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INTERPOLATION OPERATORS ON THE SPACE OF HOLOMORPHIC FUNCTIONS ON THE UNIT CIRCLE*

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Abstract. The aim of the paper is to get an estimation of the error of the general interpolation rule for functions which are real valued on the interval $[-a, a]$, $a \in (0, 1)$, have a holomorphic extension on the unit circle and are quadratic integrable on the boundary of it. The obtained estimate does not depend on the derivatives of the function to be interpolated. The optimal interpolation formula with mutually different nodes is constructed and an error estimate as well as the rate of convergence are obtained. The general extremal problem with free weights and knots is solved.

Keywords: numerical interpolation, optimal interpolatory rule with prescribed nodes, optimal interpolatory rule with free nodes, remainder estimation

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1. The space $H_2(K_1)$

Notation. By the symbol $H(K_1)$ we denote the space of all functions which are holomorphic in the open unit circle K_1. By M_2 we denote

$$M_2(f; r) = \left\{ \int_0^{2\pi} |f(re^{i\varphi})|^2 \, d\varphi \right\}^{\frac{1}{2}}, \quad r \in [0, 1). \tag{1}$$

It is known (see Rudin [10]) that the function $M_2(f; r)$ is nondecreasing as a function of the variable $r \in [0, 1)$, thus we may define

$$\|f\|_2 = \lim_{r \to 1-} M_2(f; r). \tag{2}$$

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Definition 1. By $H_2(K_1)$ we define the space of all functions from the space $H(K_1)$ for which the inequality $\|f\|_2 < +\infty$ holds.

Remark 1. (Properties of the functions from the space $H_2(K_1)$.) The main property of the space $H_2(K_1)$ is that it can be considered to be a Hilbert space which may be identified with a certain subspace of the space $L_2(\partial K_1)$. The norm of a function $g \in L_2(\partial K_1)$ is defined as

$$\|g\|_2 = \left(\int_0^{2\pi} |g(\mathrm{e}^{\mathrm{i}\varphi})|^2 \,\mathrm{d}\varphi\right)^{\frac{1}{2}}.$$

Fourier's coefficients of the function $g \in L_2(\partial K_1)$ are defined by the formulas

$$\hat{g}(n) = \int_0^{2\pi} g(\mathrm{e}^{\mathrm{i}\varphi})\mathrm{e}^{-\mathrm{i}n\varphi} \,\mathrm{d}\varphi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

A function $f \in H(K_1)$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an element of $H_2(K_1)$ iff $\sum\limits_{n=0}^{\infty} |a_n|^2 < +\infty$; in this case

$$\|f\|_2 = \left\{2\pi \sum_{n=0}^{\infty} |a_n|^2\right\}^{\frac{1}{2}}.$$

If $f \in H_2(K_1)$ then f has radial limits $(r \to -1)$ $f^*(\mathrm{e}^{\mathrm{i}\varphi})$ almost everywhere in ∂K_1, $f^* \in L_2(\partial K_1)$, the n-th Fourier's coefficient of the function f^* is $\sqrt{2\pi}\, a_n$ for $n \geqslant 0$ and it is equal to zero for $n < 0$. We have

$$\lim_{r \to 1-} \int_0^{2\pi} |f^*(\mathrm{e}^{\mathrm{i}\varphi}) - f(r\mathrm{e}^{\mathrm{i}\varphi})|^2 \,\mathrm{d}\varphi = 0.$$

For $z = r\mathrm{e}^{\mathrm{i}\varphi}$ the equality

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f^*(\zeta)}{\zeta - z} \,\mathrm{d}\zeta$$

holds. (Γ is the positively oriented unit circle.) The mapping $f \to f^*$ is an isometric one of the space $H_2(K_1)$ onto the subspace of the space $L_2(\partial K_1)$ formed by all elements $g \in L_2(\partial K_1)$ for which $\hat{g}(n) = 0$ for $n < 0$ holds.

Definition 2. We say that a function $K(z,u)$ is the reproduction kernel of the space $H_2(K_1)$ if for every function $f \in H_2(K_1)$ the identity

$$f(z) = (f^*(\,.\,), K(z,\,.\,)), \quad z \in K_1$$

holds.

Lemma. *The function*

$$K(z,u) = \frac{1}{2\pi(1-\overline{z}u)} \tag{3}$$

is the reproduction kernel of the space $H_2(K_1)$.

Proof. For every $f \in H_2(K_1)$ and $z \in K_1$ we may write

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f^*(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(\mathrm{e}^{\mathrm{i}\varphi})}{1 - z\mathrm{e}^{-\mathrm{i}\varphi}}\,\mathrm{d}\varphi \\ &= \int_0^{2\pi} f^*(\mathrm{e}^{\mathrm{i}\varphi})\overline{K(z,\mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{d}\varphi = (f^*(\,.\,), K(z,\,.\,)). \end{aligned}$$

□

2. Linear interpolants on $H_2(K_1)$

We define a linear interpolating operator on $H_2(K_1)$ in this section and give the form of the norm of the error functional for the general interpolation formula on the interval $[-a,+a]$. We will also study the problem of the optimal coefficients of the interpolating operator under the condition that the nodes of the given interpolation are fixed. We will establish a point estimate of the error of interpolation and obtain also the norm of the truncation error for non-optimal interpolation. In what follows, we denote by a a positive constant from the interval $(0,1)$.

Definition 3. We call L_n an interpolating operator on $H_2(K_1)$, if L_n is an additive, homogeneous operator of the form

$$L_n(f;x) = \sum_{k=1}^{n} A_k^{(n)}(x) f(x_k^{(n)})$$

where

$$\sum_{k=1}^{n} |A_k^{(n)}(x)| < +\infty \;\; \forall x \in [-a,a]; \tag{4}$$

$A_k^{(n)}(x)$, $k = 1, 2, \dots, n$ are continuous functions of $x \in [-a, +a]$, $a \in (0, 1)$ and $x_k^{(n)} \in [-a, +a]$, $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j = 1, 2, \dots, n$. Let a function f be real-valued on $[-a, +a]$. We write $I(f; x) = f(x)$ for every $x \in [-a, +a]$ and define the truncation-error operator by

$$R_n(f; x) = I(f; x) - L_n(f; x). \tag{5}$$

Theorem 1. *Let a point $x \in [-a, +a]$ be fixed, $a \in (0, 1)$. Then $R_n(f; x)$ is a linear, continuous functional on the Hilbert space $H_2(K_1)$ which can be written in the form*

$$R_n(f; x) = \Big(f, g_x - \sum_{k=1}^{n} \overline{A_k^{(n)}(x)}\, g_{x_k^{(n)}}\Big), \tag{6}$$

where

$$g_x = g_x(\overline{\zeta}) = \overline{K(x, \overline{\zeta})}, \quad \zeta \in K_1. \tag{7}$$

The norm on R_n can be written as

$$\|R_n\| = \Big\| g_x - \sum_{k=1}^{n} A_k^{(n)}(x) g_{x_k^{(n)}} \Big\|. \tag{8}$$

P r o o f. In view of (4) we have according to Lemma that

$$\begin{aligned} L_n(f; x) &= \sum_{k=1}^{n} A_k^{(n)}(x) f(x_k^{(n)}) = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\varphi}) \sum_{k=1}^{n} A_k^{(n)}(x) K(x_k^{(n)}, \mathrm{e}^{-\mathrm{i}\varphi})\, \mathrm{d}\varphi \\ &= \Big(f, \sum_{k=1}^{n} \overline{A_k^{(n)}(x)}\, \overline{K(x_k^{(n)}, \mathrm{e}^{-\mathrm{i}\varphi})}\Big) = \Big(f, \sum_{k=1}^{n} \overline{A_k^{(n)}(x)} g_{x_k^{(n)}}\Big). \end{aligned}$$

Further,

$$\begin{aligned} I(f; x) = f(x) &= \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\varphi}) \frac{1}{1 - x\mathrm{e}^{-\mathrm{i}\varphi}}\, \mathrm{d}\varphi \\ &= (f, \overline{K(x, \mathrm{e}^{-\mathrm{i}\varphi})}) = (f, g_x). \end{aligned}$$

Finally,

$$R_n(f; x) = \Big(f, g_x - \sum_{k=1}^{n} \overline{A_k^{(n)}(x)} g_{x_k^{(n)}}\Big),$$

which proves (6).

The function $h(\overline{\xi}) = g_x(\overline{\xi}) - \sum_{k=1}^{n} \overline{A_k^{(n)}(x)}\, g_{x_k^{(n)}}(\overline{\xi})$ is an element of $H_2(K_1)$, because $x \in [-a, +a]$, $x_k^{(n)} \in [-a, +a]$, $k = 1, 2, \dots, n$, $a \in (0, 1)$. According to the Riesz theorem $\|R_n\| = \|h\|$ and the norm $\|R_n\|$ is a function of $x \in [-a, +a]$. $\square$

Definition 4. The interpolating operator ${}^{(\mathrm{opt})}L_n^x$, where $x \in [-a, +a]$ is fixed, $a \in (0,1)$,

$$ {}^{(\mathrm{opt})}L_n(f;x) = \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x) f(x_k^{(n)}) $$

is said to be optimal if

$$ \left\| g_x - \sum_{k=1}^{n} \overline{{}^{(\mathrm{opt})}A_k^{(n)}(x)} g_{x_k^{(n)}} \right\| = \inf_{A_k^{(n)}(x)\in C,\, k=1,2,\ldots,n} \left\| g_x - \sum_{k=1}^{n} \overline{A_k^{(n)}(x)} g_{x_k^{(n)}} \right\|, \tag{9} $$

where C is the set of all complex numbers. By the symbol ${}^{(\mathrm{opt})}R_n^x$ we denote the error of the operator ${}^{(\mathrm{opt})}L_n^x$.

Remark 2. It follows immediately from Theorem 1 that the error functional $R_n(f;x)$ can be estimated as follows:

$$ |R_n(f;x)| \leqslant \|R_n\|\, \|f\|. $$

For fixed $x \in [-a,+a]$ the norm $\|R_n\|$ can be considered a quadratic function of n variables $A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}$. The next theorem implies that there exist uniquely determined numbers ${}^{(\mathrm{opt})}A_i^{(n)}$, $i = 1,2,\ldots,n$ realizing the minimum of the norm $\|R_n\|$ and thus the symbols ${}^{(\mathrm{opt})}A_i^{(n)}$ can be viewed as functions of $x \in [-a,+a]$.

Theorem 2. *Let $x \in [-a,+a]$, $a \in (0,1)$, be fixed. The optimal coefficients ${}^{(\mathrm{opt})}A_k^{(n)}(x)$, $k = 1,2,\ldots,n$, satisfy a Gram system of linear algebraic equations*

$$ \sum_{l=1}^{n} (g_{x_l^{(n)}}, g_{x_k^{(n)}}) \overline{{}^{(\mathrm{opt})}A_k^{(n)}(x)} = (g_{x_l^{(n)}}, g_{x_k^{(n)}}), \quad k = 1,2,\ldots,n. \tag{10} $$

This yields

$$ {}^{(\mathrm{opt})}A_k^{(n)}(x) = \prod_{i=1}^{n} \frac{1 - x_i^{(n)} x_k^{(n)}}{1 - x x_i^{(n)}} \prod_{i=1,\, i\neq k}^{n} \frac{x - x_i^{(n)}}{x_k^{(n)} - x_i^{(n)}} \tag{11} $$

or

$$ {}^{(\mathrm{opt})}A_k^{(n)}(x) = \frac{(x_k^{(n)})^n \omega\big(\frac{1}{x_k^{(n)}}\big)}{x^n \omega_n\big(\frac{1}{x}\big)} \frac{\omega_n(x)}{(x - x_k^{(n)})\omega_n'(x_k^{(n)})}, \tag{11'} $$

where

$$ \omega_n(x) = \prod_{i=1}^{n} (x - x_i^{(n)}). $$

Moreover

$$ {}^{(\mathrm{opt})}A_k^{(n)}(x_j^{(n)}) = \delta_{k,j}, \quad k, j = 1, 2, \ldots, n. \tag{12} $$

P r o o f. The problem (9) can be solved by using the Gram matrix (see [2]) of elements $g_{x_k^{(n)}}$, $k = 1, 2, \ldots, n$. The functions

$$ g_{x_k^{(n)}}(y) = \frac{1}{2\pi(1 - x_k^{(n)} y)}, \quad k = 1, 2, \ldots, n, \tag{13} $$

$x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j = 1, 2, \ldots, n$, are linearly independent on the interval $[-a, +a]$, $a \in (0, 1)$. They form a Chebyshev system in the interval $[-a, +a]$. This fact can be seen from the identity

$$ \frac{1}{2\pi} \sum_{k=1}^{n} \alpha_k \frac{1}{1 - x_k^{(n)} y} = 0, \quad \sum_{k=1}^{n} |\alpha_k| > 0, $$

from which it follows that

$$ \sum_{k=1}^{n} \alpha_k \prod_{j=1,\, j \neq k}^{n} (1 - x_k^{(n)} y) = 0. $$

The left-hand side of this relation is a polynomial of degree at most $n-1$ with at most $n - 1$ knots on the interval $[-a, +a]$. These considerations imply that ${}^{(\mathrm{opt})}A_k^{(n)}(x)$, $k = 1, 2, \ldots, n$ (x is fixed, $x \in [-a, +a]$), satisfy the following normal system of linear algebraic equations:

$$ \sum_{k=1}^{n} (g_{x_l^{(n)}}, g_{x_k^{(n)}}) \overline{{}^{(\mathrm{opt})}A_l^{(n)}(x)} = (g_x, g_{x_k^{(n)}}), \quad k = 1, 2, \ldots, n. $$

Hence, (10) is proved.

By (7) and (13) ($x, x_k^{(n)}$, $k = 1, 2, \ldots, n$ are real),

$$ (g_x, g_{x_k^{(n)}}) = \big(K(x, .), K(x_k^{(n)}, .)\big) = K(x, x_k^{(n)}) = \frac{1}{2\pi(1 - x x_k^{(n)})}. $$

Analogously we have

$$ (g_{x_l^{(n)}}, g_{x_k^{(n)}}) = \big(K(x_l^{(n)}, .), K(x_k^{(n)}, .)\big) = K(x_l^{(n)}, x_k^{(n)}) = \frac{1}{2\pi(1 - x_l^{(n)} x_k^{(n)})} $$

for $l, k = 1, 2, \ldots, n$.

The system (10) can be written in the form

$$\sum_{l=1}^{n} {}^{(\mathrm{opt})}A_l^{(n)}(x)\frac{1}{1-x_l^{(n)}x_k^{(n)}} = \frac{1}{1-xx_k^{(n)}}, \quad k=1,2,\dots,n. \tag{14}$$

By D_n we denote the determinant of the system (14). Then according to [3], we may write

$$D_n = \frac{\prod_{i>k=1}^{n}(x_i^{(n)}-x_k^{(n)})^2}{\prod_{i=1}^{n}(1-(x_i^{(n)})^2)\prod_{i>k=1}^{n}(1-x_i^{(n)}x_k^{(n)})^2} > 0, \tag{15}$$

where $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i,j=1,2,\dots,n$, $x_i^{(n)} \in [-a,+a]$, $i=1,2,\dots,n$, $a \in (0,1)$.

By Cramer's rule we have

$${}^{(\mathrm{opt})}A_k^{(n)}(x) = \frac{D_n^{(k)}}{D_n}, \tag{16}$$

where $D_n^{(k)}$ results by replacing the k-th column of the determinant D_n by the right-hand side of the system of equation (14). For further considerations note that

$$(g_{x_i^{(n)}}, g_{y_j^{(n)}}) = \frac{1}{2\pi(1-x_i^{(n)}y_j^{(n)})}, \quad i,j=1,2,\dots,n. \tag{17}$$

Then according to [4] we may write

$$\begin{aligned}\det\{(g_{x_i^{(n)}}, g_{y_j^{(n)}})\}_{i,j=1}^{n} = {} & \frac{1}{(2\pi)^n x_1^{(n)}x_2^{(n)}\dots x_n^{(n)}} \\ & \times \prod_{i>j=1}^{n}\left(\frac{1}{x_i^{(n)}}-\frac{1}{x_j^{(n)}}\right)(y_j^{(n)}-y_i^{(n)}) \\ & \times \left[\prod_{i,j=1}^{n}\left(\frac{1}{x_i^{(n)}}-y_j^{(n)}\right)\right]^{-1}.\end{aligned} \tag{18}$$

Setting $x_l^{(n)}$ instead of $y_l^{(n)}$, $l=1,2,\dots,n$, we get

$$\begin{aligned}D_n &= (2\pi)^n \det\{(g_{x_i^{(n)}}, g_{x_j^{(n)}})\}_{i,j=1}^{n} \\ &= \frac{1}{x_1^{(n)}x_2^{(n)}\dots x_n^{(n)}}\prod_{i>j=1}^{n}\left(\frac{1}{x_i^{(n)}}-\frac{1}{x_j^{(n)}}\right)(x_j^{(n)}-x_i^{(n)}) \\ &\quad \times \left[\prod_{i,j=1}^{n}\left(\frac{1}{x_i^{(n)}}-x_j^{(n)}\right)\right]^{-1}.\end{aligned} \tag{19}$$

We substitute the k-th column of the determinant D_n by the right-hand side of the system. Using (18), where we put $y_j^{(n)} = x_j^{(n)}$ for $j = 1, 2, \dots, k-1, k+1, \dots, n$ and $y_k^{(n)} = x$ for $k = 1, 2, \dots, n$ we obtain

$$(20)\quad D_n^{(k)}(x) = \frac{1}{x_1^{(n)} x_2^{(n)} \dots x_n^{(n)}} \prod_{i>j=1}^{n} \left(\frac{1}{x_i^{(n)}} - \frac{1}{x_j^{(n)}} \right) \times \prod_{i>j=1,\, i,j\neq k}^{n} (x_j^{(n)} - x_i^{(n)}) \prod_{i=k+1}^{n} (x - x_i^{(n)}) \times \prod_{j=1}^{k-1} (x_j^{(n)} - x) \left[\prod_{i,j=1,\, j\neq k}^{n} \left(\frac{1}{x_i^{(n)}} - x_j^{(n)} \right) \right]^{-1} \left[\prod_{i=1}^{n} \left(\frac{1}{x_i^{(n)}} - x \right) \right]^{-1}.$$

From this, (20) and (19) we get the identity

$$\frac{D_n^{(k)}}{D_n} = \frac{\prod\limits_{i>j=1}^{n} (\frac{1}{x_i^{(n)}} - \frac{1}{x_j^{(n)}}) \prod\limits_{i>j=1,\, i,j\neq k}^{n} (x_j^{(n)} - x_i^{(n)}) \prod\limits_{i=k+1}^{n} (x - x_i^{(n)})}{\prod\limits_{i,j=1,\, j\neq k}^{n} (\frac{1}{x_i^{(n)}} - x_j^{(n)}) \prod\limits_{i=1}^{n} (\frac{1}{x_i^{(n)}} - x) \prod\limits_{i>j=1}^{n} (\frac{1}{x_i^{(n)}} - \frac{1}{x_j^{(n)}})} \times \frac{\prod\limits_{j=1}^{k-1} (x_j^{(n)} - x) \prod\limits_{i,j=1}^{n} (\frac{1}{x_i^{(n)}} - x_j^{(n)})}{\prod\limits_{i>j=1}^{n} (x_j^{(n)} - x_i^{(n)})}$$

$$= \frac{\prod\limits_{i=1}^{n} (\frac{1}{x_i^{(n)}} - x_k^{(n)}) \prod\limits_{i=k+1}^{n} (x - x_i^{(n)})}{(x_k^{(n)} - x_{k+1}^{(n)})(x_k^{(n)} - x_{k+2}^{(n)}) \dots (x_k^{(n)} - x_n^{(n)})} \times \frac{\prod\limits_{j=1}^{k-1} (x_j^{(n)} - x)}{(x_1^{(n)} - x_k^{(n)})(x_2^{(n)} - x_k^{(n)}) \dots (x_{k-1}^{(n)} - x_k^{(n)}) \prod\limits_{i=1}^{n} \frac{1 - x x_i^{(n)}}{x_i^{(n)}}}$$

$$= \frac{\prod\limits_{i=1}^{n} (1 - x_i^{(n)} x_k^{(n)}) \prod\limits_{i=k+1}^{n} (x - x_i^{(n)})}{(x_k^{(n)} - x_{k+1}^{(n)})(x_k^{(n)} - x_{k+2}^{(n)}) \dots (x_k^{(n)} - x_n^{(n)})} \times \frac{\prod\limits_{j=1}^{k-1} (x_j^{(n)} - x)}{(x_1^{(n)} - x_k^{(n)})(x_2^{(n)} - x_k^{(n)}) \dots (x_{k-1}^{(n)} - x_k^{(n)}) \prod\limits_{i=1}^{n} (1 - x x_i^{(n)})}$$

$$= (-1)^{n-k} (-1)^{k-1} \prod_{i=1}^{n} \frac{1 - x_i^{(n)} x_k^{(n)}}{1 - x x_i^{(n)}} \prod_{i=1,\, i\neq k}^{n} \frac{1}{x_i^{(n)} - x_k^{(n)}} \prod_{l=1,\, l\neq k}^{n} (x - x_l^{(n)})$$

$$= \prod_{i=1}^{n} \frac{1 - x_i^{(n)} x_k^{(n)}}{1 - x x_i^{(n)}} \prod_{i=1,\, i\neq k}^{n} \frac{x - x_i^{(n)}}{x_k^{(n)} - x_i^{(n)}}.$$

This and (16) immediately imply (11) and (12).

Further,

$$\omega_n'(x_k^{(n)}) = \prod_{i=1,\, i\neq k}^{n} (x_k^{(n)} - x_i^{(n)}).$$

From (11) we see that

$${}^{(\mathrm{opt})}A_k^{(n)}(x) = \frac{(x_k^{(n)})^n \omega_n(\frac{1}{x_k^{(n)}})}{x^n \omega_n(\frac{1}{x})} \frac{\omega_n(x)}{(x - x_k^{(n)})\omega_n'(x_k^{(n)})},$$

which is (11′). Note that $\lim_{x\to 0} x^n \omega_n(\frac{1}{x}) = 1$. □

Theorem 3. *The following two expressions for the norm* $\|{}^{(\mathrm{opt})}R_n\|$ *are equivalent:*

(21) $$\|{}^{(\mathrm{opt})}R_n\|^2 = \frac{1}{2\pi(1 - x^2)} \left(\prod_{j=1}^{n} \frac{x_j^{(n)} - x}{1 - x x_j^{(n)}} \right)^2,$$

and, if we write $\omega_n(x) = \prod_{j=1}^{n} (x - x_j^{(n)})$,

(21′) $$\|{}^{(\mathrm{opt})}R_n\|^2 = \frac{1}{2\pi(1 - x^2)} \left(\frac{\omega_n(x)}{x^n \omega_n(\frac{1}{x})} \right)^2.$$

P r o o f. It is possible to write the expression $\|{}^{(\mathrm{opt})}R_n\|^2$ in the form (cf. [2])

(22) $$\|{}^{(\mathrm{opt})}R_n\|^2 = \frac{D_{n+1}(x)}{D_n},$$

where $D_{n+1}(x)$ is the determinant arising from the determinant D_n if we add to D_n the column

$$\big((g_x, g_{x_1^{(n)}}), (g_x, g_{x_2^{(n)}}), \ldots, (g_x, g_{x_n^{(n)}}), (g_x, g_x)\big)^T$$

and the row

$$\Big((g_{x_1^{(n)}}, g_x), (g_{x_2^{(n)}}, g_x), \ldots, (g_{x_n^{(n)}}, g_x), (g_x, g_x)\Big).$$

From the proof of Theorem 2 we have

$$(g_{x_k^{(n)}}, g_x) = \frac{1}{2\pi(1 - x x_k^{(n)})}, \quad k = 1, 2, \ldots, n$$

and

$$(g_x, g_x) = \frac{1}{2\pi(1-x^2)}. \tag{23}$$

In view of (19), where we put $x_{n+1}^{(n+1)} := x$ and $x_i^{(n+1)} := x_i^{(n)}$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} D_{n+1}(x) &= \frac{1}{(2\pi)^{n+1} x_1^{(n+1)} x_2^{(n+1)} \ldots x_n^{(n+1)} x_{n+1}^{(n+1)}} \\ &\quad \times \prod_{i>j=1}^{n+1} \left(\frac{1}{x_i^{(n+1)}} - \frac{1}{x_j^{(n+1)}} \right) (x_j^{(n+1)} - x_i^{(n+1)}) \\ &\quad \times \left[\prod_{i,j=1}^{n+1} \left(\frac{1}{x_i^{(n+1)}} - x_j^{(n+1)} \right) \right]^{-1} \\ &= \frac{1}{(2\pi)^{n+1} x_1^{(n)} x_2^{(n)} \ldots x_n^{(n)} x} \\ &\quad \times \prod_{i>j=1}^{n} \left(\frac{1}{x_i^{(n)}} - \frac{1}{x_j^{(n)}} \right) (x_j^{(n)} - x_i^{(n)}) \\ &\quad \times \left[\prod_{i,j=1}^{n} \left(\frac{1}{x_i^{(n)}} - x_j^{(n)} \right) \right]^{-1} \prod_{j=1}^{n} \left(\frac{1}{x} - \frac{1}{x_j^{(n)}} \right) (x_j^{(n)} - x) \\ &\quad \times \left[\prod_{j=1}^{n} \left(\frac{1}{x} - x_j^{(n)} \right) \prod_{i=1}^{n} \left(\frac{1}{x_i^{(n)}} - x \right) \left(\frac{1}{x} - x \right) \right]^{-1}. \end{aligned} \tag{24}$$

From this we get for $\|{}^{(\text{opt})}R_n^x\|^2$ according to (22), (24) and (19)

$$\begin{aligned} \|{}^{(\text{opt})}R_n^x\|^2 &= \frac{\prod_{i>j=1}^{n} (\frac{1}{x_i^{(n)}} - \frac{1}{x_j^{(n)}})(x_j^{(n)} - x_i^{(n)})}{2\pi x \prod_{i,j=1}^{n} (\frac{1}{x_i^{(n)}} - x_j^{(n)}) \prod_{j=1}^{n} (\frac{1}{x} - x_j^{(n)})} \\ &\quad \times \frac{\prod_{j=1}^{n} (\frac{1}{x} - \frac{1}{x_j^{(n)}})(x_j^{(n)} - x) \prod_{i,j=1}^{n} (\frac{1}{x_i^{(n)}} - x_j^{(n)})}{\prod_{i=1}^{n} (\frac{1}{x_i^{(n)}} - x)(\frac{1}{x} - x) \prod_{i>j=1}^{n} (\frac{1}{x_i^{(n)}} - \frac{1}{x_j^{(n)}})(x_j^{(n)} - x_i^{(n)})} \\ &= \frac{\prod_{j=1}^{n} (\frac{1}{x} - \frac{1}{x_j^{(n)}})(x_j^{(n)} - x)}{2\pi \prod_{j=1}^{n} (\frac{1}{x} - x_j^{(n)}) \prod_{i=1}^{n} (\frac{1}{x_i^{(n)}} - x)(\frac{1}{x} - x)} \end{aligned}$$

$$= \frac{1}{2\pi(1-x^2)} \frac{\prod_{j=1}^{n} \frac{(x_j^{(n)}-x)(x_j^{(n)}-x)}{xx_j^{(n)}}}{\prod_{j=1}^{n} \frac{1-xx_j^{(n)}}{x} \prod_{j=1}^{n} \frac{1-xx_j^{(n)}}{x}}$$

$$= \frac{\prod_{j=1}^{n} (x_j^{(n)}-x)^2}{2\pi(1-x^2) \prod_{j=1}^{n} (1-xx_j^{(n)})^2} = \frac{1}{2\pi(1-x^2)} \left(\prod_{j=1}^{n} \frac{x_j^{(n)}-x}{1-xx_j^{(n)}} \right)^2.$$

Setting $\omega_n(x) = \prod_{i=1}^{n} (x - x_i^{(n)})$ it is possible to write the last expression in the form of (21′). □

R e m a r k 3. Theorem 3 can be also proved in the following way: We calculate the norm of the representant of the functional ${}^{(\mathrm{opt})}R_n^x$ (x is fixed, $x \in [-a,+a]$, $a \in (0,1)$) (cf. (6), (9)):

$$\begin{aligned} \|{}^{(\mathrm{opt})}R_n\|^2 &= \left\| g_x - \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x) g_{x_k^{(n)}} \right\|^2 \\ &= \|g_x\|^2 - 2\sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x)(g_x, g_{x_k^{(n)}}) \\ &\quad + \sum_{k=1}^{n}\sum_{l=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x) {}^{(\mathrm{opt})}A_l^{(n)}(x)(g_{x_l^{(n)}}, g_{x_k^{(n)}}) \\ &= \|g_x\|^2 - \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x)(g_x, g_{x_k^{(n)}}) \\ &= \|g_x\|^2 - \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x) \frac{1}{2\pi} \frac{1}{1-xx_k^{(n)}}. \end{aligned}$$

Here we have used the identity

$$\sum_{k=1}^{n}\sum_{l=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x) {}^{(\mathrm{opt})}A_l^{(n)}(x)(g_{x_l^{(n)}}, g_{x_k^{(n)}}) = \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x)(g_x, g_{x_k^{(n)}}),$$

which follows from (10) by multiplying the k-th equation by ${}^{(\mathrm{opt})}A_k^{(n)}(x)$ and adding all equations. When we rewrite the last expression for $\|{}^{(\mathrm{opt})}R_n\|^2$ substituting for ${}^{(\mathrm{opt})}A_k^{(n)}(x)$ from (11) and (11′) we get (21) and (21′), respectively.

Our goal is to prove the following fundamental result.

Theorem 4. *Let x be fixed, $x \in [-a,+a]$, $a \in (0,1)$. Let R_n be an arbitrary error functional defined by (5), where $x_k^{(n)} \in [-a,+a]$, $k = 1,2,\ldots,n$, $x_i^{(n)} \neq x_j^{(n)}$,*

$i \neq j$, $i, j = 1, 2, \ldots, n$. Then for every function $f \in H_2(K_1)$, which is real-valued in interval $[-a, +a]$, the inequality

$$|R_n(f,x)|^2 \leqslant \|R_n\|^2 \|f\|^2 \tag{25}$$

holds, where

$$\|f\|^2 = \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\varphi})|^2 \, \mathrm{d}\varphi \tag{26}$$

and

$$\begin{aligned} 2\pi \|R_n\|^2 = {} & \frac{1}{1-x^2} - 2\sum_{k=1}^{n} A_k^{(n)}(x) \frac{1}{1 - x x_k^{(n)}} \\ & + \sum_{k=1}^{n} \sum_{l=1}^{n} A_k^{(n)}(x) A_l^{(n)}(x) \frac{1}{1 - x_k^{(n)} x_l^{(n)}}. \end{aligned} \tag{27}$$

P r o o f. The inequality (25) follows directly from (5) and from the definition of the space $H_2(K_1)$. We get the formula (27) in the following way:

$$R_n(f,x) = f(x) - \sum_{k=1}^{n} A_k^{(n)}(x) f(x_k^{(n)})$$

is for fixed x a functional on $H_2(K_1)$. According to Lemma we have

$$\begin{aligned} f(x) &= (f(\,.\,), K(x, .\,)), \\ f(x_k^{(n)}) &= (f(\,.\,), K(x_k^{(n)}, .\,)), \quad k = 1, 2, \ldots, n, \end{aligned}$$

which gives

$$R_n(f,x) = (f(\,.\,), K(x, .\,) - \sum_{k=1}^{n} A_k^{(n)}(x) K(x_k^{(n)}, .\,))$$

and

$$|R_n(f,x)|^2 \leqslant \|f\|^2 \|K(x, .\,) - \sum_{k=1}^{n} A_k^{(n)}(x) K(x_k^{(n)}, .\,)\|^2.$$

This yields

$$\begin{aligned}
\|R_n\|^2 &= \Big\| K(x,.) - \sum_{k=1}^{n} A_k^{(n)}(x) K(x_k^{(n)},.)\Big\|^2 \\
&= \Big(K(x,.) - \sum_{k=1}^{n} A_k^{(n)}(x) K(x_k^{(n)},.), K(x,.) - \sum_{l=1}^{n} A_l^{(n)}(x) K(x_l^{(n)},.)\Big) \\
&= (K(x,.), K(x,.)) - 2\sum_{k=1}^{n} A_k^{(n)}(x)\big(K(x,.), K(x_k^{(n)},.)\big) \\
&\quad + \sum_{k=1}^{n}\sum_{l=1}^{n} A_k^{(n)}(x) A_l^{(n)}(x)\big(K(x_k^{(n)},.), K(x_l^{(n)},.)\big) \\
&= \frac{1}{2\pi(1-x^2)} - \frac{1}{\pi}\sum_{k=1}^{n} A_k^{(n)}(x)\frac{1}{1-xx_k^{(n)}} \\
&\quad + \frac{1}{2\pi}\sum_{k=1}^{n}\sum_{l=1}^{n} A_k^{(n)}(x) A_l^{(n)}(x)\frac{1}{1-x_k^{(n)}x_l^{(n)}},
\end{aligned}$$

which is the formula (27). □

R e m a r k 4. Theorem 4 is the main result. In order to compute the error of the interpolation of the type (5) using (4) for a given function $f \in H_2(K_1)$ it is sufficient to know the value of the integral (26) or an estimate of this integral without any knowledge about derivatives of the function f. It is clear that the norm $\|R_n\|^2$ depends on the chosen interpolatory rule only.

R e m a r k 5. From the formula (27) we get immediately

$$2\pi\frac{\partial\|R_n\|^2}{\partial A_i^{(n)}(x)} = -\frac{2}{1-xx_i^{(n)}} + 2\sum_{k=1}^{n} A_k^{(n)}(x)\frac{1}{1-x_k^{(n)}x_i^{(n)}}, \quad k = 1,2,\dots,n, \tag{28}$$

which gives the system (14) for the optimal weights ${}^{(\mathrm{opt})}A_i^{(n)}$, $i = 1,2,\dots,n$.

The conditions

$$\frac{\partial\|R_n\|^2}{\partial A_i^{(n)}(x)} = 0, \quad i = 1,2,\dots,n$$

are necessary and sufficient for the minimum of $\|R_n\|^2$ because it is a nonnegative quadratic function of $A_i^{(n)}(x)$, $i = 1,2,\dots,n$. From the formula (27) and relations (14) we have

Theorem 5. *Let x be fixed, $x \in [-a,+a]$, $a \in (0,1)$, $x_k^{(n)} \in [-a,+a]$, $k = 1,2,\dots,n$, $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i,j = 1,2,\dots,n$. Let ${}^{(\mathrm{opt})}A_i^{(n)}(x)$, $i = 1,2,\dots,n$, be* *the weights of the optimal interpolation formula of type* (4) *on* $H_2(K_1)$. *Then we have*

$$(29)\qquad 2\pi\|^{(\mathrm{opt})}R_n^x\|^2 = \frac{1}{1-x^2} - \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_i^{(n)}(x)\,\frac{1}{1-xx_k^{(n)}}$$

or

$$(29')\qquad 2\pi\|^{(\mathrm{opt})}R_n^x\|^2 = \frac{1}{1-x^2} - \sum_{k=1}^{n}\sum_{l=1}^{n} {}^{(\mathrm{opt})}A_i^{(n)}(x)\,{}^{(\mathrm{opt})}A_l^{(n)}(x)\,\frac{1}{1-x_k^{(n)}x_l^{(n)}}.$$

P r o o f. Multiplying the k-th equation (14) by the function ${}^{(\mathrm{opt})}A_k^{(n)}(x)$, $k=1,2,\ldots,n$, and adding for $k=1,2,\ldots,n$, we get

$$\sum_{k=1}^{n}\sum_{l=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x)\,{}^{(\mathrm{opt})}A_l^{(n)}(x)\,\frac{1}{1-x_l^{(n)}x_k^{(n)}} = \sum_{k=1}^{n} {}^{(\mathrm{opt})}A_k^{(n)}(x)\,\frac{1}{1-xx_k^{(n)}}.$$

The formulae (25) and (25′) we obtain by inserting this identity for ${}^{(\mathrm{opt})}A_k^{(n)}(x)$, $k=1,2,\ldots,n$, into (27). □

R e m a r k 6. The formulae (29) and (29′) are obviously other expressions for (21) and (21′). They may be used if the optimal weights ${}^{(\mathrm{opt})}A_k^{(n)}(x)$, $k=1,2,\ldots,n$, are known. Then the value $\|^{(\mathrm{opt})}R_n^x\|^2$ for a given $x\in[-a,+a]$ is easier to compute using (29) and (29′) than using (21) and (21′).

R e m a r k 7. From relations (14) it follows that the optimal interpolatory rule of the type (4) interpolates the functions

$$\frac{1}{1-xx_i^{(n)}},\qquad i=1,2,\ldots,n,$$

exactly if $x_i^{(n)}\in[-a,+a]$, $i=1,2,\ldots,n$, $x_i^{(n)}\neq x_j^{(n)}$, $i\neq j$, $i,j=1,2,\ldots,n$.

3. The convergence of the norm of the optimal error functional

In what follows, we will study the rate of convergence of $\|{}^{(\mathrm{opt})}R_n^x\|$ for an arbitrary distribution of nodes. The estimate obtained is sharpened for the roots of Chebyshev polynomials as the nodes of interpolation.

Theorem 6. *Let x be arbitrary and fixed, $x \in [-a, +a]$, $a \in (0,1)$, $x_k^{(n)} \in [-a, +a]$, $k = 1, 2, \ldots, n$, $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j = 1, 2, \ldots, n$. Then we have*

$$\|{}^{(\mathrm{opt})}R_n\|^2 \leqslant \frac{1}{2\pi(1-a^2)} \mathrm{e}^{-2n \ln \frac{1+a^2}{2a}}, \quad \textit{for } n \to \infty \tag{30}$$

where $\|{}^{(\mathrm{opt})}R_n\|$ is a function of x.

P r o o f. We define $\psi(x_j^{(n)}, x)$ by

$$\psi(x_j^{(n)}, x) = \frac{x - x_j^{(n)}}{1 - x x_j^{(n)}}.$$

Then

$$\psi(x_j^{(n)}, x) \leqslant \frac{2a}{1+a^2}$$

for every $x \in [-a, +a]$ and $j = 1, 2, \ldots, n$. It is easy to see that

$$\frac{\partial}{\partial x_j^{(n)}} \psi(x_j^{(n)}, x) = \frac{x^2 - 1}{(1 - x x_j^{(n)})^2} < 0.$$

Hence $\psi(x_j^{(n)}, x)$ is decreasing as a function of the variable $x_j^{(n)}$ for every $j = 1, \ldots, n$ and $x \in (-a, +a)$. Further,

$$\frac{\partial}{\partial x} \psi(x_j^{(n)}, x) = \frac{1 - (x_j^{(n)})^2}{(1 - x x_j^{(n)})^2} > 0$$

so that the function $\psi(x_j^{(n)}, x)$ is increasing as a function of $x \in (-a, +a)$ for every $x_j^{(n)}$, $j = 1, \ldots, n$. This implies that the maximum of the function $\psi(x_j^{(n)}, x)$ is achieved at $[-a, +a]$. This maximum equals $\frac{2a}{1+a^2}$. The function $\psi(x_j^{(n)}, x)$ has its minimum at $[+a, -a]$ and it equals $\frac{-2a}{1+a^2}$. Thus we have

$$\left| \prod_{j=1}^{n} \frac{x - x_j^{(n)}}{1 - x x_j^{(n)}} \right| \leqslant \left(\frac{2a}{1+a^2} \right)^n \to 0$$

for $n \to \infty$ because $\frac{2a}{1+a^2} < 1$ for $a \neq 1$. Finally, from (21) we get

$$\|^{(\text{opt})}R_n\|^2 \leqslant \frac{1}{2\pi(1-a^2)}\Big(\frac{2a}{1+a^2}\Big)^{2n} = \frac{1}{2\pi(1-a^2)}\mathrm{e}^{-2n\ln\frac{1+a^2}{2a}},$$

where $\frac{1+a^2}{2a} > 1$. □

Theorem 7. *Let x be arbitrary, $x \in [-a,+a]$, $a \in (0,1)$. Let $x_i^{(n)}$, $i = 1,2,\ldots,n$, be the roots of the Chebyshev polynomials $\tilde{T}_n(x)$ at $[-a,+a]$. Then*

$$\|^{(\text{opt})}R_n\|^2 \leqslant \frac{1}{2\pi(1-a^2)}\mathrm{e}^{-2n\ln\frac{1+a^2\cos(\pi/2n)}{a(1+\cos(\pi/2n))}} \to 0 \quad \textit{for} \quad n \to \infty. \tag{31}$$

P r o o f. From the definition of Chebyshev polynomials $\tilde{T}_n(x)$ defined on the interval $[-a,+a]$ we get easily

$$\frac{\tilde{T}_n(x)}{x^n\tilde{T}_n(\frac{1}{x})} = \prod_{k=1}^{n}\frac{x - a\cos\frac{(2k-1)\pi}{2n}}{1 - xa\cos\frac{(2k-1)\pi}{2n}}, \tag{32}$$

where $x_k^{(n)} = a\cos\frac{(2k-1)\pi}{2n}$, $k = 1,2,\ldots,n$, and $\cos\frac{(2k-1)\pi}{2n}$, $k = 1,2,\ldots,n$, are the roots of the Chebyshev polynomials lying in $[-a,+a]$. Let us introduce the following notation:

$$t_k^{(n)} = \cos\frac{(2k-1)\pi}{2n} \quad \text{for} \quad k = 1,2,\ldots,n.$$

Then $t_n^{(n)} < t_{n-1}^{(n)} < \ldots < t_1^{(n)}$ and similarly as in the proof of Theorem 6 we have

$$\psi(t_k^{(n)},x) = \frac{x - at_k^{(n)}}{1 - xat_k^{(n)}}, \quad k = 1,2,\ldots,n.$$

It is obvious that

$$\frac{\partial}{\partial x}\psi(t_k^{(n)},x) > 0 \quad \text{for} \quad k = 1,2,\ldots,n.$$

Thus

$$\max_{x\in[-a,+a]}\frac{x - at_k^{(n)}}{1 - xat_k^{(n)}} = \frac{a(1 - t_k^{(n)})}{1 - a^2t_k^{(n)}} > 0,$$

$$\min_{x\in[-a,+a]}\frac{x - at_k^{(n)}}{1 - xat_k^{(n)}} = \frac{-a(1 + t_k^{(n)})}{1 + a^2t_k^{(n)}} < 0, \quad k = 1,2,\ldots,n.$$

Further,

$$a\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1-t}{1-a^2t}\Big) = a\frac{a^2-1}{(1-a^2t)^2} < 0.$$

Consequently,

$$\max_{\substack{t_k^{(n)}\in[t_n^{(n)},t_1^{(n)}]\\ k=1,2,\dots,n}} \frac{a(1-t_k^{(n)})}{1-a^2t_k^{(n)}} = \frac{a(1-t_n^{(n)})}{1-a^2t_n^{(n)}} = \frac{a(1+t_1^{(n)})}{1+a^2t_1^{(n)}},$$

$$(-a)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1+t}{1+a^2t}\right) = (-a)\frac{1-a^2}{1+a^2t} < 0,$$

which gives

$$\min_{\substack{t_k^{(n)}\in[t_n^{(n)},t_1^{(n)}]\\ k=1,2,\dots,n}} (-a)\frac{1+t_k^{(n)}}{1+a^2t_k^{(n)}} = (-a)\frac{1+t_1^{(n)}}{1+a^2t_1^{(n)}}.$$

Thus

$$\left|\prod_{k=1}^{n} \frac{x-a\cos\frac{(2k-1)\pi}{2n}}{1-xa\cos\frac{(2k-1)\pi}{2n}}\right| \leqslant \left(\frac{a(1+\cos\frac{\pi}{2n})}{1+a^2\cos\frac{\pi}{2n}}\right)^n.$$

It is very easy to verify that the following inequality holds:

$$\frac{1+a^2\cos\frac{\pi}{2n}}{a(1+\cos\frac{\pi}{2n})} > 0. \tag{33}$$

From (33) it follows that $1+a^2\cos\frac{\pi}{2n} > a+a\cos\frac{\pi}{2n} \Rightarrow a\cos\frac{\pi}{2n} < 1$, which obviously holds because of $a\in(0,1)$. With the aid of (21), (32) and (33) we easily establish the formula (31). □

4. Existence and uniqueness of the optimal nodes of the interpolation

Hitherto we have dealt with the problem of interpolation under the condition that the nodes of interpolation are mutually different and arbitrarily given in $[-a,+a]$. Then for the optimal coefficients of the interpolation rule the expressions (11) or (11′) are valid. The norm $\|^{(\mathrm{opt})}R_n\|$ for the optimal error functional is given by the formulas (21) or (21′), respectively.

Further, we get ahead starting from the formula (21) in order to minimize the norm $\|^{(\mathrm{opt})}R_n\|$ in the following sense: we shall solve the problem of finding $E_{n,n}$, where

$$E_{n,n} = \min_{\substack{x_k^{(n)}\in[-a,+a],\, k=1,2,\dots,n\\ x_i^{(n)}\neq x_j^{(n)},\, i\neq j,\, i,j=1,2,\dots,n\\ a\in(0,1),\, a \text{ fixed}}} \max_{x\in[-a,+a]} \left|\frac{1}{\sqrt{2\pi(1-x^2)}}\prod_{j=1}^{n}\frac{x_j^{(n)}-x}{1-xx_j^{(n)}}\right|. \tag{34}$$

Then we prove that the function of the form

$$\frac{(-1)^n \omega_n(x)}{\sqrt{2\pi(1-x^2)}\,x^n \omega_n(\frac{1}{x})} \tag{35}$$

possesses in the interval $[-a,+a]$, $a \in (0,1)$, the minimal deviation from zero under the condition that $x^n \omega_n(\frac{1}{x}) > 0$.

Theorem 8. *There exists only one minimal solution of the problem* (34), *namely*

$$\frac{(-1)^n \omega_n^*(x)}{\sqrt{2\pi(1-x^2)}\,x^n \omega_n^*(\frac{1}{x})}. \tag{36}$$

This solution has the following properties:

1) *All nodes of the polynomial* $\omega_n^*(x)$ *lie in the interval* $[-a,+a]$, $a \in (0,1)$, *and are mutually different.*
2) *The length of the alternant of the minimal solution of the problem* (36) *is* $n+1$.
3) *For all* n *we have*

$$E_{n,n} \geqslant \sqrt{\frac{2}{\pi}} \left(\frac{a}{2}\right)^n. \tag{37}$$

P r o o f. In the same way as in [2] and [5] we use the theory of the Chebyshev approximations.

We denote

$$S(x) = \frac{1}{\sqrt{2\pi(1-x^2)}}. \tag{38}$$

Let us suppose that n is fixed. By $F(\alpha, x)$ let us denote functions of the form

$$F(\alpha,x) = S(x)\,\frac{(-1)^n P_n(x)}{x^n P_n(\frac{1}{x})} = S(x)\,\frac{(-1)^n \sum\limits_{i=0}^{n} \alpha_i x^i}{\sum\limits_{i=0}^{n} \alpha_{n-i} x^i}. \tag{39}$$

The parameter α is a vector-parameter with real components $\alpha_0, \alpha_1, \ldots, \alpha_n$. In what follows we set $\alpha_n = 1$. In accordance with (34) we have

$$x^n P_n\left(\frac{1}{x}\right) = \sum_{i=0}^{n} \alpha_{n-i} x^i > 0 \tag{40}$$

for $[-a,+a]$, $a\in(0,1)$. By V we denote the set of functions $F(\alpha,x)$. It is easy to see that

$$\frac{\partial F(\alpha,x)}{\partial\alpha_j}=(-1)^nS(x)\,\frac{x^j\sum\limits_{i=0}^{n}\alpha_{n-i}x^i-x^{n-j}\sum\limits_{i=0}^{n}\alpha_ix^i}{\Big(\sum\limits_{i=0}^{n}\alpha_{n-i}x^i\Big)^2}=(-1)^nS(x)\,\frac{x^{n+j}P_n(\frac1x)-x^{n-j}P_n(x)}{(x^nP_n(\frac1x))^2},\quad j=0,1,\dots,n-1. \tag{41}$$

From this expression we get

$$\operatorname{grad}_\alpha F(\alpha,x)=\frac{(-1)^nS(x)}{(x^nP_n(\frac1x))^2} \tag{42}$$
$$=\Big(x^nP_n\Big(\frac1x\Big)-x^nP_n(x),x^{n+1}P_n\Big(\frac1x\Big)-x^{n-1}P_n(x),\dots,x^{2n-1}P_n\Big(\frac1x\Big)-xP_n(x)\Big)^T.$$

Obviously the degree of the polynomial $P_n(x)$ is n.

By A we denote the set of all vectors $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_{n-1})$, which is open. The set $B=[-a,+a]$, $a\in(0,1)$, is compact. The functions $F(\alpha,x)\in V$ are obviously continuous with respect to $\alpha\in A$ for every $x\in B$. By the symbol $W(\alpha)$ we denote the space of all linear combinations of the components of the vector $\operatorname{grad}_\alpha F(\alpha,x)$. Obviously $\dim W(\alpha)=n$ is independent of $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_{n-1})$. In accordance with [5] the set V fulfils the local Haar condition when it fulfils the classical Haar condition, i.e. every function $\varkappa\in W(\alpha)$, $\varkappa\neq0$, has in interval B at most $n-1$ roots.

Now let us show that the function

$$\frac{(-1)^nS(x)}{(x^nP_n(\frac1x))^2}\sum_{j=0}^{n-1}\beta_j\Big(x^{n+j}P_n\Big(\frac1x\Big)-x^{n-j}P_n(x)\Big) \tag{43}$$

has in the interval B at most $n-1$ roots if $\sum\limits_{j=0}^{n-1}|\beta_j|>0$. Obviously the degree of the polynomial

$$\sum_{j=0}^{n-1}\beta_j\Big(x^{n+j}P_n\Big(\frac1x\Big)-x^{n-j}P_n(x)\Big) \tag{44}$$

is at most $2n$ (and equals $2n$ if $\beta_0\neq0$).

Because $S(x) \neq 0$ in B we have $(x^n P_n(\frac{1}{x})) > 0$ in B. We can write

$$\begin{aligned}
&\sum_{j=0}^{n-1} \beta_j \left(x^{n+j} P_n\left(\frac{1}{x}\right) - x^{n-j} P_n(x) \right) \\
&\quad = \sum_{j=0}^{n-1} \beta_j \left(x^j \sum_{i=0}^{n} \alpha_{n-i} x^i - x^{n-j} \sum_{i=0}^{n} \alpha_i x^i \right) \\
&\quad = \sum_{j=0}^{n-1} \beta_j \left(\sum_{i=0}^{n} \alpha_{n-i} x^{i+j} - \sum_{i=0}^{n} \alpha_i x^{n+1-j} \right) \\
&\quad = \sum_{j=0}^{n-1} \sum_{i=0}^{n} \alpha_{n-i} \beta_j x^{i+j} - \sum_{k=1}^{n} \sum_{i=0}^{n} \alpha_i \beta_{n-k} x^{i+k} \\
&\quad = \beta_0 \sum_{i=0}^{n} \alpha_{n-i} x^i - \beta_0 \sum_{i=0}^{n} \alpha_i x^{n+i} + \sum_{j=1}^{n-1} \sum_{i=0}^{n} (\alpha_{n-i}\beta_j - \alpha_i \beta_{n-j}) x^{i+j}.
\end{aligned}$$

If $k = 0, 1, \ldots, n-1$ then the coefficient at x^k equals

$$\beta_0 \alpha_{n-k} + \sum_{\substack{j=1 \\ i+j=k}}^{n-1} \sum_{i=0}^{n} (\alpha_{n-i}\beta_j - \alpha_i \beta_{n-j}).$$

Analogously we get for the coefficient at x^{2n-k} $(k = 0, 1, \ldots, n-1)$:

$$\begin{aligned}
&-\beta_0 \alpha_{n-k} + \sum_{\substack{j=1 \\ i+j=2n-k}}^{n-1} \sum_{i=0}^{n} (\alpha_{n-i}\beta_j - \alpha_i \beta_{n-j}) \\
&= -\beta_0 \alpha_{n-k} + \sum_{\substack{m=1 \\ l+m=k}}^{n-1} \sum_{l=0}^{n} (\alpha_l \beta_{n-m} - \alpha_{n-l} \beta_m)
\end{aligned}$$

(here we have used the transformation $n - i = l$, $n - j = m \Rightarrow l + m = k$)

$$= -\beta_0 \alpha_{n-k} + \sum_{\substack{j=1 \\ i+j=k}}^{n-1} \sum_{i=0}^{n} (\alpha_i \beta_{n-j} - \alpha_{n-i} \beta_j).$$

If $k = 0, 1, \dots, n$ it is clear that the coefficient at x^k equals the coefficient at x^{2n-k} (except for the sign). For the coefficient by x^n we get

$$(\beta_0\alpha_0 - \beta_0\alpha_0) + \sum_{\substack{j=1 \\ i+j=n}}^{n-1} \sum_{i=0}^{n} (\alpha_{n-i}\beta_j - \alpha_i\beta_{n-j})$$

$$= \sum_{\substack{j=1 \\ i+j=n}}^{n-1} \sum_{i=1}^{n-1} (\alpha_{n-i}\beta_j - \alpha_i\beta_{n-j}) = \sum_{j=1}^{n-1} (\alpha_j\beta_j - \alpha_{n-j}\beta_{n-j})$$

$$= \sum_{j=1}^{n-1} \alpha_j\beta_j - \sum_{k=1}^{n-1} \alpha_k\beta_k = 0.$$

(We have set $n - j = k$.) It follows that the equation

$$\sum_{j=0}^{n-1} \beta_j \left(x^{n+j} P_n\left(\frac{1}{x}\right) - x^{n-j} P_n(x) \right) = 0 \tag{45}$$

is negative reciprocal of an even degree at most $2n$. If $\beta_0 \neq 0$ then the coefficient at x^{2n} equals $-\beta_0$ and that at x^0, as follows from (45), equals β_0.

Thus the equation (45) is of the form

$$-\beta_0 x^{2n} + \beta_1 x^{2n-1} + \ldots + \beta_0 = 0 \Rightarrow x^{2n} + \ldots + (-1) = 0.$$

Let us denote the roots of this equation by $\xi_1, \xi_2, \dots, \xi_{2n}$. It is well known that in this case

$$\xi_1\xi_2 \ldots \xi_{2n} = -1.$$

It follows from the theory of reciprocal equations that the equation (45) has one root $\xi_{2n} = -1$. The equation (45) has the root $\xi_{2n-1} = 1$ as well. Then

$$\xi_1\xi_2 \ldots \xi_{2n-2} = 1.$$

From these facts we may conclude that the coefficients ξ_i for $i = 1, 2, \dots, 2n-2$ are either reciprocal (real or complex) or they equal $-1, +1$, respectively. Moreover, it follows that for arbitrary β_j, $j = 1, 2, \dots, n-1$, $\beta_0 \neq 0$, the polynomial (45) has in $(-1, +1)$ at most $n - 1$ roots. If $\beta_0 = 0$, the equation (45) is reciprocal of a degree at most $2n - 1$ and it is of the form

$$-\beta_1 x^{2n-1} + \ldots + \beta_1 x = 0. \tag{46}$$

This equation has one root $x = 0 \in (-1, +1)$. When we divide the equation (46) by $x \neq 0$ we get the equation

$$-\beta_1 x^{2n-2} + \ldots + \beta_1 = 0.$$

From the above considerations it follows that this equation has at most $n-2$ roots in $(-1, +1)$ provided we put $n := n-1$. Thus the equation (46) has at most $n-1$ roots in $(-1, +1)$ altogether.

Now, summarizing the above properties, we have that the function (43) has in B at most $n-1$ roots.

We have proved that the functions

$$\frac{(-1)^n S(x)}{(x^n P_n(\frac{1}{x}))^2}\left(x^{n+j} P_n\left(\frac{1}{x}\right) - x^{n-j} P_n(x)\right), \quad j = 0, 1, \ldots, n-1 \tag{47}$$

fulfil in B the classical Haar condition and the functions (39) fulfil the local Haar condition.

In addition, $\dim W(\alpha) = n$ independently of $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$. In view of Theorem 9 in [5] and [2] we have that the length of the alternant of the only one minimal solution equals $n+1$.

Further, all roots of the minimal solution lie in B and are mutually different. Thus, we have proved the existence and uniqueness of a solution of the problem (34) which has properties 1) and 2) of Theorem 8.

Now, let us prove the inequalities (37) using Theorem 18 in [5].

We choose instead of $P_n(x)$ the polynomial $\tilde{T}_n(x)$ the roots of which lie in the interval $B = [-a, +a]$, $a \in (0, 1)$,

$$\tilde{t}_k^{(n)} = a \cos \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n$$

and we choose the points $\tilde{\xi}_k^{(n)} = a \cos \frac{k\pi}{2n}, \quad k = 0, 1, \ldots, n.$

Further, we write

$$\frac{\tilde{T}_n(x)}{x^n \tilde{T}_n(\frac{1}{x})} = \prod_{j=1}^{n} \frac{1 - \tilde{t}_j^{(n)}}{1 - x\tilde{t}_j^{(n)}}.$$

From this we get setting $k=\frac{n}{2}$ for n even

$$\tag{48} \begin{aligned} \frac{\tilde{T}_n(\tilde{\xi}_k^{(n)})}{\big(\tilde{\xi}_k^{(n)}\big)^n \tilde{T}_n\big(\frac{1}{\tilde{\xi}_k^{(n)}}\big)} &= \prod_{j=1}^{n} \frac{a(\cos\frac{k\pi}{n} - \cos\frac{(2j-1)\pi}{2n})}{a^2\cos\frac{k\pi}{n}\big(\frac{1}{a^2\cos\frac{k\pi}{n}} - \cos\frac{(2j-1)\pi}{2n}\big)} \\ &= \frac{1}{a^n\cos^n\frac{k\pi}{n}} \prod_{j=1}^{n} \frac{\cos\frac{k\pi}{n} - \cos\frac{(2j-1)\pi}{2n}}{a^{-2}\cos^{-1}\frac{k\pi}{n} - \cos\frac{(2j-1)\pi}{2n}} \\ &= \frac{(-1)^k}{2^{n-1}a^n\cos^n\frac{k\pi}{n}} \prod_{j=1}^{n} \frac{1}{a^{-2}\cos^{-1}\frac{k\pi}{n} - \cos\frac{(2j-1)\pi}{2n}} \\ &= \frac{(-1)^k}{2^{n-1}a^n\cos^n\frac{k\pi}{n}} \frac{1}{T_n(a^{-2}\cos^{-1}\frac{k\pi}{n})}, \end{aligned}$$

where $T_n(x)=\frac{1}{2^{n-1}}\cos(n\arccos x)$, $x\in[-1,+1]$, is the Chebyshev polynomial, the deviation of which is minimal under the condition that the coefficient at the highest power of x equals 1. It is well known that

$$\tag{49} 2^{n-1}T_n(x) = \frac{1}{2}\big\{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\big\}$$

for $|x|>1$. From this expression it follows for each $k=0,1,\dots,n$ with the exception of $k=\frac{n}{2}$, n even

$$\begin{aligned} &\Big(a\cos\frac{k\pi}{n}\Big)^n 2^{n-1}T_n\Big(\frac{1}{a^2\cos\frac{k\pi}{n}}\Big) \\ &\quad= \frac{1}{2}\Bigg\{\Bigg(\frac{a\cos\frac{k\pi}{n}}{a^2\cos\frac{k\pi}{n}} + \sqrt{\Big(\frac{a\cos\frac{k\pi}{n}}{a^2\cos\frac{k\pi}{n}}\Big)^2 - a^2\cos^2\frac{k\pi}{n}}\Bigg)^n \\ &\qquad+\Bigg(\frac{a\cos\frac{k\pi}{n}}{a^2\cos\frac{k\pi}{n}} - \sqrt{\Big(\frac{a\cos\frac{k\pi}{n}}{a^2\cos\frac{k\pi}{n}}\Big)^2 - a^2\cos^2\frac{k\pi}{n}}\Bigg)^n\Bigg\} \\ &\quad= \frac{1}{2}\Bigg\{\Bigg(\frac{1}{a} + \sqrt{\frac{1}{a^2} - a^2\cos^2\frac{k\pi}{n}}\Bigg)^n + \Bigg(\frac{1}{a} - \sqrt{\frac{1}{a^2} - a^2\cos^2\frac{k\pi}{n}}\Bigg)^n\Bigg\} \\ &\quad= \frac{1}{2a^n}\Bigg\{\Bigg(1 + \sqrt{1 - a^4\cos^2\frac{k\pi}{n}}\Bigg)^n + \Bigg(1 - \sqrt{1 - a^4\cos^2\frac{k\pi}{n}}\Bigg)^n\Bigg\} \\ &\quad= \frac{1}{2a^n}\{e^{n\ln(1+\sqrt{1-a^4\cos^2\frac{k\pi}{n}})} + e^{n\ln(1-\sqrt{1-a^4\cos^2\frac{k\pi}{n}})}\}. \end{aligned}$$

Obviously

$$\tag{50} 1+\sqrt{1-a^4\cos^2\frac{k\pi}{n}} > 1, \quad 0 \leqslant 1-\sqrt{1-a^4\cos^2\frac{k\pi}{n}} < 1.$$

Finally, we get for $(-1)^n S(x)\frac{\tilde{T}_n(x)}{x^n\tilde{T}_n(\frac{1}{x})}$ altogether

$$(51)\qquad (-1)^{n+k}\frac{2a^n}{\sqrt{2\pi(1-a^2\cos^2\frac{k\pi}{n})}}\,\frac{1}{\mathrm{e}^{n\ln(1+\sqrt{1-a^4\cos^2\frac{k\pi}{n}})}+\mathrm{e}^{n\ln(1-\sqrt{1-a^4\cos^2\frac{k\pi}{n}})}}$$

for $k=0,1,\ldots,n$ except for $k=\frac{n}{2}$, n even. At the points $\xi_k^{(n)}$, $k=0,1,\ldots,n$, the values (51) change their sign. Now in the same way we get the formula (48) for n even, $k=\frac{n}{2}$, $\xi_{\frac{n}{2}}=0$:

$$\frac{\tilde{T}_n(x)}{x^n\tilde{T}_n(\frac{1}{x})}\bigg|_{x=\xi_{\frac{n}{2}}^{(n)}}=\prod_{j=1}^{n}(-\tilde{t}_j^{(n)})=a^n\prod_{j=1}^{n}(-t_j^{(n)})=a^nT_n(0)=a^n\frac{(-1)^{\frac{n}{2}}}{2^{n-1}}.$$

Altogether we get for n even

$$(52)\qquad (-1)^nS(x)\frac{\tilde{T}_n(x)}{x^n\tilde{T}_n(\frac{1}{x})}\bigg|_{x=\xi_{\frac{n}{2}}^{(n)}}=(-1)^{\frac{n}{2}}\frac{a^n}{2^{n-1}\sqrt{2\pi}}.$$

It can be very easily shown that by virtue of (50) the formula (51) for n even, $k=\frac{n}{2}$ comes over to (52), setting zero instead of the second term in the denominator of the second multiplier in the formula (51). Let $k=0,1,\ldots,n$. Let us seek such a k for which the absolute value of the function (51) attains its minimum. It can be seen from (51) that it is sufficient to consider the problem

$$(53)\qquad \max_{y\in\{a\cos\frac{k\pi}{n},\,k=0,1,\ldots,n\}}\sqrt{1-y^2}\Big\{\big(1+\sqrt{1-a^2y^2}\big)^n+\big(1-\sqrt{1-a^2y^2}\big)^n\Big\}.$$

Let us consider the function $\gamma(x)=(1+x)^n+(1-x)^n$ in the interval $\big[(1-a^4)^{1/2},1\big]$ for n even and in the interval $\big[(1-a^4)^{1/2},(1-a^4\sin^2\frac{\pi}{2n})^{1/2}\big]$ for n odd because if n is even the values $a^2\cos^2\frac{k\pi}{n}$ lie in the interval $[0,a^2]\Rightarrow(1-a^2y^2)^{1/2}$ lie in the interval $[(1-a^4)^{1/2},1]$; if n is odd, the values $a^2\cos^2\frac{k\pi}{n}$ lie in the interval $\big[a^2\cos^2\frac{(n-1)\pi}{2n},a^2\big]=\big[a^2\sin^2\frac{\pi}{2n},a^2\big]\Rightarrow(1-a^2y^2)^{1/2}$ lie in the interval $\big[(1-a^4)^{1/2},(1-a^4\sin^2\frac{\pi}{2n})^{1/2}\big]$. The first derivative of the function γ is

$$\gamma'(x)=n[(1+x)^{n-1}-(1-x)^{n-1}],$$

hence with the aid of the binomial formula we obtain that the inequality

$$\gamma'(x)=n\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}(1-(-1)^{n-i-1})>0$$

for $x>0$ holds.

Then the function $\gamma(x)$ is increasing, thus it assumes its minimum at the point $x = 1$ for n even and at the point $x = (1 - a^4 \sin^2 \frac{\pi}{2n})^{1/2}$ for n odd.

For n even the point $y = 0$ corresponds to the point $x = 1$ and the maximum of (53) is 2^n. For n odd the point $y = a \sin \frac{\pi}{2n}$ corresponds to the point $x = (1-a^4 \sin^2 \frac{\pi}{2n})^{1/2}$ and the maximum in formula (53) is

$$\sqrt{1 - a^2 \sin^2 \frac{\pi}{2n}} \Big\{ \Big(1 + \sqrt{1 - a^4 \sin^2 \frac{\pi}{2n}}\Big)^n + \Big(1 - \sqrt{1 - a^4 \sin^2 \frac{\pi}{2n}}\Big)^n \Big\}.$$

For the minimum of the absolute value of the function defined in (51) we obtain immediately

$$E_{n,n} \geqslant \frac{2a^n}{\sqrt{2\pi}2^n} = \sqrt{\frac{2}{\pi}} \Big(\frac{a}{2}\Big)^n$$

for each n.

Hence, (3) is proved. □

Remark 8. The explicit solution of the problem (34) was not found. The problem may be solved through various methods numerically, for instance using Newton's method or other methods given in [5] and [7]. From the proof of Theorem 8 it follows (property 3) that the polynomial $\tilde{T}_n(x)$ may be taken as a good initial approximation for the solution for sufficiently small $a \in (0, 1)$.

Let us consider instead of (53) the expression

$$\min_{y \in \{a \cos \frac{k\pi}{n},\, k=0,1,\dots,n\}} \sqrt{1 - y^2} \{(1 + \sqrt{1 - a^2 y^2})^n + (1 - \sqrt{1 - a^2 y^2})^n\}. \tag{54}$$

The function $\gamma(x)$ attains its minimum at the point $x = (1 - a^4)^{1/2}$ (see the proof of Theorem 8) for arbitrary natural n. This point corresponds to the point $y = a$. Then we have for the expression in (54)

$$\sqrt{1 - a^2} \{(1 + \sqrt{1 - a^4})^n + (1 - \sqrt{1 - a^4})^n\},$$

from which we get for the maximum of the absolute value of (51)

$$\lambda_{\max} = \sqrt{\frac{2}{\pi}} \frac{a^n}{\sqrt{1 - a^2}} \frac{1}{\big(1 + \sqrt{1 - a^4}\big)^n + \big(1 - \sqrt{1 - a^4}\big)^n}.$$

In view of (37), we have

$$q_n = \frac{2^n}{\sqrt{1 - a^2}} \frac{1}{\big(1 + \sqrt{1 - a^4}\big)^n + \big(1 - \sqrt{1 - a^4}\big)^n} \to 1 \quad (a \to 0+)$$

for each n.

5. Minimization of the error estimate of the interpolatory rule

In this part we study the possibility of minimization of the estimate (25) for the case of optimal interpolatory rule with respect to the subspace that is generated by functions for which the optimal rule is exact.

Theorem 9. *Let* ${}^{(\mathrm{opt})}R_n$ *for fixed* $x \in [-a, +a]$, $a \in (0, 1)$, *be the error functional of the optimal interpolatory rule of the type* (4). *Let* $f \in H_2(K_1)$ *be a real-valued function on* $[-a, +a]$. *Let us assume that the nodes of interpolation are mutually different and lie in* $[-a, +a]$. *Then*

$$|{}^{(\mathrm{opt})}R_n(f, x)|^2 \leqslant \|{}^{(\mathrm{opt})}R_n\|^2(\|f\|^2 - \|P_n(f)\|^2), \tag{55}$$

where

$$\|f\|^2 = \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\varphi})|^2 \,\mathrm{d}\varphi. \tag{56}$$

The norm $\|{}^{(opt)}R_n\|$ *is given by the formula* (21) *or* (21′). *Further,* P_n *is the orthogonal projection from* $H_2(K_1)$ *onto* $\mathrm{span}(\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_n^{(n)})$ *and*

$$\|P_n(f)\|^2 = \frac{1}{\pi} \sum_{k=1}^{n} \sum_{l=1}^{n} a_k^{(n)}(f) a_l^{(n)}(f) \frac{1}{1 - x_k^{(n)} x_l^{(n)}}, \tag{57}$$

or

$$\|P_n(f)\|^2 = \sum_{k=1}^{n} a_k^{(n)}(f) f(x_k^{(n)}) \tag{57′}$$

respectively, where $a_k^{(n)}(f)$, $k = 1, 2, \ldots, n$, *are the solutions of the system of normal equations*

$$Ga = p \tag{58}$$

where the elements of the matrix G *are given by*

$$g_{k,l} = \frac{1}{2\pi(1 - x_k^{(n)} x_l^{(n)})}, \quad k, l = 1, 2, \ldots, n \tag{59}$$

and the vector $p = \{p_k\}$ *on the right-hand side of the system is given by*

$$p_k = (\Phi_k^{(n)}, f), \quad k = 1, 2, \ldots, n. \tag{60}$$

Moreover, the determinant of the matrix G *is positive.*

P r o o f. We prove this theorem in the same way as in [8] (where it is given for a case of quadrature) but in more detail.

From (14) it follows that the optimal interpolatory rule interpolates the functions

$$\Phi_i^{(n)}(x) = \frac{1}{2\pi(1 - x x_i^{(n)})}, \quad i = 1, 2, \ldots, n,$$

exactly. Then $R_n(\Phi_i^{(n)}; x) = 0$, $i = 1, 2, \ldots, n$. This implies

$$g(x) = \sum_{k=1}^{n} c_k \Phi_k^{(n)}(x)$$

for each $g \in \operatorname{span}(\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_n^{(n)})$ and thus

$$R_n(g; x) = 0$$

for each $x \in [-a, +a]$, $a \in (0, 1)$.

Let $f \in H_2(K_1)$, $g \in \operatorname{span}(\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_n^{(n)})$. It is easy to see that

$$R_n(f + g; x) = R_n(f; x).$$

Let P_n be the operator of the orthogonal projection from $H_2(K_1)$ into $\operatorname{span}(\Phi_1^{(n)}, \Phi_2^{(n)}, \ldots, \Phi_n^{(n)})$; then we have

$$R_n(f; x) = R_n(f - P_n(f); x).$$

Because P_n is the orthogonal projector, $a_k^{(n)}(f)$, $k = 1, 2, \ldots, n$, can be calculated from the expression

$$P_n(f) = \sum_{k=1}^{n} a_k^{(n)}(f) \Phi_k^{(n)} \tag{61}$$

as the solution of the system of normal equations

$$Ga = p \tag{62}$$

where

$$\begin{aligned} p &= [(\Phi_1^{(n)}, f), (\Phi_2^{(n)}, f), \ldots, (\Phi_n^{(n)}, f)]^T, \\ G &= (\Phi_i^{(n)}, \Phi_j^{(n)})_{i,j}^n, \\ a &= (a_1^{(n)}(f), a_2^{(n)}(f), \ldots, a_n^{(n)}(f))^T \end{aligned}$$

and

$$(63)\qquad (\Phi_k^{(n)}, f) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-\mathrm{e}^{\mathrm{i}\varphi}x_k^{(n)}}\overline{f(\mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{d}\varphi$$
$$= \overline{\frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\varphi})\frac{1}{1-\mathrm{e}^{-\mathrm{i}\varphi}x_k^{(n)}}\,\mathrm{d}\varphi}$$
$$= \overline{\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\varphi})\overline{K(x_k^{(n)}, \mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{d}\varphi} = f(x_k^{(n)}), \quad k = 1, 2, \ldots, n,$$
$$(\Phi_i^{(n)}, \Phi_j^{(n)}) = \frac{1}{2\pi(1 - x_i^{(n)}x_j^{(n)})}, \quad i, j = 1, 2, \ldots, n$$

(see the proof of Theorem 2).

The determinant D_n of the system is given by the formula (15). Because $x_i^{(n)} \neq x_j^{(n)}$, $i \neq j$, $i, j = 1, 2, \ldots, n$, D_n is positive (according to (15)) and the system (62) has only one solution. The solution $a_i^{(n)}(f)$, $i = 1, 2, \ldots, n$, is real and independent of $x \in [-a, +a]$ and $a \in (0, 1)$.

According to Theorem 4 (inequality (25)) we have

$$|R_n(f; x)|^2 = |R_n(f - P_n(f); x)|^2 \leqslant \|R_n\|^2 \|f - P_n(f)\|^2,$$
$$\|f - P_n(f)\|^2 = \big(f - P_n(f), f - P_n(f)\big) = \|f\|^2 - (f, P_n(f)).$$

By (63) we have

$$(f, P_n(f)) = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\varphi}) \sum_{k=1}^{n} a_k^{(n)}(f)\overline{\Phi_k^{(n)}(\mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{d}\varphi$$
$$= \sum_{k=1}^{n} a_k^{(n)}(f) f(x_k^{(n)}).$$

Finally,

$$\big(P_n(f), P_n(f)\big) = \int_0^{2\pi} \sum_{k=1}^{n} a_k^{(n)}(f)\Phi_k^{(n)}(\mathrm{e}^{\mathrm{i}\varphi}) \sum_{l=1}^{n} a_l^{(n)}(f)\overline{\Phi_l^{(n)}(\mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{d}\varphi$$
$$= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k^{(n)}(f) a_l^{(n)}(f) \int_0^{2\pi} \Phi_k^{(n)}(\mathrm{e}^{\mathrm{i}\varphi})\overline{\Phi_l^{(n)}(\mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{d}\varphi$$
$$= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k^{(n)}(f) a_l^{(n)}(f) \frac{1}{2\pi(1 - x_k^{(n)}x_l^{(n)})},$$

which we wanted to prove. □

R e m a r k 9. Theorem 7 is of theoretical importance only; the coefficients $a_k^{(n)}$, $k = 1, 2, \ldots, n$, must be calculated as the solution of the system (58).

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References

[1] *K. Yosida*: Functional Analysis. Springer-Verlag, Berlin, 1965.
[2] *N. I. Achiezer*: Lectures on the Theory of Approximation. Nauka, Moscow, 1965. (In Russian.)
[3] *H. Engels*: Ein numerisches Differentiationsverfahren mit ableitungsfreien Fehlerschranken. Z. Angew. Math. Mech. *52* (1972), 145–154.
[4] *A. Paulik*: Zur Existenz optimaler Quadraturformeln mit freien Knoten bei Integration analytischer Funktionen. Numer. Math. *27* (1977), 395–405.
[5] *G. Meinardus*: Approximation von Funktionen und ihre numerische Behandlung. Springer-Verlag, Berlin, 1964.
[6] *I. P. Natanson*: Konstruktive Funktionentheorie. Akademie-Verlag, Berlin, 1955.
[7] *A. Ralston*: A First Course in Numerical Analysis. Mc Graw-Hill, New York, 1965.
[8] *W. Knauff, A. Paulik*: A note on Davis type error bounds. BIT *18* (1978), 175–183.
[9] *J. Kofroň*: Die ableitungsfreien Fehlerabschätzungen von Interpolationsformeln. Apl. Mat. *17* (1972), 137–152.
[10] *W. Rudin*: Real and Complex Analysis. Mc Graw-Hill, New York, 1966, 1974.

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