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Talman, A.J.J.; van der Laan, G.

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INTERPRETATION OF A VARIABLE DIMENSION FIXED POINT ALGORITHM WITH AN ARTIFICIAL LEVEL*

G. VAN DER LAAN AND A. J. J. TALMAN

Free University, Amsterdam

In this paper two interpretations of the variable dimension algorithm to compute a fixed point of a continuous function from the product space S of simplices into itself, introduced in an earlier paper, are given. The first interpretation yields a subdivision, whereas the second one yields a triangulation of the convex hull of the set S on the natural level and some set on the artificial level. After labelling the vertices of the latter set in such a way that this set is completely labelled, a path of adjacent polyhedra (or simplices) can be generated with common completely labelled facets starting with the set on the artificial level and terminating with a completely labelled simplex on the natural level yielding an approximate fixed point. The intersection of the path with this level is the sequence of the simplices of variable dimension of the algorithm. So, the algorithm can be viewed as tracing zeroes of a piecewise linear homotopy function.

1. Introduction. In this paper two intepretations of a variable dimension algorithm for the computation of a fixed point on an (un)bounded set, introduced by the authors in earlier papers (see [9], [10], [12]–[14], see also [8], [15]) are given. In each interpretation a set of artificially labelled points is added. Doing so, the algorithm can be viewed as tracing zeroes of a piecewise linear homotopy function.

The main part of the paper is concerned with an algorithm introduced in [14] (see also [15]) for computing a fixed point of a continuous function on the product space S of N unit simplices. In §2 we give some preliminaries and a concise description of the algorithm. Then in §3 we present the first interpretation, which consists of constructing a subdivision of $S \times [0, 1]$, where $S \times \{0\}$ denotes the artificial level and $S \times \{1\}$ the natural level. This subdivision is formed by a collection of polyhedra. The vertices of S are the only grid points on the zero-level. Clearly, if the jth unit simplex has dimension $m_j - 1$, $j = 1, \ldots, N$, then the product space S has $\prod_{j=1}^{N} m_j$ vertices, being therefore the number of grid points on the artificial level.

In §4 we give a second interpretation with the $(M - 1)$-dimensional unit simplex on the artificial level, where $M = \sum_{j=1}^{N} m_j$. Therefore the dimension of the set on the zero-level is now $M - 1$ as compared with $\sum_{j=1}^{N}(m_j - 1) = M - N$ for the first interpretation. Again the only grid points on the zero level are the vertices of the $(M - 1)$-dimensional unit simplex. Now the subdivision of the convex hull of the natural and the artificial level is formed by a collection of M-dimensional simplices. For the special case $N = 1$ the two interpretations are identical and can also be found in Van der Laan and Talman [9] (see also Van der Laan [8] and Todd [16]).

In §5 the modifications are given to generalize the interpretations to a class of algorithms on R^n, introduced in [12] (see also [13] and [15]). Some concluding remarks are made at the end of §5.

2. Preliminaries. Let us assume we want to compute a fixed point of a continuous function f mapping the set S into itself where S is the product space of N unit

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simplices, i.e.,

$$S = \prod_{j=1}^{N} S^{m_j - 1},$$

where for $j = 1, \ldots, N$.

$$S^{m_j - 1} = \left\{ x \in R_+^{m_j} \,\middle|\, \sum_{i=1}^{m_j} x_i = 1 \right\}.$$

First we will triangulate S in $(M - N)$-dimensional simplices (see [14]), where $M = \sum_{j=1}^{N} m_j$.

DEFINITION 2.1. Let X be an m-dimensional convex subset of R^n. A collection of m-dimensional polyhedra is a subdivision of X if

(i) the union of all polyhedra is X,

(ii) the intersection of two polyhedra is a face of each,

(iii) each compact subset of X is covered by a finite number of relative interiors of faces.

A subdivision is called a triangulation if every polyhedron is an m-dimensional simplex.

Let us define the $M \times M$-matrix Q by

$$Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & Q_N \end{bmatrix},$$

where Q_j is the $m_j \times m_j$ matrix corresponding to the standard triangulation of $S^{m_j - 1}$, i.e.,

$$Q_j = \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 \\ 1 & -1 & \ddots & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

The $(k + \sum_{i=1}^{j-1} m_i)$th column of the matrix Q will be denoted by $q(j,k)$. Furthermore, let $d = (d_1, \ldots, d_N)$ be a vector of grid sizes and let D be the M-diagonal matrix with $(k + \sum_{i=1}^{j-1} m_i)$th diagonal element equal to d_j^{-1}, $k = 1, \ldots, m_j$, $j = 1, \ldots, N$. Now, a grid point of S is an element $x \in S$, such that the $(k + \sum_{i=1}^{j-1} m_i)$ th element of x, denoted by $x_{j,k}$, is a multiple of d_j^{-1}. The set of indices (j,k), $k = 1, \ldots, m_j, j = 1, \ldots, N$ is denoted by I.

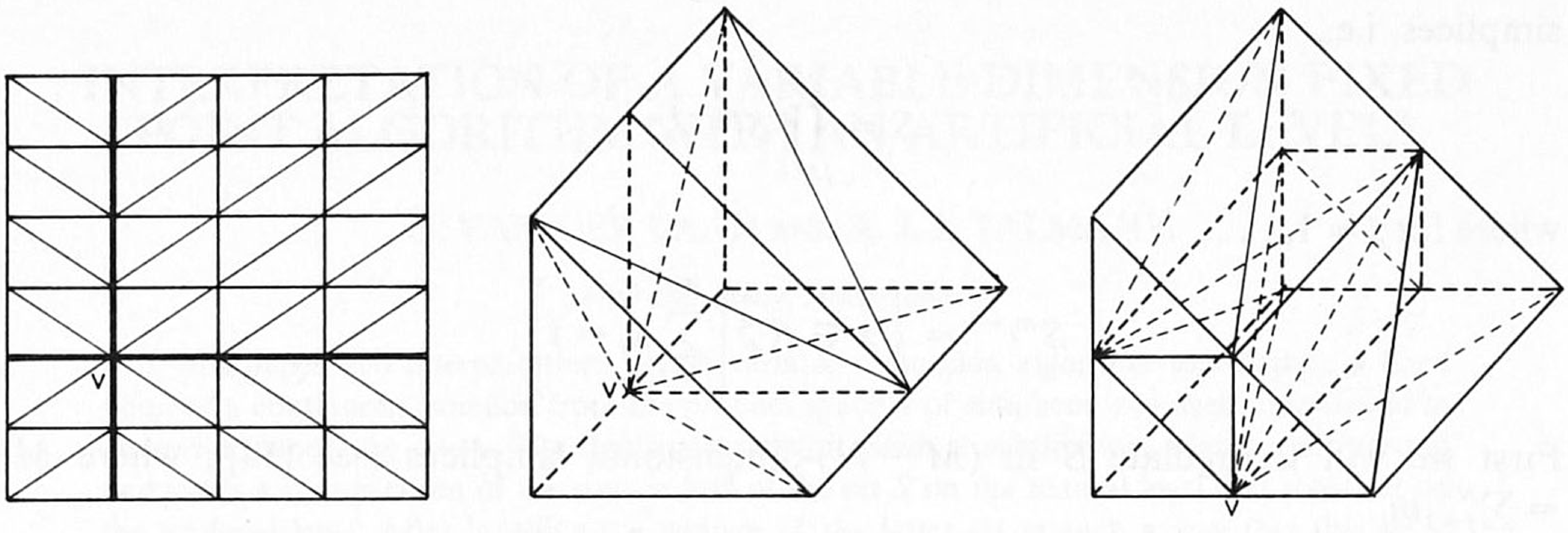
DEFINITION 2.2. A subset $\mathfrak{T}$ of I is feasible if for all j at least one (j,k) is not in $\mathfrak{T}$.

This definition means that an index set $\mathfrak{T}$ is feasible if and only if the matrix with columns $q(j,k)$, $(j,k) \in \mathfrak{T}$, has full rank.

Let v be an arbitrarily chosen gridpoint. Then we define for any feasible subset $\mathfrak{T}$, the region $A(\mathfrak{T})$ as follows.

DEFINITION 2.3.

$$A(\mathfrak{T}) = \left\{ x \in S \mid x = v + \sum_{(j,k) \in \mathfrak{T}} \alpha(j,k) D q(j,k) \right.$$
$$\left. \text{for positive numbers } \alpha(j,k),\ (j,k) \in \mathfrak{T} \right\}.$$



(a) (b) (c)

FIGURE 1. Triangulation of S,
$N = 2,\ m_1 = m_2 = 2,\ d_1 = 4,\ d_2 = 6,\ v = (3/4, 1/4, 2/3, 1/3)^T$,
$N = 2,\ m_1 = 3,\ m_2 = 2,\ d_1 = 1,\ d_2 = 2,\ v = (1, 0, 0, \frac{1}{2}, \frac{1}{2})^T$,
$N = 2,\ m_1 = 3,\ m_2 = 2,\ d_1 = 2,\ d_2 = 1,\ v = (\frac{1}{2}, \frac{1}{2}, 0, 1, 0)^T$.

If $A(\mathfrak{T})$ is nonempty, then $A(\mathfrak{T})$ is t-dimensional if t is the number of indices in $\mathfrak{T}$. Clearly the collection of these regions partitions S. Now, the closure of $A(\mathfrak{T})$, denoted by $\bar{A}(\mathfrak{T})$, is triangulated in t-simplices $\sigma(y^1, \pi(\mathfrak{T}))$ with vertices $y^1, \dots, y^{t+1}$ in S, such that

(i) $y^1 = v + \sum_{(j,k) \in \mathfrak{T}} \beta(j,k) Dq(j,k)$ for nonnegative integers $\beta(j,k)$, $(j,k) \in \mathfrak{T}$, and

(ii) $y^{i+1} = y^i + Dq(\pi_i)$, $i = 1, \dots, t$, where $\pi(\mathfrak{T}) = (\pi_1, \dots, \pi_t)$ is a permutation of the elements of $\mathfrak{T}$.

It can easily be seen that all vertices are grid points and that the collection of these simplices is indeed a triangulation of $\bar{A}(\mathfrak{T})$. Van der Laan and Talman [14] prove that the union of the triangulations (over the feasible $\mathfrak{T}$) is a triangulation of S (see Figure 1).

To generate a simplex of this triangulation which approximates a fixed point of f, a grid point x receives a vector label $l(x)$ in R^M defined by

$$l(x) = -f(x) + x + e,$$

where e denotes the M-vector $(1, \dots, 1)^T$. The vector e is added to guarantee that solutions of the system of equations given in Definition 2.4 below are bounded. Let $e(j,k)$ denote the $(k + \sum_{i=1}^{j-1} m_i)$th unit vector.

DEFINITION 2.4. For feasible $\mathfrak{T}$ a $(t-1)$-simplex $\sigma(w^1, \dots, w^t)$ is called $\mathfrak{T}$-complete if the system of M linear equations

$$\sum_{i=1}^{t} \lambda_i l(w^i) + \sum_{(j,k) \notin \mathfrak{T}} \mu(j,k) e(j,k) = e$$

has a nonnegative solution λ_i^*, $i = 1, \dots, t$, and $\mu^*(j,k)$, $(j,k) \notin \mathfrak{T}$.

In the following we assume nondegeneracy in the sense that the system of equations has always a positive solution. This can be assured by a small perturbation of the right-hand side of the system.

Note that $t \leqslant M - N + 1$. By summing up for $j = 1, \dots, N$ over the equations (j,k), $k = 1, \dots, m_j$, we obtain the following lemma.

LEMMA 2.5. *Let λ_i^*, $i = 1, \dots, t$ and $\mu^*(j,k)$, $(j,k) \notin \mathfrak{T}$ be the solution of the system of linear equations corresponding to a $\mathfrak{T}$-complete simplex.*

Define for $(j,k) \in \mathfrak{T}$, $\mu^*(j,k) = 0$. *Then for* $j = 1, \ldots, N$,

$$m_j \sum_{i=1}^{t} \lambda_i^* + \sum_{k=1}^{m_j} \mu^*(j,k) = m_j .$$

DEFINITION 2.6. An $(s-1)$-simplex $\sigma(w^1, \ldots, w^s)$ is called completely labelled if

$$\sum_{i=1}^{s} \lambda_i l(w^i) = e$$

has a nonnegative solution λ_i^*, $i = 1, \ldots, s$.

From Lemma 2.5 it follows that for a completely labelled simplex $\sum_{i=1}^{s} \lambda_i^* = 1$. As proved in [14], the point $x^* = \sum_{i=1}^{s} \lambda_i^* w^i$ is a good approximation of a fixed point of f.

Also in [14], Van der Laan and Talman present an algorithm to generate a completely labelled simplex. The method starts with an (arbitrarily chosen) zero-dimensional simplex $\{v\}$ and the initial system of linear equations $I\mu = e$. By pivoting, a path of adjacent simplices of variable dimension of the triangulation of S is generated. To be more precise, if $\mathfrak{T}$ is the set of eliminated unit vectors, a path of adjacent t-simplices $\sigma(y^1, \pi(\mathfrak{T}))$ of the triangulation of $\bar{A}(\mathfrak{T})$ is generated, such that the common facets are $\mathfrak{T}$-complete. As soon as a unit vector, say $e(i,h)$ is eliminated, i.e., a $(\mathfrak{T} \cup \{(i,h)\})$-complete simplex $\sigma(y^1, \ldots, y^{t+1})$ in $\bar{A}(\mathfrak{T})$ is found, the dimension of the simplex $\sigma(y^1, \ldots, y^{t+1})$ is increased by adding the vertex $y^{t+2} = y^{t+1} + Dq(i,h)$, the set $\mathfrak{T}$ is extended with the element (i,h) and the algorithm continues in $\bar{A}(\mathfrak{T} \cup \{(i,h)\})$. If a simplex of the path of generated simplices in $\bar{A}(\mathfrak{T})$ has a $\mathfrak{T}$-complete facet in $\bar{A}(\mathfrak{T} \setminus \{(j,k)\})$ for some $(j,k) \in \mathfrak{T}$, then the set $\mathfrak{T}$ becomes $\mathfrak{T} \setminus \{(j,k)\}$ and the dimension of the current simplex $\sigma(y^1, \ldots, y^{t+1})$ is decreased by deleting the vertex y^{t+1}, whereas the algorithm continues in the region $\bar{A}(\mathfrak{T} \setminus \{(j,k)\})$ by reintroducing $e(j,k)$ in the system of linear equations. The algorithm terminates as soon as a completely labelled simplex is found, which will be the case if $\mathfrak{T}$ becomes infeasible. Clearly, then by Lemma 2.5, $\sum \lambda_i^* = 1$. In [14] it is proved that the algorithm terminates within a finite number of iterations with a completely labelled simplex. This simplex can be used to choose a new starting point in a next application of the method for a finer grid to improve the accuracy of the approximation. In the next sections the interpretations of this algorithm with artificial points are all based on the regions $A(\mathfrak{T})$.

We remark that the application for integer labelling is straightforward and is described in full detail in [14], see also [15]. Whereas the vector labelling method terminates in general with full dimensional simplices (we assume always nondegeneracy), the dimension of the integer labelling method's terminal simplex lies between $\min_{j=1,\ldots,N} m_j - 1$ and $M - N$.

3. Geometrical interpretation with a polytope on the artificial level. To give an interpretation of the variable dimension algorithm on S, we subdivide the set $S \times [0,1]$, where zero corresponds to the artificial level and one to the natural level. We will argue below, that doing so, the algorithm can be seen as a method which traces a path of zeroes of $\bar{h}(x,\delta)$, a piecewise linear approximation with respect to the subdivision of $S \times [0,1]$ to a homotopy function $h(x,\delta)$ with

$$h(x,0) = C^{-1}(x - c) \quad \text{and} \quad h(x,1) = x - f(x),$$

where the vector $c \in S$ is defined by $c_{j,k} = m_j^{-1}$ and C is the diagonal matrix with $(k + \sum_{i=1}^{j-1} m_i)$th element equal to $c_{j,k}$, $k = 1, \ldots, m_j$, $j = 1, \ldots, N$.

Let $S \times \{1\}$ be triangulated as defined in the previous section. The set $S \times \{0\}$ consists of one piece, i.e., the $(M-N)$-dimensional artificial level $S \times \{0\}$ is not triangulated. Now $S \times [0,1]$ is subdivided as follows. Let $u(p) \equiv u(p_1, \ldots, p_N)$ be the

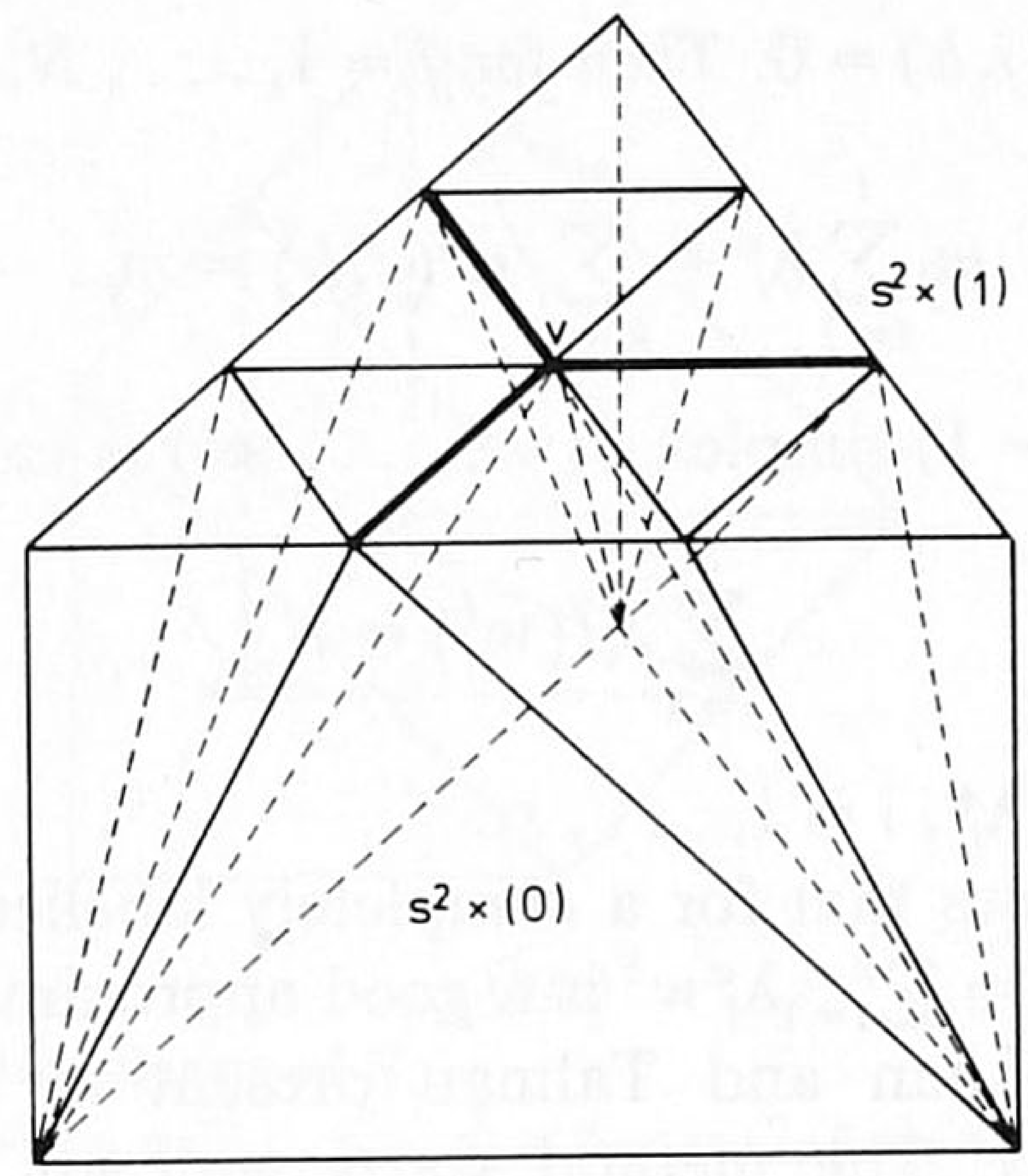


FIGURE 2. Triangulation of $S^2 \times [0, 1]$, $d = 3$, $v = (1/3, 1/3, 1/3)^T$.

vertex of S with

$$u_{j,h} = 1 \quad \text{if} \quad h = p_j, \quad j = 1, \ldots, N$$

and

$$u_{j,h} = 0 \qquad \text{otherwise.}$$

For any feasible $\mathfrak{T}$, a grid point $x \in A(\mathfrak{T})$ on the one-level is connected with all vertices $u(p)$ on the zero-level, such that for all $j = 1, \ldots, N$, $(j, p_j) \notin \mathfrak{T}$. Before we prove that indeed a subdivision is obtained, we give some properties and special cases.

First, we remark that $v \in A(\emptyset)$. Hence, we have that $v \times \{1\}$ is connected with all vertices $u(p)$ on the zero-level, which implies that a $(M - N + 1)$-dimensional polyhedron is formed (see Figures 2 and 3). On the other hand, let $\mathfrak{T}$ be a feasible subset of I, such that for all j, there is a unique h_j with $(j, h_j) \notin \mathfrak{T}$. Then a point $x \in A(\mathfrak{T})$ is only connected with $u(h_1, \ldots, h_N)$. Hence, if $\sigma(w^1, \ldots, w^{M-N+1})$ is a $(M - N)$-simplex in $\bar{A}(\mathfrak{T})$, then all vertices $w^i \times \{1\}$ are connected with $u(h_1, \ldots, h_N)$ on the zero level and a $(M - N + 1)$-dimensional simplex is formed.

We consider now two special cases. The first one is $N = 1$. Then we obtain the *triangulation* of $S^{m_1 - 1} \times [0, 1]$ as described by Van der Laan and Talman [9] and [13], Todd [16], Bárány [1] and Van der Laan [8] (see Figure 2). In the case that $m_j = 2$, $j = 1, \ldots, N$, S is the product of N 1-dimensional simplices and is the analog of the N-dimensional unit cube $C^N = \{x \in R^N \mid 0 \leqslant x_i \leqslant 1\}$. For this special case the subdivision of $S \times [0, 1]$ described above is equal to that of $C^N \times [0, 1]$ given by Todd [17], Todd and Wright [18] and Bárány [2]. In Figure 3 this case is illustrated for $N = 2$. Observe that the point v is connected with 4 points on the zero-level, a point $x \in A(\mathfrak{T})$ with $|\mathfrak{T}| = 1$ with 2, and a point $x \in A(\mathfrak{T})$ with $|\mathfrak{T}| = 2$ with 1 point. Returning to the

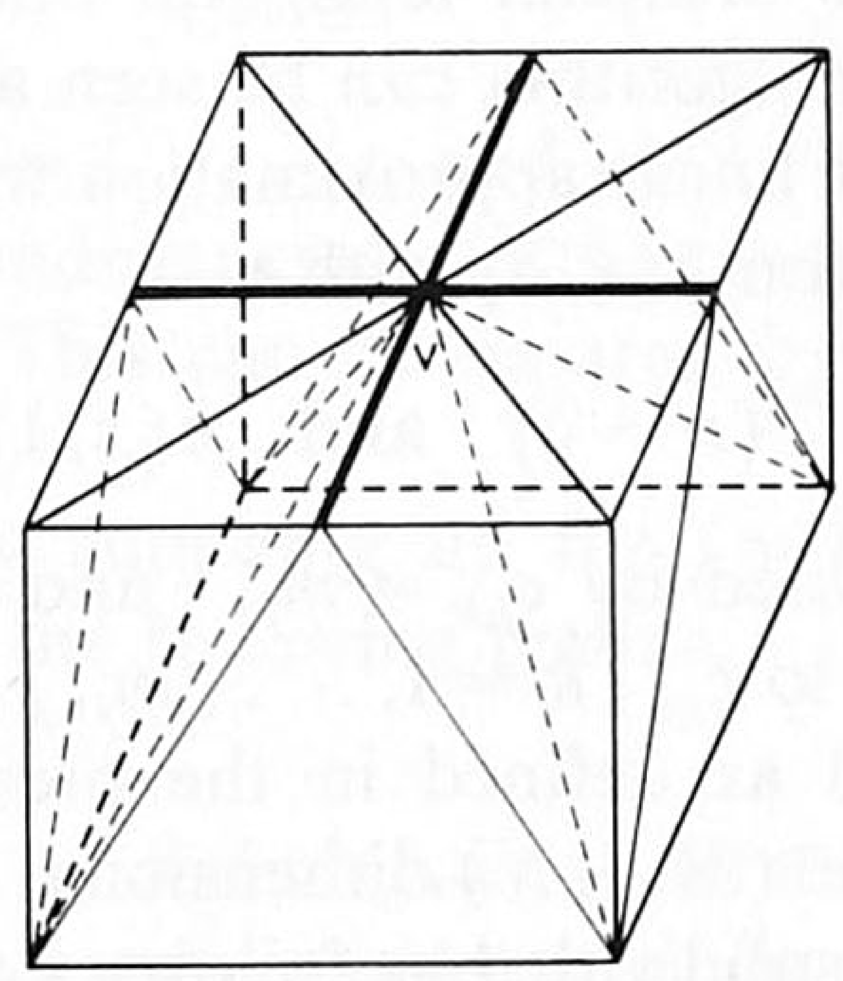


FIGURE 3. Triangulation of $S \times [0, 1]$ with $N = 2$, $m_1 = m_2 = 2$, $d_1 = d_2 = 2$, $v = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$.

general case we have that if $x \in A(\mathfrak{T})$ for some $\mathfrak{T}$ with k_j elements (j,h), $j = 1, \ldots, N$ $(0 \leq k_j \leq m_j - 1)$, then x is connected with $\Pi_j(m_j - k_j)$ points on the zero level.

In the remainder of this section we use the following notation. The set L is the collection of polyhedra of $S \times [0,1]$. A point (y,δ) denotes an element of $S \times [0,1]$ and for any feasible $\mathfrak{T}$, $E(\mathfrak{T})$ denotes the set of vertices $u(p_1, \ldots, p_N) \times \{0\}$ such that for $j = 1, \ldots, N$, $(j, p_j) \notin \mathfrak{T}$, i.e., $E(\mathfrak{T})$ is the set of vertices on the zero level with which a point $x \in A(\mathfrak{T})$ on the one-level is connected. Observe that $\bar{\mathfrak{T}} \subset \mathfrak{T}$ implies $E(\mathfrak{T}) \subset E(\bar{\mathfrak{T}})$. Furthermore we denote by $H(\mathfrak{T})$ the set of N-dimensional vectors p such that $(u(p),0) \in E(\mathfrak{T})$.

Finally, for a simplex σ in $\bar{A}(\mathfrak{T})$, $P(\sigma)$ denotes the polyhedron defined by the convex hull of the vertices of σ on the one-level and the elements of $E(\mathfrak{T})$.

To prove that L indeed subdivides $S \times [0,1]$ we have to show that each point of S lies in an element of L. To find this element is a rather technical task. To illustrate the proof below, take some element $(y,\delta) \in S \times [0,1]$ in Figure 2 or 3. Clearly, if $\delta = 0$, we have that (y,δ) belongs to the polyhedron defined by the convex hull of the point $(v,1)$ and all vertices $(u(p),0)$. If $\delta = 1$, let $\mathfrak{T}$ be the unique subset of I such that $y \in A(\mathfrak{T})$. Then $(y,1)$ belongs to a polyhedron defined by the convex hull of a t-simplex of $\bar{A}(\mathfrak{T})$ which contains y and the elements of $E(\mathfrak{T})$. More generally, if $y \in A(\mathfrak{T})$ we have that (y,δ) belongs to a polyhedron formed by the convex hull of a simplex in $\bar{A}(\bar{\mathfrak{T}}) \times \{1\}$ and the elements of $E(\bar{\mathfrak{T}})$, where $\bar{\mathfrak{T}} = \emptyset$ if $\delta = 0$, $\bar{\mathfrak{T}} = \mathfrak{T}$ if $\delta = 1$ and $\bar{\mathfrak{T}} \subset \mathfrak{T}$ if $0 < \delta < 1$. Actually, the number of elements of $\bar{\mathfrak{T}}$ is a monotone nondecreasing function of δ.

LEMMA 3.1. *The set L of polyhedra forms a subdivision of $S \times [0,1]$.*

PROOF. It is sufficient to prove the conditions (i)–(iii) of Definition 2.1. Since the total number of faces is finite it remains to prove (i) and (ii). To prove (i) let $(y,\delta^*) \in S \times [0,1]$. For $\delta^* = 0$ or 1 the proof follows immediately from the arguments given above. So, let $\delta^* \in (0,1)$. Firstly, let $\mathfrak{T}^*$ be the feasible set such that $y \in A(\mathfrak{T}^*)$. Since the regions $A(\mathfrak{T})$ partition S, this set is unique. By definition there are unique positive numbers $\bar{\alpha}_{j,k}$, $(j,k) \in \mathfrak{T}^*$, such that

$$y - v = \sum_{(j,k) \in \mathfrak{T}^*} \bar{\alpha}_{j,k} q(j,k).$$

For $\delta \in [0,1]$, let $\alpha_{j,k}(\delta)$ and $\beta_{j,k}(\delta)$ be determined as follows. For $j = 1, \ldots, N$, let (j,k_j) be an index not in $\mathfrak{T}^*$. Then, for $h = k_j + 1$ (with $k_j + 1 = 1$ if $k_j = m_j$),

$$\alpha_{j,h}(\delta) = \max(0, -y_{j,h} + \delta v_{j,h}) \quad \text{and} \quad \beta_{j,h}(\delta) = \max(0, y_{j,h} - \delta v_{j,h})$$

and for $h = k_j + 2, \ldots, m_j, 1, \ldots, k_j$,

$$\alpha_{j,h}(\delta) = \max(0, -y_{j,h} + \delta v_{j,h} + \alpha_{j,h-1}(\delta)),$$

$$\beta_{j,h}(\delta) = \max(0, y_{j,h} - \delta v_{j,h} - \alpha_{j,h-1}(\delta)),$$

where $h - 1 = m_j$ if $h = 1$. Then we have that

$$y - \delta v = \sum_{(j,k)} \alpha_{j,k}(\delta) q(j,k) + \sum_{(j,k)} \beta_{j,k}(\delta) e(j,k).$$

By definition of $A(\mathfrak{T})$ we have that $\beta_{j,k}(1) = 0$; $\alpha_{j,k}(1) = \bar{\alpha}_{j,k}$, $(j,k) \in \mathfrak{T}^*$ and $\alpha_{j,k}(1) = 0$, $(j,k) \notin \mathfrak{T}^*$. Observe that $\alpha_{j,k}(\delta)$ is nondecreasing in δ. Hence $\alpha_{j,k_j}(\delta^*) = 0$ since $(j,k_j) \notin \mathfrak{T}^*$, $j \in I_N$. Moreover, for all (j,k), we have that $\alpha_{j,k}(\delta)\beta_{j,k}(\delta) = 0$.

Consequently, there exists a feasible set $\mathfrak{T} \subset \mathfrak{T}^*$ such that

$$y - \delta^* v = \sum_{(j,k) \in \mathfrak{T}} \alpha_{j,k}(\delta^*) q(j,k) + \sum_{(j,k) \notin \mathfrak{T}} \beta_{j,k}(\delta^*) e(j,k). \tag{3.1}$$

Let $\bar{\mathfrak{T}}$ be the subset of $\mathfrak{T}$ such that $\alpha_{j,k}(\delta^*) > 0$ for all $(j,k) \in \bar{\mathfrak{T}}$. Then it follows from (3.1) that

$$y = \delta^* x + (1-\delta^*)z \tag{3.2}$$

where

$$x = v + \delta^{*-1} \sum_{(j,k)\in\bar{\mathfrak{T}}} \alpha_{j,k}(\delta^*) q(j,k)$$

and

$$z = (1-\delta^*)^{-1} \sum_{(j,k)\notin\bar{\mathfrak{T}}} \beta_{j,k}(\delta^*) e(j,k).$$

We will prove now that $x \in A(\bar{\mathfrak{T}})$ and z is a convex combination of the vertices $u(p)$ with $(u(p),0) \in E(\bar{\mathfrak{T}})$. By the structure of Q we get after adding the equations (j,k), $k = 1, \ldots, m_j$ in the system (3.1) that for fixed j

$$\beta_j = \sum_{k:(j,k)\notin\bar{\mathfrak{T}}} \beta_{j,k}(\delta^*) = \sum_{k=1}^{m_j} (y - \delta^* v)_{j,k} = (1-\delta^*), \qquad j = 1, \ldots, N,$$

which implies that z is in S. Since $z_{j,k} = 0$ for $j,k \notin \bar{\mathfrak{T}}$, z can be expressed as a linear combination of the extreme points $u(p)$, $p \in H(\bar{\mathfrak{T}})$. To prove that $x \in A(\bar{\mathfrak{T}})$, suppose the contrary. Then, since $\alpha_{i,h}(\delta^*) > 0$ for all $(i,h) \in \bar{\mathfrak{T}}$ we must have that $x_{i,h} < 0$ for some $(i,h) \in \bar{\mathfrak{T}}$. Moreover we have that $e_{i,h}(j,k) = 0$ for all $(j,k) \notin \bar{\mathfrak{T}}$. Consequently $z_{i,h} = 0$ and hence by (3.2), $y_{i,h} < 0$ which contradicts the fact that $(y,\delta) \in S \times [0,1]$. So, (3.2) implies that (y,δ^*) is a convex combination of $(x,1) \in A(\bar{\mathfrak{T}}) \times \{1\}$ and $(z,0)$, which is a convex combination of the elements of $E(\bar{\mathfrak{T}})$. Now, let σ be a simplex in $\bar{A}(\bar{\mathfrak{T}})$ which contains x. Then $(y,\delta^*) \in P(\sigma)$ which proves (i).

To prove (ii), suppose P_1 and P_2 are two polyhedra of L both containing a point (y,δ^*). We will show that (y,δ^*) belongs to a common face of P_1 and P_2. Let σ_i, $i = 1,2$, be the corresponding simplex on the 1-level, i.e., $\sigma_i \times \{1\} = P_i \cap (S \times \{1\})$ and let $E(\mathfrak{T}^i)$, $i = 1,2$, be the set of vertices of P_i on the zero level. If $\delta^* = 1$, we have that $(y,1) \in (\sigma_1 \cap \sigma_2) \times \{1\}$ which is a common face of P_1 and P_2. If $\delta^* = 0$, then $(y,0)$ is a convex combination of the elements of $E(\mathfrak{T}^i)$ for both $i = 1$ and 2 and hence $(y,0)$ lies in the convex hull of the elements of $E(\mathfrak{T}^1) \cap E(\mathfrak{T}^2)$. Since $E(\mathfrak{T}^1) \cap E(\mathfrak{T}^2) \subset E(\mathfrak{T}^i)$, $i = 1,2$, this convex hull is a common face of P_1 and P_2. For $\delta^* \in (0,1)$, let $\tilde{\mathfrak{T}}$ be the largest feasible set of indices such that (3.1) holds, i.e., $\beta_{j,k}(\delta^*) > 0$ for all $(j,k) \notin \tilde{\mathfrak{T}}$. Now, (y,δ^*) lies in the convex hull $\bar{P}$ of the vertices of $(\sigma_1 \cap \sigma_2) \times \{1\}$ and the elements of $E(\tilde{\mathfrak{T}})$. Since $(y,\delta^*) \in P_i$, (3.1) also holds for $\mathfrak{T} = \mathfrak{T}^i$. So $E(\tilde{\mathfrak{T}}) \subset E(\mathfrak{T}^i)$, $i = 1,2$. Hence the convex hull of the elements of $E(\tilde{\mathfrak{T}})$ is a common face of P_1 and P_2. The same holds for $(\sigma_1 \cap \sigma_2) \cap \{1\}$. So, $\bar{P}$ is a common face of P_1 and P_2. Q.E.D.

To conclude this section we show that the path of t-simplices generated by the variable dimension algorithm described in §2 corresponds to a path of adjacent polyhedra of L such that each polyhedron contains a line segment of zeroes of the piecewise linear approximation $\bar{h}(x,\delta)$ to $h(x,\delta)$ defined above. Observe that, although $S \times \{0\}$ is not triangulated, $\bar{h}(x,\delta)$ is well defined by the fact that $h(x,\delta)$ is linear on $S \times \{0\}$.

We label now each vertex (y,δ) of the subdivision of $S \times [0,1]$ by

$$\tilde{l}(y,\delta) = \bar{h}(y,\delta) + e.$$

Observe that for each vertex of the subdivision holds $\delta = 0$ or 1 and that the labelling rule results in

$$\tilde{l}(y,1) = \bar{h}(y,1) + e = y - f(y) + e,$$

for a grid point y on the one-level, which is consistent with the rule given in §2, and

$$\tilde{l}(y,0) = h(y,0) + e = C^{-1}(y-c) + e = C^{-1}y.$$

In particular this means that

$$\tilde{l}(u(p),0) = C^{-1}u(p), \quad \text{i.e.,}$$

$$\left.\begin{aligned} \tilde{l}_{j,k}(u(p),0) &= m_j \quad \text{if} \quad k = p_j \\ &= 0 \quad \text{otherwise} \end{aligned}\right\}, \quad j = 1, \ldots, N.$$

We show that for a $(t-1)$-facet $\tau(w^1, \ldots, w^t)$ of a simplex σ in $\bar{A}(\mathfrak{T})$ the system of linear equations

$$\sum_{i=1}^{t} \lambda_i l(w^i) + \sum_{(j,k) \notin \mathfrak{T}} \mu(j,k) e(j,k) = e \tag{3.4}$$

has a feasible solution if and only if

$$\sum_{i=1}^{t} \bar{\lambda}_i \tilde{l}(w^i, 1) + \sum_{p \in H(\mathfrak{T})} \rho(p) \tilde{l}(u(p),0) = e \tag{3.5}$$

has one. Recall that if (3.4) has a feasible solution, $\tau(w^1, \ldots, w^t)$ is $\mathfrak{T}$-complete. Moreover, the vertices in (3.5) are the vertices of $P(\tau)$, being the $(M-N)$-dimensional face of $P(\sigma)$ with vertices $(w^i, 1)$, $i = 1, \ldots, t$, and the elements of $E(\mathfrak{T})$. Assume that the system (3.5) has a feasible solution $\bar{\lambda}_i^*$, $i = 1, \ldots, t$, and $\rho^*(p)$, $(u(p), 0) \in E(\mathfrak{T})$. Set $\lambda_i^* = \bar{\lambda}_i^*$, $i = 1, \ldots, t$, and $\mu^*(j,k) = m_j \sum \rho^*(p)$, where the sum is over $p \in H(\mathfrak{T})$ such that $p_j = k$. Clearly λ_i^*, $i = 1, \ldots, t$ and $\mu^*(j,k)$, $(j,k) \notin \mathfrak{T}$ is a feasible solution of (3.4). If, on the other hand, the system (3.4) has a feasible solution λ_i^* $i = 1, \ldots, t$, $\mu^*(j,k)$, $(j,k) \notin \mathfrak{T}$, then all elements $(\bar{\lambda}^*, \rho^*)$ of the set

$$\left\{ (\bar{\lambda}, \rho) \,\middle|\, \bar{\lambda}_i = \lambda_i^*, i = 1, \ldots, t, \text{ and } \sum_{\substack{p \in H(\mathfrak{T}) \\ p_j = k}} \rho(p) = \mu^*(j,k)/m_j, (j,k) \notin \mathfrak{T} \right\} \tag{3.6}$$

are solutions of (3.5). From Lemma 2.5 it follows that this set is nonempty. For instance we can take

$$\rho^*(p) = (1 - \delta^*)^{1-N} \prod_{j=1}^{N} (\mu^*(j,k)/m_j),$$

where $\delta^* = \sum_{i=1}^{t} \lambda_i^*$.

Consequently, we have that a $(t-1)$-facet $\tau(w^1, \ldots, w^t)$ of a simplex σ in $\bar{A}(\mathfrak{T})$ is $\mathfrak{T}$-complete, if and only if the $(M-N)$-dimensional facet $P(\tau)$ of $P(\sigma)$ contains a zero point of $\bar{h}(x, \delta)$, namely the point (y^*, δ^*), where

$$y^* = \sum_{i=1}^{t} \bar{\lambda}_i^* w^i + \sum_{p \in H(\mathfrak{T})} \rho^*(p) u(p).$$

Note that $\sum_{i=1}^{t} \bar{\lambda}_i^* + \sum_{p \in H(\mathfrak{T})} \rho^*(p) = 1$, which follows from (3.5) by adding all equations.

Since the solution of the system of linear equations (3.4) corresponding to $\tau(w^1, \ldots, w^t)$ is unique, it follows from (3.6) that (y^*, δ^*) is unique. Consequently the algorithm can be interpreted as a method which generates a path of adjacent polyhedra of L starting with the polyhedron $P(v) = \text{conv}(S \times \{0\} \times \{(v,1)\})$. Such a polyhedron is a simplex if just one element (j,k) is not in $\mathcal{T}$ for at least $N-1$ indices j. The facet $S \times \{0\}$ of $P(v)$ which is on the boundary of $S \times [0,1]$ contains a zero point of $\bar{h}(x,\delta)$ viz. $(c,0)$ and the common facets of the sequence of polyhedra contain all a zero point of $\bar{h}(x,\delta)$ obtained from the solution of (3.5). So the method traces a path of zeroes of $\bar{h}(x,\delta)$ and terminates as soon as a zero point on the one level is found which is the case if and only if all $\rho^*(p)$, $p \in H(\mathcal{T})$, are zero and hence $\sum \lambda_i^* = 1$. The intersection of the path of generated polyhedra with $S \times \{1\}$ is the path of simplices of the variable dimension algorithm described in §2.

4. Interpretation with *M* points on the zero level. In the previous section we have subdivided $S \times [0,1]$, in which case we obtained a subdivision in $(M-N+1)$-dimensional polyhedra (and simplices). In this section we interpret the algorithm described in §2 as an algorithm which generates a path of M-dimensional simplices. To do so we triangulate the convex hull of the $(M-1)$-dimensional unit simplex S^{M-1} on the zero-level and the set S on the one-level, i.e., we triangulate the set

$$\tilde{S} = \operatorname{conv}\big((S^{M-1} \times \{0\}) \cup (S \times \{1\})\big).$$

So, instead of the $(M-N)$-dimensional set S (having $\prod_{j=1}^{N} m_j$ vertices) we have now on the zero level the $(M-1)$-dimensional set S^{M-1} having M vertices. Clearly, if $N=1$ we have that $S = \prod_{j=1}^{N} S^{m_j-1} = S^{M-1}$, which implies that in this case the two interpretations are identical.

Now, we define G as the set of M-simplices which are obtained by connecting a grid point $(w,1)$ in $A(\mathcal{T}) \times \{1\}$ with all the vertices $(e(j,k),0)$ of $S^{M-1} \times \{0\}$ such that $(j,k) \notin \mathcal{T}$. So, in particular we have that the zero-dimensional simplex $(v,1)$ of $A(\emptyset) \times \{1\}$ is connected with all the vertices of $S^{M-1} \times \{0\}$, whereas the vertices of a t-simplex in $\bar{A}(\mathcal{T})$ for some feasible $\mathcal{T}$ with $|\mathcal{T}| = t = M-N$ are connected with the N vertices $e(i,h)$ on the zero level where for all $i = 1, \ldots, N$, (i,h) is the unique index not in $\mathcal{T}$.

LEMMA 4.1. *The set G of M-simplices triangulates $\tilde{S}$.*

PROOF. To show that each point of $\tilde{S}$ lies in at least one simplex of G, let (y,δ) be an arbitrarily chosen point in $\tilde{S}$. If $\delta = 0$ it is immediately clear that (y,δ) lies in the simplex which is the convex hull of $S^{M-1} \times \{0\}$ and $\{(v,1)\}$. If $\delta = 1$, then $y \in S$ and there is a unique feasible $\mathcal{T}$ such that $y \in A(\mathcal{T})$. Letting σ be some simplex of $\bar{A}(\mathcal{T})$ containing y, then $(y,1)$ lies in the simplex of G being the convex hull of $\sigma \times \{1\}$ and the vertices $(e(j,k),0)$ for $(j,k) \notin \mathcal{T}$. So, it remains to consider the case that $0 < \delta < 1$. Since $(y,\delta) \in \tilde{S}$, there are points $x \in S$ and $z \in S^{M-1}$ such that

$$y = \delta x + (1-\delta)z$$

and so

$$y - \delta v = \delta(x-v) + (1-\delta)z.$$

Since x and v belong to S, we have

$$\sum_{k=1}^{m_j} (y - \delta v)_{j,k} = (1-\delta)\sum_{k=1}^{m_j} z_{j,k} \geqslant 0, \qquad j = 1, \ldots, N. \tag{4.1}$$

So, using the special structure of the matrix Q, we obtain that there is at least one feasible set $\mathcal{T}$ of I such that

$$y - \delta v = \sum_{(i,h)\in\mathcal{T}} \alpha_{i,h}^* q(i,h) + \sum_{(j,k)\notin\mathcal{T}} \beta_{j,k}^* e(j,k) \tag{4.2}$$

for unique nonnegative $\alpha^*_{i,h}$, $(i,h) \in \mathfrak{T}$, and $\beta^*_{j,k}$, $(j,k) \notin \mathfrak{T}$. Consequently,

$$y = \sum_{(j,k) \notin \mathfrak{T}} \beta^*_{j,k} e(j,k) + \delta \bar{y} \tag{4.3}$$

where $\bar{y} = v + \sum_{(i,h) \in \mathfrak{T}} \delta^{-1} \alpha^*_{i,h} q(i,h)$.

As in the proof of Lemma 3.1, we can show that $\bar{y} \in A(\bar{\mathfrak{T}})$ where

$$\bar{\mathfrak{T}} = \{(i,h) \in \mathfrak{T} \mid \alpha^*_{i,h} > 0\}.$$

In fact, supposing that $\bar{y} \notin A(\bar{\mathfrak{T}})$ we must have $\bar{y}_{i,h} < 0$ for some $(i,h) \in \bar{\mathfrak{T}}$ and hence by (4.3) $y_{i,h} < 0$, which contradicts that $y \in \tilde{S}$.

Now define $\beta^*_{j,k} = 0$ for $(j,k) \in \mathfrak{T} \setminus \bar{\mathfrak{T}}$. From (4.1) and (4.2) we now obtain that

$$\sum_{(j,k) \notin \bar{\mathfrak{T}}} \beta^*_{j,k} = \sum_{j=1}^{N} \sum_{k=1}^{m_j} (y - \delta v)_{j,k} = (1-\delta) \sum_{j=1}^{N} \sum_{k=1}^{m_j} z_{j,k} = 1 - \delta,$$

since $z \in S^{M-1}$. Hence

$$\bar{z} = (1-\delta)^{-1} \sum_{(j,k) \notin \bar{\mathfrak{T}}} \beta^*_{j,k} e(j,k)$$

is a point in S^{M-1}. Since by (4.3) $y = (1-\delta)\bar{z} + \delta\bar{y}$ we have that (y, δ) is a convex combination of the point $(\bar{y}, 1)$ in $A(\bar{\mathfrak{T}}) \times \{1\}$ and $(\bar{z}, 0)$ in $S^{M-1} \times \{0\}$. Hence (y, δ) lies in the simplex of G, whose vertices are $(e(j,k), 0)$ for $(j,k) \notin \bar{\mathfrak{T}}$ and the vertices of $\sigma \times \{1\}$, where σ is some simplex of $\bar{A}(\bar{\mathfrak{T}})$ containing $\bar{y}$.

This proves (i) of Definition 2.1. To prove (ii), let σ_1 and σ_2 be two M-simplices of G both containing some point (y, δ) and let τ_1 and τ_2 be the faces of σ_1 and σ_2 lying on the one-level, i.e., $\tau_i = \sigma_i \cap (S \times \{1\})$, $i = 1,2$.

Moreover, let $\mathfrak{T}^i$, $i = 1,2$ be such that $(e(j,k), 0)$, $(j,k) \notin \mathfrak{T}^i$ are the vertices of σ_i on the zero level. Then (4.2) holds for both $\mathfrak{T}^1$ and $\mathfrak{T}^2$. Let $\mathfrak{T}^*$ be the largest feasible set of indices such that (4.2) holds, i.e., $\beta^*_{j,k} > 0$ for all $(j,k) \notin \mathfrak{T}^*$. Now (y, δ) is in the convex hull $\bar{\sigma}$ of the vertices of $\tau_1 \cap \tau_2$ and the vertices $(e(j,k), 0)$, $(j,k) \notin \mathfrak{T}^*$. Since $\mathfrak{T}^i \subset \mathfrak{T}^*$, $i = 1,2$, the convex hull of $(e(j,k), 0)$, $(j,k) \notin \mathfrak{T}^*$ is a face of both σ_1 and σ_2. Clearly, also $\tau_1 \cap \tau_2$ is common face of σ_1 and σ_2. Hence $\bar{\sigma}$ is a common face of σ_1 and σ_2.

Finally (iii) of Definition 2.1 follows immediately from the fact that the total number of faces is finite. Q.E.D.

The variable dimension algorithm described in §2 can now be interpreted as tracing zeroes of the piecewise linear approximation $\bar{h}(x, \delta)$ with respect to the triangulation G to a homotopy function $h(x, \delta)$ which satisfies

$$h(x,1) = x - f(x) \quad \text{and} \quad h(x,0) = Mx - e.$$

Labelling each vertex (y, δ) of $\tilde{S}$ by $\tilde{l}(y, \delta) = \bar{h}(y, \delta) + e$ we obtain that for a grid point $(y, 1)$,

$$\tilde{l}(y,1) = y - f(y) + e,$$

which is consistent with the labelling l of §2, and that

$$\tilde{l}(y,0) = My,$$

which implies that $\tilde{l}(e(j,k), 0) = Me(j,k)$ for all $(j,k) \in I$. Let $\tau(w^1, \ldots, w^t)$ be a $(t-1)$-facet of a simplex $\sigma(w^1, \ldots, w^{t+1})$ in $\bar{A}(\mathfrak{T})$ and let R be the $(M-1)$-facet of the M-simplex $P = \operatorname{conv}(\sigma \times \{0\} \cup \{(e(j,k), 0) \mid (j,k) \notin \mathfrak{T}\})$ of G opposite to w^{t+1}.

Clearly, τ is $\mathcal{T}$-complete if and only if the system of linear equations corresponding to R,

$$\sum_{i=1}^{t} \bar{\lambda}_i \bar{l}(w^i) + \sum_{(j,k)\notin\mathcal{T}} \bar{\mu}(j,k) M e(j,k) = e, \tag{4.4}$$

has a feasible solution $\bar{\lambda}_i^*$, $i = 1, \ldots, t$, $\bar{\mu}^*(j,k)$, $(j,k) \notin \mathcal{T}$. To be more precise, τ is $\mathcal{T}$-complete with solution λ_i^*, $i = 1, \ldots, t$, $\mu^*(j,k)$, $(j,k) \notin \mathcal{T}$ iff $\bar{\lambda}_i^* = \lambda_i^*$, $i = 1, \ldots, t$, and $\bar{\mu}^*(j,k) = M^{-1}\mu^*(j,k)$ is a solution of (4.4). From Lemma 2.5 it follows that

$$\sum_{i=1}^{t} \bar{\lambda}_i^* + \sum_{(j,k)\notin\mathcal{T}} \bar{\mu}^*(j,k) = 1.$$

Consequently, if $\tau(w^1, \ldots, w^t)$ is $\mathcal{T}$-complete then R contains a zero of $\bar{h}(x,\delta)$, namely the point (x^*, δ^*), where

$$x^* = \sum_{i=1}^{t} \bar{\lambda}_i^* w^i + \sum_{(j,k)\notin\mathcal{T}} \bar{\mu}^*(j,k) e(j,k) \quad \text{and} \quad \delta^* = \sum_{i=1}^{t} \bar{\lambda}_i^*.$$

Observe that $(M^{-1}e, 0)$ is a zero of $\bar{h}(x,\delta)$ in $S^{M-1} \times \{0\}$. So, starting with the simplex of G, being the conv$(S^{M-1} \times \{0\} \cup \{(v,1)\})$, the method generates a path of M-simplices of G such that common facet of two adjacent generated simplices contains a zero point of $\bar{h}(x,\delta)$. The method terminates as soon as for some $\mathcal{T}$, all $\bar{\mu}^*(j,k)$, $(j,k) \notin \mathcal{T}$, become zero, i.e., as soon as for some $\mathcal{T}$ an M-simplex R is generated having a zero of $\bar{h}(x,\delta)$ in the corresponding t-face $\sigma(w^1, \ldots, w^{t+1}) = R \cap (\bar{A}(\mathcal{T}) \times \{1\})$ of R on the one level.

Assuming nondegeneracy $t = M - N$. Observe, however, that the solution on the one level is degenerated in the sense that the weights associated with the unit vectors at the artificial level become simultaneously zero. The intersection of the path of generated simplices of G with the one-level is the path of simplices of the variable dimension algorithm discussed in §2.

5. Modifications for R^n and some concluding remarks. A class of restart algorithms without an artificial level to compute a fixed point of a function f on the n-dimensional real space R^n has been presented by the authors in [12]. Before setting the algorithms in the framework of the previous sections we have to summarize the main ideas of that paper.

Let $m_1, \ldots, m_N$ be N positive integers such that $\sum_{j=1}^{N} m_j = n$, and write R^n as $R^n = \prod_{j=1}^{N} R^{m_j}$. Let Q be the n by $n + N$ matrix defined by

$$Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & Q_N \end{bmatrix},$$

where Q_j is an m_j by $(m_j + 1)$ triangulation matrix of R^{m_j} (see [11]). Let I be the set of indices (j,k), $k = 1, \ldots, m_j + 1$, $j = 1, \ldots, N$ and let the vector $q(j,k)$ denote the $(k + \sum_{i=1}^{j-1}(m_i + 1))$th column of Q, $k = 1, \ldots, m_j + 1$, $j = 1, \ldots, N$. If $x \in R^n$ the element $x_{j,k}$ denotes the $(k + \sum_{i=1}^{j-1} m_i)$th component of x. If $x \in R^{n+N}$ the element $x_{j,k}$ denotes the $(k + \sum_{i=1}^{j-1}(m_i + 1))$th component of x. We redefine for each feasible subset $\mathcal{T}$ of I the open regions $A(\mathcal{T})$ by

$$A(\mathcal{T}) = \left\{ x \in R^n \mid x = v + \sum_{(j,k)\in\mathcal{T}} \alpha(j,k) q(j,k) \text{ for positive } \alpha(j,k),\ (j,k) \in \mathcal{T} \right\}.$$

The closure $\bar{A}(\mathfrak{T})$ is triangulated by simplices $\sigma(y^1, \pi(\mathfrak{T}))$ as done in §2 with Q as defined above. To compute a fixed point of a function f from R^n into itself we define a function g from R^n to R^{n+N} by

$$\begin{aligned} g_{j,k}(x) &= f_{j,k}(x) - x_{j,k} \quad \text{if} \quad k = 1, \ldots, m_j, \\ &= \sum_{k=1}^{m_j} -g_{j,k}(x) \quad \text{if} \quad k = m_j + 1, \end{aligned} \qquad j = 1, \ldots, N.$$

Each grid point $x \in R^n$ receives a vector label $l(x) \in R^{n+N}$ defined by

$$l(x) = g(x) + e.$$

A $\mathfrak{T}$-complete simplex $\sigma(w^1, \ldots, w^t)$ and a completely labelled simplex are now defined as in Definitions 2.4 and 2.6 with $M = m + N$.

Again a completely labelled simplex $\sigma(w^1, \ldots, w^t)$ yields an approximate fixed point $\sum_{i=1}^{t} \lambda_i^* w^i$, where λ_i^*, $i = 1, \ldots, t$, is the solution of the system of linear equations (see [12]). To find such a completely labelled simplex the algorithm proceeds as described in §2 starting with the zero-dimensional simplex $\{v\}$ of $A(\emptyset)$ and the system of $n + N$ linear equations $I\mu = e$.

To interpret the algorithm as a method tracing zeroes of a piecewise linear approximation to a homotopy function we have again to add an extra level of points. This extra level will consist of a polytope and hence the extra level is bounded.

For the first interpretation, the analog of §3, we take on the zero level a polytope Z, being the product of N m_j-dimensional simplices Z_j, $j = 1, \ldots, N$, i.e., $Z = \prod_{j=1}^{N} Z_j$. Clearly Z has $\prod_{j=1}^{N}(m_j + 1)$ vertices. The simplices have to be defined in such a way that there is just one vertex, say $z(p) = z(p_1, \ldots, p_N)$, in $A(p_1, \ldots, p_N) \times \{0\}$ where $A(p_1, \ldots, p_N) = A(\mathfrak{T})$ with $\mathfrak{T} = I \backslash \{(j, p_j) | j = 1, \ldots, N\}$. Following the lines of §3, a subdivision into $(n + 1)$-dimensional polyhedra of the set

$$\bar{Z} = \operatorname{conv}(Z \times \{0\} \cup R^n \times \{1\})$$

is obtained by connecting a grid point $x \in A(\mathfrak{T})$ on the one-level with all the vertices $z(p_1, \ldots, p_N)$ such that $(j, p_j) \notin \mathfrak{T}$, $j = 1, \ldots, N$.

Observe that the subdivision is not locally finite as is required by many authors (see Eaves [3], Bárány [1]). Locally finiteness, i.e., each point x has a neighborhood meeting only a finite number of polyhedra, is to guarantee that every compact subset is covered by a finite number of relative interiors of faces (see Definition 2.1 (iii)). This is indeed the case for the subdivision presented above. By choosing an appropriate homotopy function $h(x, \delta)$ on $\bar{Z}$ we can prove that, as in §3, the algorithm traces a path of zeroes of $\bar{h}(x, \delta)$, the piecewise linear approximation to $h(x, \delta)$.

We obtain the analog of §4 if we take on the zero-level the $(n + N - 1)$-dimensional set $-S^{n+N-1}$, i.e., the "negative" unit simplex. Then $\operatorname{conv}(R^n \times \{1\} \cup -S^{n+N-1} \times \{0\})$ is triangulated in $(n + N)$-dimensional simplices by connecting a grid point $(x, 1)$ in $A(\mathfrak{T}) \times \{1\}$ with all the vertices $(-e(i, h), 0)$, $(i, h) \notin \mathfrak{T}$. Again the algorithm can be interpreted as tracing zeroes of the piecewise linear approximation $\bar{h}(x, \delta)$ to a homotopy function $h(x, \delta)$ with $h(x, 1) = g(x)$ and $h(x, 0) = -(e + (n + N)x)$.

The class of algorithms on R^n discussed above has two extreme cases. The first one, $N = 1$, is the "standard" variable dimension algorithm on R^n (see [10] and [16]). In this case the two interpretations are again identical and can be found already in Van der Laan and Talman [13], Todd [10] and Bárány [1]. For the other extreme case, $N = n$, $m_j = 1$, $j = 1, \ldots, N$, the first interpretation was found by Bárány [2] and Todd and Wright [18], see also Todd [17]. As computational results have shown (see [12]) the latter case ($N = n$) is very appropriate to compute a fixed point of a function on R^n. This is due to the fact that in this case the algorithm generates simplices of the so-called K' triangulation, which depends on the starting point and allows for making fast movements in all directions. One of the disadvantages, however, is that we have to handle with a system of $2n$ linear equations instead of $n+1$ in the standard case. Making use of the fact that the algorithm actually traces zeroes of a homotopy function from R^{n+1} to R^n (the first interpretation) Todd [17] shows that under some additional constraints the size of the system can be reduced from $2n$ to $n+1$. Since we have shown that for each intermediate case $1 < N < n$ the algorithm traces a path of zeroes of a function from R^{n+1} to R^n, in fact we can prove that for each case the size of the system of equations can be reduced to $n+1$ ($M - N + 1$ for the algorithm on S). In [14] we constructed an example of a continuous function on the product space S of some unit simplices. This example shows the importance of having an algorithm on S. It is not clear if there is any advantage in having a class of algorithms on R^n. We do not have an example for which one of the intermediate cases, $1 < N < n$, is very appropriate to use. It could be possible that an intermediate case becomes advantageous if the function on R^n has some properties, e.g., linearity or separability. For a further discussion of the properties and possibilities by using an element of the class of algorithms on R^n we refer the reader to [12] and [15].

Recently, some studies have appeared exploiting the ideas behind variable dimension restart algorithms to compute fixed points. Firstly, Kojima and Yamamoto [5] and [6] have put the simplical fixed point algorithms in a unified framework based on a primal-dual pair of subdivided manifolds (see also Kojima [7]).

Let $\sigma(w^1, \ldots, w^t)$ be a $\mathfrak{T}$-complete facet of a simplex in $\bar{A}(\mathfrak{T})$ generated by the algorithm with solution $\lambda_1^*, \ldots, \lambda_t^*, \mu^*(j,k), (j,k) \notin \mathfrak{T}$. Then it follows from Definition 2.4 that for $x^* = \sum_{i=1}^{t} \bar{\lambda}_i^* w^i$ with $\bar{\lambda}_i^* = \lambda_i^* / \sum_{i=1}^{t} \lambda_i^*$ holds

$$-\bar{f}(x^*) + x^* \in \mathcal{C}(\mathfrak{T})$$

where $\bar{f}$ is the piecewise linear approximation to f with respect to the triangulation of S and

$$\mathcal{C}(\mathfrak{T}) = \{ y \in R^M \mid \text{for some } \alpha \geq 0,\ y_{j,h} = \alpha,\ (j,h) \in \mathfrak{T},\ y_{j,h} \leq \alpha, (j,h) \notin \mathfrak{T} \}.$$

Kojima and Yamamoto interpret the algorithm as a method tracing a path of points $(x, y) \in A(\mathfrak{T}) \times \mathcal{C}(\mathfrak{T})$. The set $\mathfrak{T}$ changes if some boundary is hit, i.e., if $x \in A(\mathfrak{T} \setminus \{(j,k)\})$ for some $(j,k) \in \mathfrak{T}$ or $y_{j,h} = \alpha$, for some $(j,h) \notin \mathfrak{T}$.

Secondly, Freund [4] has given an interpretation based on a new mathematical structure, called a V-complex, and its associated H-complex. In this paper we have presented path-following interpretations of variable dimension algorithms by connecting vertices of the triangulation on the natural level with the vertices of a certain set on an extra level. In contrast with Kojima and Yamamoto, in our interpretation a path of zeroes of a homotopy function is followed. In comparison with Freund, the constructed subdivisions of $S \times [0,1]$ can be seen as being H-complexes. However, an H-complex is not necessarily a subdivision. In the first interpretation the artificial set is the product of several simplices and has the same dimension as the real set. This interpretation is an extension of work done by Todd and Wright [18], Todd [17] and Bárány [2]. Note that this interpretation yields a polyhedral subdivision. This implies that, although the path of zeroes of the piecewise linear approximation to the underlying homotopy function is uniquely determined, the solution of the corresponding system of linear equations is not necessarily unique (see (3.6)).

In the second interpretation the set on the extra level is a simplex with a higher dimension than the dimension of the real set, which results in a simplicial subdivision of the convex hull of the set on the natural level and the one on the artificial level. Hence, following the standard technique of simplicial fixed point algorithms, a path of adjacent simplices is generated. Each common facet of two adjacent simplices of this path has now a unique solution with respect to the associated system of linear equations.

The second interpretation can be easily adapted for integer labelling by labelling the (j,k)th unit vector on the extra level with (j,k). For the first interpretation the adaption to integer labelling appears complicated. One possibility is to label the face which is a convex hull of the vertices in $E(\mathfrak{T})$ with the labelset $\mathfrak{T}$. To apply index theory the second interpretation seems to be more appropriate than the first one. On the other hand index theory can be applied directly without using an extra level, as is done by Van der Laan [8] for the extreme case $N = 1$.

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G. VAN DER LAAN: DEPARTMENT OF ACTUARIAL SCIENCES AND ECONOMETRICS, FREE UNIVERSITY, DE BOELELAAN 1081, AMSTERDAM, THE NETHERLANDS

A. J. J. TALMAN: DEPARTMENT OF ECONOMETRICS, TILBURG UNIVERSITY, HOGESCHOOLLAAN 225, TILBURG, THE NETHERLANDS