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INTERPOLATION OF QUASI-NORMED SPACES

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Introduction.

The study of interpolation spaces has hitherto mainly been restricted to Banach spaces (e.g. normed and complete spaces). Krée [5] was the first to realize that large parts of the theory could be carried over to quasi-normed spaces which need not even be complete. We will here continue Krée's work. Most of our results, however, are new even for Banach spaces.

Let A_0 and A_1 be a couple of quasi-normed spaces continuously embedded into a topological vector space $\mathscr{A}$. For every $a \in A_0+A_1$ let us put

$$K(t,a) = \inf_{a_0+a_1=a}(\|a_0\|_{A_0}+t\|a_1\|_{A_1}), \quad a_i \in A_i,\ i=0,1 ,$$

where $0<t<\infty$. With the aid of $K(t,a)$ we introduce in section 1 interpolation spaces $(A_0,A_1)_{\theta,p}$, $0<\theta<1$, $0<p\leqq\infty$. In section 2 we express $K(t,a;E_0,E_1)$, where $E_i=(A_0,A_1)_{\theta_i,q_i}$, in terms of $K(t,a;A_0,A_1)$.

Our main result is

$$K(t,a;E_0,E_1) \sim \left(\int_0^{t^{1/\eta}} (s^{-\theta_0}K(s,a;A_0,A_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} + $$

$$+\, t\left(\int_{t^{1/\eta}}^{\infty} (s^{-\theta_1}K(s,a;A_0,A_1))^{q_1}\frac{ds}{s}\right)^{1/q_1} ,$$

where $\eta=\theta_1-\theta_0$, $0<\theta_0<\theta_1<1$, $0<q_0,q_1<\infty$.

From this result we derive

$$(A_0,A_1)_{\theta,p} = (E_0,E_1)_{\lambda,p}, \quad \theta = (1-\lambda)\theta_0+\lambda\theta_1 ,$$

algebraically (which Lions–Peetre [8] have shown for Banach spaces). We also get a very precise estimate of the corresponding norms (section 3), which is more precise than that of Lions–Peetre. For instance we prove a new Marcinkiewicz's interpolation theorem with the right order of

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magnitude on the constant in the "convexity inequality" [17, 112–116]. We also get, in section 4, a theorem by O'Neil [13]. In section 5 we extend to the case of quasi-normed spaces a result by Peetre [15] concerning the equivalence between $(A_0,A_1)_{\theta,p}$ and the spaces $(A_0,A_1)_{\theta,p_0,p_1}$ defined there. Our method, which is different from Peetre's, gives a very precise estimate of the norms.

The main results of this paper have been summarized in a note [3] by the author.

The problems treated in this paper have been suggested to me by professor Jaak Peetre. I wish to thank him for valuable advice and for his great interest in my work.

1. Preliminaries on interpolation spaces.

We consider couples (A_0,A_1) of topological vector spaces A_0 and A_1, which are both continuously embedded in a topological vector space $\mathscr{A}$. (In the sequel we let $\subset$ denote continuous embedding.)

If (A_0,A_1) and (B_0,B_1) are two such couples with

$$A_0,A_1 \subset \mathscr{A} \quad \text{and} \quad B_0,B_1 \subset \mathscr{B} ,$$

and if A and B are two other spaces with

$$A \subset \mathscr{A} \quad \text{and} \quad B \subset \mathscr{B} ,$$

we say that A and B are interpolation spaces with respect to the couples (A_0,A_1) and (B_0,B_1) if the following interpolation property is fulfilled: For every linear operator T such that

$$T:\ A_0 \to B_0, \qquad T:\ A_1 \to B_1 ,$$

it follows that

$$T:\ A \to B .$$

Here we let the symbol $T\colon A \to B$ denote that the restriction to A of the linear operator T is continuous.

We shall in the sequel mainly be occupied with couples (A_0,A_1) of quasi-normed spaces. Most frequent in the applications are couples of Banach spaces, but our theorems for quasi-normed spaces are also true for normed spaces. A quasi-norm $\|\cdot\|$ on a vector space A is a functional defined on A such that ([6, p. 162])

$$\|x\| > 0 \text{ if } x \neq 0 ,$$

$$\|\lambda x\| = |\lambda|\, \|x\|, \text{ where } \lambda \text{ is a real or complex number} ,$$

$$\|x+y\| \leqq k(\|x\|+\|y\|), \quad k \geqq 1 .$$

In all the following sections except section 5 we shall restrict ourselves to one very important interpolation method introduced by Peetre [14]. (An interpolation method is a method of constructing interpolation spaces from a given couple of spaces.)

Let (A_0, A_1) be a couple of quasi-normed spaces with $A_i \subset \mathscr{A}$, $i=0,1$. For every $a \in A_0+A_1$ we define the functional

$$K(t,a;A_0,A_1) = K(t,a) = \inf_{a_0+a_1=a}(\|a_0\|_{A_0}+t\|a_1\|_{A_1}), \tag{1.1}$$

where $a_i \in A_i$, $i=0,1$, and $0<t<\infty$. For every fixed t this is a quasi-norm on A_0+A_1 and from the definition it is easy to see that $K(t,a)$ is a non-negative, increasing and concave function of t.

For $0<\theta<1$, $0<p\leqq\infty$, the space

$$(A_0,A_1)_{\theta,p} = \left\{a\,;\ a\in A_0+A_1, \int_0^\infty (t^{-\theta}K(t,a))^p \frac{dt}{t} < \infty\right\} \tag{1.2}$$

with the quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,p}} = \left(\int_0^\infty (t^{-\theta}K(t,a))^p \frac{dt}{t}\right)^{1/p},$$

is an interpolation space and we have the following fundamental interpolation theorem [8], [14].

THEOREM 1.1. *If (A_0,A_1) and (B_0,B_1) are two couples of quasi-normed spaces with $A_i\subset\mathscr{A}$ and $B_i\subset\mathscr{B}$, $i=0,1$, and if T is a linear operator*

$$T:\ A_0\to B_0, \qquad T:\ A_1\to B_1,$$

with the quasi-norms M_0 and M_1 respectively, then

$$T:\ (A_0,A_1)_{\theta,p}\to(B_0,B_1)_{\theta,p}$$

is also continuous, and for its quasi-norm we have the so called convexity inequality

$$M \leqq M_0^{1-\theta}M_1^{\theta}. \tag{1.3}$$

PROOF. From the definition of $K(t,a)$ it is obvious that

$$K(t,Ta;B_0,B_1) \leqq M_0K(M_1t/M_0,a;A_0,A_1) \tag{1.4}$$

and from this inequality the theorem follows at once.

In the sequel we often write $A_{\theta,p}$ instead of $(A_0,A_1)_{\theta,p}$.

2. An estimate of $K(t,a)$.

NOTATION: $f(t) \sim g(t) \Leftrightarrow Cf(t) \leqq g(t) \leqq C^{-1}f(t),\ C>0$.

THEOREM 2.1. *Let (A_0,A_1) be a couple of quasi-normed spaces and put*

$$E_i = (A_0,A_1)_{\theta_i,q_i} = A_{\theta_i q_i}, \quad i=0,1 .$$

Then

$$(2.1) \quad K(t,a;E_0,E_1)$$

$$\sim \left(\int_0^{t^{1/\eta}} (s^{-\theta_0}K(s,a;A_0,A_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} + t\left(\int_{t^{1/\eta}}^{\infty} (s^{-\theta_1}K(s,a;A_0,A_1))^{q_1}\frac{ds}{s}\right)^{1/q_1}$$

if $\eta=\theta_1-\theta_0$, $0<\theta_0<\theta_1<1$ and $0<q_0,\ q_1 \leqq \infty$.

PROOF. For the sake of simplicity we prove the theorem only when $q_0,q_1 \geqq 1$. Put $K(t,a;A_0,A_1)=K(t,a)$ and

$$(2.2) \quad H(t,a) = \left(\int_0^{t^{1/\eta}} (s^{-\theta_0}K(s,a))^{q_0}\frac{ds}{s}\right)^{1/q_0} + t\left(\int_{t^{1/\eta}}^{\infty} (s^{-\theta_1}K(s,a))^{q_1}\frac{ds}{s}\right)^{1/q_1}$$

$$= L_1+L_2 .$$

By definition we have

$$(2.3) \quad K(t,a;E_0,E_1) = \inf_{a_0+a_1=a}(\|a_0\|_{E_0}+t\|a_1\|_{E_1})$$

$$= \inf_{a_0+a_1=a}\left[\left(\int_0^{\infty} (s^{-\theta_0}K(s,a_0))^{q_0}\frac{ds}{s}\right)^{1/q_0} + t\left(\int_0^{\infty} (s^{-\theta_1}K(s,a_1))^{q_1}\frac{ds}{s}\right)^{1/q_1}\right].$$

Suppose that

$$(2.4) \quad \|a+b\|_{A_i} \leqq k_i(\|a\|_{A_i}+\|b\|_{A_i}), \quad i = 0,1 ,$$

and put

$$(2.5) \quad k = \max(k_0,k_1) .$$

Then it is obvious that

$$(2.6) \quad K(t,a+b) \leqq k\big(K(t,a)+K(t,b)\big) .$$

We now start showing that $H(t,a) \leqq CK(t,a;E_0,E_1)$. If $a_0+a_1=a$ is any partition of $a\in E_0+E_1$ with $a_i \in E_i$, $i=0,1$, then by (2.6) and Minkowski's inequality

$$(2.7) \quad k^{-1}H(t,a) \leqq \left(\int_0^{t^{1/\eta}} (s^{-\theta_0}K(s,a_0))^{q_0}\frac{ds}{s}\right)^{1/q_0} + \left(\int_0^{t^{1/\eta}} (s^{-\theta_0}K(s,a_1))^{q_0}\frac{ds}{s}\right)^{1/q_0} +$$

$$+ t\left(\int_{t^{1/\eta}}^{\infty} (s^{-\theta_1} K(s,a_0))^{q_1} \frac{ds}{s}\right)^{1/q_1} + t\left(\int_{t^{1/\eta}}^{\infty} (s^{-\theta_1} K(s,a_1))^{q_1} \frac{ds}{s}\right)^{1/q_1}$$

$$= I_1 + I_2 + I_3 + I_4 .$$

(If $0 < q_0$, $q_1 < 1$, then the constant in the left member has to be bigger.) We define

$$J_i = \left(\int_0^{\infty} (u^{-\theta_i} K(u,a_i))^{q_i} \frac{du}{u}\right)^{1/q_i}, \quad i = 0,1 . \tag{2.8}$$

From the definition of $K(t,a)$ it is easily seen that $K(t,a)$ is increasing and that $t^{-1}K(t,a)$ is decreasing. Hence we get respectively

$$J_i^{q_i} \geqq \int_0^s u^{-\theta_i q_i - 1} K(u,a_i)^{q_i}\, du \geqq (s^{-1}K(s,a_i))^{q_i} \int_0^s u^{(1-\theta_i)q_i - 1}\, du$$

$$= K(s,a_i)^{q_i} s^{-\theta_i q_i} [(1-\theta_i)q_i]^{-1}$$

and

$$J_i^{q_i} \geqq \int_s^{\infty} u^{-\theta_i q_i - 1} K(u,a_i)^{q_i}\, du \geqq K(s,a_i)^{q_i} \int_s^{\infty} u^{-\theta_i q_i - 1}\, du$$

$$= K(s,a_i)^{q_i} s^{-\theta_i q_i} (\theta_i q_i)^{-1} ,$$

that is,

$$K(s,a_i) \leqq J_i s^{\theta_i} q_i^{1/q_i} [\min(\theta_i;\, (1-\theta_i))]^{1/q_i} = J_i s^{\theta_i} C_i, \quad i = 0,1 . \tag{2.9}$$

We can estimate I_2 and I_3 with the aid of (2.9):

$$I_2 \leqq J_1 C_1 \left(\int_0^{t^{1/\eta}} s^{(\theta_1-\theta_0)q_0 - 1}\, ds\right)^{1/q_0} = t J_1 C_1 (\eta q_0)^{-1/q_0} ,$$

$$I_3 \leqq t J_0 C_0 \left(\int_{t^{1/\eta}}^{\infty} s^{(\theta_0-\theta_1)q_1 - 1}\, ds\right)^{1/q_1} = J_0 C_0 (\eta q_1)^{-1/q_1} .$$

For I_1 and I_4 we have the trivial estimates

$$I_1 \leqq J_0 \quad \text{and} \quad I_4 \leqq tJ_1 .$$

Thus

$$H(t,a) \leqq C(J_0 + J_1) ,$$

where $C = O(\eta^{-\max(1/q_0;\, 1/q_1)})$ as $\eta \to 0$. If we now take inf over all partitions $a_0 + a_1 = a$, we get

$$H(t,a) \leqq CK(t,a;E_0,E_1)\,.$$

To show the remaining inequality of the equivalence between $H(t,a)$ and $K(t,a;E_0,E_1)$ we choose $a_i(t)\in A_i$, $i=0,1$, so that

$$(2.10)\qquad a_0(t)+a_1(t) = a \quad \text{and} \quad \|a_0(t)\|_{A_0}+t\|a_1(t)\|_{A_1} \leqq 2K(t,a)$$

for $t>0$. We now define a_0' and a_1' by

$$(2.11)\qquad a_i'(t) = a_i(t^{1/\eta}), \quad i = 0,1\,.$$

Then $a_0'+a_1'=a$ and

$$(2.12)\qquad K(s,a_0'(t)) \leqq \|a_0'(t)\|_{A_0} = \|a_0(t^{1/\eta})\|_{A_0} \leqq 2K(t^{1/\eta},a)\,,$$

$$(2.13)\qquad K(s,a_1'(t)) \leqq s\|a_1'(t)\|_{A_1} = s\|a_1(t^{1/\eta})\|_{A_1} \leqq 2st^{-1/\eta}K(t^{1/\eta},a)\,.$$

By the quasi-triangle inequality it follows that

$$(2.14)\qquad K(s,a_0'(t)) \leqq k\big(K(s,a)+K(s,a_1'(t))\big)\,,$$

$$(2.15)\qquad K(s,a_1'(t)) \leqq k\big(K(s,a)+K(s,a_0'(t))\big)\,.$$

But $a_0'+a_1'=a$ is a special partition of a. Therefore

$$(2.16)\qquad K(t,a;E_0,E_1)$$

$$\leqq \left(\int_0^\infty \big(s^{-\theta_0}K(s,a_0'(t))\big)^{q_0}\frac{ds}{s}\right)^{1/q_0} + t\left(\int_0^\infty \big(s^{-\theta_1}K(s,a_1'(t))\big)^{q_1}\frac{ds}{s}\right)^{1/q_1}$$

$$\leqq \left(\int_0^{t^{1/\eta}} \big(s^{-\theta_0}K(s,a_0'(t))\big)^{q_0}\frac{ds}{s}\right)^{1/q_0} + \left(\int_{t^{1/\eta}}^\infty \big(s^{-\theta_0}K(s,a_0'(t))\big)^{q_0}\frac{ds}{s}\right)^{1/q_0} +$$

$$+\,t\left(\int_0^{t^{1/\eta}} \big(s^{-\theta_1}K(s,a_1'(t))\big)^{q_1}\frac{ds}{s}\right)^{1/q_1} + t\left(\int_{t^{1/\eta}}^\infty \big(s^{-\theta_1}K(s,a_1'(t))\big)^{q_1}\frac{ds}{s}\right)^{1/q_1}$$

$$= K_1+K_2+K_3+K_4\,.$$

Introducing L_0 and L_1 from (2.2) we now get, in the same way as before (cf. (2.9)),

$$(2.17)\qquad K(s,a) \leqq L_0 s^{\theta_0}(q_0(1-\theta_0))^{1/q_0} \quad \text{if } s \leqq t^{1/\eta}\,,$$

$$(2.18)\qquad K(s,a) \leqq t^{-1}L_1 s^{\theta_1}(q_1\theta_1)^{1/q_1} \quad \text{if } s \geqq t^{1/\eta}\,.$$

From (2.14), (2.13) and (2.17) we get

$$(2.19)\quad k^{-1}K_1 \leqq \left(\int_0^{t^{1/\eta}} s^{-\theta_0 q_0-1}K(s,a)^{q_0}ds\right)^{1/q_0} + \left(\int_0^{t^{1/\eta}} s^{-\theta_0 q_0-1}K(s,a_1'(t))^{q_0}ds\right)^{1/q_0}$$

$$\leqq L_0 + 2K(t^{1/\eta},a)t^{-\theta_0/\eta}(q_0(1-\theta_0))^{-1/q_0} \leqq 3L_0 .$$

From (2.15), (2.12) and (2.18) we get

$$(2.20)\quad k^{-1}K_4 \leqq t\left(\int_{t^{1/\eta}}^{\infty} s^{-\theta_1 q_1-1}K(s,a)^{q_1}ds\right)^{1/q_1} + t\left(\int_{t^{1/\eta}}^{\infty} s^{-\theta_1 q_1-1}K(s,a_0'(t))^{q_1}ds\right)^{1/q_1}$$

$$\leqq L_1 + 2K(t^{1/\eta},a)t^{1-\theta_1/\eta}(q_1\theta_1)^{-1/q_1} \leqq 3L_1 .$$

From (2.12), (2.13), (2.17) and (2.18) we get

$$(2.21)\qquad k^{-1}K_2 \leqq 2K(t^{1/\eta},a)t^{-\theta_0/\eta}(q_0\theta_0)^{-1/q_0} \leqq CL_0$$

$$(2.22)\qquad k^{-1}K_2 \leqq 2K(t^{1/\eta},a)t^{1-\theta_1/\eta}(q_1(1-\theta_1))^{-1/q_1} \leqq CL_1 ,$$

where $C=O(1)$ as $\eta \to 0$. Thus we finally have

$$(2.23)\qquad K(t,a;E_0,E_1) \leqq CH(t,a) .$$

REMARK 2.1. With exactly the same technique we can estimate $K(t,a;E_0,E_1)$ in the two extreme cases $K(t,a;A_0,A_{\theta_1 q_1})$ and $K(t,a;A_{\theta_0 q_0},A_1)$. The result in these two cases is

$$(2.24)\qquad K(t,a;A_0,A_{\theta_1 q_1}) \sim t\left(\int_{t^{1/\theta_1}}^{\infty} (s^{-\theta_1}K(s,a))^{q_1}\frac{ds}{s}\right)^{1/q_1},$$

$$(2.25)\qquad K(t,a;A_{\theta_0 q_0},A_1) \sim \left(\int_0^{t^{1/(1-\theta_0)}} (s^{-\theta_0}K(s,a))^{q_0}\frac{ds}{s}\right)^{1/q_0}.$$

3. Interpolation theorems.

THEOREM 3.1. *If* (A_0,A_1) *is a couple of quasi-normed spaces and* (E_0,E_1) *is a couple of interpolation spaces, where*

$$E_i = (A_0,A_1)_{\theta_i,q_i},\quad 0<\theta_i<1,\ \theta_0 \neq \theta_1,\ 0<q_i\leqq\infty,\quad i=0,1 ,$$

then

$$(3.1)\qquad (E_0,E_1)_{\lambda,p} = (A_0,A_1)_{\theta,p}$$

and

$$(3.2)\qquad C\lambda^{-\min(1/p;1/q_0)}(1-\lambda)^{-\min(1/p;1/q_1)}\|a\|_{(A_0A_1)_{\theta,p}} \leqq \|a\|_{(E_0,E_1)_{\theta,p}} \leqq C^{-1}\lambda^{-\max(1/p;1/q_0)}(1-\lambda)^{-\max(1/p;1/q_1)}\|a\|_{(A_0,A_1)_{\theta,p}}.$$

Here $\theta=(1-\lambda)\theta_0+\lambda\theta_1$, $0<\lambda<1$ *and* $0<p\leqq\infty$.

REMARK 3.1. This theorem is an improvement of the so called reiteration theorem of Lions–Peetre (se [8]). Besides that our theorem is true even for quasi-normed spaces, the constants in the estimates of the norms are better, in fact as we will show later on, they are the best possible what concerns their dependence on λ.

PROOF OF THEOREM 3.1. We first suppose that $p\geqq 1$ and $\theta_0<\theta_1$. From theorem 2.1 we get

$$(3.3)\qquad \|a\|_{(E_0,E_1)_{\lambda,p}} = \left(\int_0^\infty (t^{-\lambda}K(t,a;E_0,E_1))^p\frac{dt}{t}\right)^{1/p}$$
$$\sim \left(\int_0^\infty\left(t^{-\lambda}\left(\int_0^{t^{1/\eta}}(s^{-\theta_0}K(s,a))^{q_0}\frac{ds}{s}\right)^{1/q_0}\right)^p\frac{dt}{t}\right)^{1/p} +$$
$$+\left(\int_0^\infty\left(t^{1-\lambda}\left(\int_{t^{1/\eta}}^\infty(s^{-\theta_1}K(s,a))^{q_1}\frac{ds}{s}\right)^{1/q_1}\right)^p\frac{dt}{t}\right)^{1/p}$$
$$= I_0+I_1\,.$$

The constants occuring in the equivalence in (3.3) are of course independent of λ. We now make two changes of variables in I_0 and I_1. We first put $s=t^{1/\eta}\sigma$ and then $t=\tau^\eta$. We get

$$(3.4)\qquad I_0 = \eta^{1/p}\left(\int_0^\infty\left(\tau^{-\theta}\left(\int_0^1(\sigma^{-\theta_0}K(\sigma\tau,a))^{q_0}\frac{d\sigma}{\sigma}\right)^{1/q_0}\right)^p\frac{d\tau}{\tau}\right)^{1/p},$$

$$(3.5)\qquad I_1 = \eta^{1/p}\left(\int_0^\infty\left(\tau^{-\theta}\left(\int_1^\infty(\sigma^{-\theta_1}K(\sigma\tau,a))^{q_1}\frac{d\sigma}{\sigma}\right)^{1/q_1}\right)^p\frac{d\tau}{\tau}\right)^{1/p}.$$

For the further estimates we distinguish between several cases.

1° $q_0\leqq p$. Jessen's inequality (cf. [2, p. 148]), implies

$$(3.6)\qquad I_0 \leqq \eta^{1/p}\left(\int_0^1\left(\int_0^\infty(\sigma^{-\theta_0}\tau^{-\theta}K(\sigma\tau,a))^p\frac{d\tau}{\tau}\right)^{q_0/p}\frac{d\sigma}{\sigma}\right)^{1/q_0}$$

$$= \eta^{1/p}\left(\int_0^1 \sigma^{-q_0(\theta_0-\theta)}\left(\int_0^\infty (t^{-\theta}K(t,a))^p \frac{dt}{t}\right)^{q_0/p}\frac{d\sigma}{\sigma}\right)^{1/q_0}$$

$$= \eta^{1/p}\left(\int_0^1 \sigma^{-q_0(\theta_0-\theta)}\frac{d\sigma}{\sigma}\right)^{1/q_0}\|a\|_{A_{\theta,p}} = C\lambda^{-1/q_0}\|a\|_{A_{\theta,p}},$$

where C is independent of λ.

2°. If $q_1 \leqq p$, we get in the same way

$$I_1 \leqq C(1-\lambda)^{-1/q_1}\|a\|_{A_{\theta,p}}. \tag{3.7}$$

3°. If $q_0 \geqq p$,

$$A = \left(\int_0^1 (\sigma^{-\theta_0}K(\sigma\tau,a))^{q_0}\frac{d\sigma}{\sigma}\right)^{p/q_0} \leqq C\int_0^1 (\sigma^{-\theta_0}K(\sigma\tau,a))^p\frac{d\sigma}{\sigma} = BC. \tag{3.8}$$

For, when $0 \leqq s \leqq 1$,

$$B \geqq \int_0^s (\sigma^{-\theta_0}K(\sigma\tau,a))^p\frac{d\sigma}{\sigma} \geqq \left(\frac{K(s\tau,a)}{s\tau}\right)^p \tau^p\int_0^s \sigma^{p(1-\theta_0)-1}\,d\sigma$$

$$= K(s\tau,a)^p\, s^{-p\theta_0}(p(1-\theta_0))^{-1},$$

that is,

$$K(s\tau,a) \leqq B^{1/p}s^{\theta_0}(p(1-\theta_0))^{1/p}. \tag{3.9}$$

But

$$A = \left(\int_0^1 \sigma^{-\theta_0 p-1}K(\sigma\tau,a)^p\sigma^{\theta_0(p-q_0)}K(\sigma\tau,a)^{q_0-p}\,d\sigma\right)^{p/q_0},$$

so with the estimate (3.9) we get

$$A \leqq \left(\int_0^1 \sigma^{-\theta_0 p-1}K(\sigma\tau,a)^p\,d\sigma\right)^{p/q_0}\left(B^{(q_0-p)/p}(p(1-\theta_0))^{(q_0-p)/p}\right)^{p/q_0}$$

$$= B(p(1-\theta_0))^{(q_0-p)/q_0}.$$

If we use the inequality (3.8) in formula (3.4), we get

$$I_0 \leqq C\left(\int_0^\infty\int_0^1 (\sigma^{-\theta_0}\tau^{-\theta}K(\sigma\tau,a))^p\frac{d\sigma}{\sigma}\frac{d\tau}{\tau}\right)^{1/p} \tag{3.10}$$

$$= C\left(\int_0^1 \sigma^{(\theta-\theta_0)p}\frac{d\sigma}{\sigma}\right)^{1/p}\|a\|_{A_{\theta,p}} = C\lambda^{-1/p}\|a\|_{A_{\theta,p}}.$$

4°. If $q_1 \geqq p$, we get in the same way

$$I_1 \leqq C(1-\lambda)^{-1/p} \|a\|_{A_{\theta,p}} . \tag{3.11}$$

From (3.6), (3.7), (3.10) and (3.11) we now get

$$\|a\|_{(E_0,E_1)_{\lambda,p}} \leqq C\lambda^{-\max(1/p;1/q_0)} (1-\lambda)^{-\max(1/p;1/q_1)} \|a\|_{(A_0,A_1)_{\theta,p}} . \tag{3.12}$$

With exactly the same methods one can then show that

$$\|a\|_{(E_0,E_1)_{\lambda,p}} \geqq C\lambda^{-\min(1/p;1/q_0)} (1-\lambda)^{-\min(1/p;1/q_1)} \|a\|_{(A_0,A_1)_{\theta,p}} . \tag{3.13}$$

If $p<1$ the same principles for estimating will work, the constants C however will be worse depending on the fact that L_p, $0<p<1$, is quasi-normed. The dependence on λ will not be affected. We can get rid of the assumption $\theta_0<\theta_1$ by observing that

$$K(t,a;A_0,A_1) = tK(1/t,a;A_1,A_0)$$

which implies that

$$(A_0,A_1)_{\theta,p} = (A_1,A_0)_{1-\theta,p} \quad \text{with} \quad \|a\|_{(A_0,A_1)_{\theta,p}} = \|a\|_{(A_1,A_0)_{1-\theta,p}} . \tag{3.14}$$

Thus if $\theta_0>\theta_1$ we get from (3.14) and from that part of theorem (3.1) which is already proven

$$\|a\|_{(A_{\theta_0,q_0},A_{\theta_1,q_1})_{\lambda,p}} = \|a\|_{(A_{\theta_1,q_1},A_{\theta_0,q_0})_{1-\lambda,p}} = \|a\|_{(A_0,A_1)_{\theta',p}} \tag{3.15}$$

with

$$\theta' = (1-(1-\lambda))\theta_1 + (1-\lambda)\theta_0 = \theta .$$

Remark 3.2. Remark 2.1 shows that theorem 3.1 is true even in the two extreme cases, i.e.,

$$(A_0,E_1)_{\lambda,p} = (A_0,A_1)_{\theta,p} \quad \text{with } \theta=\lambda\theta_1 \text{ and } \theta_0=0 , \tag{3.16}$$

$$(E_0,A_1)_{\lambda,p} = (A_0,A_1)_{\theta,p} \quad \text{with } \theta=(1-\lambda)\theta_0+\lambda \text{ and } \theta_1=1 . \tag{3.17}$$

Remark 3.3. The constants of theorem 3.1 are the best possible with respect to their dependence on λ and $1-\lambda$, for if $A_0=L_1$ and $A_1=L_\infty$, it is well known (see also section 4) that every increasing, concave function $f(t)$ with $f(0)=0$, is a $K(t,a)$. Let therefore $a_1,a_2 \in L_1+L_\infty$ be such that

$$\begin{aligned} K(t,a_1;L_1,L_\infty) &= t \quad \text{for } 0\leqq t\leqq 1 , \\ &= 1 \quad \text{for } 1<t , \end{aligned}$$

and

$$\begin{aligned} K(t,a_2;L_1,L_\infty) &= t^{\theta_1} \quad \text{for } 0\leqq t\leqq 1 , \\ &= t^{\theta_0} \quad \text{for } 1\leqq t . \end{aligned}$$

Rather simple computations now show that

$$\|a_1\|_{(E_0,E_1)_{\lambda,p}}(\|a_1\|_{(A_0,A_1)_{\theta,p}})^{-1} = O(\lambda^{-1/p}(1-\lambda)^{-1/p}),$$

$$\|a_2\|_{(E_0,E_1)_{\lambda,p}}(\|a_2\|_{(A_0,A_1)_{\theta,p}})^{-1} = O(\lambda^{-1/q_0}(1-\lambda)^{-1/q_1}),$$

as $\lambda \to 0$ and $\lambda \to 1$.

THEOREM 3.2. *If (A_0,A_1) and (B_0,B_1) are two couples of quasi-normed spaces and T a linear operator such that*

$$T:\ (A_0,A_1)_{\eta_0,p_0} \to (B_0,B_1)_{\theta_0,q_0} \quad \textit{with the norm } M_0,$$

$$T:\ (A_0,A_1)_{\eta_1,p_1} \to (B_0,B_1)_{\theta_1,q_1} \quad \textit{with the norm } M_1,$$

and if $\eta=(1-\lambda)\eta_0+\lambda\eta_1$, $\theta=(1-\lambda)\theta_0+\lambda\theta_1$, $0<\lambda<1$ and $p\leqq q$, then

$$T:\ (A_0,A_1)_{\eta,p} \to (B_0,B_1)_{\theta,q} \quad \textit{with the norm } M,$$

where

$$M \leqq C M_0^{1-\lambda} M_1^{\lambda} \lambda^{\alpha_0}(1-\lambda)^{\alpha_1}$$

and

$$\alpha_i = \min(1/q;\, 1/q_i) - \max(1/p;\, 1/p_i) + 1/p - 1/q, \quad i=0,1,$$

PROOF. If $p\leqq q$, then

(3.18) $$\|a\|_{(A_0\,A_1)_{\theta,q}} \leqq C\|a\|_{(A_0,A_1)_{\theta,p}}[\theta(1-\theta)]^{1/p-1/q}.$$

For $K(t,a)$ is increasing so that

(3.19) $$\|a\|_{\theta,p}^p = \int_0^\infty t^{-\theta p-1}K(t,a)^p\,dt \geqq \int_t^\infty s^{-\theta p-1}K(s,a)^p\,ds$$

$$\geqq K(t,a)^p t^{-\theta p}(\theta p)^{-1},$$

and $K(t,a)t^{-1}$ is decreasing so that

(3.20) $$\|a\|_{\theta,p}^p \geqq \int_0^t s^{-\theta p-1}K(s,a)^p\,ds \geqq K(t,a)^p t^{-p}\int_0^t s^{(1-\theta)p-1}\,ds$$

$$= K(t,a)^p t^{-\theta p}(1-\theta)^{-1}p^{-1},$$

thus

(3.21) $$K(t,a) \leqq C\|a\|_{\theta,p} t^{\theta}\theta^{1/p}(1-\theta)^{1/p}.$$

If $q\geqq p$, then

(3.22) $$\|a\|_{\theta,q}^q = \int_0^\infty t^{-\theta p-1}K(t,a)^p t^{-\theta(q-p)}K(t,a)^{q-p}\,dt$$

$$\leqq \|a\|_{\theta,p}^{p} C^{q-p} \|a\|_{\theta,p}^{q-p} \theta^{(q-p)/p} (1-\theta)^{(q-p)/p} ,$$

and (3.18) is proved.

By theorem 3.1 we get

$$(3.23) \qquad \|Ta\|_{(B_0,B_1)_{\theta,q}} \leqq C\lambda^{\min(1/q;\,1/q_0)}(1-\lambda)^{\min(1/q;\,1/q_1)} \|Ta\|_{(B_{\theta_0,q_0},B_{\theta_1,q_1})_{\lambda,q}}$$

and from (3.18)

$$(3.24) \quad \|Ta\|_{(B_{\theta_0,q_0},B_{\theta_1,q_1})_{\lambda,q}} \leqq C\lambda^{1/p-1/q}(1-\lambda)^{1/p-1/q} \|Ta\|_{(B_{\theta_0,q_0},B_{\theta_1,q_1})_{\lambda,p}} .$$

Further the interpolation theorem 1.1 yields

$$(3.25) \qquad \|Ta\|_{(B_{\theta_0,q_0},B_{\theta_1,q_1})_{\lambda,p}} \leqq M_0^{1-\lambda} M_1^{\lambda} \|a\|_{(A_{\eta_0,p_0},A_{\eta_1,p_1})_{\lambda,p}}$$

and finally form theorem 3.1 we get

$$(3.26) \quad \|a\|_{(A_{\eta_0,p_0},A_{\eta_1,p_1})_{\lambda,p}} \leqq C\lambda^{-\max(1/p;\,1/p_0)}(1-\lambda)^{-\max(1/p;\,1/p_1)} \|a\|_{(A_0,A_1)_{\theta,p}} .$$

Combining (3.23), (3.24), (3.25) and (3.26) we get the convexity inequality of the theorem.

4. Concrete examples.

4.1. Lebesgue and Lorentz spaces. Let (X,μ) be a measure space. The Lebesgue space $L_p = L_p(X,\mu)$, $0<p\leqq\infty$, is the space of all μ-measurable functions such that

$$(4.1) \qquad \|a\|_{L_p} = \left(\int_X |a(x)|^p \, d\mu\right)^{1/p} < \infty .$$

In this space $\|a\|_{L_p}$ is a norm if $1\leqq p\leqq\infty$ and a quasi-norm if $0<p<1$. The Lorentz space $L_{p,q} = L_{p,q}(X,\mu)$, $0<p$, $q\leqq\infty$, is the space of all measurable functions such that

$$(4.2) \qquad \|a\|_{L_{p,q}} = \left(\int_0^\infty (t^{1/p} a^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty ,$$

where $a^*(t)$ is the decreasing rearrangement of $|a(x)|$ on the intervall $0\leqq t<\infty$. (See [2, pp. 260–299].) Here $\|a\|_{L_{p,q}}$ is a quasi-norm. Observe that $L_{p,p} = L_p$ and $\|a\|_{L_{p,p}} = \|a\|_{L_p}$.

Peetre [14] has shown that

$$(4.3) \qquad K(t,a;L_1,L_\infty) = \int_0^t a^*(s)\, ds .$$

This result has been generalized by Krée [5] to yield

$$K(t,a;L_r,L_\infty) \sim \left(\int_0^{t^r} a^*(s)^r\,ds\right)^{1/r}, \quad 0<r<\infty . \tag{4.4}$$

From (4.4) it is easy to derive the following lemma.

LEMMA 4.1. *If* $0<r<p<\infty$ *and* $\theta=1-r/p$, *then*

$$(L_r,L_\infty)_{\theta,p} = L_p .$$

The norms $\|a\|_{(L_r,L_\infty)_{\theta,p}}$ *and* $\|a\|_{L_p}$ *are equivalent.*

Now we can further generalize (4.4).

THEOREM 4.1. *If* $0<p_0<p_1\leqq\infty$ *and* $1/\alpha=1/p_0-1/p_1$, *then*

$$K(t,a;L_{p_0},L_{p_1}) \sim \left(\int_0^{t^\alpha} a^*(s)^{p_0}\,ds\right)^{1/p_0} + t\left(\int_{t^\alpha}^{\infty} a^*(s)^{p_1}\,ds\right)^{1/p_1} .$$

PROOF. For the sake of simplicity we prove the theorem only when $1\leqq p_0<p_1\leqq\infty$. By (4.3), lemma 4.1, and theorem 2.1 we have

$$K(t,a;L_{p_0},L_{p_1}) \sim K(t,a;(L_1,L_\infty)_{1-1/p_0,p_0},(L_1,L_\infty)_{1-1/p_1,p_1}) \tag{4.5}$$

$$\sim \left(\int_0^{t^\alpha}\left(s^{-1+1/p_0}\int_0^s a^*(u)\,du\right)^{p_0}\frac{ds}{s}\right)^{1/p_0} + t\left(\int_{t^\alpha}^{\infty}\left(s^{-1+1/p_1}\int_0^s a^*(u)\,du\right)^{p_1}\frac{ds}{s}\right)^{1/p_1} .$$

As $a^*(s)$ is decreasing, $\int_0^s a^*(u)du \geqq s a^*(s)$, thus

$$\int_0^{t^\alpha}\left(s^{-1+1/p_0}\int_0^s a^*(u)\,du\right)^{p_0}\frac{ds}{s} \geqq \int_0^{t^\alpha} a^*(s)^{p_0}\,ds . \tag{4.6}$$

From Hardy's inequality (see [2, pp. 239–243]) we get

$$\left(\int_0^{t^\alpha}\left(s^{-1+1/p_0}\int_0^s a^*(u)\,du\right)^{p_0}\frac{ds}{s}\right)^{1/p_0} \leqq {p_0}^{1/p_0}(p_0-1)^{-1/p_0}\left(\int_0^{t^\alpha} a^*(s)^{p_0}\,ds\right)^{1/p_0} . \tag{4.7}$$

The remaining term of (4.5) is treated in the same way and the proof is complete in the case $1\leqq p_0<p_1\leqq\infty$. If $0<p_0<p_1\leqq\infty$ we can either use Krée's formula (4.4), or copy the proof of theorem 2.1.

For the Lorentz spaces $L_{p,q}$ we have an analogous result, which is proven exactly as theorem 4.1. For similar results see also Oklander [10] and [11].

THEOREM 4.2. *If* $0<p_0<p_1\leqq\infty$, $0<q_0,q_1\leqq\infty$ *and* $1/\alpha=1/p_0-1/p_1$. *then*

$$K(t,a;L_{p_0,q_0},L_{p_1,q_1}) \sim \left(\int_0^{t^\alpha}\left(s^{1/p_0}a^*(s)\right)^{q_0}\frac{ds}{s}\right)^{1/q_0}+t\left(\int_{t^\alpha}^{\infty}\left(s^{1/p_1}a^*(s)\right)^{q_1}\frac{ds}{s}\right)^{1/q_1}.$$

As a special case we get

$$K(t,a;L_{r,\infty},L_\infty) \sim \sup_{s\leq t^r} s^{1/r}a^*(s)\ . \tag{4.8}$$

It is rather simple to sharpen (4.8) if $r=1$, then $K(t,a;L_{1,\infty},L_\infty)$ is equal to the least concave majorant of $ta^*(t)$.

For Lorentz spaces we have an analogue of lemma 4.1.

LEMMA 4.2. *If* $0<r<p<\infty$, $0<q<\infty$ *and* $\theta=1-r/p$, *then*

$$(L_{r,\infty}L_\infty)_{\theta,q} = L_{p,q}\ .$$

Theorem 3.1 and lemma 4.2 give us the following, generalization of lemma 4.1 and 4.2.

THEOREM 4.3. *If* $1/p=(1-\lambda)/p_0+\lambda/p_1$, $0<p_0$, $p_1<\infty$, $p_0\neq p_1$ *and* $0<q_0,q_1,q\leqq\infty$, *then*

$$(L_{p_0,q_0},L_{p_1,q_1})_{\lambda,q} = L_{p,q}\ ,$$

$$\begin{aligned} C\lambda^{-\min(1/q;\,1/q_0)}(1-\lambda)^{-\min(1/q;\,1/q_1)}\|a\|_{L,qp} &\\ \leqq \|a\|_{(L_{p_0,q_0},L_{p_1,q_1})_{\lambda,p}} &\\ \leqq C^{-1}\lambda^{-\max(1/q;\,1/q_0)}(1-\lambda)^{-\max(1/q;\,1/q_1)}\|a\|_{L_{p,q}}\ . \end{aligned} \tag{4.9}$$

REMARK 4.1. In general the interpolation parameter θ in $(A_0,A_1)_{\theta,p}$ cannot be 0 or 1, but in the case of Lebesgue and Lorentz spaces it is easy to see that the following formulas are true:

$$(L_r,L_\infty)_{0,\infty} = L_r,\qquad (L_{r,\infty},L_\infty)_{0,\infty} = L_{r,\infty}\ , \tag{4.10}$$

and

$$(L_r,L_\infty)_{1,\infty} = L_\infty,\qquad (L_{r,\infty},L_\infty)_{1,\infty} = L_\infty\ , \tag{4.11}$$

where $\theta<r\leqq\infty$.

As an application of the above results on L_p and $L_{p,q}$-spaces we shall prove Riesz's and Marcinkiewicz's interpolation theorems and also Calderon's extension of the Marcinkiewicz's theorem. All the theorems will be true if $0 < p, q \leqq \infty$.

THEOREM 4.4. (M. Riesz's interpolation theorem [16]). *If T is a linear operator such that*

$$T : L_{p_i} \to L_{q_i} \text{ with the norm } M_i, \quad i = 0,1 ,$$

and if $1/p = (1-\lambda)/p_0 + \lambda/p_1$, $1/q = (1-\lambda)/q_0 + \lambda/q_1$, $0 < \theta < 1$ *and* $0 < p \leqq q \leqq \infty$, *then*

$$T : L_p \to L_q \text{ with the norm } M ,$$

where

$$M \leqq C M_0^{1-\lambda} M_1^{\lambda} .$$

REMARK 4.2. Riesz's theorem is true without the assumption $p \leqq q$. Our method like most other pure real proofs does not work if $p > q$.

PROOF OF THEOREM 4.4. By theorem 4.3, (3.18), theorem 1.1 and finally theorem 4.3 again we get

$$\|Ta\|_{L_q} \leqq C \|Ta\|_{(L_{q_0}, L_{q_1})_{\lambda,q}} \lambda^{\min(1/q;\,1/q_0)} (1-\lambda)^{\min(1/q;\,1/q_1)} , \tag{4.12}$$

$$\|Ta\|_{(L_{q_0}, L_{q_1})_{\lambda,q}} \leqq C \|Ta\|_{(L_{q_0}, L_{q_1})_{\lambda,p}} \lambda^{1/p-1/q} (1-\lambda)^{1/p-1/q} , \tag{4.13}$$

$$\|Ta\|_{(L_{q_0}, L_{q_1})_{\lambda,p}} \leqq M_0^{1-\lambda} M_1^{\lambda} \|a\|_{(L_{p_0}, L_{p_1})_{\lambda p}} , \tag{4.14}$$

$$\|a\|_{(L_{p_0}, L_{p_1})_{\lambda,p}} \leqq C \|a\|_{L_p} \lambda^{-\max(1/p;\,1/p_0)} (1-\lambda)^{-\max(1/p;\,1/p_1)} . \tag{4.15}$$

We now combine (4.12)–(4.15) to

$$\|Ta\|_{L_q} \leqq C M_0^{1-\lambda} M_1^{\lambda} \|a\|_{L_p} \lambda^{\min(0, q_0^{-1}-q^{-1}) + \min(0, p^{-1}-p_0^{-1})} (1-\lambda)^{\min(0, q_1^{-1}-q^{-1}) + \min(0, p^{-1}-p_1^{-1})} , \tag{4.16}$$

but $q_0^{-1} - q^{-1} = \lambda(q_0^{-1} - q_1^{-1})$, $q_1^{-1} - q^{-1} = (1-\lambda)(q_1^{-1} - q_0^{-1})$ and analogously for p so that the constant in the right member of (4.16) is $O(1)$ as $\lambda \to 0$ and $\lambda \to 1$. The above proof will only work if $p_0 \neq p_1$ and $q_0 \neq q_1$. The cases when $p_0 = p_1$ or $q_0 = q_1$ follow from the fact that

$$\|a\|_{L_p}^p = p\lambda(1-\lambda) \|a\|_{(L_p, L_p)_{\lambda,p}} .$$

THEOREM 4.5. (Marcinkiewicz's interpolation theorem [9].) *T is a linear operator such that*

$$T : L_{p_i} \to L_{q_i,\infty} \text{ with the norm } M_i, \quad i = 0,1 ,$$

then, if $q_0 \neq q_1$, $1/p=(1-\lambda)/p_0+\lambda/p_1$, $1/q=(1-\lambda)/q_0+\lambda/q_1$, $0<\lambda<1$, *and* $0<p\leqq q\leqq\infty$,

$$T : L_p \to L_q \text{ with the norm } M ,$$

where

$$M \leqq CM_0^{1-\lambda}M_0^{-\lambda}\lambda^{-1/q}(1-\lambda)^{-1/q} .$$

PROOF. In the same way as in the preceding theorem we get

$$(4.17)\quad \|Ta\|_{L_q} \leqq CM_0^{1-\lambda}M_1^{\lambda}\|a\|_{L_p}\lambda^{-1/q+\min(0,1/p-1/p_0)}(1-\lambda)^{-1/q+\min(0,1/p-1/p_1)},$$

where $\lambda^{1/p-1/p_0}$ and $(1-\lambda)^{1/p-1/p_1}$ are $O(1)$ as $\lambda\to 0$ and $\lambda\to 1$.

REMARK 4.3. The dependence on λ and $(1-\lambda)$ in the "convexity inequalities" of theorem 4.4 and 4.5 is the best possible. See Zygmund [17, chap. XII].

THEOREM 4.6. (Calderon's interpolation theorem [1]). *T is a linear operator such that*

$$T : L_{p_i,1} \to L_{q_i,\infty} \text{ with the norm } M_i, \quad i = 0,1 ,$$

then, if $p_0 \neq p_1$, $q_0 \neq q_1$, $1/p=(1-\lambda)/p_0+\lambda/p_1$, $1/q=(1-\lambda)/q_0+\lambda/q_1$, $0<\lambda<1$, *and* $r<s$,

$$T : L_{p,r} \to L_{q,s} \text{ with the norm } M ,$$

where

$$M \leqq CM_0^{1-\lambda}M_1^{\lambda}\lambda^{1/r-1/s-1}(1-\lambda)^{1/r-1/s-1} .$$

This theorem is proven exactly in the same way as the theorems 4.4 and 4.5.

4.2. Lip spaces. As another application of theorem 3.2 we will prove a theorem by O'Neil [13] about interpolation of Lip spaces.

THEOREM 4.7. *If T is a linear operator such that*

$$T : \mathrm{Lip}\,\alpha_i \to \mathrm{Lip}\,\beta_i \text{ with the norm } M_i, \quad i = 0,1 ,$$

then, if $0\leqq\alpha_0\leqq\alpha_1\leqq 1$, $0\leqq\beta_0,\beta_1\leqq 1$, $0<\lambda<1$, $\alpha=(1-\lambda)\alpha_0+\lambda\alpha_1$, *and* $\beta=(1-\lambda)\beta_0+\lambda\beta_1$,

$$T : \mathrm{Lip}\,\alpha \to \mathrm{Lip}\,\beta \text{ with the norm } M ,$$

where

$$M \leqq CM_0^{1-\lambda}M_1^{\lambda} .$$

PROOF. It is well known in the theory of interpolation spaces (see [14]) that

$$\operatorname{Lip}\alpha = (C_0, C_1)_{\alpha,\infty}, \tag{4.18}$$

where C_0 is the space of continuous functions and C_1 is the space of continuously differentiable functions. The constant in the equivalence of the norms of the spaces $\operatorname{Lip}\alpha$ and $(C_0,C_1)_{\alpha,\infty}$ is independent of α, so the theorem follows at once from theorem 3.2.

5. The equivalence between $(A_0,A_1)_{\theta,p_0,p_1}$ and $(A_0,A_1)_{\theta,p}$.

For any couple (A_0,A_1) of quasi-normed spaces we define the space $(A_0,A_1)_{\theta,p_0,p_1}$ (see [8], [15]) to consist of all $a \in A_0+A_1$ for which

$$\|a\|_{(A_0,A_1)_{\theta,p_0,p_1}} \tag{5.1}$$

$$= \inf_{a_0(t)+a_1(t)=a} \max\left[\left(\int_0^\infty (t^{-\theta}\|a_0(t)\|_{A_0})^{p_0}\frac{dt}{t}\right)^{1/p_0}; \left(\int_0^\infty (t^{1-\theta}\|a_1(t)\|_{A_1})^{p_1}\frac{dt}{t}\right)^{1/p_1}\right] < \infty,$$

where $a_i(t) \in A_i$, $i=0,1$, $0<\theta<1$ and $0<p_0,p_1 \leqq \infty$. In this space we have the quasi-norm $\|\cdot\|_{(A_0,A_1)_{\theta,p_0,p_1}}$ defined by (5.1). The main result of this section is the following theorem.

THEOREM 5.1. *If* $1/p=(1-\theta)/p_0+\theta/p_1$, $0<\theta<1$ *and* $0<p_0,p_1,p \leqq \infty$, *then*

$$(A_0,A_1)_{\theta,p_0,p_1} = (A_0,A_1)_{\theta,p} \tag{5.3}$$

and

$$C_0\|a\|_{(A_0,A_1)_{\theta,p}} \leqq \|a\|_{(A_0,A_1)_{\theta,p_0,p_1}} \leqq C_1\|a\|_{(A_0,A_1)_{\theta,p}}, \tag{5.4}$$

where C_0 *and* C_1 *are independent of* θ.

REMARK 5.1. Our theorem is an improvement of a theorem by Peetre [15]. The constants C_0 and C_1 are better, besides our theorem is true even for quasi-normed spaces.

In the sequel we write A_{θ,p_0,p_1} and $\|a\|_{\theta,p_0,p_1}$ instead of $(A_0,A_1)_{\theta,p_0,p_1}$ and $\|a\|_{(A_0,A_1)_{\theta,p_0,p_1}}$, respectively.

LEMMA 5.1. $A_{\theta,p,p}=A_{\theta,p}$ *and*

$$\|a\|_{\theta,p,p} \leqq \|a\|_{\theta,p} \leqq 2C\|a\|_{\theta,p,p}, \tag{5.5}$$

where

$$(5.6) \qquad C = 1 \quad \text{if } p \geqq 1\,, \qquad = 2^{-1+1/p} \quad \text{if } 0<p<1\,.$$

PROOF. It is obvious that

$$(5.7) \qquad \|a\|_{\theta,p_0,p_1} \leqq \inf_{a_0+a_1=a}\left[\left(\int_0^\infty (t^{-\theta}\|a_0\|)^{p_0}\frac{dt}{t}\right)^{1/p_0} + \left(\int_0^\infty (t^{1-\theta}\|a_1\|_{A_1})^{p_1}\frac{dt}{t}\right)^{1/p_1}\right] \leqq 2\|a\|_{\theta,p_0,p_1}\,.$$

By the definition of $K(t,a;A_0,A_1)$ there are $a_0(t) \in A_0$ and $a_1(t) \in A_1$ with $a_0(t)+a_1(t)=a$ such that

$$\|a_0(t)\|_{A_0} \leqq K(t,a) \quad \text{and} \quad t\|a_1(t)\|_{A_1} \leqq K(t,a)\,,$$

thus

$$(5.8) \qquad \left(\int_0^\infty (t^{-\theta}\|a_0(t)\|_{A_0})^p\frac{dt}{t}\right)^{1/p} \leqq \left(\int_0^\infty (t^{-\theta}K(t,a))^p\frac{dt}{t}\right)^{1/p},$$

$$(5.9) \qquad \left(\int_0^\infty (t^{1-\theta}\|a_1(t)\|_{A_1})^p\frac{dt}{t}\right)^{1/p} \leqq \left(\int_0^\infty (t^{-\theta}K(t,a))^p\frac{dt}{t}\right)^{1/p},$$

that is, $\|a\|_{\theta,p,p} \leqq \|a\|_{\theta,p}$.

Now let $a_0(t)+a_1(t)=a$ be an arbitrary partition of a, then

$$(5.10) \qquad \|a\|_{\theta,p} = \left(\int_0^\infty (t^{-\theta}K(t,a))^p\frac{dt}{t}\right)^{1/p}$$

$$\leqq \left(\int_0^\infty \left(t^{-\theta}(\|a_0(t)\|_{A_0}+t\|a_1(t)\|_{A_1})\right)^p\frac{dt}{t}\right)^{1/p}$$

$$\leqq C\left[\left(\int_0^\infty (t^{-\theta}\|a_0(t)\|_{A_0})^p\frac{dt}{t}\right)^{1/p} + \left(\int_0^\infty ({}^{1-\theta}\|a_1(t)\|_{A_1})^p\frac{dt}{t}\right)^{1/p}\right],$$

where C is defined by (5.6). Taking the inf over all partitions $a_0+a_1=a$ we get by (5.9)

$$\|a\|_{\theta,p} \leqq 2C\|a\|_{\theta,p,p}\,,$$

and the proof is complete.

To prove theorem 5.1 it suffices, according to lemma 5.1, to show that $A_{\theta,p_0,p_1}=A_{\theta,p,p}$ if $1/p=(1-\theta)/p_0+\theta/p_1$, which is an immediate consequense of lemma 5.3 below. To prove this lemma we need a reformulation of the definition (5.2) of the quasi-norm $\|a\|_{A_{\theta,p_0,p_1}}$ (see 5.11 and 5.12).

For every $a = a_0 + a_1$, $a_i \in A_i$, $i = 0,1$, and for every $x \geqq 0$ we now define the function

(5.11) $$f(a,x) = \inf_{\|a_0\|_{A_0} \leqq x} \|a_1\|_{A_1}, \quad a = a_0 + a_1 .$$

From the definition of $f(a,x)$ it is easy to see that $f(a,x)$ is non-negative, decreasing and convex function of x. If we use the function f, we get the following definition of $\|a\|_{\theta,p_0,p_1}$:

(5.12) $$\|a\|_{\theta,p_0,p_1} = \inf_{w(t)} \max \left[\left(\int_0^\infty (t^{-\theta} w(t))^{p_0} \frac{dt}{t} \right)^{1/p_0} ; \left(\int_0^\infty (t^{1-\theta} f(a,w(t))^{p_1}) \frac{dt}{t} \right)^{1/p_1} \right],$$

where the inf is to be taken over all non-negative measurable functions $w(t)$.

The main idea is now to show that we will come close to the inf in (5.12), if we choose $w(t)$ so that the two integrands $t^{-\theta p_0 - 1} w(t)^{p_0}$ and $t^{(1-\theta)p_1 - 1} f(a,w(t))^{p_1}$ become proportional. For this purpose we define

(5.13) $$\alpha(a) = \theta^{(p_0-p_1)/p} \left(\int_0^\infty (t^{1-\theta} f(a,t)^{\theta})^p \frac{dt}{t} \right)^{(p_1-p_0)/p} .$$

(In the sequel we keep a fixed and so we do not write a in the formulas). The function $v^{-1}(s)$ defined by

(5.14) $$v^{-1}(s) = \alpha^{p/p_0 p_1} s^{p/p_1} f(s)^{-p/p_0}, \quad s \geqq 0 ,$$

is increasing, continuous and $v^{-1}(0) = 0$. Accordingly v^{-1} has an inverse function v with the same properties. Thus (5.14) is equivalent to

(5.15) $$t = \alpha^{p/p_0 p_1} v(t)^{p/p_1} f(v(t))^{-p/p_0}$$

which obviously is equivalent to

(5.16) $$\alpha t^{-\theta p_0 - 1} v(t)^{p_0} = t^{(1-\theta)p_1 - 1} f(v(t))^{p_1} .$$

Lemma 5.2. *With the assumptions of theorem* 5.1 *we have*

$$\left(\int_0^\infty (t^{-\theta} v(t))^{p_0} \frac{dt}{t} \right)^{1/p_0} = \left(\int_0^\infty (t^{1-\theta} f(v(t)))^{p_1} \frac{dt}{t} \right)^{1/p_1}$$

$$= \theta^{-1/p} \left(\int_0^\infty (t^{1-\theta} f(t)\, \theta)^p \frac{dt}{t} \right)^{1/p} .$$

PROOF. Making a change of variables by putting $t=v^{-1}(s)$ we get

$$\int_0^\infty (t^{-\theta}v(t))^{p_0}\frac{dt}{t} = \int_0^\infty v^{-1}(s)^{-p_0\theta-1}s^{p_0}\,d(v^{-1}(s)), \tag{5.17}$$

which after an integration by parts yields

$$\int_0^\infty (t^{-\theta}v(t))^{p_0}\frac{dt}{t} = [-p_0^{-1}\theta^{-1}v^{-1}(s)^{-p_0\theta}s^{p_0}]_0^\infty + \theta^{-1}\int_0^\infty v^{-1}(s)^{-p_0\theta}s^{p_0-1}\,ds. \tag{5.18}$$

If $\int_0^\infty(t^{-\theta}v(t))^{p_0}dt/t<\infty$, it is easy to see that $\lim_{t\to 0}t^{-\theta}v(t)=0$ and $\lim_{t\to\infty}t^{-\theta}v(t)=0$, so the term within brackets in (5.18) vanishes. From the definition of $v^{-1}(s)$ (5.14) and (5.13) we finally get

$$\begin{aligned}\theta^{-1}\int_0^\infty v^{-1}(s)^{-p_0\theta}s^{p_0-1}\,ds &= \theta^{-1}\alpha^{-p\theta/p_1}\int_0^\infty f(s)^{p\theta}s^{-p_0p\theta/p_1+p_0-1}\,ds \\ &= \theta^{-1}\alpha^{-p\theta/p_1}\int_0^\infty (s^{1-\theta}f(s)^\theta)^p\frac{ds}{s} \\ &= \theta^{-p_0/p}\left(\int_0^\infty (s^{1-\theta}f(s)^\theta)^p\frac{ds}{s}\right)^{p_0/p}.\end{aligned} \tag{5.19}$$

This proves one half of the lemma. The other one is obtained in exactly the same way.

LEMMA 5.3. *With the assumptions of theorem* 5.1 *we have*

$$C\theta^{-1/p}\left(\int_0^\infty (t^{1-\theta}f(t)^\theta)^p\frac{dt}{t}\right)^{1/p} \leqq \|a\|_{\theta,p_0,p_1} \leqq \theta^{1/p}\left(\int_0^\infty (t^{1-\theta}f(t)^\theta)^p\frac{dt}{t}\right)^{1/p},$$

where

$$\begin{aligned}C &= 2^{-\max(p_0/p_1;\,p_1/p_0)} && \textit{if } p_0,p_1\geqq 1, \\ &= 2^{-\max(p_0/p_1^2;\,p_1/p_0^2)} && \textit{if } p_0 \text{ or } p_1<1.\end{aligned} \tag{5.20}$$

PROOF. From lemma 5.2 and (5.12) we get

$$\|a\|_{\theta,p_0,p_1} \leqq \theta^{-1/p}\left(\int_0^\infty (t^{1-\theta}f(t)^\theta)^p\frac{dt}{t}\right)^{1/p}. \tag{5.21}$$

Let now $w(t)$ be an arbitrary measurable function and put

$$(5.22)\qquad M = M(w) = \{t\,;\ t \geqq 0 \text{ and } v(t) \leqq w(t)\} = \{t\,;\ t \geqq 0 \text{ and } f(v(t)) \geqq f(w(t))\}$$

and $\complement M$ the complement of M. Then by lemma 5.2, Minkowski's inequality and (5.16)

$$(5.23)\quad I = \left(\int_0^\infty (t^{1-\theta} f(t)^\theta)^p \frac{dt}{t}\right)^{1/p}$$

$$= 2^{-1}\theta^{1/p}\left[\left(\int_0^\infty (t^{-\theta}v(t))^{p_0}\frac{dt}{t}\right)^{1/p_0} + \left(\int_0^\infty (t^{1-\theta}f(v(t)))^{p_1}\frac{dt}{t}\right)^{1/p_1}\right]$$

$$\leqq 2^{-1}\theta^{1/p}C'\left[\left(\int_M (t^{-\theta}v(t))^{p_0}\frac{dt}{t}\right)^{1/p_0} + \left(\int_{\complement M} (t^{-\theta}v(t))^{p_0}\frac{dt}{t}\right)^{1/p_0} + \right.$$

$$\left. + \left(\int_M (t^{1-\theta}f(v(t)))^{p_1}\frac{dt}{t}\right)^{1/p_1} + \left(\int_{\complement M} (t^{1-\theta}f(v(t)))^{p_1}\frac{dt}{t}\right)^{1/p_1}\right]$$

$$= 2^{-1}\theta^{1/p}C'\left[\left(\int_M (t^{-\theta}v(t))^{p_0}\frac{dt}{t}\right)^{1/p_0} + \alpha^{-1/p_0}\left(\int_{\complement M} (t^{1-\theta}f(v(t)))^{p_1}\frac{dt}{t}\right)^{1/p_0} + \right.$$

$$\left. + \alpha^{1/p_1}\left(\int_M (t^{-\theta}v(t))^{p_0}\frac{dt}{t}\right)^{1/p_1} + \left(\int_{\complement M} (t^{1-\theta}f(v(t)))^{p_1}\frac{dt}{t}\right)^{1/p_1}\right].$$

The constant C' in Minkowski's inequality is

$$(5.24)\qquad \begin{aligned} C' &= 1 && \text{if } p_0, p_1 \geqq 1\,, \\ &= 2^{\max(1/p_0;\,1/p_1)-1} && \text{if } p_0 \text{ or } p_1 < 1\,. \end{aligned}$$

If we now substitute $w(t)$ for $v(t)$ in the four last integrals of (5.23), all these integrals will increase, and if we put

$$(5.25)\qquad I_0 = \left(\int_0^\infty (t^{-\theta}w(t)^{p_0}\frac{dt}{t}\right)^{1/p_0} \quad\text{and}\quad I_1 = \left(\int_0^\infty (t^{1-\theta}f(w(t)))^{p_1}\frac{dt}{t}\right)^{1/p_1},$$

we get with the aid of (5.13)

$$(5.26)\qquad I \leqq 2^{-1}\theta^{1/p}C'(I_0 + \theta^{-(p_0-p_1)/pp_0} I^{-(p_1-p_0)/p_0} I_1^{p_1/p_0} + \theta^{(p_0-p_1)/pp_1} I^{(p_1-p_0)/p_1} I_0^{p_0/p_1} + I_1)\,.$$

If we put $K_0 = I_0\theta^{1/p}I^{-1}$, $K_1 = I_1\theta^{1/p}I^{-1}$ and $p_0/p_1 = q$ in (5.26), we get

$$(5.27)\qquad 1 \leqq 2^{-1}C'(K_0 + K_1^{1/q} + K_0^q + K_1)\,.$$

It is easy to see that (5.27) implies

$$\max(K_0, K_1) \geqq (2C')^{-\max(p_0/p_1; p_1/p_0)}, \tag{5.28}$$

that is,

$$I \leqq \theta^{1/p}(2C')^{\max(p_0/p_1; p_1/p_0)} \max(I_0; I_1) \tag{5.29}$$

or

$$\left(\int_0^\infty (t^{1-\theta} f(t)^\theta)^p \frac{dt}{t}\right)^{1/p} \tag{5.30}$$

$$\leqq \theta^{1/p}(2C')^{\max(p_0/p_1; p_1/p_0)} \max\left[\left(\int_0^\infty (t^{-\theta} w(t))^{p_0} \frac{dt}{t}\right)^{1/p_0}; \left(\int_0^\infty (t^{1-\theta} f(w(t)))^{p_1} \frac{dt}{t}\right)^{1/p_1}\right].$$

The inequality (5.29) is true for all measurable functions $w(t)$. Taking the inf over all such functions we get the remaining inequality of lemma 5.3.

PROOF OF THEOREM 5.1. From lemma 5.3 we get, if

$$J = \theta^{-1/p}\left(\int_0^\infty (t^{1-\theta} f(t)^\theta)^p \frac{dt}{t}\right)^{1/p},$$

the inequalities

$$C_2 J \leqq \|a\|_{\theta, p_0, p_1} \leqq J \tag{5.31}$$

and

$$C_3 J \leqq \|a\|_{\theta, p, p} \leqq J, \tag{5.32}$$

where C_2 is the constant C defined by (5.20) and

$$C_3 = 2^{-1} \quad \text{if } p \geqq 1, \qquad = 2^{-1/p} \quad \text{if } p < 1. \tag{5.33}$$

(5.31)–(5.33) and lemma 5.1 obviously imply the theorem.

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