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Interpretations of Belief Functions in the Theory of Rough Sets

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ABSTRACT

This paper reviews and examines interpretations of belief functions in the theory of rough sets with finite universe. The concept of standard rough set algebras is generalized in two directions. One is based on the use of non-equivalence relations. The other is based on relations over two universes, which leads to the notion of interval algebras. Pawlak rough set algebras may be used to interpret belief functions whose focal elements form a partition of the universe. Generalized rough set algebras using non-equivalence relations may be used to interpret belief functions which have less than $|U|$ focal elements, where $|U|$ is the cardinality of the universe U on which belief functions are defined. Interval algebras may be used to interpret any belief functions.

1. INTRODUCTION

The relationships between the theory of belief functions and rough sets continue to receive attention in the research community. Many studies on this topic have been focused on the Pawlak rough set algebras [14, 25, 26, 27]. One of the results of these studies is that a belief function can be derived from a Pawlak rough set algebra. In this formulation, belief functions are considered as extensions of probabilities from definable or measurable sets to undefinable or non-measurable sets. That is, belief functions are inner probabilities [14]. However, the converse, that every belief function can be derived from a Pawlak rough set algebra, has not been considered in these studies. Harmanec *et al.* [7] referred to such an issue as the completeness of an interpretation. In contrast, many other studies on interpretations of belief functions have treated the problem of

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completeness extensively [5, 7, 21, 30].

Ruspini [17] interpreted belief functions by adopting an epistemic interpretation of modal logic S5, which corresponds to Pawlak rough set algebra [32]. Fagin and Halpern [5] examined the interpretation of belief functions as inner measures. Given a belief function Bel on a universe U, in general it is impossible to find a probability function P on U such that $P_* = Bel$, where P_* is the inner measure induced by P. To resolve this problem, they viewed belief and probability functions as functions not just on sets, but on formulas. In their study, they implicitly used several universes [1]. This is related to the generalization of the concept of rough set algebras to two universes, called interval algebras [30]. Resconi *et al.* [16], Harmanec *et al.* [7], and Klir [8] provided a systematic investigation of belief functions based on the standard semantics of modal logic. They have proved that such an interpretation is complete. Yao and Lin [32] suggested that one can generalize Pawlak rough sets by considering non-equivalence relations. Generalized rough set algebras can be derived using results from modal logic. This suggests that one may consider interpretations of belief functions in a wider context of generalized rough set algebras. Lingras [11] showed the usefulness of belief and plausibility functions derived from a non-transitive rough set algebras for generating decision rules in incomplete databases.

Many authors offered possible interpretations of the Dempster rule of combination, we will not address this problem in this paper. Our main objective is to review and examine interpretations of belief functions in rough set algebras. Although new results are provided, we will focus on the synthesis of many existing interpretations developed by many authors using different terminologies and notations. In this study, we will consider the algebraic aspects involved in the interpretation belief functions in the theory of rough sets, without dealing with semantic interpretations extensively. In order to show the completeness of our interpretations, the concept of standard rough set algebras is extended in two directions. Generalized rough set algebras are obtained by using non-equivalence relations. Interval algebras are obtained by using relations over two universes. Pawlak rough set algebras may be used to interpret belief functions whose focal elements form a partition of the universe. A special class of generalized rough set algebras, i.e., serial rough set algebra, may be used to interpret belief functions which have less than $|U|$ focal elements. Serial rough set algebra is a counterpart of modal logic KD [2, 32]. Interval algebras may be used to interpret any belief functions. The results of this study justify the need for generalizing Pawlak rough set algebras. They may extend the domain of applications of both theory of rough sets and theory of belief functions.

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2. ROUGH SET ALGEBRAS

The section reviews the Pawlak rough set algebras and its two extensions, rough set algebras and interval algebras.

2.1. PAWLAK ROUGH SET ALGEBRAS

Let U be a finite and nonempty set called universe. In the development of Pawlak rough set algebra, an equivalence relation $\Re \subseteq U \times U$ is used to describe the relationships between elements of U. The equivalence relation $\Re$ partitions the universe into a family of disjoint subsets, denoted by $U/\Re$. Elements of $U/\Re$ are called the elementary sets. The emptyset $\emptyset$ and the union of one or more elementary sets are called definable, observable, or composed sets. The family of all definable sets is denoted by $\sigma(U/\Re)$. For a finite universe, it is an σ-algebra of subsets of U generated by the family of equivalence classes $U/\Re$. In addition, $U/\Re$ is the basis of the σ-algebra $\sigma(U/\Re)$. Members of $\sigma(U/\Re)$ may be interpreted as measurable sets [5, 12].

Suppose we only have information and knowledge about the measurable sets. In order to carry out necessary inference, one must represent or approximate any subet of U using measurable sets. For any subset $A \subseteq U$, the greatest measurable set contained in A is called the lower approximation of A, written $\underline{apr}(A)$, while the least measurable set containing A is called the upper approximation of A, written $\overline{apr}(A)$. They can be expressed as:

$$\begin{aligned} \underline{apr}(A) &= \bigcup\{X \mid X \in \sigma(U/\Re), X \subseteq A\}, \\ \overline{apr}(A) &= \bigcap\{X \mid X \in \sigma(U/\Re), X \supseteq A\}. \end{aligned} \tag{1}$$

The set A lies between its lower and upper approximations. By definition, a measurable set has the same lower and upper approximations. Let

$$[x]_\Re = \{y \mid x\Re y\}, \tag{2}$$

denote the equivalence class containing x. In terms of equivalence classes, approximations of $A \subseteq U$ can be written as:

$$\begin{aligned} \underline{apr}(A) &= \{x \in U \mid [x]_\Re \subseteq A\}, \\ \overline{apr}(A) &= \{x \in U \mid [x]_\Re \cap A \neq \emptyset\}. \end{aligned} \tag{3}$$

The lower approximation consists of those elements whose equivalence classes are contained in A, and the upper approximation consists of those elements whose equivalence classes have a nonempty intersection with A.

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One may interpret the lower and upper approximations as a pair of unary set-theoretic operators [10, 31]. The system $(2^U, \cap, \cup, \sim, \underline{apr}, \overline{apr})$ is called a Pawlak rough set algebra [31]. It is an extension of the standard set algebra $(2^U, \cap, \cup, \sim)$.

Researchers [4, 27] have shown that Shafer [20] used a similar notion, i.e., a calculus of partitions, in the study of belief functions. Specifically, $U/\Re$ may be interpreted as a coarsening of U, while U is a refinement of $U/\Re$. The lower and upper approximations of a subset A are the inner and outer reductions as defined by Shafer [20].

2.2. SERIAL ROUGH SET ALGEBRAS

The concept of Pawlak rough set algebra can be easily generalized by considering an arbitrary binary relation $\Re \subseteq U \times U$ on U. The relation $\Re$ can be conveniently expressed as a mapping from U to 2^U:

$$r(x) = \{y \in U \mid x\Re y\}, \tag{4}$$

by collecting all $\Re$-related elements for each element $x \in U$. The set $r(x)$ may be viewed as a $\Re$-neighborhood of x defined by the binary relation $\Re$ [9, 32]. The $\Re$-neighborhood $r(x)$ becomes an equivalence class containing x if $\Re$ is an equivalence relation. For an arbitrary relation $\Re$, by substituting equivalence class $[x]_\Re$ with $\Re$-neighborhood $r(x)$ in equation (3), we have:

$$\begin{aligned} \underline{apr}(A) &= \{x \in U \mid r(x) \subseteq A\}, \\ \overline{apr}(A) &= \{x \in U \mid r(x) \cap A \neq \emptyset\}. \end{aligned} \tag{5}$$

The set $\underline{apr}(A)$ consists of those elements whose $\Re$-neighborhoods are contained in A, and $\overline{apr}(A)$ consists of those elements whose $\Re$-neighborhoods have a nonempty intersection with A. The system $(2^U, \cap, \cup, \sim, \underline{apr}, \overline{apr})$ is called a rough set algebra [31]. By using different types of binary relations, one may define various classes of rough set algebras [32].

The following properties hold for $\underline{apr}$, independent of the properties of $\Re$: for subsets $A, B \subseteq U$,

$$\begin{array}{ll} \text{(L0)} & \underline{apr}(A) = \ \sim\overline{apr}(\sim A), \\ \text{(L1)} & \underline{apr}(U) = U, \\ \text{(L2)} & \underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B), \\ \text{(L3)} & \underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B). \end{array}$$

Property (L0) states that $\underline{apr}$ and $\overline{apr}$ are dual to each other with respect to $\sim$. From (L0), dual properties of $\overline{apr}$ can be easily derived. Property (L3) is implied by (L2).

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A binary relation $\Re$ is called a serial relation if for every element $x \in U$, there exists at least one element $y \in U$ such that $x\Re y$. In other words, for every element $x \in U$, $r(x) \neq \emptyset$. This is equivalent to saying that every element of U has a nonempty $\Re$-neighborhood. A rough set algebra constructed from a serial relation is called a serial rough set algebra. In such a algebra, the following properties hold:

$$\text{(L4)} \qquad \underline{apr}(A) \subseteq \overline{apr}(A),$$
$$\text{(L5)} \qquad \underline{apr}(\emptyset) = \emptyset.$$

In the subsequent discussions, we only consider rough set algebras that are at least serial. According to property (L4), we may therefore call $\underline{apr}$ and $\overline{apr}$ a pair of lower and upper approximation operators. A serial rough set algebra is characterized by three independent axioms consisting of (L1), (L2), and (L4). Another set of independent axioms can be derived by replacing (L4) with (L5).

From the binary relation $\Re$, we construct a mapping $j : 2^U \longrightarrow 2^U$:

$$j(A) = \{x \in U \mid r(x) = A\}, \tag{6}$$

by grouping the elements with the same $\Re$-neighborhood. It is referred to as the basic set assignment. A set $A \subseteq U$ with $j(A) \neq \emptyset$ is called a focal set [30, 33]. Conversely, from j we can obtain the binary relation by:

$$r(x) = A, \ \text{for all } x \in j(A). \tag{7}$$

In a serial rough set algebra, j obeys the following properties:

$$\text{(S1)} \qquad j(\emptyset) = \emptyset,$$
$$\text{(S2)} \qquad \bigcup_{A \subseteq U} j(A) = U,$$
$$\text{(S3)} \qquad A \neq B \Longrightarrow j(A) \cap j(B) = \emptyset.$$

Properties (S2) and (S3) hold for any binary relation. The condition of a serial relation guarantees that property (S1) holds. Using the basic set assignment j, the lower and upper approximations can be equivalently defined by:

$$\text{(S4)} \qquad \underline{apr}(A) = \bigcup_{B \subseteq A} j(B),$$
$$\text{(S4}'\text{)} \qquad \overline{apr}(A) = \bigcup_{A \cap B \neq \emptyset} j(B).$$

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Conversely, from a basic set assignment satisfying (S1)-(S3), one can define a lower approximation operator by [28, 29]:

$$j(A) = \underline{apr}(A) - \bigcup_{B \subset A} \underline{apr}(B). \tag{8}$$

Therefore, one can define a lower approximation operator indirectly by providing a basic set assignment. An advantage of employing the basic set assignment is that properties (S1)-(S3) can be easily checked in comparison with the properties (L1), (L2), and (L4).

Based on the focal sets of a lower approximation operator, we define a family of subsets of U as follows:

$$M = \{j(A) \mid A \subseteq U, j(A) \neq \emptyset\}. \tag{9}$$

By property (S1)-(S3), one can see that M forms a partition of the universe. Using the rough set terminology, we regard elements of M as elementary sets. The empty set and the unions of one or more elementary sets are called measurable sets. The family of all measurable sets formed from M is denoted by $\sigma(M)$. Since U is finite, $\sigma(M)$ is the σ-algebra generated by M. The family of subsets M is the basis of $\sigma(M)$. The number of equivalence classes is at most equal to the number of elements in the universe, we immediately have the following lemma.

LEMMA 1. *For a given lower approximation operator, there are at most $|U|$ focal sets, where $|\cdot|$ denotes the cardinality of a set.*

In the case where $\Re$ is an equivalence relation, focal sets are in fact equivalence classes of $\Re$. The basic set assignment maps every equivalence class of $\Re$ to itself and all other sets to the empty set $\emptyset$. According to (S4) and (S4$'$), we have another expression of Pawlak's lower and upper approximation operators:

$$\begin{aligned} \underline{apr}(A) &= \bigcup_{[x]_\Re \subseteq A} [x]_\Re, \\ \overline{apr}(A) &= \bigcup_{[x]_\Re \cap A \neq \emptyset} [x]_\Re. \end{aligned} \tag{10}$$

For an arbitrary relation, the replacement of $[x]_\Re$ by the $\Re$-neighborhood does not produce a pair of operators that are equivalent to the ones defined by equation (5).

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2.3. INTERVAL ALGEBRAS

The notion of interval algebras or interval structures provides an alternative generalization of the concept of rough set algebras [30]. In contrast to a rough set algebra, approximation operators are based on a multi-valued mapping between two universes [3] or a compatibility relation on two universes [22, 23].

Let W and U be two finite and nonempty sets, and let $\Re \subseteq W \times U$ be a binary relation from W to U. The binary relation can be regarded as a multi-valued mapping from W to U: for $w \in W$,

$$r(w) = \{x \in U \mid w\Re x\}. \tag{11}$$

With the binary relation $\Re$, we define a pair of approximation operators, $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^W$, by modifying equation (3): for $A \subseteq U$,

$$\begin{aligned} \underline{apr}(A) &= \{w \in W \mid r(w) \subseteq A\}, \\ \overline{apr}(A) &= \{w \in W \mid r(w) \cap A \neq \emptyset\}. \end{aligned} \tag{12}$$

Both $\underline{apr}(A)$ and $\overline{apr}(A)$ are subsets of W. To be consistent with serial rough set algebras, we assume that for every element $w \in W$, $r(w) \neq \emptyset$. Under this assumption, the approximation operators have properties:

(IL0) $\underline{apr}(A) = \ \sim\overline{apr}(\sim A)$,
(IL1) $\underline{apr}(U) = W$,
(IL2) $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$,
(IL3) $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$,
(IL4) $\underline{apr}(A) \subseteq \overline{apr}(A)$,
(IL5) $\underline{apr}(\emptyset) = \emptyset$.

If a pair of approximation operators satisfies (IL0) and (IL4), we call them lower and upper approximation operators. Properties (IL1), (IL2), and (IL5) are independent and imply other properties. Wong *et al.* [30] used these axioms to define the notion of interval structures. In this study, the system $(2^W, 2^U, \cap, \cup, \sim, \underline{apr}, \overline{apr})$ is called an interval algebra, where $\underline{apr}$ and $\overline{apr}$ are dual operators and $\underline{apr}$ satisfies axioms (IL1), (IL2), and (IL5). It may be regarded as a rough set algebra over two universes.

Following the same argument in the development of rough set algebras, we can define a basic set assignment, $j : 2^U \longrightarrow 2^W$, for $A \subseteq U$,

$$j(A) = \{w \in W \mid r(w) = A\}. \tag{13}$$

It obeys axioms:

$$\begin{aligned} &\text{(IS1)} && j(\emptyset) = \emptyset, \\ &\text{(IS2)} && \bigcup_{A\subseteq U} j(A) = W, \\ &\text{(IS3)} && A \neq B \Longrightarrow j(A)\cap j(B) = \emptyset. \end{aligned}$$

A subset $A \subseteq U$ with $j(A) \neq \emptyset$ is called a focal set. Similarly, we can construct a family of subsets of W, $M = \{j(A) \mid A \subseteq U, j(A) \neq \emptyset\}$. It generates an σ-algebra $\sigma(M)$ with M as its basis. The notion of basic set assignment was implicitly used by Fagin and Halpern [5], and by Harmanec *et al.* [7]. The relationships between a basic set assignment and a lower approximation operator established for rough set algebras also hold for interval algebras.

An interval algebra can be alternatively formulated by a serial rough set algebra on W and a single-valued onto mapping from W to U. Let $\pi : W \longrightarrow U$ be a single-valued onto mapping, i.e., π associates every element $w \in W$ with exactly one element $x \in U$ and for every $x \in U$ there exists a $w \in W$ such that $\pi(w) = x$. Based on π, one can associate each subset $A \subseteq U$ with a unique subset $\Pi^{-1}(A) \subseteq W$ as follows:

$$\Pi^{-1}(A) = \{w \mid \pi(w) \in A\}. \tag{14}$$

The inverse mapping mapping $\Pi^{-1} : 2^U \longrightarrow 2^W$ is also called an incidence mapping [1]. It satisfies the following properties:

$$\begin{aligned} &(\Pi 1) && \Pi^{-1}(\emptyset) = \emptyset, \\ &(\Pi 2) && \Pi^{-1}(U) = W, \\ &(\Pi 3) && \Pi^{-1}(\sim A) = \ \sim\Pi^{-1}(A), \\ &(\Pi 4) && \Pi^{-1}(A\cap B) = \Pi^{-1}(A)\cap\Pi^{-1}(B), \\ &(\Pi 5) && \Pi^{-1}(A\cup B) = \Pi^{-1}(A)\cup\Pi^{-1}(B). \end{aligned}$$

Consider a serial rough set algebra $(2^W, \cap, \cup, \sim, \underline{apr}_W, \overline{apr}_W)$ on W. For a set $A \subseteq U$, we have $\Pi^{-1}(A)$ and a pair of lower and upper approximations $\underline{apr}_W(\Pi^{-1}(A))$ and $\overline{apr}_W(\Pi^{-1}(A))$. Thus, we can define a pair of approximation operators $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^W$ by:

$$\begin{aligned} \underline{apr}(A) &= \underline{apr}_W(\Pi^{-1}(A)), \\ \overline{apr}(A) &= \overline{apr}_W(\Pi^{-1}(A)). \end{aligned} \tag{15}$$

From properties (Π1)-(Π5), one can conclude that if $\underline{apr}_W$ and $\overline{apr}_W$ obey axioms (L0)-(L2) and (L5), $\underline{apr}$ and $\overline{apr}$ as defined by equation (15) satisfy

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axioms (IL0)-(IL2) and (IL5). In other words, we can construct an interval algebra from a serial rough set algebra on W and a single-valued onto mapping from W to U. A Pawlak rough set algebra is a serial rough set algebra. A special kind of interval algebras can be defined by Pawlak rough set algebras on W and single-valued onto mappings from W to U.

For any interval algebra, it is always possible to construct a multi-valued mapping from W to U that produces the same interval algebra. This is not true in the framework using a serial rough set algebra and a single-valued onto mapping. The condition of a single-valued onto mapping $\pi : W \longrightarrow U$ implies $|W| \geq |U|$. Consequently, for an arbitrary interval algebra, there may not exist a serial rough set algebra on W and a mapping π from W to U that produce the same interval algebra. The framework for formulating an interval algebra from a serial rough set algebra and a single-valued onto mapping is therefore less general than the one based on a multi-valued mapping.

3. BELIEF FUNCTIONS

A belief function is a mapping from 2^U to the unit interval $[0, 1]$ and satisfies the following axioms:

(F1) $Bel(\emptyset) = 0$,

(F2) $Bel(U) = 1$,

(F3) For every positive integer n and every collection $A_1, \ldots, A_n \subseteq U$,

$$Bel(A_1 \cup A_2 \ldots \cup A_n) \geq \sum_i Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) \pm \ldots + (-1)^{n+1} Bel(A_1 \cap \ldots \cap A_n).$$

Axioms (F1) and (F2) may be considered as normalization conditions. Axiom (F3) is a weaker version of the commonly known additivity axiom of probability functions. It is referred to as the axiom of superadditivity. The dual of a belief function, called a plausibility function Pl, is defined by:

$$Pl(A) = 1 - Bel(\sim A). \quad (16)$$

For any subset $A \subseteq U$, $Bel(A) \leq Pl(A)$.

A belief function can be equivalently defined by another mapping, $m : 2^U \longrightarrow [0, 1]$, which is called a basic probability assignment and satisfies two axioms:

(M1) $m(\emptyset) = 0$,

(M2) $\sum_{A \subseteq U} m(A) = 1$.

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A subset $A \subseteq U$ with $m(A) > 0$ is called a focal element. Using the basic probability assignment, belief and plausibility of A are expressed as:

$$\text{(M3)} \qquad Bel(A) = \sum_{B \subseteq A} m(B),$$

$$\text{(M3}'\text{)} \qquad Pl(A) = \sum_{B \cap A \neq \emptyset} m(B).$$

Conversely, a basic probability assignment can be defined from a belief function as:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} Bel(B). \tag{17}$$

4. INTERPRETATIONS OF BELIEF FUNCTIONS

From the discussion of the last two sections, one can see that the notions of lower approximation and basic set assignment are closely related to belief functions and basic probability assignment [28]. Based on such a connection, this section reviews and examines various interpretations of belief functions in the Pawlak and generalized rough set algebras.

4.1. PAWLAK ROUGH SET ALGEBRA

In a Pawlak rough set algebra, the qualities of lower and upper approximations of a subset $A \subseteq U$ are defined by [6, 13]:

$$\underline{q}(A) = |\underline{apr}(A)|/|U|, \qquad \overline{q}(A) = |\overline{apr}(A)|/|U|. \tag{18}$$

From properties (L0)-(L5) of approximation operators, one can easily prove the following theorem [6, 24, 25]. For completeness and subsequent discussion, a proof is also provided.

THEOREM 1. *The quality of lower approximation $\underline{q}$ is a belief function.*

Proof. From (L5) and (L1), one can see that $\underline{q}$ satisfies (F1) and (F2). Consider a collection $A_1, \ldots, A_n \subseteq U$, according to (L3) and (L2), we have:

$$\begin{aligned} \underline{q}(A_1 \cup \ldots \cup A_n) &= \frac{|\underline{apr}(A_1 \cup \ldots \cup A_n)|}{|U|} \\ &\geq \frac{|\underline{apr}(A_1) \cup \ldots \cup \underline{apr}(A_n)|}{|U|} \end{aligned}$$

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$$
\begin{aligned}
&= \sum_i \frac{|\underline{apr}(A_i)|}{|U|} - \sum_{i<j} \frac{|\underline{apr}(A_i) \cap \underline{apr}(A_j)|}{|U|} \\
&\quad \pm \ldots + (-1)^{n+1} \frac{|\underline{apr}(A_1) \cap \ldots \cap \underline{apr}(A_n)|}{|U|} \\
&= \sum_i \frac{|\underline{apr}(A_i)|}{|U|} - \sum_{i<j} \frac{|\underline{apr}(A_i \cap A_j)|}{|U|} \\
&\quad \pm \ldots + (-1)^{n+1} \frac{|\underline{apr}(A_1 \cap \ldots \cap A_n)|}{|U|} \\
&= \sum_i \underline{q}(A_i) - \sum_{i<j} \underline{q}(A_i \cap A_j) \\
&\quad \pm \ldots + (-1)^{n+1} \underline{q}(A_1 \cap \ldots \cap A_n).
\end{aligned}
$$

Therefore, $\underline{q}$ satisfies (F3). ■

The set of focal elements of $\underline{q}$ is the same as the set of focal sets of $\underline{apr}$. They consist of equivalence classes of $\Re$. The basic probability assignment of $\underline{q}$ is:

$$m(E) = |E|/|U|, \tag{19}$$

for all $E \in U/\Re$, and $m(A) = 0$ for all other subsets of U. Theorem 1 can be proved more easily by verifying that m satisfies (M1)-(M3) as shown by Skowron [24, 25].

There are several limitations to such a formulation. The belief and plausibility of every subset of U must be rational numbers that are equivalent to the ones with $|U|$ as the denominator. For an arbitrary rational-valued belief function Bel on U, it may not be possible to build a Pawlak rough set algebra such that $\underline{q}(A) = Bel(A)$ for all $A \subseteq U$. The condition that the family of focal sets of $\underline{q}$ is a partition of U further restricts the set of derived rational-valued belief functions. A much weaker version of the converse of Theorem 1 can be stated.

THEOREM 2. *Suppose Bel is a belief function on U satisfying two conditions:*

(i). *the set of focal elements of Bel is a partition of U,*

(ii). $m(A) = |A|/|U|$ *for every focal element A of Bel,*

where m is the basic probability assignment of Bel. There exists a Pawlak rough set algebra such that $\underline{q}(A) = Bel(A)$ for every $A \subseteq U$.

Proof. According to condition (i), one can construct a Pawlak rough algebra using focal elements of Bel. In this rough set algebra, the set of focal elements of $\underline{q}$ is the same as that of Bel. Condition (ii) guarantees that basic probability assignments of $\underline{q}$ and Bel are the same. ■

A number of proposals have been made to remove some of the above limitations [5, 14, 27]. Pawlak suggested that one may start from a probability function on the σ-algebra $\sigma(U/\Re)$, instead of using the cardinality of sets. Fagin and Halpern [5] interpreted a belief function using a σ-algebra $\mathcal{X}$ of subsets of U and a probability function on $\mathcal{X}$. For a finite universe, given an σ-algebra $\mathcal{X}$, one can construct a Pawlak rough set algebra such that $\mathcal{X} = \sigma(U/\Re)$, and vice versa. Therefore, both formulations may be considered as being equivalent.

Suppose P is a probability function defined on $\sigma(U/\Re)$. It is not defined for subsets of U which are not members of $\sigma(U/\Re)$. One can extend P to 2^U in two standard ways by defining functions P_* and P^*, traditionally called the inner measure and outer measure induced by P. For an arbitrary subset $A \subseteq U$, we define:

$$\begin{aligned} P_*(A) &= \sup\{P(X) \mid X \in \sigma(U/\Re), X \subseteq A\} = P(\underline{apr}(A)), \\ P^*(A) &= \inf\{P(X) \mid X \in \sigma(U/\Re), X \supseteq A\} = P(\overline{apr}(A)). \end{aligned} \quad (20)$$

Pawlak [14] referred to the interval $[P_*(A), P^*(A)]$ as rough probability of A. The inner and outer probabilities P_* and P^* are a pair of belief and plausibility functions [5].

THEOREM 3. *Suppose P is a probability function on $\sigma(U/\Re)$. The inner probability P_* is a belief function.*

The above theorem just presents one interpretation of belief functions. Under this view, a belief function is derived by extending a probability function from measurable sets to non-measurable sets. The basic probability assignment of P_* is defined by:

$$m(E) = P(E), \quad (21)$$

for all $E \in U/\Re$, and $m(A) = 0$ for all other subsets of U. That is, the set of all focal elements of P_* is a partition of the universe. This implies that given an arbitrary belief function Bel on U, there may not exist a Pawlak algebra and a probability function P on $\sigma(U/\Re)$ such that $P_*(A) = Bel(A)$ and $P^*(A) = Pl(A)$. In other words, such an interpretation of belief functions is still restrictive.

THEOREM 4. *Suppose Bel is a belief function on U satisfying condition (i). There exist a Pawlak rough set algebra and a probability function on $\sigma(U/\Re)$ such that $P_*(A) = Bel(A)$ for every $A \subseteq U$.*

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Proof. Suppose F is the set of focal elements of Bel which is a partition of the universe U. It defines a Pawlak rough set algebra with F as the set of all elementary sets. For any subset set $A \in F$, let $P(A) = m(A)$. The family F is the basis of the σ-algebra $\sigma(F)$. The function P can be extended to all elements in $\sigma(F)$ by additivity. By the properties of basic probability assignment, it follows that P is a probability function. It can be easily verified that $P_*(A) = Bel(A)$ for every $A \subseteq U$. ■

An important implication of this theorem is that the Pawlak rough set algebra may only be used to interpret belief functions whose focal elements form a partition of the universe.

Pawlak rough set algebra corresponds to modal logic S5 [31, 32]. The lower and upper approximation operators correspond to the necessity and possibility operators. Probability related interpretation of belief functions may be alternatively explained using a possible-worlds model based on modal logic S5. Ruspini [17, 18, 19] adopted an epistemic interpretation of modal logic S5. Each proposition is related to a subset of possible worlds. The necessity operator is viewed as epistemic operator $\mathbf{K}$. It represents the knowledge of a rational agent. The notation $\mathbf{K}p$ is used to denote the set of possible worlds at which the proposition p may be known or proved to be true. The belief function can be interpreted using a probability function:

$$Bel(p) = P(\mathbf{K}p). \tag{22}$$

This interpretation views belief functions as probabilities of provability [15]. It is essentially the same as the interpretation given by equation (20). A more elaborate comparison of these interpretations has been given by Fagin and Halpern [5].

4.2. SERIAL ROUGH SET ALGEBRAS

In a serial rough set algebra constructed from a serial relation $\Re$, we can similarly define qualities of lower and upper approximations as:

$$\underline{Q}(A) = |\underline{apr}(A)|/|U|, \qquad \overline{Q}(A) = |\overline{apr}(A)|/|U|. \tag{23}$$

They are a pair of belief and plausibility functions as shown by the following generalization of Theorem 1.

THEOREM 5. *The quality of lower approximation $\underline{Q}$ in a serial rough set algebra is a belief function.*

This theorem can be proved from properties (L0)-(L5) of approximation operators by using the technique developed in the proof of Theorem 1. The

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family of focal sets of $\underline{apr}$ is the same as the family of focal elements of $\underline{Q}$. The function:

$$m(A) = |j(A)|/|U|. \tag{24}$$

is the basic probability assignment of $\underline{Q}$. Properties (M1) and (M2) of m follow from the properties (S1)-(S3) of the basic set assignment j.

With serial rough set algebras, we have a generalized version of Theorem 2.

THEOREM 6. *Suppose Bel is a belief function on U satisfying condition:*

(iii). $\quad$ *$Bel(A)$ is equivalent to a rational number with $|U|$ as its denominator for every $A \subseteq U$.*

There exists a serial rough set algebra such that $\underline{Q}(A) = Bel(A)$ for every $A \subseteq U$.

Proof. In order to prove this theorem, we must first prove that condition (iii) is equivalent to the condition:

(iii′). $\quad$ *$m(A)$ is equivalent to a rational number with $|U|$ as its denominator for every $A \subseteq U$.*

The implication (iii′) ⇒ (iii) trivially follows from (M3) stating the transformation from m to Bel. The implication (iii) ⇒ (iii′) can be proved inductively as follows. Suppose A is focal element of Bel such that none of its subset is a focal element. We have $Bel(A) = m(A)$. By condition (iii), $Bel(A)$ is equivalent to a number $a/|U|$, where a is a non-negative integer. Thus, (iii′) holds for A. Assume now that A is a focal element such that (iii′) holds for all its subset. We have:

$$Bel(A) = \sum_{B \subset A} m(B) + m(A).$$

By assumption and condition (iii), the above equation can be expressed as:

$$\frac{a}{|U|} = \frac{b}{|U|} + m(A),$$

where a and b are non-negative integers. Hence $m(A) = (a-b)/|U|$, which implies that (iii′) holds for A. By induction, condition (iii′) holds.

Since the smallest positive rational number with $|U|$ as the denominator is $1/|U|$, it follows from axiom (M2) that there are at most $|U|$ focal elements for Bel. By summarizing the above results, the family of focal

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elements of Bel can be expressed by $F = \{F_1, \ldots, F_n\}$, where $n \leq |U|$. Each $m(F_i)$ can be expressed by $m_i/|U|$, where $m_i \leq |U|$ are positive integers and $\sum_{i=1}^{n} m_i = |U|$. According to the values of n and m_i's, we can construct a partition, $M = \{M_1, \ldots, M_n\}$ of U such that $|M_i| = m_i$. From the families F and M, we define a basic set assignment j as follows:

$$j(F_i) = M_i.$$

It is straightforward to verify that in the serial rough set algebra defined by j we have $m(F_i) = m_i/|U| = |j(F_i)|/|U|$. Therefore, $\underline{Q}(A) = Bel(A)$ for every $A \subseteq U$. ■

From the proof of Theorem 6, one can see that condition (iii) is a much weaker version of condition (ii). Moreover, condition (i) is no longer needed. Thus, serial rough set algebras can be used to interpret a larger class of belief functions.

Example 1. Consider a set $U = \{a, b, c, d\}$ and a belief function Bel defined as follows:

$$\begin{array}{lll}
Bel(\emptyset) = 0, & Bel(\{a\}) = 0, & Bel(\{b\}) = 0, \\
Bel(\{c\}) = 0, & Bel(\{d\}) = \frac{1}{4}, & Bel(\{a,b\}) = 0, \\
Bel(\{a,c\}) = 0, & Bel(\{a,d\}) = \frac{1}{4}, & Bel(\{b,c\}) = \frac{1}{4}, \\
Bel(\{b,d\}) = \frac{1}{4}, & Bel(\{c,d\}) = \frac{1}{4}, & Bel(\{a,b,c\}) = \frac{1}{4}, \\
Bel(\{a,b,d\}) = \frac{1}{4}, & Bel(\{a,c,d\}) = \frac{3}{4}, & Bel(\{b,c,d\}) = \frac{2}{4}, \\
Bel(U) = 1. & &
\end{array}$$

The set of focal elements of Bel is $F = \{\{d\}, \{b, c\}, \{a, c, d\}\}$. The basic probability assignment is:

$$m(\{d\}) = \frac{1}{4}, \qquad m(\{b,c\}) = \frac{1}{4}, \qquad m(\{a,c,d\}) = \frac{2}{4},$$

and $m(A) = 0$ for all other subsets of U. Based on these values, we construct a partition $M = \{\{a\}, \{b\}, \{c, d\}\}$ of U. From F and M, we define a basic set assignment:

$$j(\{d\}) = \{a\}, \qquad j(\{b,c\}) = \{b\}, \qquad j(\{a,c,d\}) = \{c,d\}.$$

The corresponding binary relation is given be:

$$r(a) = \{d\}, \qquad r(b) = \{b,c\}, \qquad r(c) = \{a,c,d\}, \qquad r(d) = \{a,c,d\}.$$

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It is a serial binary relation. The lower approximation operator is given by:

$$\begin{array}{lll}
\underline{apr}(\emptyset) = \emptyset, & \underline{apr}(\{a\}) = \emptyset, & \underline{apr}(\{b\}) = \emptyset, \\
\underline{apr}(\{c\}) = \emptyset, & \underline{apr}(\{d\}) = \{a\}, & \underline{apr}(\{a,b\}) = \emptyset, \\
\underline{apr}(\{a,c\}) = \emptyset, & \underline{apr}(\{a,d\}) = \{a\}, & \underline{apr}(\{b,c\}) = \{b\}, \\
\underline{apr}(\{b,d\}) = \{a\}, & \underline{apr}(\{c,d\}) = \{a\}, & \underline{apr}(\{a,b,c\}) = \{b\}, \\
\underline{apr}(\{a,b,d\}) = \{a\}, & \underline{apr}(\{a,c,d\}) = \{a,c,d\}, & \underline{apr}(\{b,c,d\}) = \{a,b\}, \\
\underline{apr}(U) = U. & &
\end{array}$$

It can be easily verified that $m(A) = |j(A)|/|U|$ and $\underline{Q}(A) = Bel(A)$ for all $A \subseteq U$.

In a serial rough set algebra, consider a probability function P defined on the σ-algebra $\sigma(M)$, where $M = \{j(A) \mid A \subseteq U, j(A) \neq \emptyset\}$. Based on lower and upper approximation operators, a pair of lower and upper probability functions can be defined as:

$$\underline{P}(A) = P(\underline{apr}(A)), \qquad \overline{P}(A) = P(\overline{apr}(A)). \tag{25}$$

They are in fact belief and plausibility functions.

THEOREM 7. *In a serial rough set algebra, if P is a probability function on $\sigma(M)$, the lower probability $\underline{P}$ is a belief function.*

Similar to the proof of Theorem 1, this theorem can be proved from properties (L0)-(L5) of approximation operators and the properties of probability functions. The corresponding basic probability assignment is:

$$m(A) = P(j(A)). \tag{26}$$

The family of focal elements of the belief function $\underline{P}$ may only be a subset of the family of focal sets of $\underline{apr}$. If the probability $P(A) \neq 0$ for all $A \in M$, these two families are the same. By Lemma 1, this implies that rough set algebras cannot be used to interpret belief functions with more than $|U|$ focal elements.

EXAMPLE 2. Consider a set $U = \{a, b\}$ and a belief function Bel defined as follows:

$$Bel(\emptyset) = 0, \quad Bel(\{a\}) = 0.3, \quad Bel(\{b\}) = 0.3, \quad Bel(U) = 1.$$

For this belief function, there are three focal elements and the basic probability assignment is given by:

$$m(\emptyset) = 0, \quad m(\{a\}) = 0.3, \quad m(\{b\}) = 0.3, \quad m(U) = 0.4.$$

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With a two element universe, one can only construct a lower approximation operator having at most two focal sets. It is impossible to construct a serial rough set algebra to interpret this belief function.

On the other hand, if a belief function has at most $|U|$ focal elements, one can interpret it using a serial rough set algebra.

THEOREM 8. *Suppose Bel is a belief function on U satisfying the condition:*

(iv) *Bel has at most $|U|$ focal elements.*

There exists a serial rough set algebra such that $\underline{P}(A) = Bel(A)$ for every $A \subseteq U$.

Condition (iv) is weaker than condition (iii). This theorem can be proved using the technique developed in the proof of Theorem 6.

The notion of serial rough set algebras provides a more general framework for probability related interpretation of belief functions. A belief function is considered to be an approximation of a probability function when the relationships between elements of the universe are considered. The interpretation given in the Pawlak rough set algebra, i.e., belief functions as an inner probabilities, is a special case of this interpretation. A disadvantage of this framework is that only belief functions with at most $|U|$ focal elements can be interpreted.

4.3. INTERVAL ALGEBRAS

In an interval algebra $(2^W, 2^U, \cap, \cup, \sim, \underline{apr}, \overline{apr})$, the qualities of lower and upper approximations define a pair of belief and plausibility functions:

$$\underline{Q}(A) = |\underline{apr}(A)|/|W|, \qquad \overline{Q}(A) = |\overline{apr}(A)|/|W|. \tag{27}$$

The corresponding basic probability assignment is defined by:

$$m(A) = |j(A)|/|W|. \tag{28}$$

The notion of interval algebras can be used to interpret the class of *all* rational-valued belief functions. The main results are summarized in the following two theorems.

THEOREM 9. *The quality of lower approximation $\underline{Q}$ in an interval algebra is a belief function.*

Theorem 9 can be proved from (IL0)-(IL5) similar to the proof of Theorem 1 by using probabilities of sets, instead of their cardinalities.

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THEOREM 10. *Suppose Bel is a rational-valued belief function on U. There exist a set W and an interval algebra such that $\underline{Q}(A) = Bel(A)$ for every $A \subseteq U$.*

Proof. Since U is finite, there are only a finite number of focal elements for a belief function Bel on U. Let $F = \{F_1, \dots, F_n\}$ be the family of focal elements of Bel. From the proof of Theorem 6, one can similarly show that the basic probability assignment of Bel must be rational-valued. Thus, we have $m(F_i) = m_i/k$, where m_i's and k are positive integers, and $m(F_i)$'s are expressed as rational numbers with the same denominator k. We can form a set $W = \bigcup_{i=1}^{n} M_i$ such that the family of M_i's is a partition of W, $|M_i| = m_i$, and $|W| = \sum_{i=1}^{n} |M_i| = k$. It can be easily verified that

$$j(F_i) = M_i$$

is a basic set assignment of an interval algebra. Moreover, $m(F_i) = m_i/k = |j(F_i)|/|W|$. Therefore, $\underline{Q}(A) = Bel(A)$ for every $A \subseteq U$. ■

One can use a probability function P on the σ-algebra $\sigma(M)$, where $M = \{j(A) \mid A \subseteq U, j(A) \neq \emptyset\}$. In this case a pair of lower and upper probability functions are defined by:

$$\underline{P}(A) = P(\underline{apr}(A)), \qquad \overline{P}(A) = P(\overline{apr}(A)). \tag{29}$$

This provides a more general framework for the interpretation of the class of *all* belief functions.

THEOREM 11. *In an interval algebra, if P is a probability function on $\sigma(M)$, the lower probability $\underline{P}$ is a belief function.*

THEOREM 12. *Suppose Bel is a belief function on U. There exist a set W, an interval algebra, and a probability function on $\sigma(M)$ such that $\underline{P}(A) = Bel(A)$ for every $A \subseteq U$.*

The significance of these theorems is that the class of all belief functions can be associated with a probability related interpretation based on the notion of interval algebras. They can be proved by using the techniques developed earlier. The following example illustrates the proof of Theorem 12.

EXAMPLE 3. Consider the belief function given in Example 2. We define a set $W = \{w_1, w_2, w_3\}$ and a relation from W to U as:

$$r(w_1) = \{a\}, \quad r(w_2) = \{b\}, \quad r(w_3) = U.$$

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It defines the following lower approximation operator:

$$\underline{apr}(\emptyset) = \emptyset, \quad \underline{apr}(\{a\}) = \{w_1\}, \quad \underline{apr}(\{b\}) = \{w_2\}, \quad \underline{apr}(U) = W.$$

The basic set assignment is:

$$j(\emptyset) = \emptyset, \quad j(\{a\}) = \{w_1\}, \quad j(\{b\}) = \{w_2\}, \quad j(U) = \{w_3\}.$$

In this case, $\sigma(M) = 2^W$. Consider a probability function P defined on 2^W with distribution:

$$p(w_1) = 0.3, \quad p(w_2) = 0.3, \quad p(w_3) = 0.4.$$

The probability of a subset $Q \subseteq W$ is computed by $P(Q) = \sum_{w \in Q} p(w)$. It can be seen that $m(A) = P(j(A))$ and $Bel(A) = P(\underline{apr}(A))$ for all $A \subseteq U$.

In interpreting belief functions with interval algebras, there is no restriction on the universe W. By properly constructing the universe W, one may interpret any belief function based on an interval algebra defined by a Pawlak rough set algebra on W and a single-valued onto mapping from W to U.

THEOREM 13. *Suppose* $(2^W, \cap, \cup, \sim, \underline{apr}_W, \overline{apr}_W)$ *is a Pawlak rough set algebra, and* $\pi : W \longrightarrow U$ *is a single-valued onto mapping. Then the quality of lower approximation,*

$$\underline{Q}(A) = \frac{|\underline{apr}(A)|}{|W|} = \frac{|\underline{apr}_W(\Pi^{-1}(A))|}{|W|}, \tag{30}$$

is a belief function.

THEOREM 14. *Suppose* Bel *is a rational-valued belief function on* U. *There exist a set* W, *a Pawlak rough set algebra* $(2^W, \cap, \cup, \sim, \underline{apr}_W, \overline{apr}_W)$, *and a single-valued onto mapping* $\pi : W \longrightarrow U$ *such that* $\underline{Q}(A) = Bel(A)$ *for every* $A \subseteq U$, *where* $\underline{Q}$ *is defined by equation (30).*

THEOREM 15. *Suppose* $(2^W, \cap, \cup, \sim, \underline{apr}_W, \overline{apr}_W)$ *is a Pawlak rough set algebra,* $\pi : W \longrightarrow U$ *is a single-valued onto mapping, and* P *is a probability function on the* σ*-algebra* $\sigma(W/\Re)$. *Then the lower probability,*

$$\underline{P}(A) = P(\underline{apr}(A)) = P(\underline{apr}_W(\Pi^{-1}(A))), \tag{31}$$

is a belief function.

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THEOREM 16. *Suppose Bel is a belief function on U. There exist a set W, a Pawlak rough set algebra $(2^W, \cap, \cup, \sim, \underline{apr}_W, \overline{apr}_W)$ which is defined by an equivalence relation $\Re$, a single-valued onto mapping $\pi : W \longrightarrow U$, and a probability function P on the σ-algebra $\sigma(W/\Re)$ such that $\underline{P}(A) = Bel(A)$ for every $A \subseteq U$, where $\underline{P}$ is defined by equation (31).*

Theorems 13 and 15 can be easily seen from the proofs of similar theorems discussed earlier. The proof of Theorem 14 can be obtained by adopting the proof of Theorem 3 in Harmanec *et al* [7, pp. 948-949]. Theorem 16 can be proved similarly. It can also be proved by adopting the technique used in the proof of Theorem 3.3 in Fagin and Halpern [5, pp. 165-166] as illustrated in the following example.

EXAMPLE 4. Consider the belief function given in Example 2. From the set W given in Example 3, we define a set $W' = \{(w_1, a), (w_2, b), (w_3, a), (w_3, b)\}$. Let π be a single-valued onto mapping from W' to U:

$$\pi(w_1, a) = a, \quad \pi(w_2, b) = b, \quad \pi(w_3, a) = a, \quad \pi(w_3, b) = b.$$

Then we have:

$$\begin{aligned} &\Pi^{-1}(\emptyset) = \emptyset, && \Pi^{-1}(\{a\}) = \{(w_1, a), (w_3, a)\}, \\ &\Pi^{-1}(\{b\}) = \{(w_2, b), (w_3, b)\}, && \Pi^{-1}(U) = W'. \end{aligned}$$

We now construct a Pawlak rough set algebra $(2^{W'}, \cap, \cup, \sim, \underline{apr}_{W'}, \overline{apr}_{W'})$ by using a partition $W'/\Re = \{\{(w_1, a)\}, \{(w_2, b)\}, \{(w_3, a), (w_3, b)\}\}$. The lower approximations of subsets of U are given by:

$$\begin{aligned} &\underline{apr}(\emptyset) = \underline{apr}_{W'}(\Pi^{-1}(\emptyset)) = \emptyset, \\ &\underline{apr}(\{a\}) = \underline{apr}_{W'}(\Pi^{-1}(\{a\})) = \{(w_1, a)\}, \\ &\underline{apr}(\{b\}) = \underline{apr}_{W'}(\Pi^{-1}(\{b\})) = \{(w_2, b)\}, \\ &\underline{apr}(U) = \underline{apr}_{W'}(\Pi^{-1}(U)) = W'. \end{aligned}$$

Let P be a probability function defined on the σ-algebra $\sigma(W'/\Re)$ with distribution:

$$p(\{(w_1, a)\}) = 0.3, \quad p(\{(w_2, b)\}) = 0.3, \quad p(\{(w_3, a), (w_3, b)\}) = 0.4.$$

It follows that $Bel(A) = P(\underline{apr}(A))$ for all $A \subseteq U$.

Probability related interpretations of belief functions based on interval algebras have been examined by many authors under different terminologies and notations. The basic idea of approximating one subset of a universe U

by two subsets of another universe W was first introduced by Dempster as follows [3]:

$$\begin{aligned} A_* &= \{w \in W \mid r(w) \neq \emptyset, r(w) \subseteq A\}, \\ A^* &= \{w \in W \mid r(w) \cap A \neq \emptyset\}. \end{aligned} \tag{32}$$

The set A^* equals $\overline{apr}(A)$ as defined by equation (12), while A_* is a subset of $\underline{apr}(A)$. Under the condition $r(w) \neq \emptyset$ for all $w \in W$, both definitions are equivalent. Our definition of the interval algebras is adopted from the proposal of Wong *et al.* [28, 29, 30]. They defined an interval structure axiomatically by using a set of three axioms consisting of (IL1), (IL2), and (IL5). The method suggested for constructing an interval structure is based on the use of a multi-valued mapping. In contrast, other proposals on the interpretation of belief functions are based on an interval algebra constructed from a rough set algebra on W, typically a Pawlak rough set algebra, and a single-valued onto mapping π from W to U. Harmanec *et al.* [7] used modal logic and interpreted elements of W as possible worlds, elements of U as atomic propositions, and subset of U as propositions. The mapping π, called an evaluation function, associates each possible world $w \in U$ with a unique element $x \in U$ such that the atomic proposition corresponding to $\{x\}$ is true. Ruspini [17, 18, 19], and Fagin and Halpern [5] adopted a similar framework to provide semantics to belief functions based on possible-worlds model.

Although different semantics and technical methods are used in these studies, they are algebraically the same. In this study, we only considered the algebraic aspects. The fundamental result has been in fact given by Shafer [21]. Consider the interpretation of belief functions given by equation (29), namely,

$$Bel(A) = P(\underline{apr}(A)) = P \circ \underline{apr}(A). \tag{33}$$

The σ-algebra $\sigma(M)$ is a multiplicative subclass of 2^W, and 2^U is a multiplicative subclass of 2^U. The lower approximation operator of an interval algebra is an $\cap$-homomorphism from 2^U to $\sigma(M)$. The results of Theorems 11 and 12 state that a belief function can be interpreted as the composition of a probability function and an $\cap$-homomorphism, and vice versa. That is, every belief function is of the form $P \circ \underline{apr}$, and every function of the form $P \circ \underline{apr}$ is a belief function [5, 21, 30]. Theorem 16 shows that a belief function can also be expressed in the form of $P \circ \underline{apr}_W \circ \Pi^{-1}$, where composition of a lower approximation operator and a single-valued onto mapping, $\underline{apr}_W \circ \Pi^{-1}$, is an $\cap$-homomorphism.

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5. CONCLUSION

In this paper, we have discussed interpretations of belief functions in Pawlak rough set algebras and their extensions. The notion of Pawlak rough set algebras leads to a probability related interpretation of belief functions whose focal elements form a partition of the universe. Serial rough set algebras may be used to interpret belief functions having at most $|U|$ focal elements. In general, interval algebras can be used to interpret any belief functions. Algebraically speaking, many existing probability related interpretations of belief functions can be expressed in the developed framework. The relationship between belief functions and rough sets may enhance our understanding of each theory.

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