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Intersecting Loop Model as a Solvable Super Spin Chain

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In this paper, we investigate an integrable loop model and its connection with a supersymmetric spin chain. The Bethe ansatz solution allows us to study some properties of the ground state. When the loop fugacity q lies in the physical regime, we conjecture that the central charge is $c = q - 1$ for q integer <2. Low-lying excitations are examined, supporting a superdiffusive behavior for $q = 1$. We argue that these systems are interesting examples of integrable lattice models realizing $c \leq 0$ conformal field theories. [S0031-9007(98)06495-3]

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In statistical mechanics, the basic difference between an ordinary local model (vertex models, spin chain systems) and a loop model is that for the latter the total weight configuration cannot be written as a product of local weights. Being intrinsically nonlocal, loop models appear as ideal paradigms for studying statistical properties of extended objects such as polymers [1].

In this Letter, we investigate some critical properties of an integrable intersecting loop model in a two dimensional square lattice [2]. The fact that intersections between the polygon configurations are allowed makes this loop model very interesting. In this case, it is not clear how to find a transformation to a standard local model [3,4]. The model can be considered as a Lorentz gas of particles moving through a set of scatterers randomly distributed in the nodes of the two dimensional square lattice [5,6]. The scatterers are double-sided mirrors allowing right-angle reflections, and they are placed along the diagonals of the square lattice. The particles move along the edges of the lattice and, arriving on a node, can be scattered to the left, to the right, or pass freely in the case of the absence of a scatterer. We denote by w_a, w_b, and w_c the Boltzmann weights corresponding to these three possibilities, respectively. By imposing periodic boundary conditions, each particle follows a closed path. For every closed loop we assign a fugacity q, and the partition function is given by

$$Z = \sum_{\text{scatterer configurations}} w_a^{n_a} w_b^{n_b} w_c^{n_c} q^{\#\text{paths}}, \qquad (1)$$

where n_a, n_b, and n_c are the number of weights w_a, w_b, and w_c appearing in a configuration, respectively. Only when $w_c = 0$ the closed loop configurations no longer intersect, and this limit corresponds to a graphical representation of the critical q^2-state Potts model [3].

Despite the inherent nonlocality of this loop model, it is still possible to formulate a purely local condition that two different transfer matrices commute for arbitrary system size. This is a sufficient condition for integrability, and it imposes a restriction on the manifold of possible weights w_a, w_b, and w_c. It turns out that the intersecting loop model is integrable [2] if the Boltzmann weights are parametrized as follows:

$$w_a(\mu) = 1 - \mu, \qquad w_b(\mu) = \mu, \\ w_c(\mu) = (1 - q/2)\mu(1 - \mu). \qquad (2)$$

Here we argue that this integrable intersecting loop model with $q \in \mathbf{Z}$ can be realized in terms of a local supersymmetric $O\,sp(n|2m)$ spin chain. This not only provides a means for investigating the physical regime of the loop gas but also allows us to establish, for the first time in the literature, a theoretical framework to study the diffusive behavior of the Lorentz lattice gas itself. In particular, we use the fact that the $O\,sp(n|2m)$ super spin chain is solvable by the Bethe ansatz in order to find that their critical properties are governed by $c \leq 0$ conformal field theories. We remark that $c \leq 0$ theories appear to be useful in condensed matter systems such as the quantum Hall effect [7,8], disordered models [9], and polymer field theories [10]. In fact, we present evidence that the $O\,sp(1|2)$ spin chain is a strong lattice candidate for describing the low energy behavior of the Haldane-Rezayi *fractional* quantum Hall state [11].

Essential to our approach is to observe that the weights (2) can be derived in the context of a standard Yang-Baxter solution for a local vertex model. These weights are in one-to-one correspondence to the generators of a degenerated point of the Birman-Wenzel-Murakami algebra [12]. This algebra is generated by the identity I_i, a braid b_i, and a Temperely-Lieb operator E_i acting on sites i and $i + 1$ of a quantum spin chain of length L. On the degenerated point, the braid operator becomes a generator of the symmetric group, namely,

$$b_i b_{i\pm1} b_i = b_{i\pm1} b_i b_{i\pm1}, \qquad b_i^2 = I_i, \\ b_i b_j = b_j b_i \quad \text{if } |i - j| \geq 2, \qquad (3)$$

and the other set of relations closing the degenerated point of the braid-monoid algebra (see, e.g., Ref. [13]) are

$$E_i E_{i\pm1} E_i = E_i, \qquad E_i^2 = qE_i, \\ E_i E_j = E_j E_i \quad \text{if } |i - j| \geq 2, \qquad (4)$$

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and

$$b_i E_i = E_i b_i = E_i, \quad E_i b_{i\pm 1} b_i = b_{i\pm 1} b_i E_{i\pm 1} = E_i E_{i\pm 1}. \tag{5}$$

It is not difficult to see that relations (3)–(5) can be made to provide us with a rational solution of the Yang-Baxter equation having precisely the weights w_a, w_b, and w_c. To make further progress, we search for a representation of the algebraic relations (3)–(5). At least for integer q, such representation can be found in terms of the invariants of the superalgebra $\mathrm{Osp}(n|2m)$ [13]. This superalgeba combines the $\mathrm{O}(n)$ symmetry and the simplectic $\mathrm{Sp}(2m)$ algebra, and the integers n and $2m$ play the role of the number of bosonic and fermionic degrees of freedom. The braid operator b_i becomes the graded permutation between the $(n + 2m)$ degrees of freedom which is defined by [14]

$$b_i = \sum_{a,b=1}^{n+2m} (-1)^{p(a)p(b)} e_{ab} \otimes e_{ba}, \tag{6}$$

where $p(a)$ is the parity distinguishing the bosonic $[p(a) = 0$ for $a = 1, \ldots, n]$ and the fermionic $[p(a) = 1$ for $a = n + 1, \ldots, n + 2m]$ elements. Explicit expressions for the monoids E_i have been recently discussed in detail in Ref. [13]. The important point here is that the fugacity q is precisely the difference between the number of bosonic and fermionic degrees of freedom; namely,

$$q = n - 2m. \tag{7}$$

The formulation of the corresponding transfer matrix has to respect the bosonic and the fermionic graduations [14]. This is possible by writing $T(\lambda)$ as the supertrace of an auxiliary monodromy operator, $T(\lambda) = \sum_{a\in\mathcal{A}} (-1)^{p(a)} \mathcal{T}_{aa}$, where $\mathcal{A}$ stands for the horizontal space of $(n + 2m)$ variables of the vertex model. As usual, the monodromy matrix is composed by a product of vertex operators $\mathcal{L}_{\mathcal{A}i}(\lambda)$ which are given by

$$\mathcal{L}_{\mathcal{A}i}(\lambda) = (1 - q/2 - \lambda) b_i + \lambda(1 - q/2 - \lambda) I_i + \lambda E_i. \tag{8}$$

Performing the scale $\lambda \rightarrow \mu(1 - q/2)$ in Eq. (8), it is straightforward to see the correspondence between the weights (2) and the operators I_i, b_i, and E_i. The corresponding local $\mathrm{Osp}(n|2m)$ spin chain $\mathcal{H}$ is proportional to the logarithmic derivative of the transfer matrix around the regular point $\lambda = 0$, and its expression is given by

$$\mathcal{H} = \pm \sum_{i=1}^{L} \left\{ b_i + \frac{1}{1 - q/2} E_i \right\}, \tag{9}$$

where the sign in (9) is chosen to select the antiferromagnetic regime of the theory. This supersymmetric Hamiltonian, with periodic boundary conditions imposed, admits a Bethe ansatz solution. This means that the eigenenergies $E(L)$ on a ring of size L are parametrized in terms of a complex set of variables $\{\lambda_j^a\}$, satisfying coupled nonlinear Bethe ansatz equations. These equations are equivalent to the analyticity of the eigenvalues of the corresponding transfer matrix and also reflect the underlying $\mathrm{Osp}(n|2m)$ group symmetry. We derived that they are given by

$$\left[\frac{\lambda_j^{(a)} - i\frac{\delta_{a,1}}{2\eta_a}}{\lambda_j^{(a)} + i\frac{\delta_{a,1}}{2\eta_a}} \right]^L = \prod_{b=1}^{r} \prod_{k=1,k\neq j}^{m_b} \frac{\lambda_j^{(a)} - \lambda_k^{(b)} - i\frac{C_{a,b}}{2\eta_a}}{\lambda_j^{(a)} - \lambda_k^{(b)} + i\frac{C_{a,b}}{2\eta_a}}, \quad j = 1, \ldots, m_a; \quad a = 1, \ldots, r, \tag{10}$$

and the eigenenergies are parametrized by $\{\lambda_j^{(1)}\}$:

$$E(L) = -\sum_{i=1}^{m_1} \frac{1}{[\lambda_i^{(1)}]^2 + \frac{1}{4}} + L, \tag{11}$$

where $C_{a,b}$ is the Cartan matrix, r is the number of roots, and η_a is the normalized length of the ath root of the $\mathrm{Osp}(n|2m)$ superalgebra. For an algebraic Bethe ansatz derivation of Eqs. (10) and (11) for some classes of $\mathrm{Osp}(n|2m)$ models as well as for further technical details, we refer to Ref. [13].

We now turn to the study of the critical behavior of the super spin chain (9). The existence of a Bethe ansatz solution allows us to calculate the eigenvalues $E(L)$ for quite large values of L, providing us with reasonable estimates of the finite size effects. For a conformally invariant system, the universality can then be determined by exploiting a set of important relations satisfied by the eigenvalues on a strip of size L [15]. For example, the central charge c is related to the ground state energy $E_0(L)$ by [16]

$$\frac{E_0(L)}{L} = e_\infty - \frac{\pi v_s c}{6L^2} + \mathrm{O}(L^{-2}), \tag{12}$$

where e_∞ is the ground state energy per particle in the thermodynamic limit, and v_s is the sound velocity. These two parameters can be determined exactly from the unitarity and the crossing properties (around $\lambda = 1 - q/2$) of the transfer matrix $T(\lambda)$. In fact, these properties together imply that, in the thermodynamic limit, the largest eigenvalue $\Lambda_0(\lambda)$ of the transfer matrix satisfies the relations

$$\Lambda_0(\lambda)\Lambda_0(-\lambda) = [1 - \lambda^2][(1 - q/2)^2 - \lambda^2], \quad \Lambda_0(\lambda) = \Lambda_0(1 - q/2 - \lambda). \tag{13}$$

Solving these equations with the restriction that the solution is free of zeros in the physical strip $0 < \lambda < 1 - q/2$ and taking its logarithmic derivative at $\lambda = 0$, we find that

$$e_\infty = -\frac{1}{1 - q/2} \left\{ \psi\left(\frac{1}{2} + \frac{1}{2 - q}\right) - \psi\left(\frac{1}{2 - q}\right) + 2\ln(2) \right\} + 1, \tag{14}$$

where $\psi(x)$ is the Euler function. The sound velocity measures how the energy scales with the low momenta. If we recall that Eqs. (13) are identical to the one we solve to find the crossing factors in a relativistic S-matrix theory, we can obtain that the appropriate relativistic scale is given by

$$v_s = \frac{\pi}{1 - q/2}. \tag{15}$$

We now have the basic ingredients to begin our analysis of the finite size effects for the ground state energy. Let us first consider the case when the fugacity is 1. For this value of q, the partition function of the intersecting loop model is trivial, $E_0(L)/L$ is exactly e_∞ for any size L, and therefore $c = 0$. However, from the spin chain point of view, this scenario is far from being trivial, and provides us with an important check concerning the loop model $\leftrightarrow$ spin chain mapping. The simplest spin chain giving us $q = 1$ is the $O\,sp(3|2)$ model. Its spectrum is given in terms of one level nested Bethe ansatz, and the Bethe equations are parametrized by two sets of variables $\{\lambda_j^{(1,2)}\}$. The ground state is characterized by a complex root distribution, forming "fractional" strings of the following type:

$$\lambda_j^{(1,2)} = \xi_j^{(1,2)} \pm i/4 + O(e^{-L}). \tag{16}$$

By solving numerically the corresponding Bethe ansatz equation for some values of L and substituting the value of $\{\lambda_j^{(1)}\}$ in Eq. (11), it is remarkable to see how the Bethe ansatz roots conspire together in order to produce the simple result $E_0(L) = -3L$ exactly. Although a similar effect has been observed before in fine-tuned anisotropic spin chains [17], to the best of our knowledge, this is the first time that such simplification is noted in a free-parameter (isotropic) set of Bethe ansatz equations. This gives confidence to investigate the super spin chains for other values of $q < 2$.

For $q \neq 1$, the procedure described above can also be used, and we have analyzed Eqs. (10) and (11) for sizes up to $L = 80$. In Table I, we show our estimates for the central charge c for the $O\,sp(2|2)$, $O\,sp(1|2)$, $O\,sp(1|4)$, and $O\,sp(1|6)$ supersymmetric spin chains. In our numerical analysis, we already have considered the presence of logarithmic contributions of $O(1/L^2 \ln(L))$ to the finite size corrections of the ground state. We remark that this kind of correction typically cannot be overcome by a standard transfer matrix or Hamiltonian diagonalization due to size limitations. All of the results lead us to the following conjecture for the central charge behavior for these models when q is an integer <2:

$$c = q - 1. \tag{17}$$

This formula also reproduces the central charge in the limit $q = 2$. For this point, the weight w_c vanishes and, hence, we expect the critical behavior of the isotropic six-vertex model. Furthermore, the ground state of the $O\,sp(1|2n)$ models ($q = 1 - 2n$) are parametrized by real roots, and by using an analytical method developed in Ref. [18] we obtain $c = -2n$, in accordance with Eq. (17). Finally, we note that it is possible to derive formula (17) in the context of a super Sugawara construction of the Virasoro algebra by Goddard *et al.* [19].

TABLE I. Finite size sequences for the extrapolation of the central charge for some super spin chains.

L	$O\,sp(2\|2)$	$O\,sp(1\|2)$	$O\,sp(1\|4)$	$O\,sp(1\|6)$
40	−1.084 01	−1.950 41	−3.997 53	−6.104 83
48	−1.078 96	−1.953 69	−3.992 28	−6.076 58
56	−1.075 00	−1.956 17	−3.989 17	−6.058 63
64	−1.071 79	−1.958 12	−3.987 21	−6.046 43
72	−1.069 11	−1.959 73	−3.985 93	−6.037 71
80	−1.066 83	−1.961 07	−3.985 08	−6.031 25
Extr.	−1.01 (1)	−1.996 (2)	−3.985 (3)	−5.997 (1)

In order to provide extra physical insight in these models, we turn to the analysis of the excitations. This study is of particular physical relevance for the Lorentz lattice gas model ($q = 1$). In this case, the fractal dimension of the loops $d_f = 2 - 2h$, where h is a conformal weight, characterizes the superdiffusive properties of particles through the lattice [5,6]. Recent numerical simulations [6] predicted logarithmic superdiffusive behavior (corresponding to $d_f = 2$), when the density of mirrors is smaller than one ($w_c \neq 0$). To lend some theoretical support to this class of universality we have studied the finite size corrections for the lowest excitation present in the $q = 1$ model. We find that this excitation is of spin wave type, made by taking out one root λ_j^2 from the ground state configuration. In Table II, we present the finite size sequences for the exponent $x = 2h$. We see that the extrapolated value for h is very small, indicating the presence of a zero conformal weight and consequently predicting $d_f = 2 - 2h = 2$ within reasonable precision [20]. We remark that better numerical data was prevented by strong logarithmic corrections [6].

A second interesting example is the supersymmetric $O\,sp(1|2)$ spin chain. This model has $c = -2$ which is precisely the central charge of the conformal theory describing the Haldane-Rezayi state [7,8]. To see that this relationship goes beyond the central charge behavior, we follow Ref. [8] and study the excitations in the twisted $O\,sp(1|2)$ chain. The twisting plays the role of a fermionic index, and we select the Ramond sector

TABLE II. Finite size sequences for the extrapolation of the lowest dimension $x = 2h$ of the $q = 1$ model.

L	$x = 2h$
48	$6.032\,298 \times 10^{-2}$
64	$5.592\,822 \times 10^{-2}$
80	$5.288\,533 \times 10^{-2}$
96	$5.060\,282 \times 10^{-2}$
112	$4.880\,02 \times 10^{-2}$
128	$4.732\,45 \times 10^{-2}$
Extr.	1.33×10^{-2}

imposing antiperiodic boundary conditions in the even charge sectors of the Hamiltonian [21]. We found that the lowest Ramond dimension is $h = -\frac{1}{8}$, and the first physical excitation over the ground state has $h = \frac{3}{8}$, in accordance with the quasiparticle exponent predicted in Ref. [22]. More generally, the scaling dimensions can be interpreted in terms of a $c = 1$ Coulomb gas with radius $\rho = \sqrt{2}$, and also we verified the SU(2) invariance of the twisted Hamiltonian. This scenario remarkably agrees with the one proposed in Refs. [7,8,22] to describe the quasiparticle excitations of the Haldane-Rezayi *fractional* quantum Hall state. It should be noted that the integrability of this supersymmetric spin chain provides a means to calculate nonperturbatively extra physical quantities such as thermodynamic properties of the excitations. We also recall that other super spin chains have recently been investigated, though in the context of the plateau-to-plateau transition of the *integer* quantum Hall effect [23].

Finally, very recently relations between $c \leq 0$ and $c > 0$ theories have been investigated in the literature [24]. We have seen that the $Osp(1|2)$ super spin chain is such an example; i.e., its operator content can be seen either as a $c = -2$ or as a $c = 1$ conformal field theory. Similarly, we observed that the $Osp(2|2)$ model realizes both $c = -1$ and $c = 2$ field theories, suggesting that this is a common feature of the whole family of lattice models of this paper. This means that their Hilbert space can be positive definite [8,24], bringing us hope that this hierarchy of nonunitary models are physically meaningful, too.

In summary, a solvable loop model has been mapped onto a super spin chain, and we have found its central charge behavior for integer values of the fugacity in the physical regime. Our discussions support the idea that these models are ideal lattice candidates for describing conformal properties of relevant condensed matter systems. They also provide an interesting bridge between integrable models, motion in a random environment, and the quantum Hall effect.

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