)t \wedge f^{**}(t)t\mu(E)\\ &\leqslant \mu(E)tf^{**}(\mu(E))^\theta f^{**}(t)^{1-\theta}\\ &\leqslant \mu(E)tk/\varphi_X(\mu(E))^\theta\varphi_X(t)^{1-\theta}.\end{aligned}$$

Thus

$$\varphi_X(t)^{1-\theta}(T\chi_E)^{**}(t) \leqslant k\mu(E)/\varphi_X(\mu(E))^\theta$$

and (4.10) follows.

Choose $0 \leqslant \theta_1, \theta_2 \leqslant 1$ appropriately and interpolate. In this case $\psi(t) = \eta^{-1}(\xi(t)) = t$ and therefore

$$\varphi_Z(t) = \frac{\varphi_Y(t)}{\varphi_{Y_{\theta_1}}(t)}\varphi_{Z_{\theta_1}}(t) = \varphi_Y(t)/\varphi_{X'}(t).$$

The proof is thus complete.

When restricted to Orlicz spaces the theorem implies $L_A * L_B \subset L_C$, where $C^{-1}(t)t = B^{-1}(t)A^{-1}(t)$. This is due to O'Neil [18], Theorem 2.5, where the fractional integration results are obtained as convolution results by noting that $f(x) = 1/|x|^{n(1-a)}$ satisfies $f^{**}(t) \leqslant kt^{a-1}$.

A more general type of transformation is given by

$$(4.11) \qquad Tf(x) = \int_N k(x, y)f(y)\,d\nu(y)$$

from ν-measurable function f into μ-measurable functions $Tf(x)$ defined for $x \in M$. $k(x, y)$ is a $\mu\times\nu$ measurable function defined on $M\times N$.

This is a particular instance of the so-called "bilinear operators" of O'Neil and it satisfies a fundamental inequality, also due to O'Neil, which we now show.

(4.12) Lemma. *If* T *is given by* (4.11), *then*

$$(Tf)^{**}(t) \leqslant \int_0^\infty k^*(ts)f^*(s)\,ds.$$

Proof. Assume that $f(y) = \chi_E(y)$, where E is a ν-measurable set of finite measure. Then if B is a μ-measurable set of measure $\leqslant t$ we have

$$\begin{aligned}\int \chi_B(x)|T\chi_E(x)|\,d\mu(x) &\leqslant \int \chi_B(x)\int |k(x, y)|\chi_E(y)\,d\nu(y)\,d\mu(x)\\ &\leqslant \mu(B)\nu(E)k^{**}(\mu(B)\nu(E)) = \int_0^{\mu(B)\nu(E)} k^*(s)\,ds\\ &\leqslant \int_0^{t\nu(E)} k^*(s)\,ds = t\int_0^\infty k^*(ts)(\chi_E)^*(s)\,ds\end{aligned}$$

because $(\chi_E)^*(s) = \chi_{[0,\nu(E)]}(s)$.

Consequently

$$(T\chi_E)^{**}(t) \leqslant \int_0^\infty k^*(ts)(\chi_E)^*(s)\,ds.$$

For general functions f the result follows readily from this inequality. See [20], pp. 202–203.

(4.13) Corollary. *Let*

$$s\varphi_Y(t) \leqslant \delta\varphi_Z(st)\varphi_X(s) \quad \textit{for all } s, t > 0.$$

Then if $C(t)/t$ *increases we have*

$$|Tf|_{\Lambda(\varphi_Y, C)} \leqslant \theta\|k\|_{\Lambda(Z)}|f|_{\Lambda(\varphi_X, C)}.$$

Proof. By (4.10) we have

$$\begin{aligned}(4.14) \qquad (Tf)^{**}(t)\varphi_Y(t) &\leqslant \delta\int_0^\infty k^*(ts)\varphi_Z(st)f^*(s)\varphi_X(s)\frac{ds}{s}\\ &= \delta\int_0^\infty k^*(s)\varphi_Z(s)f^*(s/t)\varphi_X(s/t)\frac{ds}{s}.\end{aligned}$$

Therefore by Minkowski's inequality it follows that

$$\begin{aligned}\|(Tf)^{**}\varphi_Y\|_C &\leqslant \delta\int_0^\infty k^*(s)\varphi_Z(s)\|f^*(s/\cdot)\varphi_X(s/\cdot)\|_C\frac{ds}{s}\\ &= \theta\|f^*\varphi_X\|_C\|k\|_{\Lambda(Z)}.\end{aligned}$$

This extends some results of [20] and [16], Theorem 3. A similar result for C concave follows by interpolation from the statements

(i) $\|Tf\|_{M(Y)} \leqslant \theta \|k\|_{M(Z)} \|f\|_{A(Y)}$,

(ii) $\|Tf\|_{M(Y)} \leqslant \theta \|k\|_{A(Z)} \|f\|_{M(X)}$,

(iii) $\|Tf\|_{A(Y)} \leqslant \theta \|k\|_{A(Z)} \|f\|_{A(X)}$.

Indeed (i)–(iii) are easy consequences of (4.12) (cf. [16], Theorem 2). The following lemma also follows easily from (4.12).

(4.15) LEMMA. *Let T be given by* (4.9) *and let* $s\varphi_Y(t) \leqslant \delta\varphi_Z(st)\varphi_X(s)$. *Then*

$$\|(Tf)^{**}\varphi_Y\|_{L^p} \leqslant \gamma \|k^{**}\varphi_Z\|_{L^r} \|f^*\varphi_X\|_{L^s}, \qquad \gamma \text{ constant},$$

where $1/p+1 = 1/r+1/s$.

Indeed the proof follows with the aid of Hölder's inequality (cf. [20], 10.1).

Similar results hold for the general $A(\varphi_X, C)$ classes.

We conclude the applications given in this paper with a generalization of (4.13). For the L^p classes this is done in [25] (see also [27]). Similar results have some interesting applications, see for instance [1].

(4.16) THEOREM. *Let A, B, C, D be generalized Young's functions such that*

$$\int_t^\infty \frac{ds}{A(s)} \leqslant \frac{t}{A(t)},$$

similarly for B, C, D.

Let T be defined as in 4.11 *and assume that the kernel k has the following properties (mixed weak type), to wit:*

$$\mu\big(\{x \in M:\ |k(x, y)| > t\}\big) \leqslant 1/A\big(t/\varphi(y)\big),$$

where

$$\nu\big(\{y \in N:\ \varphi(y) > t\}\big) \leqslant 1/B(t);$$

and further assume that

$$\nu\big(\{y \in N:\ |k(x, y)| > t\}\big) \leqslant 1/C\big(t/\psi(x)\big),$$

where

$$\mu\big(\{x \in M:\ \psi(x) > t\}\big) \leqslant 1/D(t).$$

Then if S is ν-measurable.

$$\tfrac{1}{8}(T\chi_S)^{**}(t) \leqslant \frac{A^{-1}(1/t)}{\bar{B}^{-1}\big(1/\nu(S)\big)} \wedge \frac{D^{-1}(1/t)}{\bar{C}^{-1}\big(1/\nu(S)\big)}. \tag{4.17}$$

Proof. Let Q be a μ-measurable set with $\mu(Q) \leqslant t$. Then

$$\int_Q (T\chi_S)(x)\,d\mu(x) \leqslant I = \int \chi_Q(x) \int \chi_S(y)|k(x, y)|\,d\nu(y)\,d\mu(x).$$

Now for any $\delta > 0$ we have

$$\int \chi_S(y)\,|k(x, y)|\,d\nu(y) = \int_0^\infty \nu\big(\{y \in S:\ |k(x, y)| > t\}\big)\,dt$$

$$\leqslant \int_0^\delta \nu(S)\,dt + \int_\delta^\infty 1/C\big(t/\psi(x)\big)\,dt \leqslant \delta\nu(S) + \delta/C\big(\delta/\psi(x)\big) = J.$$

So if we set $\delta = \psi(x)C^{-1}\big(1/\nu(S)\big)$ we obtain

$$J \leqslant 2\psi(x)\nu(S)C^{-1}\big(1/\nu(S)\big).$$

Consequently

$$I \leqslant \nu(S)C^{-1}\big(1/\nu(S)\big) \int \chi_Q(x)\psi(x)\,d\mu(x).$$

Also as above it follows that

$$\int \chi_Q(x)\psi(x)\,d\mu(x) \leqslant 2tD^{-1}(1/t).$$

That

$$(T\chi_S)^{**}(t) \leqslant 8D^{-1}(1/t)/\bar{C}^{-1}\big(1/\nu(S)\big)$$

now readily follows from (1.3). That (4.15) holds is obvious, and the proof is complete.

We have thus shown that T is an operation simultaneously of weak type (X_0, Y_0) and (X_1, Y_1), where $\varphi_{X_0}(t) = 1/\bar{B}^{-1}(1/t)$, $\varphi_{Y_0}(t) = 1/A^{-1}(1/t)$, $\varphi_{X_1}(t) = 1/\bar{C}^{-1}(1/t)$ and $\varphi_{Y_1}(t) = 1/D^{-1}(1/t)$, for instance.

A particularly interesting case is when the Young's functions involved are powers. In this case if $A(t) = t^{r_0}$, $B(t) = t^{s_0}$, $C(t) = t^{r_1}$, $D(t) = t^{s_1}$, then T is of weak types (s_0', r_0) and (r_1', s_1). Thus

$$\varepsilon = \frac{1/r_0 - 1/s_1}{1/r_1 - 1/s_0} \quad \text{and} \quad \gamma = \frac{s_0'/r_0 - r_1'/s_1}{s_0' - r_1'}$$

and if $A^{-1}(t)$ satisfies the hypothesis of Theorem 2.3 we then have that for a Young's function C,

$$\int_0^\infty C\big(A^{-1}(t^\varepsilon)t^\gamma (Tf)^*(t)\big)\frac{dt}{t} \leqslant K \int_0^\infty C\big(A^{-1}(t)(Tf)^*(t)\big)\frac{dt}{t}.$$

Statements of this nature generalize [25], Theorem 2, and also remove some of the restrictions imposed in the original proof.

Additional remark. After this article was submitted for publication the author received a personal communication from Messrs M. Milman and R. Sharpley informing him that they were in the process of completing an article on interpolation of bilinear operations on r.e.s. that considerably improves the results announced in [16] and consequently contains, among other interesting results, the applications given in Corollary 4.13 (when $C(t) = t^p, p \geq 1$) and Lemma 4.15. R. Sharpley also considers results similar to (4.7) and (4.8) (*Fractional integration in Orlicz spaces*, to appear in P. A. M. S.) and M. Milman applications of operators of the type defined in (4.11) to the study of qualitative theory of integral equations (*Stability results for integral operators*, preprint).

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