INTERPOLATION SERIES IN LOCAL FIELDS OF PRIME CHARACTERISTIC

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1. Introduction. In 1944 Dieudonné [3] proved a p-adic analogue of the Weierstrass Approximation Theorem by an inductive argument involving the polynomial approximation of certain continuous characteristic functions. In 1958 Mahler [4] proved the sharper result that each continuous p-adic function f defined on the p-adic integers is the uniform limit of the "interpolation series"

$$f(t) = \sum_{n=0}^{\infty} \Delta^n f(0)\binom{t}{n},$$

where

$$\Delta^n f(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k).$$

The crucial step in Mahler's proof involves showing that $\lim_{n\to\infty} \Delta^n f(0) = 0$ for the p-adic topology, and he demonstrates this by passing to a certain cyclotomic extension of the rationals. In fact, this follows quickly from Dieudonné's theorem for if $p(t)$ is a polynomial of degree r for which $|f(t) - p(t)|_p < \epsilon$ for $t \in Z_p$, then $|\Delta^n f(0) - \Delta^n p(0)|_p < \epsilon$ for all n. Hence if $n > r$, $\Delta^n p(0) = 0$ and $|\Delta^n f(0)|_p < \epsilon$.

In the present paper we use the above idea to simplify our earlier proof of a Mahler type theorem for continuous functions on the ring V of formal power series over a finite field $GF(q)$ [5]. Although the proof by Dieudonné admits a straightforward generalization to any locally compact non-archimedean field, in this case we accomplish the polynomial approximation of the relevant characteristic functions without recourse to induction by using certain powers of the Carlitz polynomials $G'_{q^r-1}(t)/g_{q^r-1}$ [1]. We conclude by giving a sufficient condition for the differentiability of a function f defined on V.

2. Preliminaries. Let $GF[q, x]$ be the ring of polynomials over the finite field $GF(q)$ of characteristic p and let $GF(q, x)$ be the quotient field of $GF[q, x]$. Denote by V the ring of formal power series over $GF(q)$ and by F the field of formal power series over $GF(q)$. Set $|0| = 0$. If $\alpha \in F - \{0\}$ is given by

$$\alpha = \sum_{i=-\infty}^{\infty} a_i x^i, \tag{2.1}$$

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where $a_i \in GF(q)$ and all but a finite number of the a_i's vanish for $i < 0$, then set $v(\alpha) = k$ and

$$|\alpha| = b^{v(\alpha)}, \tag{2.2}$$

where $0 < b < 1$ and k is the smallest subscript i in (2.1) for which $a_i \neq 0$. Then $|\ |$ is a discrete, non-archimedean absolute value on F and F is complete with respect to this absolute value. Obviously $GF[q, x]$ is dense in V as is $GF(q, x)$ in F. The valuation ring of F is V, and V is compact and open in F [5; 392]. Also, addition and multiplication are continuous operations in F so that polynomials over F define continuous functions.

Following Carlitz we define a sequence of polynomials $\Psi_n(t)$ over $GF[q, x]$ by

$$\Psi_n(t) = \prod_{\deg m < n} (t - m), \tag{2.3}$$

where the above product extends over all $m \in GF[q, x]$ of degree less than n (including 0). Then [2; 140]

$$\Psi_n(t) = \sum_{i=0}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} t^{q^i}, \tag{2.4}$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{F_n}{F_i L_{n-i}^{q^i}}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \frac{F_n}{L_n}, \quad \begin{bmatrix} n \\ n \end{bmatrix} = 1 \tag{2.5}$$

and

$$\begin{aligned} F_n &= [n][n-1]^q \cdots [1]^{q^{n-1}}, \qquad F_0 = 1 \\ L_n &= [n][n-1] \cdots [1], \qquad L_0 = 1 \\ [r] &= x^{q^r} - x. \end{aligned} \tag{2.6}$$

Following [1] we define polynomials $G_n(t)$ and $G_n'(t)$ over $GF[q, x]$ and $g_n \in GF[q, x]$ as follows. If

$$n = e_0 + e_1 q + \cdots + e_s q^s, \qquad 0 \leq e_i < q, \tag{2.7}$$

then set

$$G_n(t) = \Psi_0^{e_0}(t) \cdots \Psi_s^{e_s}(t) \tag{2.8}$$

and

$$G_n'(t) = \prod_{i=0}^{s} G'_{e_i q^i}(t), \tag{2.9}$$

where

$$G'_{eq^i}(t) = \begin{cases} \Psi_i^e(t) & 0 \leq e < q - 1 \\ \Psi_i^e(t) - F_i^e & e = q - 1 \end{cases} \tag{2.10}$$

and

$$g_n = F_1^{e_1} \cdots F_s^{e_s}, \qquad g_0 = 1. \tag{2.11}$$

We mention that $G_n(t)/g_n$ and $G_n'(t)/g_n$ are integral valued polynomials over $GF(q, x)$, i.e., for all $m \in GF[q, x]$, $G_n(m)/g_n$, $G_n'(m)/g_n \in GF[q, x]$ [1; 503].

If H is any extension field of $GF(q, x)$, since deg $G_n(t) = n$, it follows that $(G_n(t)/g_n)$ is an ordered basis of the H-vector space $H[t]$. Indeed for any $h(t) \in H[t]$ of degree $\leq n$ we have [1; 491] the unique representation

$$h(t) = \sum_{i=0}^{n} A_i \frac{G_i(t)}{g_i}, \tag{2.12}$$

where

$$A_i = (-1)^r \sum_{\deg m < r} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}} h(m), \qquad m \in GF[q, x] \tag{2.13}$$

and $i < q^r$. We emphasize that for $i > n$ Formula (2.13) yields $A_i = 0$, so we could have written the sum in (2.12) with upper limit ∞. In the sequel we shall expand an arbitrary continuous function $f: V \to F$ in a (genuinely) infinite series resembling (2.12).

3. Characteristic functions. For all nonnegative integers k define a function χ_k on V by $\chi_k(t) = 1$ if $|t| \leq b^k$ and $\chi_k(t) = 0$ if $b^k < |t| \leq 1$. As the characteristic function of an open-closed ball about 0, χ_k is continuous. The following theorem shows that it may be uniformly approximated by polynomials over $GF(q, x)$.

THEOREM A. *For $k \geq 0$ let*

$$C_k(t) = (-1)^k G'_{q^k-1}(t)/g_{q^k-1}. \tag{3.1}$$

Then for all $t \in V$ and for all natural numbers s

$$|C_k^{p^s}(t) - \chi_k(t)| \leq b^{p^s}, \tag{3.2}$$

where p is the characteristic of F.

Proof. By [2; 141] $G'_{q^k-1}(t) = \Psi_k(t)/t$. If $|t| \leq b^k$, then $t = x^k\mu$, where $\mu \in V$. It follows from (2.4), (2.5), (2.6) and (2.11) that $C_k(0) = 1$, and so we may assume that $\mu \neq 0$. Then by these same four formulae

$$C_k(x^k\mu) = (-1)^k \frac{L_k \Psi_k(x^k\mu)}{F_k x^k \mu} = 1 + \sum_{j=1}^{k} (-1)^{2k-j} \frac{(x^k\mu)^{q^j-1} L_k}{F_j L_{k-j}^{q^j}}. \tag{3.3}$$

But each of the terms other than 1 in (3.3) is congruent to zero (mod x) for if $1 \leq j \leq k$, then

$$v((x^k\mu)^{q^j-1} L_k / F_j L_{k-j}^{q^j}) \geq (q^j - 1)k + k - (1 + q + \cdots + q^{j-1}) - q^j(k - j)$$
$$= jq^j - (1 + q + \cdots + q^{j-1}) > 0.$$

Hence there exists a $\beta \in V$ such that

$$C_k(x^k\mu) = 1 + \beta x$$

and so for all $s \geq 1$

$$C_k^{p^s}(x^k\mu) = 1 + (\beta x)^{p^s}$$

from which (3.2) follows for $|t| \leq b^k$.

If $b^k < |t| < 1$ and since $|\Psi_k(t)/F_k| \leq 1$ for all $t \in V$ [6; §3], then

$$|C_k^{p^s}(t) - \chi_k(t)| = |C_k(t)|^{p^s} = \left|\frac{L_k\Psi_k(t)}{tF_k}\right|^{p^s} \leq b^{p^s}.$$

Remark. It follows from (3.2) by translation that for all $\alpha \in V$

$$(3.4) \qquad |C_k^{p^s}(t - \alpha) - \chi_k(t - \alpha)| \leq b^{p^s}.$$

Hence the characteristic function of any open-closed ball in V may be uniformly approximated by polynomials.

4. THEOREM B. *Let $f: V \to F$ be continuous and for all $i \geq 0$ set*

$$(4.1) \qquad A_i = (-1)^r \sum_{\deg m < r} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m),$$

where $i < q^r$ (any such r yields the same value for A_i [1; 492]) *and the sum in* (4.1) *extends over all $m \in GF[q, x]$ of degree $< r$. Then*

$$(4.2) \qquad \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i}$$

converges uniformly to $f(t)$ for all $t \in V$.

Proof. Since $|G_i(t)/g_i| \leq 1$ for all $t \in V$ [6; §3] and $|\ |$ is non-archimedean, the uniform convergence of (4.2) would follow from a proof that $\lim_{i\to\infty} A_i = 0$. Hence, given $s \geq 0$, we seek $N = N(s)$ such that $i > N$ implies that $|A_i| \leq b^s$. Since V is compact, f is bounded, and we may assume with no loss of generality that $f: V \to V$. Also f is uniformly continuous, and so there exists a $k = k(s)$ such that $|t_1 - t_2| \leq b^k$ implies $|f(t_1) - f(t_2)| \leq b^s$ for $t_1, t_2 \in V$.

For $m \in GF[q, x]$ suppose that $f(m) = \sum_{i=0}^{\infty} a_i x^i$. Set $f_s(m) = a_0 + a_1 x + \cdots + a_{s-1}x^{s-1}$. This defines a function $f_s: GF[q, x] \to GF[q, x]$ for which

$$(4.3) \qquad |f_s(m) - f(m)| \leq b^s$$

for all $m \in GF[q, x]$. Furthermore, f is periodic (mod x^k) for if $m_1 \equiv m_2$ (mod x^k), i.e., if $|m_1 - m_2| \leq b^k$, then by (4.3) and the uniform continuity of f it follows that $|f_s(m_1) - f_s(m_2)| \leq b^s$. Hence $f_s(m_1) = f_s(m_2)$ since distinct values of f_s are incongruent (mod x^s).

Corresponding to (4.1) we define a sequence (B_i) in $GF[q, x]$ by

$$(4.4) \qquad B_i = (-1)^r \sum_{\deg m < r} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}} f_s(m),$$

where $i < q^r$. Since $G'_{q^r-1-i}(m)/g_{q^r-1-i} \varepsilon GF[q, x]$, it follows from (4.3) that for all $i \geq 0$

$$|A_i - B_i| \leq b^s. \tag{4.5}$$

By (4.4) and the periodicity (mod x^k) of f_s it follows that

$$B_i = (-1)^r \sum_{\deg m_1 < k} f_s(m_1) \sum_{\substack{\deg m<k \\ m \equiv m_1 \pmod{x^k}}} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}}. \tag{4.6}$$

Now for each $m_1 \varepsilon GF[q, x]$ with $\deg m_1 < k$

$$(-1)^r \sum_{\substack{\deg m<r \\ m \equiv m_1 \pmod{x^k}}} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}} = (-1)^r \sum_{\deg m<r} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}} \chi_k(m - m_1), \tag{4.8}$$

where χ_k is as in §3. For each such m_1 and for all $i \geq 0$ set

$$D_i(m_1) = (-1)^r \sum_{\deg m<r} \frac{G'_{q^r-1-i}(m)}{g_{q^r-1-i}} C_k^{p^s}(m - m_1), \tag{4.9}$$

where $C_k(t)$ is defined by (3.1) and $i < q^r$. Then by (3.4), (4.6), (4.8) and (4.9)

$$|B_i - \sum_{\deg m_1<k} f_s(m_1) D_i(m_1)| \leq b^{p^s} \leq b^s. \tag{4.10}$$

But for each m_1, $\deg C_k^{p^s}(t - m_1) = p^s(q^k - 1)$ and so by (4.9) and the remarks following (2.13), $D_i(m_1) = 0$ if $i > p^s(q^k - 1)$. It follows that for such i, $|B_i| \leq b^s$ which, along with (4.5), implies that $|A_i| \leq b^s$.

It remains to be shown that (4.2) actually converges to the function f. As the uniform limit of (continuous) polynomial functions (4.2) represents a continuous function on V. Since $GF[q, x]$ is dense in V, it suffices to show that

$$f(m^*) = \sum_{i=0}^{\infty} A_i \frac{G_i(m^*)}{g_i} \tag{4.11}$$

for all $m^* \varepsilon GF[q, x]$. Suppose that $\deg m^* < d$. Then by (2.3) and (2.8) $G_i(m^*) = 0$ for $i \geq q^d$, and so the series in (4.11) is actually finite. Let $f_d(t)$ be the unique polynomial over V of degree $<q^d$ such that $f_d(m) = f(m)$ for all $m \varepsilon GF[q, x]$ of degree $<d$. Then application of (2.12) and (2.13) to $f_d(t)$ yields (4.11). The polynomials $f_d(t)$ also yield a simple proof of the uniqueness of the coefficients A_i in (4.2) [5; 404].

5. Differentiability. The following propositions will be used to discuss differentiability criteria for continuous functions on V.

PROPOSITION 1. *For all nonnegative integers j and k*

$$\binom{j+k}{j} g_{j+k} = \binom{j+k}{j} g_j g_k, \tag{5.1}$$

where g_i is defined by (2.11).

Proof. Let $j = j_0 + j_1 q + \cdots + j_s q^s$ and let $k = k_0 + k_1 q + \cdots + k_s q^s$, where $0 \leq j_i, k_i < q$. If $j_i + k_i < q$ for each i, $1 \leq i \leq s$, then $g_{j+k} = g_j g_k$ by (2.11). If $j_i + k_i \geq q$ for some i, let n be the smallest such i. Then $j_n + k_n = q + r$, where $0 \leq r < q$ and $r < j_n$. Then by a familiar congruence for binomial coefficients $\binom{j+k}{j}$ is congruent (mod p) to a product of binomial coefficients, one of which is $\binom{r}{j_n} = 0$. Hence in this case (5.1) reduces to the identity $0 = 0$.

PROPOSITION 2. *For all* $n \geq 1$

$$\frac{G_n(t)}{tg_{n-1}} = \frac{G'_{q^{e(n)}-1}(t)}{g_{q^{e(n)}-1}} \frac{G_{n-q^{e(n)}}(t)}{g_{n-q^{e(n)}}}, \tag{5.2}$$

where $q^{e(n)} \mid n$ *and* $q^{e(n)+1} \nmid n$.

Proof. Let $n = n_0 + n_1 q + \cdots + n_s q^s$, where $0 \leq n_i < q$. If $n_0 > 0$, then $e(n) = 0$, and so by (2.8), (2.11) and the fact that $\Psi_0(t) = t$

$$\frac{G_n(t)}{tg_{n-1}} = \frac{\Psi_n^{n_0-1}(t)\Psi_1^{n_1}(t) \cdots \Psi_s^{n_s}(t)}{g_{n-1}} = \frac{G_{n-1}(t)}{g_{n-1}}. \tag{5.3}$$

If $n_0 = 0$, let $j = e(n)$ be the first nonzero coefficient in the q-adic expansion of n. Then $n - 1 = (q-1) + (q-1)q + \cdots + (q-1)q^{j-1} + (n_j - 1)q^j + n_{j+1}q^{j+1} + \cdots + n_s q^s$ and $n - q^j = (n_j - 1)q^j + n_{j+1}q^{j+1} + \cdots + n_s q^s$ so that

$$\begin{aligned} \frac{G_n(t)}{tg_{n-1}} &= \frac{\Psi_j(t)}{tF_1^{q-1} \cdots F_{j-1}^{q-1}} \frac{\Psi_j^{n_j-1}(t)\Psi_{j+1}^{n_{j+1}}(t) \cdots \Psi_s^{n_s}(t)}{F_j^{n_j-1}F_{j+1}^{n_{j+1}} \cdots F_s^{n_s}} \\ &= \frac{G'_{q^j-1}(t)}{g_{q^j-1}} \frac{G_{n-q^j}(t)}{g_{n-q^j}} \end{aligned} \tag{5.4}$$

since $\Psi_j(t)/t = G'_{q^j-1}(t)$ [2; 141]. It follows from (5.2) that $G_n(t)/tg_{n-1}$ is an integral valued polynomial over $GF(q, x)$ and, since $GF[q, x]$ is dense in V, that

$$\left|\frac{G_n(t)}{tg_{n-1}}\right|_{t=\alpha} \leq 1 \tag{5.5}$$

if $|\alpha| \leq 1$.

PROPOSITION 3. *For all* $n \geq 1$

$$\left(\frac{G_n(t)}{tg_{n-1}}\right)_{t=0} = \begin{cases} (-1)^k & \text{if } n = q^k \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Proof. This follows from (5.2), the fact that $G_i(0) = 0$ for $i > 0$ and the fact that $G'_{q^k-1}(0)/g_{q^k-1} = (-1)^k$ [6; §5].

PROPOSITION 4. *For all* $n \geq 1$

$$\frac{g_{n-1}}{g_n} = \frac{1}{L_{e(n)}}, \tag{5.7}$$

where L_i *is defined by* (2.6) *and* $e(n)$ *is as in* (5.2).

Proof. This follows immediately from (2.6) and (2.11).

We may now give a sufficient condition for the differentiability of a continuous function $f:V \to V$ at $u \in V$.

THEOREM C. *Let* $f:V \to V$ *continuously and suppose that*

$$f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i} \tag{5.8}$$

is the interpolation series for f *constructed from the Carlitz polynomials. For all* $u \in V$ *set*

$$A_j(u) = \sum_{k=0}^{\infty} \binom{j+k}{j} A_{j+k} \frac{G_k(u)}{g_k}. \tag{5.9}$$

If $\lim_{j\to\infty} A_j(u)/L_{e(j)} = 0$, *then* f *is differentiable at* u *and*

$$f'(u) = \sum_{n=0}^{\infty} (-1)^n \frac{A_{q^n}(u)}{L_n}. \tag{5.10}$$

Proof. By (5.8), [1; 488, (2.3)] and Proposition 1

$$\begin{aligned} f(t+u) &= \sum_{i=0}^{\infty} A_i \frac{G_i(t+u)}{g_i} = \sum_{i=0}^{\infty} \frac{A_i}{g_i} \sum_{j=0}^{i} \binom{i}{j} G_j(t) G_{i-j}(u) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} A_i \frac{G_j(t)}{g_j} \frac{G_{i-j}(u)}{g_{i-j}} \end{aligned} \tag{5.11}$$

for all $t, u \in V$. Since (A_i) is a null sequence, we may reverse the order of summation in the last series in (5.11). This yields

$$f(t+u) = \sum_{j=0}^{\infty} A_j(u) \frac{G_j(t)}{g_j}, \tag{5.12}$$

where

$$A_j(u) = \sum_{k=0}^{\infty} \binom{j+k}{j} A_{j+k} \frac{G_k(u)}{g_k}. \tag{5.13}$$

Note that $(A_j(u))$ is a null sequence and that $A_0(u) = f(u)$; so for all nonzero $t \in V$

$$\frac{f(t+u) - f(u)}{t} = \sum_{j=1}^{\infty} A_j(u) \frac{G_j(t)}{tg_j} = \sum_{j=1}^{\infty} \frac{A_j(u)}{L_{e(j)}} \frac{G_j(t)}{tg_{j-1}} \tag{5.14}$$

by Proposition 3.

Now if $(A_j(u)/L_{e(j)})$ is a null sequence, then by (5.5) the rightmost series in (5.14) converges for all $t \in V$ (including zero) to a continuous function on V. Hence $f'(u)$ exists and by Proposition 3

$$f'(u) = \sum_{j=1}^{\infty} \left(\frac{A_j(u)}{L_{e(j)}} \frac{G_j(t)}{tg_{j-1}}\right)_{t=0} = \sum_{n=0}^{\infty} (-1)^n \frac{A_{q^n}(u)}{L_n}. \tag{5.15}$$

We remark that the function f of (5.8) is a linear operator on the $GF(q)$-vector space V precisely when $A_i = 0$ for i not a power of q [5; 406]. Hence if f is linear, then

$$A_j(u) = \sum_{k=0}^{\infty} \binom{j+k}{j} A_{j+k} \frac{G_k(u)}{g_k} = A_j \tag{5.16}$$

so that the condition $\lim_{j\to\infty} A_j(u)/L_{e(j)} = 0$ is equivalent to $\lim_{n\to\infty} A_{q^n}/L_n = 0$. This latter condition is, in the linear case, both necessary and sufficient for f to be everywhere differentiable on V [6; §5].

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