Interpreting the truncated pentagonal number theorem using partition pairs

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Abstract

In 2012 Andrews and Merca gave a new expansion for partial sums of Euler's pentagonal number series and expressed

$$\sum_{j=0}^{k-1}(-1)^j(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1))=(-1)^{k-1}M_k(n)$$

where $M_k(n)$ is the number of partitions of n where k is the least integer that does not occur as a part and there are more parts greater than k than there are less than k. We will show that $M_k(n)=C_k(n)$ where $C_k(n)$ is the number of partition pairs (S,U) where S is a partition with parts greater than k, U is a partition with $k-1$ distinct parts all of which are greater than the smallest part in S, and the sum of the parts in $S \cup U$ is n. We use partition pairs to determine what is counted by three similar expressions involving linear combinations of pentagonal numbers. Most of the results will be presented analytically and combinatorially.

Keywords: Partitions, Euler's pentagonal number theorem, Partition pairs.

1 Introduction

Euler's pentagonal number theorem gives an easy recurrence for the number of partitions of n, denoted by $p(n)$. Namely,

$$p(n)=\sum_{j=1}^{\infty}(-1)^{j+1}(p(n-j(3j-1)/2)+p(n-j(3j+1)/2))$$

*The results in Section 3 are based on Michael Burnette's undergraduate research project at UT Martin. He is currently a graduate student at Tennessee Tech.

where $p(k) = 0$ if $k < 0$. An interesting question is to determine how far off from $p(n)$ we are if we truncate this recurrence sum before we reach $n - j(3j-1)/2 < 0$ or $n - j(3j+1)/2 < 0$. In [1] Andrews and Merca answered this question when we stop the recurrence sum after an odd number of terms. In [3] Kolitsch gave an answer when we stop the recurrence sum with $p(n-1)+p(n-2)$. In Section 2 we will use generating functions to prove the general results. In section 3 we will interpret the results combinatorially.

2 A Generating Function Proof

If we define $B_k(n)$ for $k \geqslant 0$ to be the number of partition pairs (S,T) where S is a partition with parts greater than k, T is a partition with k distinct parts all of which are greater than the smallest part in S, and the sum of the parts in $S \cup T$ is n, then the generating function for $B_k(n)$ is given by

Theorem 1.

$$\sum_{n=0}^{\infty} B_k(n)q^n = \frac{1}{(q;q)_\infty} \sum_{j=k+1}^{\infty} (-1)^{j+k+1}(q^{j(3j-1)/2} + q^{j(3j+1)/2}).$$

As an immediate consequence of Theorem 1 and Euler's pentagonal number theorem we get

Corollary 2.

$$p(n) + \sum_{j=1}^{k}(-1)^j(p(n-j(3j-1)/2) + p(n-j(3j+1)/2)) = (-1)^k B_k(n)$$

since

$$\begin{aligned}\sum_{n=0}^{\infty} B_k(n)q^n &= \frac{1}{(q;q)_\infty} \sum_{j=k+1}^{\infty} (-1)^{j+k+1}(q^{j(3j-1)/2} + q^{j(3j+1)/2}) \\ &= \frac{1}{(q;q)_\infty}((-1)^{k+1}(q;q)_\infty - (-1)^{k+1} - \sum_{j=1}^{k}(-1)^{j+k+1}(q^{j(3j-1)/2} + q^{j(3j+1)/2})) \\ &= (-1)^{k+1} - (-1)^{k+1}(\sum_{n=0}^{\infty} p(n)q^n)(1 + \sum_{j=1}^{k}(-1)^j(q^{j(3j-1)/2} + q^{j(3j+1)/2})).\end{aligned}$$

Comparing coefficients of q^n, we get the desired corollary.

To prove Theorem 1 we note that for $j > k$ the generating function for partitions that fulfill the criterion to be a partition S as described above with smallest part j is $\frac{q^j}{(q^j;q)_\infty}$ and the corresponding generating function for partitions that fulfill the criterion to be a partition T as described above is $\frac{q^{k(j+1)+\binom{k}{2}}}{(q;q)_k}$.

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} B_k(n)q^n &= \sum_{j=k+1}^{\infty}\left(\frac{q^j}{(q^j;q)_\infty}\cdot\frac{q^{k(j+1)+\binom{k}{2}}}{(q;q)_k}\right)\\
&= \frac{1}{(q;q)_\infty}\sum_{j=k+1}^{\infty}\frac{q^{k(j+1)+\binom{k}{2}+j}(q;q)_{j-1}}{(q;q)_k}\\
&= \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty}\frac{q^{k(j+k+2)+\binom{k}{2}+j+k+1}(q;q)_{j+k}}{(q;q)_k}\\
&= \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty} q^{(3k^2+5k+2)/2}(q^{k+1})^j(q^{k+1};q)_j\\
&= \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}(-1)^{j+1}(q^{(j+k)(3(j+k)-1)/2}+q^{(j+k)(3(j+k)+1)/2})\\
&= \frac{1}{(q;q)_\infty}\sum_{j=k+1}^{\infty}(-1)^{j+k+1}(q^{j(3j-1)/2}+q^{j(3j+1)/2}).
\end{aligned}$$

To rewrite

$$\sum_{j=0}^{\infty} q^{(3k^2+5k+2)/2}(q^{k+1})^j(q^{k+1};q)_j$$

as

$$\sum_{j=1}^{\infty}(-1)^{j+1}(q^{(j+k)(3(j+k)-1)/2}+q^{(j+k)(3(j+k)+1)/2})$$

we are using identity 10 on page 29 in [2] with $x = q^k$.

If we define $C_k(n)$ for $k > 0$ to be the number of partition pairs (S, U) where S is a partition with parts greater than k, U is a partition with $k-1$ distinct parts all of which are greater than the smallest part in S, and the sum of the parts in $S \cup U$ is n then we have the following theorem.

Theorem 3.

$$\sum_{n=0}^{\infty} C_{k+1}(n)q^n = \sum_{n=0}^{\infty} B_k(n)q^n - \frac{q^{k(3k+5)/2+1}}{(q;q)_\infty}.$$

From the proof of Theorem 1 we have

$$\begin{aligned}
\sum_{n=0}^{\infty} C_{k+1}(n)q^n &= \sum_{j=k+2}^{\infty}\left(\frac{q^j}{(q^j;q)_\infty}\cdot\frac{q^{k(j+1)+\binom{k}{2}}}{(q;q)_k}\right)\\
&= \sum_{j=k+1}^{\infty}\left(\frac{q^j}{(q^j;q)_\infty}\cdot\frac{q^{k(j+1)+\binom{k}{2}}}{(q;q)_k}\right) - \frac{q^{k(3k+5)/2+1}}{(q;q)_\infty}
\end{aligned}$$

which gives the desired result. As an immediate consequence of Theorem 3 we get

Corollary 4.

$$\sum_{j=0}^{k}(-1)^j(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1))=(-1)^kC_{k+1}(n).$$

This corollary follows immediately from Corollary 2 by observing that Theorem 3 gives

$$C_{k+1}(n)=B_k(n)-p(n-k(3k+5)/2-1).$$

From Theorem 1 in [1] we get

Corollary 5.

$$C_k(n)=M_k(n)$$

where $M_k(n)$ is the number of partitions of n where k is the least integer that does not occur as a part and there are more parts greater than k than there are less than k.

Corollary 6. *For $k \geqslant 1$,*

$$\sum_{j=0}^{k-1}(-1)^j(p(n-j(3j+1)/2)-p(n-j(3j+5)/2-1))+(-1)^{k+1}p(n-k(3k+5)/2-1)$$

$$=(-1)^{k-1}(C_k(n)+D_k(n))$$

where $D_k(n)$ is the number of partition pairs (S,T) where S is a partition with parts greater than k containing at least one part equal to $k+1$, T is a partition with k distinct parts all of which are greater than $k+1$, and the sum of the parts in $S\cup T$ is n.

This corollary follows immediately by observing that

$$\sum_{n=0}^{\infty}(C_k(n)+D_k(n))q^n=\sum_{j=k+1}^{\infty}\left(\frac{q^j}{(q^j;q)_\infty}\cdot\frac{q^{(k-1)(j+1)+\binom{k-1}{2}}}{(q;q)_k-1}\right)+\frac{q^{k+1}}{(q^{k+1};q)_\infty}\cdot\frac{q^{k(k+2)+\binom{k}{2}}}{(q;q)_k}.$$

Corollary 7. *For $k \geqslant 2$,*

$$p(n)+\sum_{j=1}^{k-1}(-1)^j(p(n-j(3j-1)/2)+p(n-j(3j+1)/2))+(-1)^{k+1}p(n-(k+1)(3k+4)/2)$$

$$=(-1)^{k+1}(C_k(n)+E_k(n))$$

where $E_k(n)$ is the number of partition pairs (S,U) where S is a partition with one part equal to k and all other parts greater than k, U is a partition with $k-1$ distinct parts all of which are greater than k, and the sum of the parts in $S\cup U$ is n.

This corollary follows immediately by observing that

$$\sum_{n=0}^{\infty}(C_k(n)+E_k(n))q^n=\sum_{j=k+1}^{\infty}\left(\frac{q^j}{(q^j;q)_\infty}\cdot\frac{q^{(k-1)(j+1)+\binom{k-1}{2}}}{(q;q)_k-1}\right)+\frac{q^k}{(q^{k+1};q)_\infty}\cdot\frac{q^{(k-1)(k-1)+\binom{k-1}{2}}}{(q;q)_{k-1}}.$$

3 A Combinatorial Look at Our Results

In this section we will combinatorially verify the result observed from Theorem 3 that was used to prove Corollary 4 and the companion result that relates $C_k(n)$ and $B_k(n)$. These two relationships are stated in the next theorem.

Theorem 8. *For $k > 0$,*

$$(i)\ B_k(n) - p(n - k(3k+5)/2 - 1) = C_{k+1}(n)$$
$$(ii)\ p(n - k(3k+1)/2) = C_k(n) + B_k(n).$$

To prove part (i) of Theorem 8 we need to show how the partitions of $n-k(3k+5)/2-1$ bijectively correspond to the partition pairs (S, T) for n where S is a partition with all parts greater than k and $k+1$ is included as a part and T is a partition into k distinct parts greater than k. Given a partition $P = \{a_1, a_2, \ldots, a_r\}$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$ and $\sum_{i=1}^{r} a_i = n - k(3k+5)/2 - 1$, we will construct a partition pair $(\{k+1\} \cup \{a_i : a_i > k\}, \{t_1, t_2, \ldots, t_k\})$ by defining $t_i = (k+1+i) + \sum_{j=1}^{i} \alpha(k+1-j)$ where $\alpha(m)$ is the number of parts equal to m in P. This gives a partition pair of the desired type since $k + 1 + \sum_{j=1}^{k}(k+1+i) = k(3k+5)+1$ and $\sum_{i=1}^{k}\sum_{j=1}^{i} \alpha(k+1-j) = \sum_{a_i \in P, a_i \leqslant k} a_i$. Thus $B_k(n) - p(n - k(3k+5)/2 - 1)$ counts the number of partition pairs for n of the type counted by $C_{k+1}(n)$.

We illustrate the correspondence used to prove part (i) below:

Let $k = 2$ and $n = 25$. The partition $P = \{1, 2, 2, 3, 5\}$ corresponds to the partition pair $(\{3, 3, 5\}, \{6, 8\})$ and the partition pair $(\{3, 4, 6\}, \{5, 7\})$ corresponds to the partition $P = \{1, 2, 4, 6\}$.

To prove part (ii) of Theorem 8 we need to show that the partitions of $n-k(3k+1)/2$ bijectively correspond to the partition pairs counted by $B_k(n)+C_k(n)$. Given a partition $P = \{a_1, a_2, \ldots, a_r\}$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$ and $\sum_{i=1}^{r} a_i = n-k(3k+1)/2$, we will define $t_i = (k+i)+\sum_{j=1}^{i} \alpha(k+1-j)$ for $i = 1, 2, \ldots, k$. If t_1 is less than the smallest part in P that is larger than k, our partition pair will be given by $(\{t_1\} \cup \{a_i : a_i > k\}, \{t_2, t_3, \ldots, t_k\})$ (note if $k = 1$ then $T = \{\ \}$). These partition pairs are counted by $C_k(n)$. If t_1 is greater than or equal to the smallest part in P that is larger than k, our partition pair will be given by $(\{a_i : a_i > k\}, \{t_1, t_2, t_3, \ldots, t_k\})$. These partition pairs are counted by $B_k(n)$.

We illustrate the correspondence used to prove part (ii) below:

Let $k = 2$ and $n = 25$. The partition $P = \{1, 1, 1, 2, 3, 5, 5\}$ corresponds to the partition pair $(\{3, 5, 5\}, \{4, 8\})$ and the partition pair $(\{3, 4, 5, 6\}, \{7\})$ corresponds to the partition $P = \{1, 1, 1, 4, 5, 6\}$.

We now present a combinatorial proof of Corollary 5 by showing how the partitions of n counted by $M_k(n)$ bijectively correspond to the partition pairs counted by $C_k(n)$. Given a partition $P = \{a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v\}$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_u < k < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_v\}$, $u < v$ and $\sum_{i=1}^{u} a_i + \sum_{i=1}^{v} b_i = n$, we will define $x_i = b_{v-u+i} + a_i$ for $i = 1, 2, \ldots, u$. The corresponding partition pair will be (S, T) where the $k-1$ elements of T are defined by $t_j =$ smallest value among the x_i's where $a_i = j$ for $j = 1, 2, \ldots, k-1$ and $S = \{b_i : i \leqslant v - u\} \cup (\{x_i : i = 1, 2, \ldots, u\} - T)$.

We illustrate the correspondence used to prove Corollary 5 below: Let $k = 3$ and $n = 31$. The partition $P = \{1, 1, 2, 4, 5, 5, 6, 7\}$ will be transformed to $\{5, ⑥, 7, ⑨\}$ where the t_j's have been circled. The corresponding partition pair will be $(\{4, 5, 7\}, \{6, 9\})$. If we look at the bijection in the other direction and start with the partition pair $(\{4, 5, 5, 6\}, \{5, 6\})$ we will first transform it to $\{4, ⑤, 5, 5, ⑥, 6\}$. This will then become the partition $\{1, 1, 1, 2, 2, 4, 4, 4, 4, 4, 4\}$ counted by $M_3(31)$.

We now define $N_k(n)$ for $k > 0$ to be the number of of partitions of n where $1, 2, \ldots, k$ all occur as a part and there are more parts greater than k than there are less than or equal to k. The following theorem holds.

Theorem 9. *For $k > 0$, $N_k(n) = B_k(n)$.*

We can use a correspondence similar to the one used to prove Corollary 5 to prove Theorem 9. Given a partition $P = \{a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_v\}$ with

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_u = k < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_v,$$

$u < v$ and $\sum_{i=1}^{u} a_i + \sum_{i=1}^{v} b_i = n$, we will define $x_i = b_{v-u+i} + a_i$ for $i = 1, 2, \ldots, u$. The corresponding partition pair will be (S, T) where the k elements of T are defined by $t_j =$ smallest value among the x_i's where $a_i = j$ for $j = 1, 2, \ldots, k$ and $S = \{b_i : i \leqslant v - u\} \cup (\{x_i : i = 1, 2, \ldots, u\} - T)$.

We illustrate the correspondence used to prove Theorem 9 below: Let $k = 3$ and $n = 31$. The partition $P = \{1, 1, 2, 3, 4, 5, 5, 5, 5\}$ will be transformed to $\{4, ⑥, 6, ⑦, ⑧\}$ where the t_j's have been circled. The corresponding partition pair will be $(\{4, 6\}, \{6, 7, 8\})$. If we look at the bijection in the other direction and start with the partition pair $(\{4, 4, 5\}, \{5, 6, 7\})$ we will first transform it to $\{4, 4, ⑤, 5, ⑥, ⑦\}$. This will then become the partition $\{1, 1, 2, 3, 4, 4, 4, 4, 4, 4\}$ counted by $N_3(31)$.

As an immediate consequence of Theorem 9 we have

Corollary 10.

$$p(n) + \sum_{j=1}^{k} (-1)^j (p(n - j(3j-1)/2) + p(n - j(3j+1)/2)) = (-1)^k N_k(n).$$

4 Concluding Remarks

A natural question that arises from this paper is whether or not partition pairs can be used to interpret other truncated series. In particular, can they be used to answer question 2 posed by Andrews and Merca in [1]?

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