Intersecting families in the alternating group and direct product of symmetric groups

Cheng Yeaw Ku

Department of Mathematics, California Institute of Technology
Pasadena, CA 91125, USA
cyk@caltech.edu.

Tony W. H. Wong

Department of Mathematics,
The Chinese University of Hong Kong, Hong Kong
tonywhwong@yahoo.com.hk.

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Abstract

Let S_n denote the symmetric group on $[n] = \{1, \ldots, n\}$. A family $I \subseteq S_n$ is *intersecting* if any two elements of I have at least one common entry. It is known that the only intersecting families of maximal size in S_n are the cosets of point stabilizers. We show that, under mild restrictions, analogous results hold for the alternating group and the direct product of symmetric groups.

1 Introduction

Let S_n (or $\mathrm{Sym}([n])$) denote the symmetric group on the symbol-set $[n] = \{1, \ldots, n\}$. Throughout, the product (or composition) of two permutations $g, h \in S_n$, denoted by gh, will always mean 'do h first followed by g'. We say that a family $I \subseteq S_n$ of permutations is *intersecting* if $\{x : g(x) = h(x)\} \neq \emptyset$ for every $g, h \in I$, i.e. the Hamming distance $d_H(g, h) = |\{x : g(x) \neq h(x)\}| \leq n - 1$ for every $g, h \in I$. In a setting of coding theory, Deza and Frankl [5] studied extremal problems for permutations with given maximal or minimal Hamming distance. Among other results, they proved that if I is an intersecting family in S_n then $|I| \leq (n-1)!$. Recently, Cameron and Ku [4] showed that equality holds if and only if $I = \{g \in S_n : g(x) = y\}$ for some $x, y \in [n]$, i.e. I is a coset of a point stabilizer. This can also be deduced from a more general theorem of Larose and Malvenuto [8] about Kneser-type graphs.

Theorem 1.1 ([5], [4], [8]) *Let $n \geq 2$ and I be an intersecting family in S_n. Then $|I| \leq (n-1)!$. Moreover, equality holds if and only if $I = \{g \in S_n : g(x) = y\}$ for some $x, y \in [n]$.*

Here we extend the study of intersecting families of S_n to that of the alternating group A_n and the direct product of symmetric groups $S_{n_1} \times \cdots \times S_{n_q}$. We say that a family $I \subseteq A_n$ (or respectively $I \subseteq S_{n_1} \times \cdots \times S_{n_q}$) is *intersecting* if $\{x : g(x) = h(x)\} \neq \emptyset$ for any $g, h \in I$ (or respectively if, for every $(g_1, \ldots, g_q), (h_1, \ldots, h_q) \in I$, we have $\{x : g_i(x) = h_i(x)\} \neq \emptyset$ for some i). Our main results characterize intersecting families of maximal size in these groups.

Theorem 1.2 *Let $n \geq 2$ and I be an intersecting family in A_n. Then $|I| \leq (n-1)!/2$. Moreover, if $n \neq 4$, then equality holds if and only if $I = \{g \in A_n : g(x) = y\}$ for some $x, y \in [n]$.*

The following example shows that the condition $n \neq 4$ in Theorem 1.2 is necessary for the case of equality: $\{(1,2,3,4),\ (1,3,4,2),\ (2,3,1,4)\}$ (we use the notation $(a_1, \ldots, a_n)$ to denote the permutation that maps i to a_i).

Theorem 1.3 *Let $2 \leq m \leq n$ and I be an intersecting family in* $\mathrm{Sym}(\Omega_1) \times \mathrm{Sym}(\Omega_2)$, *$\Omega_1 = [m]$, $\Omega_2 = [n]$. Then $|I| \leq (m-1)!n!$. Moreover, for $m < n$ such that $(m, n) \neq (2, 3)$, equality holds if and only if $I = \{(g, h) : g(x) = y\}$ for some $x, y \in \Omega_1$, while for $m = n$ such that $(m, n) \neq (3, 3)$, equality holds if and only if $I = \{(g, h) : g(x) = y\}$ for some $x, y \in \Omega_1$ or $I = \{(g, h) : h(x) = y\}$ for some $x, y \in \Omega_2$.*

The following examples show that the conditions $(m, n) \neq (2, 3), (3, 3)$ in Theorem 1.3 are necessary for the case of equality:

- $J_{23} = \{((1,2),\ (2,3,1)),\ ((1,2),\ (1,2,3)),\ ((1,2),\ (3,1,2)),\ ((2,1),\ (2,1,3)),\ ((2,1),\ (3,2,1)),\ ((2,1),\ (1,3,2))\}$.
- $J_{33} = \{((1,3,2),\ (1,2,3)),\ ((2,1,3),\ (1,2,3)),\ ((2,1,3),\ (1,3,2)),\ ((2,1,3),\ (2,1,3)),\ ((2,1,3),\ (3,2,1)),\ ((2,3,1),\ (1,2,3)),\ ((2,3,1),\ (2,3,1)),\ ((2,3,1),\ (3,1,2)),\ ((3,1,2),\ (1,3,2)),\ ((3,1,2),\ (2,1,3)),\ ((3,1,2),\ (3,2,1)),\ ((3,2,1),\ (1,2,3))\}$.

For the direct product of finitely many symmetric groups, we prove

Theorem 1.4 *Let $2 \leq n_1 = \cdots = n_p < n_{p+1} \leq \cdots \leq n_q$, $1 \leq p \leq q$. Let $G = S_{n_1} \times \cdots \times S_{n_q}$ be the direct product of symmetric groups S_{n_i} acting on $\Omega_i = \{1, \ldots, n_i\}$. Suppose I is an intersecting family in G. Then*

$$|I| \leq (n_1 - 1)! \prod_{i=2}^{q} n_i!.$$

Moreover, except for the following cases:

- $n_1 = \cdots = n_p = 2 < n_{p+1} = 3 \leq n_{p+2} \leq \cdots \leq n_q$ *for some* $1 \leq p < q$,
- $n_1 = n_2 = 3 \leq n_3 \leq \cdots \leq n_q$,
- $n_1 = n_2 = n_3 = 2 \leq n_4 \leq \cdots \leq n_q$,

equality holds if and only if $I = \{(g_1, \ldots, g_q) : g_i(x) = y\}$ for some $i \in \{1, \ldots, p\}$, $x, y \in \Omega_i$.

The following examples show that the conditions for the case of equality are necessary:

- $S_{n_1} \times \cdots \times S_{n_{p-1}} \times J_{23} \times S_{n_{p+2}} \times \cdots \times S_{n_q}$ where $n_1 = \cdots = n_{p-1} = 2$,
- $J_{33} \times S_{n_3} \times \cdots \times S_{n_q}$,
- $J_{222} \times S_{n_4} \times \cdots \times S_{n_q}$,

where $J_{23} \subseteq S_2 \times S_3$ and $J_{33} \subseteq S_3 \times S_3$ are defined above and $J_{222} \subseteq S_2 \times S_2 \times S_2$ is given by

$$\{((1,2),(1,2),(1,2)),((1,2),(2,1),(2,1)),((1,2),(1,2),(2,1)),((2,1),(1,2),(2,1))\}.$$

In Section 2, we deduce Theorem 1.2 from a more general result by following an approach similar to [8], except that we utilize GAP share package GRAPE to establish the base cases needed for induction.

In Section3, we prove a special case of Theorem 1.4, namely when $n_i = n \geq 4$ for all $1 \leq i \leq q$. This is also a special case of a more general problem of determining independent sets of maximal size in tensor product of regular graphs, see [3] and [9] for recent interests in this area. For similar problems in extremal set theory, we refer the reader to [1] and [6].

In Section 4, we first prove Theorem 1.3, followed by a proof of Theorem 1.4.

We shall require the following tools from the theory of graph homomorphisms. Recall that a *clique* in a graph is a set of pairwise adjacent vertices, while an *independent set* is a set of pairwise non-adjacent vertices. For a graph Γ, let $\alpha(\Gamma)$ denote the size of the largest independent set in Γ. For any two graphs Γ_1 and Γ_2, a map ϕ from the vertex-set of Γ_1, denoted by $V(\Gamma_1)$, to the vertex-set $V(\Gamma_2)$ is a *homomorphism* if $\phi(u)\phi(v)$ is an edge of Γ_2 whenever uv is an edge of Γ_1, i.e. ϕ is an edge-preserving map.

Proposition 1.5 (Corollary 4 in [4]) *Let C be a clique and A be an independent set in a vertex-transitive graph on n vertices. Then $|C| \cdot |A| \leq n$. Equality implies that $|C \cap A| = 1$.*

The following fundamental result of Albertson and Collins [2], also known as the 'No-Homomorphism Lemma', will be useful.

Proposition 1.6 *Let Γ_1 and Γ_2 be graphs such that Γ_2 is vertex transitive and there exists a homomorphism $\phi : V(\Gamma_1) \to V(\Gamma_2)$. Then*

$$\frac{\alpha(\Gamma_1)}{|V(\Gamma_1)|} \geq \frac{\alpha(\Gamma_2)}{|V(\Gamma_2)|}. \tag{1}$$

Furthermore, if equality holds in (1), then for any independent set I of cardinality $\alpha(\Gamma_2)$ in Γ_2, $\phi^{-1}(I)$ is an independent set of cardinality $\alpha(\Gamma_1)$ in Γ_1.

2 Intersecting families in the alternating group

Throughout, A_n denotes the group of all even permutations of $[n]$. Let $\Gamma(A_n)$ be the graph whose vertex-set is A_n such that two vertices g, h are adjacent if and only if they do not intersect, i.e. $g(x) \neq h(x)$ for all $x \in [n]$. Clearly, left multiplication by elements of A_n is a graph automorphism; so $\Gamma(A_n)$ is vertex-transitive. By Proposition 1.5, the bound in Theorem 1.2 is attained provided there exists a clique of size n in $\Gamma(A_n)$, i.e. a Latin square whose rows are even permutations. Indeed, such a Latin square can be constructed as follows: consider the cyclic permutations $(1, 2, \dots, n)$, $(n, 1, 2, \dots, n-1)$, $\dots$, $(2, 3, \dots, n, 1)$. If n is odd then these permutations form the rows a Latin square as desired. If n is even then exactly half of these permutations are odd. Now, interchange the entries containing the symbols $n-2$ and n in these odd permutations. Together with the remaining even ones, they form a desired Latin square.

It remains to prove the case of equality of Theorem 1.2. It is feasible, by using GAP [7], to establish Theorem 1.2 for $n = 2, 3, 5, 6, 7$. For $n \geq 8$, we shall deduce Theorem 1.2 from the more general Theorem 2.1. The inductive argument in our proof is similar to [8] which we reproduce here for the convenience of the reader, except that we verify our base cases (see Lemma 2.4 and Lemma 2.5) with the help of a computer instead of proving them directly by hand, as in Lemma 4.5 of [8].

Define $A_n(b_1, \dots, b_r) = \{g \in A_n : \exists u \in \{0, 1, \dots, n-1\}$ such that $g(i+u) = b_i \;\; \forall i = 1, \dots, r\}$ where $i + u$ is in modulo n. For example, $A_5(1, 2, 3)$ consists of all even permutations of the form $(1, 2, 3, *, *)$, $(*, 1, 2, 3, *)$, $(*, *, 1, 2, 3)$, $(3, *, *, 1, 2)$, $(2, 3, *, *, 1)$.

Theorem 2.1 *For $n \geq 8$, let I be an intersecting family of maximal size in $A_n(b_1, \dots, b_r)$ where $1 \leq r \leq n-5$. Then $I = I_p^q \cap A_n(b_1, \dots, b_r)$ for some $p, q \in \{1, \dots, n\}$ where $I_p^q = \{g \in A_n : g(p) = q\}$.*

Lemma 2.2 *Let $\Gamma(A_n)(b_1, \dots, b_r)$ denote the subgraph of $\Gamma(A_n)$ induced by $A_n(b_1, \dots, b_r)$. Then, for $1 \leq r \leq n-3$,*

(i) *$\Gamma(A_n)(b_1, \dots, b_r)$ contains a clique of size n;*

(ii) *the graphs $\Gamma(A_n)(b_1, \dots, b_r)$ and $\Gamma(A_n)(1, \dots, r)$ are isomorphic, under an isomorphism which preserves the independent sets of the form $I_p^q \cap \Gamma(A_n)(b_1, \dots, b_r)$.*

(iii) *$\Gamma(A_n)(b_1, \dots, b_r)$ is vertex-transitive.*

Proof. (i) Let $\{b_1, \dots, b_n\} = [n]$. The construction is similar to that given above for the graph $\Gamma(A_n)$. Indeed, choose an even permutation w such that $w(i) = b_i$ for all $1 \leq i \leq n$ (the existence of such a permutation is guaranteed by the condition $n - r \geq 3$) and let $W = \{w, wc, wc^2, \dots, wc^{n-1}\}$ where $c = (n, 1, 2, \dots, n-1)$. If n is odd then W is the desired clique; otherwise wc^i is odd if and only if i is odd. For these odd permutations, interchange the entries containing b_{n-2} and b_n so that they become even. Together with the even permutations in W, they are now as required.

(ii) Let $h \in A_n$ such that $h(b_i) = i$ for all $1 \leq i \leq r$. Then the map $g \mapsto hg$ is the required isomorphism.

(iii) Let $g, h \in \Gamma(A_n)(1, \dots, r)$. Suppose $g(i) = h(j) = 1$ for some $i, j \in \{1, \dots, n\}$. Express g and h as $g'(n, 1, 2, \dots, n-1)^{i-1}$ and $h'(n, 1, 2, \dots, n-1)^{j-1}$ respectively such that g' and h' are permutations fixing $1, \dots, r$. Then the map $\phi : \Gamma(A_n(1, \dots, r)) \rightarrow$

$\Gamma(A_n(1,\ldots,r))$ given by $w \mapsto h'g'^{-1}w(n,1,2,\ldots,n-1)^{j-i}$ is a graph automorphism sending g to h. ■

Lemma 2.3 *Let $r \leq n-4$. If I is an independent set of $\Gamma(A_n)(b_1,\ldots,b_r)$ of maximal size then $I \cap \Gamma(A_n)(b_1,\ \ldots,\ b_r,\ b_{r+1})$ is an independent set of $\Gamma(A_n)(b_1,\ldots,b_r,b_{r+1})$ of maximal size.*

Proof. Applying Lemma 2.2 to $\Gamma_1 = \Gamma(A_n)(b_1,\ldots,b_{r+1})$ and $\Gamma_2 = \Gamma(A_n)(b_1,\ldots,b_r)$, we have the inclusions

$$K_n \hookrightarrow \Gamma_1 \hookrightarrow \Gamma_2 \hookrightarrow \Gamma(A_n)$$

so that

$$\frac{1}{n} \geq \frac{\alpha(\Gamma_1)}{|V(\Gamma_1)|} \geq \frac{\alpha(\Gamma_2)}{|V(\Gamma_2)|} \geq \frac{\alpha(\Gamma(A_n))}{|V(\Gamma(A_n))|} = \frac{1}{n}.$$

The result follows from Proposition 1.6. ■

Lemma 2.4 *Let $n \geq 8$ and $r = n-5$. Decompose $A_n(1,\ldots,r)$ into $B_n(u) = \{g \in A_n(1,\ldots,r) : g(1+u) = 1\}$, $u = 0,1,\ldots,n-1$.*
Suppose $I \subseteq C_n = B_n(0) \cup \left(\bigcup_{u=1}^{4} B_n(u) \cup B_n(n-u)\right)$ is an intersecting family. Then $|I| \leq 60$ with equality if and only if I consists of g such that $g(p) = q$ for some $p,q \in \{1,\ldots,n\}$.

Proof. It is readily checked (by using GAP) that the result holds for $8 \leq n \leq 14$. So let $n \geq 15$ and proceed by induction on n. Suppose n is odd. Let Γ_1 denote the graph whose vertex-set V_1 is C_{n-2} such that two vertices are adjacent if and only if they do not intersect. Similarly, Γ_2 denotes such a graph on $V_2 = C_n$. Define a map $\phi : C_{n-2} \to C_n$ such that if $g \in B_{n-2}(u)$ then

$$\phi(g)(i) = \begin{cases} g(i)+2 & \text{if } 1 \leq i \leq u, \\ 1 & \text{if } i = u+1, \\ 2 & \text{if } i = u+2, \\ g(i-2)+2 & \text{if } u+3 \leq i \leq n. \end{cases}$$

Since ϕ is a graph isomorphism (for $n \geq 15$) which also preserves independent sets of the form $I_p^q \cap C_{n-2}$, the result holds by induction for odd $n \geq 15$. The case for even n is similar. ■

Lemma 2.5 *Let $n \geq 8$ and $r = n-5$. Suppose $I \subseteq A_n(b_1,\ldots,b_r)$ is an intersecting family. Then $|I| \leq 60$ with equality if and only if I consists of g such that $g(p) = q$ for some $p,q \in \{1,\ldots,n\}$.*

Proof. By (ii) of Lemma 2.2, we assume, without loss of generality, that $A_n(b_1,\ldots,b_r) = A_n(1,\ldots,r)$ and the identity $(1,2,\ldots,n) \in I$. Since every other element of I must intersect the identity element, we deduce that $I \subseteq C_n = B_n(0) \cup \left(\bigcup_{u=1}^{4} B_n(u) \cup B_n(n-u)\right)$. The result now follows from Lemma 2.4. ■

Proof of Theorem 2.1. We shall imitate the proof of Theorem 4.2 in [8] by Larose and Malvenuto. For the argument to work for even permutations, we require a slightly greater degree of freedom, i.e $k = n-r \geq 5$, which is assumed by the theorem. As before, we may assume that $\Gamma(A_n)(b_1,\dots,b_r) = \Gamma(A_n)(1,\dots,r)$. Recall that $I_p^q = \{g \in A_n : g(p) = q\}$.

For $r = n-5$, this is Lemma 2.5. Assuming $1 \leq r \leq n-6$, we proceed by induction on $k = n-r$.

Case I. There exists $\beta \notin \{1,\dots,r\}$ with the property that $I \cap \Gamma(A_n)(1,\dots,r,\beta) = I_p^q \cap \Gamma(A_n)(1,\dots,r,\beta)$ for some $q \notin \{1,\dots,r,\beta\}$.

Let $g \in I$. Then there exists some u such that $g(i+u) = i$ for all $1 \leq i \leq r$. It is enough to show that $g(p) = q$. Now, construct another permutation $h \in I$ in the following order:

(i) set $h(p) = q$,

(ii) since $n-r \geq 6$, there are at least 5 choices of v such that $p \notin \{1+v, 2+v\dots,(r+1)+v\}$. Pick one of such v so that $v \neq u$ and $g((r+1)+v) \neq \beta$. Next, define $h(i+v) = i$ for all $1 \leq i \leq r$ and $h((r+1)+v) = \beta$.

(iii) there are at least 4 entries of h which have not yet been defined. Choose the remaining entries of h so that it is even and has no intersections with g in these entries.

Since both $g, h \in I$, we deduce that $g(p) = h(p) = q$.

By the inductive hypothesis and Lemma 2.3, it remains to consider:

Case II. For every $\beta \notin \{1,\dots,r\}$ there exists p and $q \in \{1,\dots,r,\beta\}$ such that $I \cap \Gamma(A_n)(1,\dots,r,\beta) = I_p^q \cap \Gamma(A_n)(1,\dots,r,\beta)$.

By permuting and relabeling entries, we may assume that the identity $id = (1,\dots,n) \in I$. Thus, $id \in I \cap \Gamma(A_n)(1,\dots,r,r+1) = I_p^q \cap \Gamma(A_n)(1,\dots,r,r+1)$. Without loss of generality, we may assume that $p = q = 1$ so that I now contains all even permutations which fix $1,\dots,r,r+1$. We shall prove that $I = I_1^1 \cap \Gamma(A_n)(1,\dots,r)$. Suppose, for a contradiction, that there exists $g \in I$ such that $g(1) \neq 1$, i.e. $g(i+u) = i$, $1 \leq i \leq r$, for some $u \neq 0$. Note that $g((r+1)+u) = \beta \neq r+1$, otherwise $g \in \Gamma(A_n)(1,\dots,r+1)$, forcing $g \in I_1^1 \cap \Gamma(A_n)(1,\dots,r+1)$. By induction again, we have

$$g \in I \cap \Gamma(A_n)(1,\dots,r,\beta) = I_{p'}^{q'} \cap \Gamma(A_n)(1,\dots,r,\beta)$$

for some $q' \in \{1,\dots,r,\beta\}$. As above, we conclude that I contains all even permutations h such that $h(i+u) = i$ for all $1 \leq i \leq r$ and $h((r+1)+u) = \beta$. If $\beta \neq (r+1)+u$, then we can find such a permutation h which is fixed-point free, contradicting the fact that $id \in I$. So $\beta = (r+1)+u$. Since now $\beta \notin \{1,\dots,r,r+1\}$ and $n-r \geq 6$, we can always find an even permutation $w \in I$ which fixes all $1 \leq i \leq r+1$ but does not intersect with h, a contradiction. ∎

3 A special case of Theorem 1.4

In this section we give the proof of a special case of Theorem 1.4, namely when all the n_i's are equal to $n \geq 4$. Throughout, G denotes the direct product of q copies of the symmetric group S_n acting on $[n]$.

Theorem 3.1 *Let $q \geq 1, n \geq 4$. Suppose I is an intersecting family of maximal size in G. Then*

$$|I| = (n-1)!n!^{q-1}.$$

Moreover, $I = \{(g_1, \ldots, g_q) : g_i(x) = y\}$ for some $1 \leq i \leq q$ and $x, y \in [n]$.

For our purpose, it is useful to view G as a subgroup of $\mathrm{Sym}(\Omega)$, where $\Omega = \{1, \ldots, qn\}$, which preserves a partition of Ω in the following way: let Σ be the partition of Ω into equal-sized subsets $\Omega_i = [(i-1)n+1, in]$, $i = 1, \ldots, q$, then G consists of $g \in \mathrm{Sym}(\Omega)$ such that $\Omega_i^g = \Omega_i$ for each i. For example, we identify the identity element $Id = (id, \ldots, id) \in G$ with $(1, 2, \ldots, qn) \in \mathrm{Sym}(\Omega)$. Therefore, a family $I \subseteq G$ is intersecting if and only if it is an intersecting family of $\mathrm{Sym}(\Omega)$. Moreover, for any $g \in G$ and $I \subseteq G$, we can now define $\mathrm{Fix}(g) = \{x \in \Omega : g(x) = x\}$ and $\mathrm{Fix}(I) = \{\mathrm{Fix}(g) : g \in I\}$ by regarding them as permutations of Ω.

For a proof of Theorem 3.1, we shall consider the cases $4 \leq n \leq 5$ and $n \geq 6$ separately. Indeed, when $n = 4, 5$, the result can be deduced from the following theorem of Alon et al. [3]. Recall that the *tensor product* of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \times \Gamma_2$, is defined as follows: the vertex-set of $\Gamma_1 \times \Gamma_2$ is the Cartesian product of $V(\Gamma_1)$ and $V(\Gamma_2)$ such that two vertices (u_1, v_1), (u_2, v_2) are adjacent in $\Gamma_1 \times \Gamma_2$ if u_1u_2 is an edge of Γ_1 and v_1v_2 is an edge of Γ_2. Let Γ^q denote the tensor product of q copies of Γ.

Theorem 3.2 (Theorem 1.4 in [3]) *Let Γ be a connected d-regular graph on n vertices and let $d = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be its eigenvalues. If*

$$\frac{\alpha(\Gamma)}{n} = \frac{-\mu_n}{d - \mu_n} \qquad (2)$$

then for every integer $q \geq 1$,

$$\frac{\alpha(\Gamma^q)}{n^q} = \frac{-\mu_n}{d - \mu_n}.$$

Moreover, if Γ is also non-bipartite, and if I is an independent set of size $\frac{-\mu_n}{d-\mu_n} n^q$ in Γ^q, then there exists a coordinate $i \in \{1, \ldots, q\}$ and a maximum-size independent set J in Γ, such that

$$I = \{(v_1, \ldots, v_q) \in V(\Gamma^q) : v_i \in J\}.$$

Theorem 3.3 *Theorem 3.1 holds for $n = 4, 5$.*

Proof. Let $n \in \{4,5\}$ and $\Gamma_n = \Gamma(S_n)$ be the graph whose vertex-set is S_n such that two vertices are adjacent if they do not intersect. It is easy to check that Γ_n is non-bipartite, connected and $d(n)$-regular where $d(n)$ is the number of derangements in S_n. In particular $d(4) = 9$ and $d(5) = 44$. Moreover, an independent set in Γ_n^q is an intersecting family in G. A MAPLE computation shows that the smallest eigenvalue of Γ_4 and Γ_5 are -3 and -11 respectively. The result now follows from Theorem 1.1 and Theorem 3.2. ■

We believe that relation (2) holds for $\Gamma(S_n)$ in general so that Theorem 3.1 follows immediately from Theorem 1.1 and Theorem 3.2. However, it seems difficult to compute the smallest eigenvalue of this graph. We conjecture the following:

Conjecture 1 *Let $n \geq 2$. Then the smallest eigenvalue of $\Gamma(S_n)$ is $-\frac{d(n)}{n-1}$.*

The rest of the proof of Theorem 3.1 is combinatorial. Our method combines ideas from [4] and an application of the 'No-Homomorphism Lemma'.

3.1 Closure under fixing operation

Let $x \in \{1, \ldots, n\}$, $g \in S_n$. We define the *x-fixing* of g to be the permutation $\lhd_x g \in S_n$ such that

(i) if $g(x) = x$, then $\lhd_x g = g$,

(ii) if $g(x) \neq x$, then

$$\lhd_x g(y) = \begin{cases} x & \text{if } y = x, \\ g(x) & \text{if } y = g^{-1}(x), \\ g(y) & \text{otherwise.} \end{cases}$$

Note that we can apply the fixing operation to an element $g \in G$ by regarding g as an element of $\mathrm{Sym}(\Omega)$. We also say that a family $I \subseteq S_n$ is *closed under the fixing operation* if

$$\text{for every } x \in \{1, \ldots, n\} \text{ and } g \in I, \text{ we have } \lhd_x g \in I.$$

Let $D_{S_n}(g) = \{w \in S_n : \ w(i) \neq g(i) \ \ \forall i = 1, ..., n\}$. The authors of [4] proved the following:

Lemma 3.4 (Proposition 6 in [4]) *Let $n \geq 2k$. Then, for any $g_1, g_2, ..., g_k \in S_n$, we have $D_{S_n}(g_1) \cap D_{S_n}(g_2) \cap ..... \cap D_{S_n}(g_k) \neq \emptyset$.*

Lemma 3.5 (Theorem 8 in [4]) *Let $n \geq 6$ and $I \subseteq S_n$ be an intersecting family of maximal size such that the identity element $id \in I$. Then I is closed under the fixing operation.*

Lemma 3.6 (Theorem 10 in [4]) *Let $S \subseteq S_n$ be an intersecting family of permutations which is closed under the fixing operation. Then $\mathrm{Fix}(S)$ is an intersecting family of subsets.*

The proof of Lemma 3.5 given in [4] can be easily modified to yield a similar result for G. For the convenience of the reader, we include the proof below.

Proposition 3.7 *Let $n \geq 6$ and $I \subseteq G$ be an intersecting family of maximal size such that $Id \in I$, $q \geq 1$. Then I is closed under the fixing operation.*

Proof. Let $\mathcal{L}$ denote the set of all n-subsets L of $\mathrm{Sym}(\Omega)$ such that for each i, the elements of L restricted to Ω_i form the rows of a Latin square of order n. Clearly, $\mathcal{L} \neq \emptyset$. By Proposition 1.5, for every $L \in \mathcal{L}$,

$$|L \cap I| \;=\; 1. \tag{3}$$

Assume, for a contradiction, that I is not closed under the fixing operation. Then there exists $g \in I$ such that $g(x) \neq x$ and $\lhd_x g \notin I$ for some $i \in \{1, \ldots, q\}$, $x \in \Omega_i$. Without loss of generality, we may assume that $i = x = 1$ (so $\lhd_1 g \notin I$) and consider the following cases:

Case I. $g(1) = 2$ and $g(2) = 1$.

Let $\Omega_1^* = \Omega_1 \setminus \{1, 2\}$. Consider the identity element Id restricted to Ω_1^*, denoted by $Id^* = Id|_{\Omega_1^*}$, and the permutation g restricted to Ω_1^*, denoted by $g^* = g|_{\Omega_1^*}$, which belong to $\mathrm{Sym}(\Omega_1^*) = G^*$. By Lemma 3.4, there exists $h^* \in D_{G^*}(Id^*) \cap D_{G^*}(g^*)$. Construct a new permutation $h' \in G^*$ as follows:

$$h'(y) = \begin{cases} h^*(y) & \text{if } y \in \Omega_1^*, \\ 2 & \text{if } y = 1, \\ 1 & \text{if } y = 2. \end{cases}$$

Applying Lemma 3.4 to each block Ω_i for $i = 2, \ldots, q$, we find a permutation $h'' \in D_{G''}(Id'') \cap D_{G''}(g'')$ where $Id'' = Id|_{\Omega_2 \cup \cdots \cup \Omega_q}$ and $g'' = g|_{\Omega_2 \cup \cdots \cup \Omega_q}$, $G'' = \mathrm{Sym}(\Omega_2 \cup \cdots \cup \Omega_q)$. Now, define $h \in G$ by

$$h(y) = \begin{cases} h'(y) & \text{if } y \in \Omega_1, \\ h''(y) & \text{otherwise}. \end{cases}$$

Then $\lhd_1 g$ and h form a Latin rectangle of order $2 \times qn$ which can now be completed to an element $L \in \mathcal{L}$ (since every Latin rectangle of order $2 \times n$ on Ω_i can be completed to a Latin square of order n on Ω_i). It is readily checked that no rows of L can lie in I, contradicting (3).

Case II. $g(1) = 2$ and $g(3) = 1$.

Let Ω_1^*, Id^*, G^* and h'' be defined as above. Now define $g^* \in G^*$ by

$$g^*(y) = \begin{cases} g(y) & \text{if } y \in \Omega_1^* \setminus \{3\}, \\ g(2) & \text{if } y = 3. \end{cases}$$

By Lemma 3.4, there is a permutation $h^* \in D_{G^*}(Id^*) \cap D_{G^*}(g^*)$.

Construct $h' \in \mathrm{Sym}(\Omega_1)$ as follows:

$$h'(y) = \begin{cases} 2 & \text{if } y = 1, \\ h^*(3) & \text{if } y = 2, \\ 1 & \text{if } y = 3, \\ h^*(y) & \text{otherwise}. \end{cases}$$

Again, defining $h \in G$ as above yields a contradiction. ■

It now follows immediately from Lemma 3.6 that

Proposition 3.8 *Let $q \geq 1, n \geq 6$ and $I \subseteq G$ be an intersecting family of maximal size such that $Id \in I$. Then* $\mathrm{Fix}(I)$ *is an intersecting family of subsets of* Ω.

3.2 Proof of Theorem 3.1

By Theorem 3.3, we may assume that $n \geq 6$. For $1 \leq i \leq n$, define $c_{(\to i)}$, $c_{(\leftarrow i)} \in S_n$ by:

$$\begin{aligned} c_{(\to i)}(j) &= n - i + j,\ 1 \leq j \leq n \\ c_{(\leftarrow i)}(j) &= i + j,\ 1 \leq j \leq n \end{aligned}$$

where the right hand side is in modulo n and 0 is written as n. In fact, we have already seen such cyclic permutations in Section 2, namely $c_{(\to 1)} = (n, 1, 2, \ldots, n-1)$, $c_{(\to i)} = c^i_{(\to 1)}$ for all $1 \leq i \leq n$, and $c_{(\to n)}$ is the identity. Observe that by right multiplication, $c_{(\to i)}$ acts on S_n by cyclicly (modulo n) moving each entry of g in i number of steps to the right. For example, if $g = (1, 3, 4, 2, 5)$, then $gc_{(\to 2)} = (2, 5, 1, 3, 4)$.

We proceed with induction on q. Let Γ' and Γ be the graphs formed on the vertex sets $G' = \mathrm{Sym}(\Omega_1) \times \cdots \times \mathrm{Sym}(\Omega_{q-1})$ and $G = \mathrm{Sym}(\Omega_1) \times \cdots \times \mathrm{Sym}(\Omega_q)$ respectively such that two vertices are adjacent if and only if none of their entries agree. Clearly,

$$\begin{aligned} \phi_* \ :\ & V(\Gamma') \to V(\Gamma), \\ & (g_1, \ldots, g_{q-1}) \mapsto (g_1, \ldots, g_{q-1}, g_1), \end{aligned} \tag{4}$$

defines a homomorphism from Γ' to Γ.

As before, let $\mathcal{L}$ denote the set of all n-subsets L of $\mathrm{Sym}(\Omega)$ such that for each i, the elements of L restricted to Ω_i form a Latin square of order n. By Proposition 1.5, I has the right size. Also, $\frac{\alpha(\Gamma')}{|V(\Gamma')|} = \frac{\alpha(\Gamma)}{|V(\Gamma)|}$.

Now, Proposition 1.6 implies that $\phi_*^{-1}(I)$ is an independent set of maximal size in Γ'. Without loss of generality, we may assume that the identity $Id = (id, \ldots, id) \in I$ so that, by the inductive hypothesis, we only need to consider the following cases:

Case I. $\phi_*^{-1}(I) = \{(g_1, \ldots, g_{q-1}) \in G' : g_u(z) = z\} = J^z_z$, for some $u \neq 1$, $z \in \Omega_u$.

Let $\Phi_1 = \phi_*(J^z_z) = \{(g_1, \ldots, g_{q-1}, g_1) \in G : g_u(z) = z\} \subseteq I$. Clearly we can find a permutation $g_u \in \mathrm{Sym}(\Omega_u)$ with $g_u(z) = z$ such that $g_u(x) \neq x$ for all $x \neq z$. Moreover, for $i \neq u$, we can choose $g_i \in \mathrm{Sym}(\Omega_i)$ such that it has no fixed points. Therefore our choice of the permutation $g = (g_1, \ldots, g_{q-1}, g_1) \in \Phi_1$ fixes a unique point, namely z. It follows from Proposition 3.8 that all permutations in I must fix z.

Case II. $\phi_*^{-1}(I) = \{(g_1, \ldots, g_{q-1}) \in G' : g_1(1) = 1\} = J^1_1$.

As above, let $\Phi_1 = \phi_*(J^1_1) = \{(g_1, \ldots, g_{q-1}, g_1) \in G : g_1(1) = 1\} \subseteq I$. We define another homomorphism from Γ' to Γ as follows:

$$\begin{aligned} \phi_{**} \ :\ & V(\Gamma') \to V(\Gamma), \\ & (g_1, \ldots, g_{q-1}) \mapsto (g_1, \ldots, g_{q-1}, g_1 c_{(\to 1)}). \end{aligned} \tag{5}$$

By induction, there exists $i \in \{1, \ldots, q-1\}$ such that

$$\phi_{**}^{-1}(I) \ = \ \{(g_1, \ldots, g_{q-1}) \in G' : g_i(u) = v\} = J^v_u,$$

for some u, $v \in \Omega_i$. Let

$$\Phi_2 = \phi_{**}(J_u^v) \subseteq I.$$

Suppose that $i \neq 1$. Then it is easy to see that there exist permutations $g \in \Phi_1$, $h \in \Phi_2$ such that $\mathrm{Fix}(g^{-1}h) = \emptyset$, that is they do not intersect, thus contradicting the intersection property of I. Therefore it suffices to consider the following cases where u, $v \in \Omega_1$.

Subcase i. $u \neq 1$, $v = 1$.

Assume for a moment that $u \neq n$. Let $g = (g_1, \ldots, g_{q-1}, g_1) \in \Phi_1$ where $g_1 = (1, a_2, \cdots, a_u, \cdots, a_n) \in \mathrm{Sym}(\Omega_1)$. Then there exists a permutation $h = (h_1, \ldots, h_{q-1}, h_1 c_{(\to 1)}) \in G$ where $h_1 = g_1 c_{(\to u-1)}$ and $\mathrm{Fix}(g_j^{-1}h_j) = \emptyset$ for all $j = 2, \ldots, q-1$. Obviously, $h \in \Phi_2 \subseteq I$ and $\mathrm{Fix}(g_1^{-1}h_1) = \mathrm{Fix}(g_1^{-1}h_1 c_{(\to 1)}) = \emptyset$. Hence $\mathrm{Fix}(g^{-1}h) = \emptyset$, which is a contradiction. So $u = n$.

Choose $h = (h_1, \ldots, h_{q-1}, h_1 c_{(\to 1)}) \in \Phi_2$ such that $h_1 = (n-1, n, 2, 3, \cdots, n-2, 1)$ and $\mathrm{Fix}(id_j^{-1}h_j) = \emptyset$ for all $j = 2, \ldots, q-1$ (id_j denotes the identity in $\mathrm{Sym}(\Omega_j)$). Moreover $h_1 c_{(\to 1)}$ fixes exactly one point since $n > 3$. Hence $|\mathrm{Fix}(h)| = 1$ and so by Proposition 3.8, all permutations in I must fix a common point.

Subcase ii. $u = v = 1$.

Choose $h = (h_1, \ldots, h_{q-1}, h_1 c_{(\to 1)}) \in \Phi_2$ such that $h_1 = (1, n, 2, 3, \cdots, n-1)$ and $\mathrm{Fix}(id_j^{-1}h_j) = \emptyset$. Clearly h fixes exactly one point and so we are done as before.

Subcase iii. $u \neq 2$, $v \neq 1$.

Take any permutation $h_1 \in S_n$ with $h_1(2) = 1$ and $h_1(u) = v$, say $h_1 = (a_1, 1, a_3, \cdots, a_{u-1}, v, a_{u+1}, \cdots, a_n)$. Let $g_1 = h_1 c_{(\leftarrow 1)} = (1, a_3, \cdots, a_{u-1}, v, a_{u+1}, \cdots, a_n, a_1)$ so that $g = (g_1, g_1, \ldots, g_1) \in \Phi_1 \subseteq I$ and $h = (h_1, \ldots, h_1, h_1 c_{(\to 1)}) \in \Phi_2 \subseteq I$. But it is easy to see that both g and h cannot agree in any entry, which is a contradiction.

Subcase iv. $u = 2$, $v \neq 1$.

Choose $h_1 = (a_1, v, 1, a_4, a_5, \cdots, a_n) \in S_n$. Let $g_1 = h_1 c_{(\leftarrow 2)} = (1, a_4, a_5, \cdots, a_n, a_1, v)$ so that $g = (g_1, \ldots, g_1) \in \Phi_1 \subseteq I$ and $h = (h_1, \ldots, h_1, h_1 c_{(\to 1)}) \in \Phi_2 \subseteq I$. Again, both g and h do not intersect, which is a contradiction.

This concludes the proof. ■

4 Intersecting families in the direct product of symmetric groups

Let S_m and S_n denote the symmetric groups acting on the symbol-set $\Omega_1 = \{1, 2, \ldots, m\}$ and $\Omega_2 = \{1, 2, \ldots, n\}$ respectively. The group $S_m \times S_n$ consists of ordered pairs (g, h) where $g \in S_m, h \in S_n$. Recall that a family $I \subseteq S_m \times S_n$ is intersecting if, for any $(g_1, h_1), (g_2, h_2) \in I$, either $\{x : g_1(x) = g_2(x)\} \neq \emptyset$ or $\{x : h_1(x) = h_2(x)\} \neq \emptyset$.

Proof of Theorem 1.3 Let Γ denote the graph whose vertex-set is $S_m \times S_n$ such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if $\{x : g_1(x) = g_2(x)\} = \emptyset$ and $\{x : h_1(x) = h_2(x)\} = \emptyset$. Clearly, Γ is vertex-transitive. As before, to obtain the upper bound of $|I|$, it is enough to show that there exists a clique of size m. Indeed, this is given by a Latin square of order m on Ω_1 and a Latin rectangle of order $m \times n$ on Ω_2.

For a proof of the characterization, we first form $(m-1)!$ Latin squares $L^1, \ldots, L^{(m-1)!}$ on the symbol-set Ω_1 as follows: for each $g \in \{g \in S_m : g(1) = 1\}$ in the point stabilizer of 1, form a Latin square whose rows consist of g and all its cyclic shifts. Clearly, these Latin squares partition S_m.

For each L^l, denote the i-th row by r_i^l. Let $T_i^l = \{h \in S_n : (r_i^l, h) \in I\}$. Further, decompose T_i^l into $T_{i1}^l, \ldots, T_{in}^l$ where $T_{ij}^l = \{h \in T_i^l : h(j) = 1\}$. Now, consider the following cases:

Case I. There exist $k, \alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ such that $T^k_{\alpha_1\beta_1} \neq \emptyset$ and $T^k_{\alpha_2\beta_2} \neq \emptyset$.

Suppose $\{\alpha_1, \ldots, \alpha_m\} = \Omega_1$. Choose pairwise distinct elements $\beta_1, \ldots, \beta_m \in \Omega_2$. Consider the sets $U_j^l = \bigcup_{i=1}^m T^l_{\alpha_i(\beta_i+j)}$, $0 \leq j \leq n-1$, where $\beta_i + j$ is in modulo n. Then

$$(m-1)!n! \;=\; |I| \;=\; \sum_{l=1}^{(m-1)!} \sum_{j=0}^{n-1} |U_j^l|. \tag{6}$$

Since U_j^l is intersecting, we have $|U_j^l| \leq (n-1)!$. In fact, it follows from (6) that $|U_j^l| = (n-1)!$ so that U_j^l must be a coset of a point stabilizer for every $0 \leq j \leq n-1$ and $1 \leq l \leq (m-1)!$ (by Theorem 1.1).

Suppose $m < n-1$. Since 1 appears in at least two (e.g. the β_1- and β_2-entry) but in at most $m \leq n-2$ different entries in U_0^k, we deduce that it cannot be a coset of a point stabilizer. Suppose $m = n-1$. Then $U_0^k = \{h \in S_n : h(\beta_n) = \gamma\}$ for some $\beta_n, \gamma \in \Omega_2$ where $\beta_n = \Omega_2 \setminus \{\beta_1, \ldots, \beta_{n-1}\}$ and $\gamma \neq 1$. Moreover, since $m = n-1 > 2$, we must have $T^k_{\alpha_{n-1}\beta_n} = \emptyset$ in order to preserve intersection with elements in U_0^k (note that this conclusion is not true if $(m, n) = (2, 3)$). Replacing our choice of β_{n-1} by β_n, the symbol 1 now appears in exactly $n-2$ different entries in the new U_0^k so that it cannot be a coset of a point stabilizer, a contradiction.

So, we may assume that $m = n$. It is readily checked that the result holds for $n = 2$. For $n \geq 4$, the result follows from Theorem 3.1.

Case II. For all k, there exist α_k, β_k such that $T^k_{\alpha_k\beta_k} \neq \emptyset$ and $T^k_{ij} = \emptyset$ for all $i \neq \alpha_k$.

If $T^k_{\alpha_k} \neq S_n$ for some k then $|I| < (m-1)!n!$, which is a contradiction. So $T^k_{\alpha_k} = S_n$ for all k. In order to preserve intersection, the maximality of I (by Theorem 1.1) implies that $I = \{(g, h) : g(x) = y\}$ for some $x, y \in \Omega_1$.

Case III. For all k, there exist α_k, β_k such that $T^k_{\alpha_k\beta_k} \neq \emptyset$ and $T^k_{ij} = \emptyset$ for all $j \neq \beta_k$.

If $T^k_{i\beta_k}$ is not a coset of the stabilizer of 1 in S_n for some i, k, then $|I| < (m-1)!m(n-1)! \leq (m-1)!n!$, contradicting the maximality of I. So $|I| = (m-1)!m(n-1)!$. Again, the maximality of I implies that $m = n$ and so I has the required shape as above. ■

Proof of Theorem 1.4 As before, the upper bound of $|I|$ is given by the existence of Latin squares of order n_1 and Latin rectangles of order $n_1 \times n_i$ for all $n_1 < n_i$. It remains to consider the case of equality with the following possibilities:

P1. $4 \leq n_1 \leq \cdots \leq n_q$;

P2. $3 = n_1 < n_2 \leq \cdots \leq n_q$;

P3. $2 = n_1 < n_2 \leq \cdots \leq n_q$ with $4 \leq n_2$;

P4. $2 = n_1 = n_2 < n_3 \leq \cdots \leq n_q$ with $4 \leq n_3$.

By Theorem 3.1, we may assume that $2 \leq n_1 = \cdots = n_p < n_{p+1} \leq \cdots \leq n_q$ for some $1 \leq p < q$ subject to the above possibilities. Set $m = n_1 = \cdots = n_p$ and $n = n_{p+1}$ so that $m < n$. For each $1 \leq i \leq p$, we first partition S_{n_i} into $(m-1)!$ Latin squares L^{il}, $1 \leq l \leq (m-1)!$, whose rows are $r_1^{il}, \ldots, r_m^{il}$. Next, for every choice of $\tilde{l} = (l_1, \ldots, l_p)$ where $1 \leq l_1, \ldots, l_p \leq (m-1)!$, construct m^{p-1} Latin rectangles as follows: fix π to be the cyclic permutation $(m, 1, 2, \ldots, m-1)$, then for every choice of $\tilde{j} = (j_2, j_3, \ldots, j_p)$ where $0 \leq j_2, \ldots, j_p \leq m-1$, construct a Latin rectangle whose rows consist of the following permutations from $S_{n_1} \times \cdots \times S_{n_p}$:

$$\begin{gathered}(r_1^{1l_1}, r_{\pi^{j_2}(1)}^{2l_2}, \ldots, r_{\pi^{j_p}(1)}^{pl_p}),\\ (r_2^{1l_1}, r_{\pi^{j_2}(2)}^{2l_2}, \ldots, r_{\pi^{j_p}(2)}^{pl_p}),\\ \vdots\\ (r_m^{1l_1}, r_{\pi^{j_2}(m)}^{2l_2}, \ldots, r_{\pi^{j_p}(m)}^{pl_p}).\end{gathered}$$

Denote this Latin rectangle by $L(\tilde{l}, \tilde{j})$ and its i-th row by $r_i(\tilde{l}, \tilde{j}) = (r_i^{1l_1}, r_{\pi^{j_2}(i)}^{2l_2}, \ldots, r_{\pi^{j_p}(i)}^{pl_p})$. Observe that these Latin rectangles partition $S_{n_1} \times \cdots \times S_{n_p}$ and there are $(m-1)!^p m^{p-1} = (m-1)! m!^{p-1}$ such Latin rectangles. Now, for each row $r_i(\tilde{l}, \tilde{j})$, define

$$T(r_i(\tilde{l}, \tilde{j})) = \{(h_{p+1}, \ldots, h_q) \in S_{n_{p+1}} \times \cdots S_{n_q} : (r_i^{1l_1}, r_{\pi^{j_2}(i)}^{2l_2}, \ldots, r_{\pi^{j_p}(i)}^{pl_p}, h_{p+1}, \ldots, h_q) \in I\}.$$

Further, partition $T(r_i(\tilde{l}, \tilde{j}))$ into

$$T(r_i(\tilde{l}, \tilde{j}))_j = \{(h_{p+1}, \ldots, h_q) \in T(r_i(\tilde{l}, \tilde{j})) : h_{p+1}(j) = 1\}, \quad 1 \leq j \leq n.$$

We shall prove the theorem by induction on $q \geq 2$. The base case $q = 2$ is the statement of Theorem 1.3. By the inductive hypothesis, we may assume that the result is true for $S_{n_{p+1}} \times \cdots \times S_{n_q}$ where $4 \leq n_{p+1} = \cdots = n_r < n_{r+1} \leq \cdots \leq n_q$ for some $p+1 \leq r \leq q$. We proceed by considering the following cases:

Case I. There exist $\tilde{l}, \tilde{j}$, $u \neq u'$, $v \neq v'$ such that $T(r_u(\tilde{l}, \tilde{j}))_v \neq \emptyset$ and $T(r_{u'}(\tilde{l}, \tilde{j}))_{v'} \neq \emptyset$.

Suppose $\{u_1 = u, u_2 = u', u_3, \ldots, u_m\} = \{1, \ldots, m\}$. Choose m pairwise distinct elements $v_1 = v, v_2 = v', v_3, \ldots, v_m$ from $\Omega_{p+1} = \{1, \ldots, n\}$. Consider the sets

$$U_w^{(\tilde{l}, \tilde{j})} = \bigcup_{i=1}^{m} T(r_{u_i}(\tilde{l}, \tilde{j}))_{v_i + w}, \quad 0 \leq w \leq n-1,$$

where $v_i + w$ is in modulo n. Then

$$(m-1)! m!^{p-1} \prod_{i=p+1}^{q} n_i! = |I| \;=\; \sum_{(\tilde{l}, \tilde{j})} \sum_{w=0}^{n-1} |U_w^{(\tilde{l}, \tilde{j})}|. \tag{7}$$

Since $U_w^{(\tilde{l},\tilde{j})}$ is intersecting, it follows from (7) that $|U_w^{(\tilde{l},\tilde{j})}| = (n-1)! \prod_{i=p+2}^q n_i!$ so that, by the inductive hypothesis, each $U_w^{(\tilde{l},\tilde{j})}$ has the form $\{(h_{p+1}, \dots, h_q) \in S_{n_{p+1}} \times \cdots \times S_{n_q} : h_s(x) = y\}$ for some $p+1 \leq s \leq r$, $x, y \in \Omega_s$.

Suppose $m < n-1$ (this covers the possibilities **P3** and **P4**). Since 1 appears in at least two (e.g. the v_1- and v_2-entry) but in at most $m \leq n-2$ different entries in the $S_{n_{p+1}}$-coordinate of elements in $U_0^{(\tilde{l},\tilde{j})}$, it cannot be a coset of a point stabilizer. So $m = n-1 > 2$ (since the possibilities **P3** and **P4** are now excluded). Since 1 appears in exactly $n-1$ different entries in the $S_{n_{p+1}}$-coordinate of elements in $U_0^{(\tilde{l},\tilde{j})}$, we deduce that $U_0^{(\tilde{l},\tilde{j})} = \{(h_{p+1}, \dots, h_q) \in S_{n_{p+1}} \times \cdots \times S_{n_q} : h_{p+1}(v_n) = z\}$ for some $v_n = \Omega_{p+1} \setminus \{v_1, \dots, v_{n-1}\}$ and $z \neq 1$. Moreover, since $m = n-1 > 2$, we must have $T(r_{u_{n-1}}(\tilde{l},\tilde{j}))_{v_n} = \emptyset$ in order to preserve intersection with elements in $U_0^{(\tilde{l},\tilde{j})}$. Replacing our choice of v_{n-1} by v_n, the symbol 1 now appears in exactly $n-2$ different entries in the $S_{n_{p+1}}$-coordinate of elements in the new $U_0^{(\tilde{l},\tilde{j})}$ so that it cannot be a coset of a point stabilizer, a contradiction.

Case II. For all $\tilde{l},\tilde{j}$, there exist u, v such that $T(r_u(\tilde{l},\tilde{j}))_v \neq \emptyset$ and $T(r_{u'}(\tilde{l},\tilde{j}))_{v'} = \emptyset$ for all $u' \neq u$.

If $T(r_u(\tilde{l},\tilde{j}))_v \neq S_{n_{p+1}} \times \cdots \times S_{n_q}$ for some $(\tilde{l},\tilde{j})$, then $|I| < (m-1)!m!^{p-1} \prod_{i=p+1}^q n_i!$, which is a contradiction. So $T(r_u(\tilde{l},\tilde{j}))_v = S_{n_{p+1}} \times \cdots \times S_{n_q}$ for all $(\tilde{l},\tilde{j})$. In order to preserve intersection, the maximality of I (using Theorem 3.1 if **P1** occurs or Theorem 1.1 if **P2** or **P3** occurs or Theorem 1.3 if **P4** occurs) implies that $I = \{(h_1, \dots, h_q) : h_i(x) = y\}$ for some $i \in \{1, \dots, p\}$, $x, y \in \Omega_i$.

Case III. For all $\tilde{l},\tilde{j}$, there exist u, v such that $T(r_u(\tilde{l},\tilde{j}))_v \neq \emptyset$ and $T(r_{u'}(\tilde{l},\tilde{j}))_{v'} = \emptyset$ for all $v' \neq v$.

If $T(r_{u'}(\tilde{l},\tilde{j}))_v \neq \{(h_{p+1}, \dots, h_q) : h_{p+1}(v) = 1\}$ for some u' and $(\tilde{l},\tilde{j})$, then $|I| < (m-1)!m!^{p-1} \cdot m \cdot (n-1)! \cdot \prod_{i=p+2}^q n_i!$, contradicting the maximality of I. So $|I| = (m-1)!m!^{p-1} \cdot m \cdot (n-1)! \cdot \prod_{i=p+2}^q n_i!$. Again, the maximality of I implies that $m = n$, a contradiction. ■

References

[1] R. Ahlswede, H. Aydinian and L. H. Khachatrian, The intersection theorem for direct products, *European Journal of Combinatorics* **19** (1998), 649–661.

[2] M. O. Albertson and K. L. Collins, Homomorphisms of 3-chromatic graphs, *Discrete Math.* **54** (1985), 127–132.

[3] N. Alon, I. Dinur, E. Friedgut and B. Sudakov, Graph products, fourier analysis and spectral techniques, *Geomatric And Functional Analysis* **14** (2004) 913–940.

[4] P. J. Cameron and C. Y. Ku, Intersecting families of permutations, *European Journal of Combinatorics* **24** (2003), 881–890.

[5] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, *Journal of Combinatorial Theory Ser A* **22** (1977), 352–360.

[6] P. Frankl, An Erdős-Ko-Rado theorem for direct products, *European Journal of Combinatorics* **17** (1996), 727–730.

[7] The GAP Group (2002), GAP–Groups, Algorithms and Programming, Aachen, St Andrews, Available from: http://www-gap.dcs.st-and.ac.uk/ gap.

[8] B. Larose and C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, *European Journal of Combinatorics* **25** (2004), 657–673.

[9] B. Larose and C. Tardif, Projectivity and independent sets in powers of graphs, *Journal of Graph Theory* **40** Issue 3 (2002), 162–171.