Interpolation on lattices generated by cubic pencils

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Dedicated to C. A. Micchelli for his mathematical contributions and friendship in occasion of his sixtieth birthday

Abstract: Principal lattices are distributions of points in the plane obtained from a triangle by drawing equidistant parallel lines to the sides and taking the intersection points as nodes. Interpolation on principal lattices leads to particularly simple formulae. These sets were generalized by Lee and Phillips considering three-pencil lattices, generated by three linear pencils. Inspired by the addition of points on cubic curves and using duality, we introduce an addition of lines as a way of constructing lattices generated by cubic pencils. They include three-pencil lattices and then principal lattices. Interpolation on lattices generated by cubic pencils has the same good properties and simple formulae as on principal lattices.

1. Introduction

In bivariate polynomial interpolation it is well-known that the Lagrange problem for a set X of $\binom{n+2}{2}$ interpolation points (nodes) is unisolvent in $\Pi_n(\mathbf{R}^2)$ if and only if the points do not lie on an algebraic curve of degree n. However a permanent and nontrivial question is the explicit construction of sets X satisfying this condition. Closely related to this question is the construction of the solution in a simple way. See [11], [12] for details.

Some sets giving rise to unisolvence in $\Pi_n(\mathbf{R}^2)$ are the ones formed by $n+1$ points lying on a straight line L_0, n points on another line L_1 but not on L_0, $n-1$ points on another line L_2 but not on $L_0 \cup L_1$, and so on. For brevity, these sets will be referred to as DL_n sets (decreasing lines sets). In [10] it was proved that a Newton-type formula solves the interpolation problem on DL_n sets in a very simple way. A little earlier, Chung and Yao [7] found a geometric characterization (GC_n) of sets of $\binom{n+2}{2}$ points with unisolvent interpolation problem in $\Pi_n(\mathbf{R}^2)$ whose Lagrange polynomials are products of n polynomials of first degree. As it will be formally defined in Section 2, they are sets X such that for each $x \in X$ all the rest of the points of X lie on n lines which do not contain x. Obviously, for these sets the Lagrange formula is a very simple way of solving the corresponding interpolation problem.

In [10] it was conjectured that every set satisfying GC_n has $n+1$ collinear points. This conjecture has been proved for $n \le 4$ in different forms [2], [4], but there is neither a counterexample nor a proof for upper degrees. Its interest stems from the fact that once excluded the $n+1$ collinear points one would get a GC_{n-1} set, with n collinear points and so on. If the conjecture holds for all degrees less than or equal to n, all GC_n sets would be DL_n sets. So, the interpolation problem on GC_n sets would be easily solvable by Newton and by Lagrange formulae.

It has been recently proved in [5] that, if the conjecture holds for all degrees less than or equal to n, then for any GC_n set there exist at least three different lines L_0, L_1, L_2,

* Partially supported by the Spanish Research Grant BFM2003-03510, by Gobierno de Aragón and Fondo Social Europeo.

each one containing $n+1$ points of X. Moreover, the intersections $L_i \cap L_j$, $i \neq j$, belong to X. All known examples of GC_n sets have this property. Observe that $X_i = X \setminus L_i$, $i = 0, 1, 2$, are GC_{n-1} sets whose corresponding Lagrange interpolation problem is unisolvent in $\Pi_{n-1}(\mathbf{R}^2)$ and so the problem for X can be solved by an Aitken-Neville formula for bivariate interpolation using the solutions of the three simpler problems associated to the sets X_i (see [9], [11], [12], [16], [18], [19]).

Perhaps the GC_n sets most used in practice, for example in finite elements [17], are the ones obtained from a triangle by drawing equidistant parallel lines to the sides and taking the intersection points as nodes, as it will be precised in the next section. These sets, which appear in classical texts of Numerical Analysis [13], [14], were called principal lattices by Nicolaides in [17] and Chung and Yao in [7], and were generalized from a projective point of view by Lee and Phillips [15] by using the intersections of three linear pencils of lines. Interpolation problems whose nodes form three-pencil lattices can be easily solved by Lagrange, Newton and Aitken-Neville formulae.

Our aim in this paper is to extend the idea of [15] by constructing sets of $\binom{n+2}{2}$ points which are intersection of three lines each, in such a form that the Lagrange interpolation problem is unisolvent in $\Pi_n(\mathbf{R}^2)$ and can be easily solved by any of the three classical formulae.

In Section 2 we recall the construction of three-pencil lattices, define *generalized principal lattices* (GPL) and show the Lagrange formula for problems associated to them. It is well-known [1], [20], that the nonsingular points of an algebraic cubic curve form an abelian group under a binary operation called addition of points. Inspired by this idea and using duality, we define in Section 3 an addition on some sets of lines as a way of constructing GPL sets. In Section 4 we show how three-pencil lattices can be interpreted in terms of addition of lines in the set formed by the union of three linear pencils. In Section 5, we see that the addition of lines can be defined on any cubic pencil. Three-pencil lattices can be considered as a particular case. We also provide a classification of all cubic pencils showing how to obtain GPL sets. As a final example of the general theory, we study in Section 6 a particular class of cubic pencils: the set of lines tangent to a deltoid.

More examples of generalized principal lattices, namely those generated by reducible cubic pencils of lines are shown in [6]

2. A generalization of principal lattices

Along this paper we shall use capital letters L to refer to lines in a geometric sense, while the corresponding lower case letter l will denote any polynomial of first degree such that $l = 0$ is the equation of L. Obviously l is determined by L only up to a constant factor.

We shall denote $\mathbf{N}_n = \{0, 1, \ldots, n\} \subset \mathbf{Z}$ and

$$\mathcal{S}_n := \{(i,j,k) \mid i,j,k \in \mathbf{N}_n, i+j+k = n\} \subseteq \mathbf{Z}^3. \tag{2.1}$$

Principal lattices in the plane are the distributions of points of the form

$$\frac{i}{n}a + \frac{j}{n}b + \frac{k}{n}c, \quad (i,j,k) \in \mathcal{S}_n,$$

where a, b, c are noncollinear points, that is, the vertices of a triangle (see [12] for details on the history of these sets). In the case of the standard triangle (i.e., with vertices $(1,0),(0,1),(0,0)$) this set of points is $X = \{(i/n, j/n) \mid i, j \in \mathbf{N}_n,\ i+j \le n\}$. It can be described as the set of points lying on exactly one line from each of the pencils consisting of $n+1$ parallel lines defined by

$$
\begin{aligned}
l_i^1(x,y) &= x - \frac{i}{n}, \quad i \in \mathbf{N}_n,\\
l_j^2(x,y) &= y - \frac{j}{n}, \quad j \in \mathbf{N}_n,\\
l_k^3(x,y) &= x + y - \frac{n-k}{n}, \quad k \in \mathbf{N}_n.
\end{aligned}
$$

Lee and Phillips [15] generalized this idea introducing lattices generated by $k+1$ pencils of hyperplanes in $\mathbf{R}^k$. In the bivariate case, a *three-pencil lattice* is a set of $\binom{n+2}{2}$ points determined by three pencils of lines (each pencil consisting of $n+1$ concurrent lines or parallel lines) such that each point lies on exactly one line from each pencil. Figure 1 shows a three-pencil lattice for interpolation with quartic polynomials.

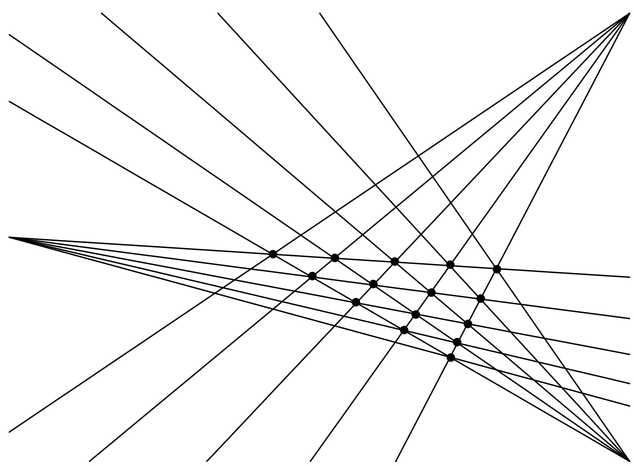


Figure 1. A three-pencil lattice

With a suitable choice of the projective coordinates, the three-pencil lattices with noncollinear vertices considered in [15] are given by

$$
\begin{aligned}
l_i^1(x,y) &= \mu^i x - 1, \quad i \in \mathbf{N}_n,\\
l_j^2(x,y) &= \mu^j y - x, \quad j \in \mathbf{N}_n,\\
l_k^3(x,y) &= \mu^{n-k} y - 1, \quad k \in \mathbf{N}_n,
\end{aligned}
$$

where $\mu \in \mathbf{R} \setminus \{-1, 0, 1\}$. Three-pencil lattices with collinear vertices arise in the limit case $\mu \to 1$. In fact, principal lattices are three-pencil lattices such that the vertices of the pencils lie on the ideal line.

Let us observe that the set of points lying exactly on one line of each pencil is $X = \{L_i^1 \cap L_j^2 \cap L_k^3 \mid (i,j,k) \in \mathcal{S}_n\}$, where L_i^1, L_j^2 and L_k^3 are the lines with equations $l_i^1 = 0$, $l_j^2 = 0$ and $l_k^3 = 0$, respectively. This motivates the following definition:

Definition 2.1. Let

$$L_i^1,\ i\in \mathbf{N}_n, \quad L_j^2,\ j\in \mathbf{N}_n, \quad L_k^3,\ k\in \mathbf{N}_n,$$

be three families of lines containing $3n+3$ distinct lines such that

$$L_i^1, L_j^2, L_k^3 \text{ are concurrent } \iff (i,j,k)\in \mathcal{S}_n. \tag{2.2}$$

A *generalized principal lattice* GPL_n is the set of points

$$X=\{x_{ijk} \mid x_{ijk} := L_i^1\cap L_j^2 \cap L_k^3, (i,j,k)\in \mathcal{S}_n\}, \tag{2.3}$$

lying on exactly one line of each family.

Let us see that a point lying on one line of each family cannot lie on any other line of the three families. If $x_{ijk}=L_i^1\cap L_j^2\cap L_k^3$, $(i,j,k)\in\mathcal{S}_n$, lies on $L_{i'}^1$, $i'\in\mathbf{N}_n$, then $L_{i'}^1, L_j^2, L_k^3$ are concurrent. By (2.2), $(i',j,k)\in\mathcal{S}_n$. Then we have $i'+j+k=n=i+j+k$ and so, $i'=i$. Analogously, if $x_{ijk}\in L_{j'}^2$, then $j=j'$ and if $x_{ijk}\in L_{k'}^3$, then $k=k'$. From this observation, it follows that points corresponding to different indices in $\mathcal{S}_n$ are distinct. In fact, if $x_{ijk}=x_{i'j'k'}$ with $(i,j,k),(i',j',k')\in\mathcal{S}_n$, we have $i'=i$, $j'=j$ and $k'=k$ because a point lying on one line of each family cannot lie on any other line. Therefore the cardinal of the set of points X defined in (2.3) is $|\mathcal{S}_n|=\binom{n+2}{2}$.

Let us observe that, in contrast to [15], we do not impose that any family of lines is composed by lines of the same linear pencil (that is concurrent or parallel lines). For instance, in section 6, we shall describe a construction of a GPL_n set where each family is not contained in a linear pencil.

Let us describe the three-pencil lattices in the terms of Definition 2.1. We take for $\mu\neq 1$

$$l_i^1=\mu^i p_1-p_0, \quad l_j^2=\mu^j p_2-p_1, \quad l_k^3=\mu^{n-k}p_2-p_0, \tag{2.4}$$

and for $\mu=1$

$$l_i^1=p_1-ip_0, \quad l_j^2=p_2-jp_0, \quad l_k^3=p_1+p_2-(n-k)p_0, \tag{2.5}$$

where p_0,p_1,p_2 is a basis of the space of polynomials of degree not greater than one. Principal lattices correspond to (2.5), taking p_0 a constant polynomial.

A suitable change of coordinates allows us to view all projective transformations of a principal lattice (2.5) as the limit case $\mu\to 1$ of (2.4). First we choose a basis p_0,p_1,p_2 of the space of polynomials of degree not greater than one. A change of basis (dependent on the parameter μ) allows us to replace p_0,p_1,p_2 by $(1-\mu)^{-1}p_0-p_1, (1-\mu)^{-1}p_0, (1-\mu)^{-1}p_0+p_2$ and then the lines are given by the formulae

$$\begin{aligned}
l_i^1 &= \frac{\mu^i}{1-\mu}p_0-\Big(\frac{1}{1-\mu}p_0-p_1\Big)=p_1-\frac{1-\mu^i}{1-\mu}p_0\to p_1-ip_0,\\
l_j^2 &= \mu^j\Big(\frac{1}{1-\mu}p_0+p_2\Big)-\frac{1}{1-\mu}p_0=\mu^j p_2-\frac{1-\mu^j}{1-\mu}p_0\to p_2-jp_0,\\
l_k^3 &= \mu^{n-k}\Big(\frac{1}{1-\mu}p_0+p_2\Big)-\Big(\frac{1}{1-\mu}p_0-p_1\Big)=p_1+\mu^{n-k}p_2-\frac{1-\mu^{n-k}}{1-\mu}p_0\\
&\to p_1+p_2-(n-k)p_0.
\end{aligned}$$

Generalized principal lattices satisfy the geometric characterization of Chung and Yao [7]. This property characterizes sets of $(n+2)(n+1)/2$ nodes in the plane which are unisolvent for the Lagrange interpolation problem in $\Pi_n(\mathbf{R}^2)$ and whose Lagrange polynomials are products of linear factors.

Definition 2.2. A set of $\binom{n+2}{2}$ nodes $X \subseteq \mathbf{R}^2$ satisfies the geometric characterization GC_n if for each node $x \in X$, there exist n lines containing all nodes in $X \setminus \{x\}$ but not x.

Proposition 2.3. *Let X be a GPL_n set. Then X satisfies GC_n and is unisolvent for the Lagrange interpolation problem in $\Pi_n(\mathbf{R}^2)$. A Lagrange formula for the interpolant is given by*

$$p = \sum_{(i,j,k)\in\mathcal{S}_n} f(x_{ijk}) \prod_{i'=0}^{i-1} \frac{l^1_{i'}}{l^1_{i'}(x_{ijk})} \prod_{j'=0}^{j-1} \frac{l^2_{j'}}{l^2_{j'}(x_{ijk})} \prod_{k'=0}^{k-1} \frac{l^3_{k'}}{l^3_{k'}(x_{ijk})}.$$

Proof: Given $(i,j,k) \in \mathcal{S}_n$, the n lines with equations

$$l^1_{i'} = 0,\ 0 \le i' \le i-1, \quad l^2_{j'} = 0,\ 0 \le j' \le j-1, \quad l^3_{k'} = 0,\ 0 \le k' \le k-1,$$

contain all nodes of $X \setminus \{x_{ijk}\}$ but not x_{ijk}. ■

3. Addition of lines

In order to construct new GPL_n configurations, we need to control the concurrence properties of the set of lines $\{L^1_i \mid i \in \mathbf{N}_n\} \cup \{L^2_j \mid j \in \mathbf{N}_n\} \cup \{L^3_k \mid k \in \mathbf{N}_n\}$ in Definition 2.1. This corresponds, by duality, to control the collinearity of a given set of points. It is a well-known fact [1], [20], that the collinearity properties of points lying on an irreducible plane cubic curve is governed by a composition law called addition of points. The underlying abelian group can be used to construct and describe complicated patterns of points satisfying prescribed collinearity conditions. For this reason, we introduce a group structure on certain families of lines (addition of lines) as a tool for obtaining sets of lines with the concurrence properties of GPL_n configurations. Let us remark that the existence of an addition on a family of lines is a restrictive condition. In fact, all examples that we provide are families satifying some algebraic relations.

Assume that we have a set Λ of lines of the plane and a binary operation $\oplus$ defined on this set of lines, such that Λ is an abelian group. Let $V \subset \mathbf{R}^2$ be a set of points which we call *vertices*. Let us assume that, if L_1, L_2 and L_3 are distinct lines of Λ, then

$$L_1 \oplus L_2 \oplus L_3 = 0,\ L_1 \cap L_2 \cap L_3 \cap V = \emptyset \Leftrightarrow L_1, L_2, L_3 \text{ are concurrent at a point of } \mathbf{R}^2 \setminus V, \tag{3.1}$$

in other words, three distinct lines of Λ not containing the same vertex are concurrent if and only if they add up to zero. In general, V will be the empty set. However, we shall also use sets of vertices with 1 or 3 elements in order to deal with reducible cases. The binary operation $\oplus$ satisfying (3.1) will be called an *addition of lines*. The zero line will be denoted by 0 and the opposite of $L \in \Lambda$ by $\ominus L$. For m a positive integer, the sum of m repeated summands $L \oplus \cdots \oplus L$ will be denoted by mL. If $m = 0$, $mL = 0$. If m is a negative number, mL denotes $\ominus(-m)L$. The sum of L_1 and $\ominus L_2$ will be written $L_1 \ominus L_2$.

The addition of two lines on a set Λ is not completely determined by property (3.1). In fact, the zero line can be arbitrarily chosen, and therefore the meaning of an opposite line may change depending on the context. So, the addition of lines needs not to have a clear geometric meaning. The reader interested in a better understanding of this operation in this moment is invited to visit the examples in sections 4 and 6.

The following theorem shows how the addition of lines is useful to generate generalized principal lattices.

Theorem 3.1. *Let $\oplus$ be an addition of lines defined on Λ with a set of vertices V.*

(i) Let H, K_1, K_2 be three lines of Λ. Then the $3n+3$ lines

$$\begin{aligned} L_i^1 &= K_1 \oplus iH, \quad i \in \mathbf{N}_n, \\ L_j^2 &= K_2 \oplus jH, \quad j \in \mathbf{N}_n, \\ L_k^3 &= \ominus K_1 \ominus K_2 \oplus (k-n)H, \quad k \in \mathbf{N}_n, \end{aligned} \tag{3.2}$$

are distinct if and only if

$$mH \neq 0, \quad 0 < m \leq n,$$

and

$$K_1 \ominus K_2 \oplus nH,\ \ominus 2K_1 \ominus K_2,\ \ominus K_1 \ominus 2K_2 \notin \{mH \mid m \in \mathbf{N}_{2n}\}. \tag{3.3}$$

Moreover, if

$$mH \neq 0, \quad 0 < m \leq 2n, \tag{3.4}$$

then $L_i^1 \oplus L_j^2 \oplus L_k^3 = 0$ if and only if $(i,j,k) \in \mathcal{S}_n$.

(ii) Let H, K_1, K_2 be three lines of Λ satisfying (3.3) and (3.4). Let X be the set of points lying on one line of each family of (3.2). If $(i,j,k) \in \mathcal{S}_n$, then L_i^1, L_j^2 and L_k^3 are concurrent at a point $x_{ijk} := L_i^1 \cap L_j^2 \cap L_k^3 \in X$ and

$$X \setminus V \subseteq \{x_{ijk} \mid (i,j,k) \in \mathcal{S}_n\} \subseteq X. \tag{3.5}$$

If, in addition, $X \cap V = \emptyset$, then

$$X = \{x_{ijk} \mid (i,j,k) \in \mathcal{S}_n\} \tag{3.6}$$

is a GPL_n set.

(iii) Let X be a GPL_n set contained in $\mathbf{R}^2 \setminus V$, defined by lines in Λ, according to Definition 2.1. Then there exist $H, K_1, K_2 \in \Lambda$ satisfying (3.3) and (3.4) such that X is the set of points lying exactly on one line of each family of (3.2).

Proof: (i) If $L_i^1 = L_{i'}^1$, then $(i-i')H = 0$. So, the lines of the first family L_i^1, $i = 0, \ldots, n$, are all distinct if and only if $mH \neq 0$, $0 < m \leq n$. Analogously for the other families. If $L_i^1 = L_j^2$ (resp., $L_i^1 = L_k^3$, $L_j^2 = L_k^3$), then $K_1 \ominus K_2 \oplus nH = (j+n-i)H$ (resp., $\ominus 2K_1 \ominus K_2 = (i+n-k)H$, $\ominus K_1 \ominus 2K_2 = (j+n-k)H$). So, the lines of different families are all distinct if and only if $mH \neq 0$, $0 < m \leq n$, and (3.3) holds.

If $(i,j,k) \in \mathcal{S}_n$ then $L_i^1 \oplus L_j^2 \oplus L_k^3 = (i+j+k-n)H = 0$. Conversely, let $i,j,k \in \mathbf{N}_n$ such that $L_i^1 \oplus L_j^2 \oplus L_k^3 = 0$. Then $0 = (i+j+k-n)H$ and, by (3.4), $i+j+k = n$, that is, $(i,j,k) \in \mathcal{S}_n$.

(ii) If $(i,j,k) \in \mathcal{S}_n$, then $L_i^1 \oplus L_j^2 \oplus L_k^3 = (i+j+k-n)H = 0$, which means that L_i^1, L_j^2 and L_k^3 are concurrent at a point $x_{ijk} \in X$. Therefore $\{x_{ijk} \mid (i,j,k) \in \mathcal{S}_n\} \subseteq X$.

If $x \in X \setminus V$, then $x = L_i^1 \cap L_j^2 \cap L_k^3 \notin V$ for some indices $i,j,k \in \mathbf{N}_n$. By (3.1) and (3.2), we have $L_i^1 \oplus L_j^2 \oplus L_k^3 = 0$ and, by (3.4), $(i,j,k) \in \mathcal{S}_n$. So we have seen that (3.5) holds.

Let us assume that $X \cap V = \emptyset$. From (3.5), formula (3.6) follows and, by (i), all lines (3.2) are distinct. So, X is a GPL_n set.

(iii) Let $X \subset \mathbf{R}^2 \setminus V$ be a GPL_n set. According to Definition 2.1, there exist lines $L_i^1 \in \Lambda$, $i \in \mathbf{N}_n$, $L_j^2 \in \Lambda$, $j \in \mathbf{N}_n$, $L_k^3 \in \Lambda$, $k \in \mathbf{N}_n$, such that x_{ijk} is the common intersection of L_i^1, L_j^2 and L_k^3 for $(i,j,k) \in \mathcal{S}_n$ and X is the set (2.3). By (3.1), we must have

$$L_i^1 \oplus L_j^2 \oplus L_k^3 = 0 \Leftrightarrow i+j+k = n. \tag{3.7}$$

Let $K_1 := L_0^1$, $K_2 := L_0^2$. Since $L_0^1 \oplus L_1^2 \oplus L_{n-1}^3 = L_1^1 \oplus L_0^2 \oplus L_{n-1}^3 = 0$ we deduce that $L_1^1 \ominus L_0^1 = L_1^2 \ominus L_0^2$. Let $H := L_1^1 \ominus L_0^1 = L_1^2 \ominus L_0^2$. From

$$L_1^1 \oplus L_j^2 \oplus L_{n-j-1}^3 = L_0^1 \oplus L_{j+1}^2 \oplus L_{n-j-1}^3 = 0,$$

for all $j = 0, \ldots, n-1$, we deduce that $L_{j+1}^2 = L_j^2 \oplus H$. So we have $L_j^2 = K_2 \oplus jH$, $j = 0, \ldots, n$. Analogously, we have $L_i^1 \oplus L_1^2 = L_{i+1}^1 \oplus L_0^2$, for all $i = 0, \ldots, n-1$ and obtain $L_i^1 = K_1 \oplus iH$, $i \in \mathbf{N}_n$. Finally, from

$$0 = L_{n-k}^1 \oplus L_0^2 \oplus L_k^3 = K_1 \oplus K_2 \oplus (n-k)H \oplus L_k^3$$

we obtain $L_k^3 = \ominus K_1 \ominus K_2 \oplus (k-n)H$. Therefore, we have shown that (3.2) holds. Since X is a GPL_n all lines (3.2) must be distinct and from (i), condition (3.3) follows. Finally (3.7) implies that $(i+j+k-n)H = 0$ only if $i+j+k = n$ and so (3.4) holds. ∎

4. Addition on three pencils

In this section, we are going to introduce an addition of lines on the set formed by three pencils and show that the construction of three-pencil lattices can be seen as a particular case of Theorem 3.1. For the description of the groups associated to three-pencil lattices, let us denote, as usual, the additive group of integers modulo p by $\mathbf{Z}_p = \{0, 1, \ldots, p-1\}$.

Let $V = \{v_0, v_1, v_2\}$ be the set of vertices corresponding to the three pencils. If we want to deal with pencils of parallel lines, we have just to consider ideal vertices representing directions and the ideal line containing all ideal points.

Let us first analyze the case where v_0, v_1, v_2 are noncollinear. Then we can choose a basis of the space of polynomials of degree not greater than one $\{p_0, p_1, p_2\}$, such that $\lambda p_1 - \mu p_2 = 0$, $\lambda p_2 - \mu p_0 = 0$, $\lambda p_0 - \mu p_1 = 0$, $(\lambda, \mu) \neq (0,0)$, are the equations of the three pencils, that is, $p_0 = 0, p_1 = 0, p_2 = 0$, are the equations of the lines lying on exactly two pencils.

Let Λ_r be the set of lines of the pencil with vertex v_r, excluding the lines containing two vertices. Let $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ be the set of lines containing exactly one vertex. Then each line of Λ_0, Λ_1, Λ_2 can be respectively defined by

$$\begin{aligned} l_0(t,s) &= p_1 - (-1)^s \exp(t) p_2, \quad t \in \mathbf{R}, s \in \mathbf{Z}_2, \\ l_1(t,s) &= p_2 - (-1)^s \exp(t) p_0, \quad t \in \mathbf{R}, s \in \mathbf{Z}_2, \\ l_2(t,s) &= p_0 - (-1)^s \exp(t) p_1, \quad t \in \mathbf{R}, s \in \mathbf{Z}_2. \end{aligned}$$

We can write the above formulae indexing each of the families by an integer modulo 3

$$l_r(t,s) = p_{r+1} - (-1)^s \exp(t) p_{r+2}, \quad t \in \mathbf{R}, s \in \mathbf{Z}_2, r \in \mathbf{Z}_3. \tag{4.1}$$

Let us denote by $L_r(t,s)$ the line with equation $l_r(t,s) = 0$. Then the lines $L_0(t_0,s_0)$, $L_1(t_1,s_1)$, $L_2(t_2,s_2)$ are concurrent if and only if

$$\begin{vmatrix} 0 & 1 & -(-1)^{s_0}\exp(t_0) \\ -(-1)^{s_1}\exp(t_1) & 0 & 1 \\ 1 & -(-1)^{s_2}\exp(t_2) & 0 \end{vmatrix} = 0,$$

that is,

$$L_0(t_0,s_0), L_1(t_1,s_1), L_2(t_2,s_2) \text{ are concurrent} \Leftrightarrow t_0+t_1+t_2 = 0, s_0+s_1+s_2 = 0. \tag{4.2}$$

This condition can be interpreted in terms of Ceva's Theorem (see section 13.7 of [8]). Condition (4.2) suggests the following addition of lines

$$L_{r_1}(t_1,s_1) \oplus L_{r_2}(t_2,s_2) := L_{r_1+r_2}(t_1+t_2, s_1+s_2), \quad t_1,t_2 \in \mathbf{R}, s_1,s_2 \in \mathbf{Z}_2, r_1,r_2 \in \mathbf{Z}_3. \tag{4.3}$$

So, Λ is a group isomorphic to $\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_3$. The 0 element of this group is $L_0(0,0)$, a line of the first pencil.

Before proving that (4.3) is an addition of lines, let us describe some consequences of the definition. The sum of a line of Λ_0 and a line of Λ_1 (resp., Λ_2) is a line of Λ_1 (resp., Λ_2). The sum of a line of Λ_1 and Λ_2 is a line of Λ_0. The sum of two lines of Λ_0 (resp., Λ_1, Λ_2) is a line of Λ_0 (resp., Λ_2, Λ_1). The opposite of a line of Λ_0 (resp., Λ_1, Λ_2) is a line of Λ_0 (resp., Λ_2, Λ_1).

Observe that if we take lines $L, M \in \Lambda$ belonging to different pencils then $\ominus L \ominus M$ is the line of the remaining pencil passing through their intersection. However the addition of lines of the same pencil has not a clear interpretation in terms of concurrences. If L, M belong to the same pencil Λ_r, then $\ominus L \ominus M$ is again a line of the same pencil Λ_r and the three lines $L, M, \ominus L \ominus M$ meet at the vertex v_r.

Proposition 4.1. *Let $V := \{v_0, v_1, v_2\}$ be a set of noncollinear vertices and let Λ be the set of lines containing exactly one vertex*

$$\Lambda := \{L_r(t,s) \mid t \in \mathbf{R}, s \in \mathbf{Z}_2, r \in \mathbf{Z}_3\},$$

where $L_r(t,s)$ is the line with equation $l_r(t,s) = 0$ defined in (4.1). Then $\oplus$ defined in (4.3) is an addition of lines on Λ with set of vertices V.

Proof: We have to verify (3.1) for the set $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ and the vertices $V = \{v_0, v_1, v_2\}$. If two lines of Λ belong to the same pencil Λ_r, they meet at the vertex $v_r \in V$. If three lines $L_{r_0}(t_0,s_0), L_{r_1}(t_1,s_1), L_{r_2}(t_2,s_2)$ in Λ are concurrent at a point which is not in V, then each line belongs to a different pencil, that is, r_0, r_1, r_2 are distinct. By (4.2), the concurrence condition is equivalent to $t_0+t_1+t_2=0$ and $s_0+s_1+s_2=0$ and from the definition (4.3) this is equivalent to the fact that the three lines add up to 0.

Conversely, assume that three lines add up to 0, $L_{r_0}(t_0,s_0) \oplus L_{r_1}(t_1,s_1) \oplus L_{r_2}(t_2,s_2) = 0$. Then from the definition of addition we have that $t_0+t_1+t_2=0$, $r_0+r_1+r_2=0$ and $s_0+s_1+s_2=0$. From $r_0+r_1+r_2=0$, we deduce that either $r_0=r_1=r_2$ or r_0, r_1, r_2 are distinct. If $r_0=r_1=r_2$, then the three lines belong to the same pencil. So, they are concurrent at a vertex. Otherwise, the indices r_0, r_1 and r_2 are distinct and (4.2) implies that the lines are concurrent. Since the sides of the triangle with vertices V are excluded, the lines of different pencils cannot meet at a vertex. So, if r_0, r_1, r_2 are distinct the three lines adding up to 0 must be concurrent at a point of $\mathbf{R}^2 \setminus V$. Therefore we have an addition of lines with set of vertices V. ■

Let us observe that the examples of formula (2.4) given in [15] correspond to the choice

$$
\begin{aligned}
L_i^1 &= L_2(i \log|\mu|, i\sigma(\mu)), \quad i \in \mathbf{N}_n, \\
L_j^2 &= L_0(j \log|\mu|, j\sigma(\mu)), \quad j \in \mathbf{N}_n, \\
L_k^3 &= L_1((k-n) \log|\mu|, (k-n)\sigma(\mu)), \quad k \in \mathbf{N}_n,
\end{aligned}
$$

where $\sigma : \mathbf{R} \to \mathbf{Z}_2$ is the characteristic function of the interval $(-\infty, 0)$,

$$
\sigma(\mu) := \begin{cases} 1, & \text{if } \mu < 0, \\ 0, & \text{if } \mu \geq 0. \end{cases}
$$

The lines (2.4) define a GPL_n and we have an addition of lines on Λ with set of vertices V. According to Theorem 3.1, formula (3.2) holds. Let us observe that K_1 is the line $L_0^1 = L_2(0,0)$ with equation $p_0 - p_1 = 0$ and K_2 is the line $L_0^2 = L_0(0,0)$ with equation $p_1 - p_2 = 0$. Finally H is the line $L_0(\log|\mu|, \sigma(\mu))$ with equation $p_1 - \mu p_2 = 0$.

Let us now analyze the case where the vertices v_0, v_1, v_2 are collinear. Then we can choose a basis of the space of polynomials of degree not greater than one $\{p_0, p_1, p_2\}$ such that $p_0 = 0$ is the equation of the line containing all the vertices, $p_1 = 0$ is a line containing v_1, $p_2 = 0$ is a line containing v_2 and $p_1 + p_2 = 0$ is a line containing v_0. For the case where we deal with pencils of parallel lines, we consider ideal vertices and ideal lines. In the general case, $\{p_0, p_1, p_2\}$ is a basis of the space of linear polynomials, such that $\lambda(p_1+p_2) + \mu p_0 = 0$, $\lambda p_1 - \mu p_0 = 0$, $\lambda p_2 - \mu p_0 = 0$, $(\lambda, \mu) \neq (0,0)$ are the equations of the three pencils.

Let Λ_r be the set of lines of the pencil with vertex v_r, excluding the line $p_0 = 0$ containing the three vertices. Let $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ be the set of lines containing exactly one vertex. Then each line of Λ_r is of the form $L_r(t)$, with corresponding equation $l_r(t) = 0$ given by

$$
\begin{aligned}
l_0(t) &= (p_1 + p_2) + t p_0, \quad t \in \mathbf{R}, \\
l_1(t) &= p_1 - t p_0, \quad t \in \mathbf{R}, \\
l_2(t) &= p_2 - t p_0, \quad t \in \mathbf{R}.
\end{aligned}
\tag{4.4}
$$

The lines $L_0(t_0), L_1(t_1), L_2(t_2)$ are concurrent if and only if

$$\begin{vmatrix} t_0 & 1 & 1 \\ -t_1 & 1 & 0 \\ -t_2 & 0 & 1 \end{vmatrix} = 0,$$

that is,

$$L_0(t_0), L_1(t_1), L_2(t_2) \text{ are concurrent} \Leftrightarrow t_0 + t_1 + t_2 = 0. \tag{4.5}$$

Then we have the following proposition. Its proof is completely analogous to that of Proposition 4.1.

Proposition 4.2. *Let $V := \{v_0, v_1, v_2\}$ be a set of collinear vertices and let Λ be the set of lines containing exactly one vertex*

$$\Lambda := \{L_r(t) \mid t \in \mathbf{R}, r \in \mathbf{Z}_3\},$$

where $L_r(t)$ is the line with equation $l_r(t) = 0$ defined in (4.4). Then

$$L_{r_1}(t_1) \oplus L_{r_2}(t_2) := L_{r_1+r_2}(t_1 + t_2), \quad t_1, t_2 \in \mathbf{R}, r_1, r_2 \in \mathbf{Z}_3, \tag{4.6}$$

defines an addition of lines on Λ with set of vertices V.

In this case Λ is a group isomorphic to $\mathbf{R} \times \mathbf{Z}_3$. Let us remark the curious fact that the addition of a line of Λ_1 and a line of Λ_2 corresponds to adding its equations, that is, the sum of the lines with equations $p_1 - t_1 p_0 = 0$ and $p_2 - t_2 p_0 = 0$ is the line with equation $(p_1 + p_2) - (t_1 + t_2)p_0 = 0$.

We observe that the examples of formula (2.5) correspond to the choice

$$L_i^1 = L_1(i),\ i \in \mathbf{N}_n, \quad L_j^2 = L_2(j),\ j \in \mathbf{N}_n, \quad L_k^3 = L_0(k - n),\ k \in \mathbf{N}_n.$$

The lines (2.5) define a GPL_n set and we have an addition of lines on Λ with set of vertices V. According to Theorem 3.1, formula (3.2) holds. In this case, K_1 is the line with equation $p_1 = 0$, K_2 the line with equation $p_2 = 0$ and H is the line $L_0(1)$ with equation $p_0 + p_1 + p_2 = 0$.

5. Addition on cubic pencils

Now, we consider algebraic pencils of lines, that is, sets of lines $ax + by + c = 0$ whose coefficients satisfy a homogeneous polynomial equation. Algebraic pencils are natural extensions of the usual linear pencils because the coefficients of all lines belonging to the same linear pencil satisfy a linear equation.

Definition 5.1. A *cubic pencil* is a set of lines $ax + by + c = 0$ such that a, b, c satisfy

$$F(a, b, c) = 0,$$

where F is a homogeneous cubic polynomial.

Associated to any cubic pencil $F(a, b, c) = 0$, we have the cubic curve Γ,

$$F(x, y, 1) = 0.$$

Each line $ax + by + c = 0$ of the pencil $F(a, b, c) = 0$ corresponds to the point $(a/c, b/c)$ of Γ. If $c = 0$, then we associate to (a, b, c) an ideal point of the projective cubic Γ.

Clearly, three lines of a cubic pencil $F(a, b, c)$ are concurrent if and only if the corresponding points of the curve Γ are collinear.

It is well-known that the set of nonsingular points Γ^* of an irreducible cubic Γ can be equipped with a binary operation $\oplus$ called *addition of points*. Then $(\Gamma^*, \oplus)$ is an abelian group and three points in Γ^* are collinear if and only if they add up to zero.

Let $\mathbf{S}^1$ be the multiplicative group of complex numbers of modulus 1,

$$\mathbf{S}^1 = \{z \in \mathbf{C} \mid |z| = 1\}.$$

The group of a complex irreducible and nonsingular cubic is the toroidal group $\mathbf{S}^1 \times \mathbf{S}^1$. The real points form a subgroup such that the connected component containing 0 is isomorphic to $\mathbf{S}^1$ (see Proposition 4.2 of [3]). Then the group Γ^* of the cubic is $\mathbf{S}^1$ or $\mathbf{S}^1 \times \mathbf{Z}_2$, depending on the fact that the cubic has 1 or 2 connected components.

If Γ is irreducible but singular, then Γ^* is isomorphic to one of the groups $\mathbf{R}$, $\mathbf{R} \times \mathbf{Z}_2$ or $\mathbf{S}^1$, depending on the fact that Γ is a cuspidal cubic, a nodal cubic or a cubic with an isolated point (see page 191 of [1] and Proposition 4.2 of [3]).

In the case of a reducible cubic, an addition $\oplus$ can be defined on the set Γ^* of nonsingular points of Γ such that $(\Gamma^*, \oplus)$ is an abelian group and three points in Γ^* not in a line contained in Γ are collinear if and only if they add up to zero. According to Proposition 4.1 of [3] we have the following groups. If Γ is composed of an irreducible conic C and a line L, then the group is $\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_2$, $\mathbf{R} \times \mathbf{Z}_2$, $\mathbf{S}^1 \times \mathbf{Z}_2$, depending on the fact that $C \cap L$ consists of 2, 1 or 0 points. If the cubic is composed of three lines, the group is $\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_3$ or $\mathbf{R} \times \mathbf{Z}_3$ depending on the fact that the lines are in general position or concurrent.

Let us observe that the addition on three pencils of Section 4, corresponds to the addition of points defined in Proposition 4.1 of [3] for cubics composed of 3 lines. The set of vertices V corresponds to the lines which are components of the cubic. The three pencils are indeed cubic pencils with equation

$$(ax_0 + by_0 + c)(ax_1 + by_1 + c)(ax_2 + by_2 + c) = 0,$$

where (x_0, y_0), (x_1, y_1), (x_2, y_2) are the vertices of the three pencils. The set Λ is the set of *nonsingular lines* of the cubic pencil, that is excluding the lines belonging to two linear pencils. The description of the group has been detailed in the precedent section (see also Proposition 4.1 of [3])

We can also think of cubic pencils composed by a linear pencil and a quadratic pencil (that is, the set of tangents to a conic). In this case, the pencil has equation

$$(ax_0 + by_0 + c)\, G(a, b, c) = 0,$$

where (x_0, y_0) is the vertex and G is a homogeneous quadratic polynomial. The description of the group can be seen in Proposition 4.1 of [3] for the canonical curves

$$(y - x^2)x = 0, \quad (y - x^2)y = 0, \quad (y - x^2 - 1)y = 0.$$

Again, Λ is the set of nonsigular lines of the pencil, that is, excluding the lines satisfying $ax_0 + by_0 + c = 0$ and $G(a, b, c) = 0$. In other words, the tangents to the conic from the vertex are excluded. Three different cases arise depending on the relative situation of the vertex and the conic. These cases are studied in more detail in [4].

In the case of irreducible cubic pencils, the set of vertices is empty. In the singular case, the pencils have simple parameterizations. In the next section, we shall analyze one of these pencils. The nonsingular case is more complicated and the addition can be described in terms of a parameterization involving elliptic functions.

Table 1 shows the group of the nonsingular points of each cubic curve up to projective transformations. For each type of cubic, we have chosen a canonical equation and depicted the graphic. In the case of irreducible nonsingular curves the equations have a free parameter $w \in (1, \infty)$ and $k \in (-2, 2)$ for cubic curves with 2 or 1 connected components, respectively. In the table, NCC denotes the number of connected components which must agree with the number of connected components of the group.

In all cases, we can parameterize the set Λ of nonsingular lines of a cubic pencil by a bijection with its associated group G, $L : G \to \Lambda$ such that $L(g_1 + g_2) = L(g_1) \oplus L(g_2)$. Now we can apply Theorem 3.1 to obtain GPL_n sets. First we choose $g_1, g_2, h \in G$. Not every choice of g_1, g_2 and h will lead to a GPL_n set. We have to take into account that all the lines

$$L(g_1 + ih),\ i \in \mathbf{N}_n, \quad L(g_2 + jh),\ j \in \mathbf{N}_n, \quad L(-g_1 - g_2 + (k - n)h),\ k \in \mathbf{N}_n$$

must be different. For this purpose, we need $mh \neq 0$, $0 < m \leq n$. Furthermore, according to (3.4), we need that

$$mh \neq 0, \quad 0 < m \leq 2n.$$

This means that we have to take an element of the group of order greater than $2n$. This can be achieved in all the groups. In the case that the connected component containing the 0 of group G is $\mathbf{S}^1$ we must take care not choosing $h = \exp(i\theta)$, with $\theta = 2k\pi/m$, $m \leq 2n$. The other condition

$$g_1 - g_2 + nh, -2g_1 - g_2, -g_1 - 2g_2 \notin \{mh \mid m \in \mathbf{N}_{2n}\}$$

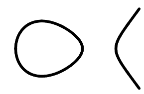
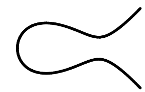
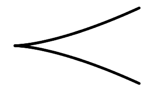
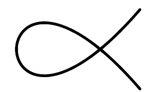
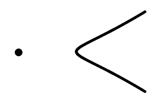
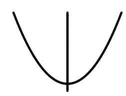
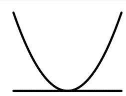
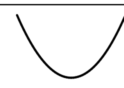
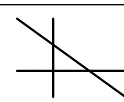
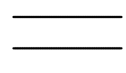
TYPE OF CUBIC	NCC	EQUATION	GRAPHIC	GROUP
irreducible nonsingular	2	$y^2 = x(x-1)(x-w)$ $w > 1$		$\mathbf{S}^1 \times \mathbf{Z}_2$
irreducible nonsingular	1	$y^2 = x(x^2+kx+1)$ $-2 < k < 2$		$\mathbf{S}^1$
irreducible singular	1	$y^2 = x^3$		$\mathbf{R}$
irreducible singular	2	$y^2 = x^2(x+1)$		$\mathbf{R} \times \mathbf{Z}_2$
irreducible singular	1	$y^2 = x^2(x-1)$		$\mathbf{S}^1$
reducible: $C \cup L$ $\lvert C \cap L \rvert = 2$	4	$(y-x^2)x = 0$		$\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_2$
reducible: $C \cup L$ $\lvert C \cap L \rvert = 1$	2	$(y-x^2)y = 0$		$\mathbf{R} \times \mathbf{Z}_2$
reducible: $C \cup L$ $C \cap L = \emptyset$	2	$(y-x^2-1)y = 0$		$\mathbf{S}^1 \times \mathbf{Z}_2$
red.: $L_1 \cup L_2 \cup L_3$ $L_1 \cap L_2 \cap L_3 = \emptyset$	6	$xy(x+y-1) = 0$		$\mathbf{R} \times \mathbf{Z}_2 \times \mathbf{Z}_3$
red.: $L_1 \cup L_2 \cup L_3$ $\lvert L_1 \cap L_2 \cap L_3 \rvert = 1$	3	$y(y-1)(y-2) = 0$		$\mathbf{R} \times \mathbf{Z}_3$

Table 1. The group structure of the set of nonsingular points of cubic curves

is also easy to ensure, taking into account the topological properties of these groups. If a choice of g_1, g_2, h produces repeated elements of the group, a slight modification will produce distinct elements. Finally, it is important to choose g_1 and g_2 in appropriate connected components of the group in order to ensure that no point of X is a vertex .

6. An example

As an example, we shall consider a cubic pencil, whose equation corresponds to a cubic with an isolated singularity. The associated group will be $\mathbf{S}^1$ and in this case it is known that the cubic has a parameterization by means of trigonometric functions. With a suitable projective transformation, the equation of the cubic can be written in the form

$$x^2 + y^2 + 2xy^2 = 0,$$

with the singular point at the origin $(0, 0)$. The corresponding pencil is

$$(a^2 + b^2)c + 2ab^2 = 0. \tag{6.1}$$

Let us observe that the singular line is the ideal line $a = b = 0$. Excluding the ideal line, the lines satisfying (6.1) are either $x = 0$ or lines with equation $ax + y + c = 0$ satisfying

$$c = \frac{-2a}{1 + a^2}.$$

With the change of variables $a = -\tan(t/2)$, we have $c = \sin(t)$ and the set Λ of nonsingular lines of the cubic pencil can be described by

$$l(t) = y - \tan(t/2)x + \sin(t), \quad t \in \mathbf{R}.$$

This cubic pencil of lines is depicted in Figure 2.

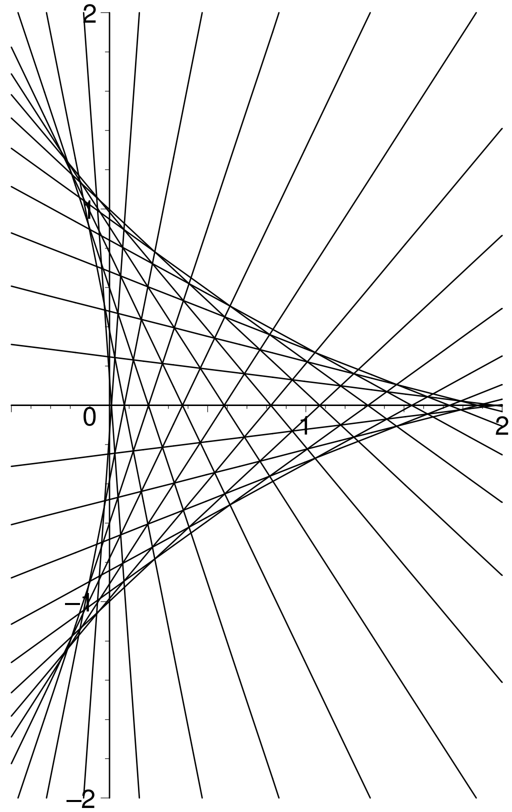


Figure 2. A cubic pencil of lines

Let us denote by $L(t)$ the line with equation $l(t) = 0$. We take as $L(\pi)$ the line with equation $x = 0$. Since $L(t + 2\pi) = L(t)$, we can parameterize all lines by the elements of the group $\mathbf{R}/2\pi\mathbf{Z}$, which is isomorphic to the group $\mathbf{S}^1$ by the mapping $t + 2\pi\mathbf{Z} \mapsto \exp(it)$.

The set of lines Λ can be described as the set of tangents of the parametric curve

$$x(t) = 2\frac{1 - \tan(t/2)^2}{(1 + \tan(t/2)^2)^2}, \quad y(t) = 4\frac{\tan(t/2)^3}{(1 + \tan(t/2)^2)^2}, \tag{6.2}$$

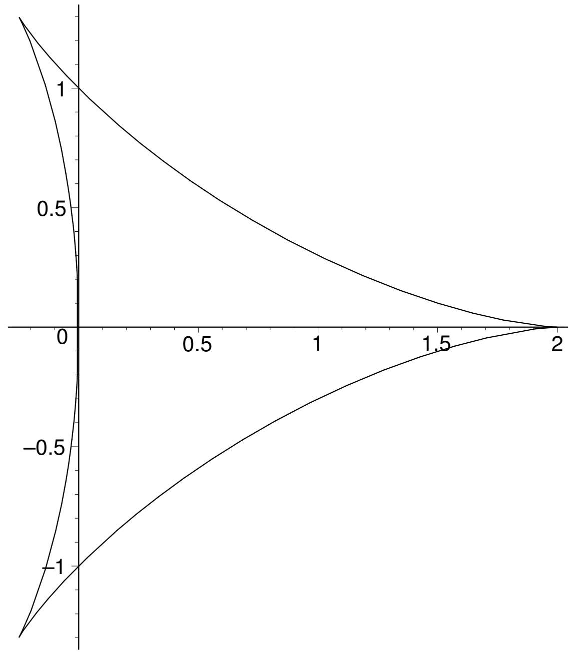


Figure 3. The deltoid envolvent of the cubic pencil

which is the translated deltoid $(x(t), y(t)) = (1/2, 0) + \Delta(t)$, where $\Delta(t) = (\cos(t) + \cos(2t)/2, \sin(t) - \sin(2t)/2)$, shown in Figure 3. We can eliminate the parameter t in (6.2) and deduce that the envolvent of Λ is an algebraic quartic curve with 3 cusps.

The lines $L(t_0)$, $L(t_1)$ and $L(t_2)$ are concurrent if and only if

$$\begin{vmatrix} -\tan(t_0/2) & 1 & \sin(t_0) \\ -\tan(t_1/2) & 1 & \sin(t_1) \\ -\tan(t_2/2) & 1 & \sin(t_2) \end{vmatrix} = 0,$$

that is,

$$\frac{\sin((t_1 - t_0)/2)\sin((t_2 - t_1)/2)\sin((t_0 - t_2)/2)\sin((t_0 + t_1 + t_2)/2)}{\cos(t_0/2)\cos(t_1/2)\cos(t_2/2)} = 0.$$

So, three distinct lines are concurrent if and only if $t_0 + t_1 + t_2 \in 2\pi\mathbf{Z}$.

Therefore, we have deduced that the addition of lines corresponds to the addition of their parameters, that is,

$$L(t_1) \oplus L(t_2) = L(t_1 + t_2),$$

and we can establish an isomorphism between the group $\mathbf{S}^1$ and the group Λ by $\exp(it) \mapsto L(t)$.

Let us apply Theorem 3.1 in order to construct GPL_n sets. Let us take t_1, t_2 and $\tau \in \mathbf{R}$ and consider the $3n + 3$ lines

$$L^1_i = L(t_1 + i\tau),\ i \in \mathbf{N}_n, \quad L^2_j = L(t_2 + j\tau),\ j \in \mathbf{N}_n, \quad L^3_k = L(-t_1 - t_2 + (k - n)\tau),\ k \in \mathbf{N}_n.$$

We need that (3.3) and (3.4) hold. To this aim, we have to choose t_1, t_2 and τ satisfying

$$m\tau \notin 2\pi\mathbf{Z}, \quad 0 < m \leq 2n,$$

and

$$t_1 - t_2 + n\tau,\ -2t_1 - t_2,\ -t_1 - 2t_2 \notin m\tau + 2\pi\mathbf{Z}, \quad 0 \le m \le 2n.$$

Obviously these conditions are easy to obtain by modifying slightly, if necessary, the values of t_1, t_2 of τ.

Figure 4 represents the GPL_4 set obtained taking $t_1 = 0.08$, $t_2 = 2.00$ and $\tau = 0.15$.

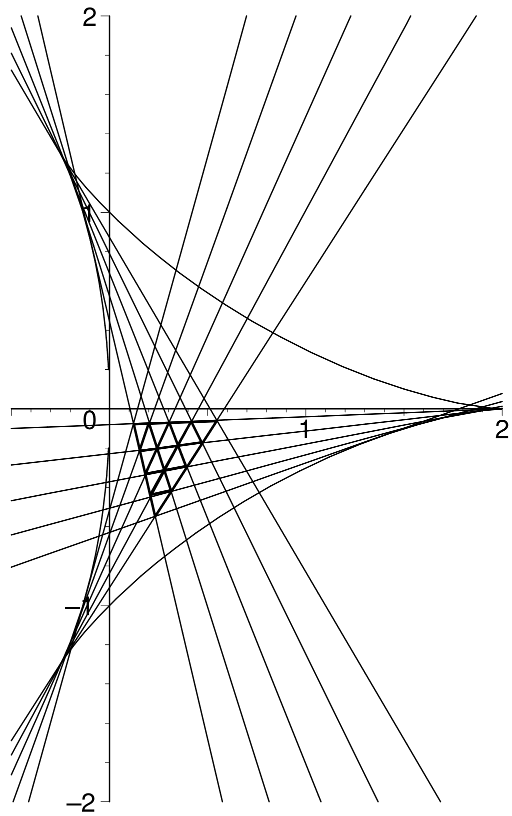


Figure 4. A GPL_4 set obtained with $t_1 = 0.08$, $t_2 = 2.00$ and $\tau = 0.15$

References

[1] R. Bix, *Conics and Cubics: a concrete introduction to algebraic curves.* (Springer, New York, 1998).

[2] J. R. Busch, A note on Lagrange interpolation in $\mathbf{R}^2$, Revista de la Unión Matemática Argentina 36 (1990) 33–38.

[3] J. M. Carnicer and M. García-Esnaola, Lagrange interpolation on conics and cubics, Computer Aided Geometric Design 19 (2002) 313–326.

[4] J. M. Carnicer and M. Gasca, A conjecture on multivariate polynomial interpolation, Rev. R. Acad. Cien. Serie A. Mat. 95 (2001) 145–153.

[5] J. M. Carnicer and M. Gasca, On Chung and Yao's geometric characterization for bivariate polynomial interpolation, to appear in *Curve and Surface Fitting: St. Malo 2002*, eds. C. Rabut and L. L. Schumaker (Nashboro Press, Nashville, 2003).

[6] J. M. Carnicer and M. Gasca, Generation of lattices of points for bivariate interpolation, Preprint.

[7] K. C. Chung and T. H. Yao, On lattices admitting unique Lagrange interpolations, SIAM J. Numer. Anal. 14 (1977) 735–743.
[8] H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed. (John Wiley and Sons, New York, 1969).
[9] M. Gasca and E. Lebrón, On Aitken-Neville formulae for multivariate interpolation, *Numerical approximation of partial differential equations*, ed. E. L. Ortiz (North Holland, Amsterdam, 1987) 133-140.
[10] M. Gasca and J. I. Maeztu, On Lagrange and Hermite interpolation in $\mathbf{R}^n$, Numer. Math. 39 (1982) 1–14.
[11] M. Gasca and T. Sauer, Polynomial interpolation in several variables, Adv. Comp. Math. 12 (2000) 377–410.
[12] M. Gasca and T. Sauer, On the history of multivariate polynomial interpolation, JCAM, 122 (2000) 23–35.
[13] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods* (John Wiley and Sons, New York, 1966).
[14] K. S. Kunz, *Numerical Analysis* (Mc-Graw-Hill, New York, 1957).
[15] S. L. Lee and G. M. Phillips, Construction of Lattices for Lagrange Interpolation in Projective Space, Constr. Approx. 7 (1991), 283–297.
[16] C. A. Micchelli, T. Sauer and Y. Xu, A construction of refinable sets for interpolating wavelets. Results Math. 34 (1998) 359–372.
[17] R. A. Nicolaides, On a class of finite elements generated by Lagrange interpolation, SIAM J. Numerical Analysis 9 (1972) 435–445.
[18] T. Sauer and Y. Xu, The Aitken-Neville scheme in several variables. *Approximation theory, X.* Innov. Appl. Math. (Vanderbilt Univ. Press, Nashville, TN, 2002) 353–366.
[19] H. C. Thacher Jr. and W. E. Milne, Interpolation in several variables, J. SIAM 8 (1960) 33-42.
[20] R. J. Walker, *Algebraic curves*, (Springer, New York, 1978).