Intersection bodies and valuations

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Dedicated to Prof. Erwin Lutwak on the occasion of his sixtieth birthday

Abstract

All $\mathrm{GL}(n)$ covariant star-body-valued valuations on convex polytopes are completely classified. It is shown that there is a unique non-trivial such valuation. This valuation turns out to be the so called 'intersection operator'– an operator that played a critical role in the solution of the Busemann-Petty problem.

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A function Z defined on the set $\mathcal{K}$ of convex bodies (that is, of convex compact sets) in $\mathbb{R}^n$ or on a certain subset $\mathcal{C}$ of $\mathcal{K}$ and taking values in an abelian semigroup is called a *valuation* if

$$\mathrm{Z}\,K + \mathrm{Z}\,L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{C}$. *Real valued* valuations are classical and Blaschke obtained the first classification of such valuations that are $\mathrm{SL}(n)$ invariant in the 1930s. This was greatly extended by Hadwiger in his famous classification of continuous, rigid motion invariant valuations and characterization of elementary mixed volumes. See [13], [17], [32], [33] for information on the classical theory and [1]–[4], [15], [16], [25], [26], [28] for some of the more recent results.

In [24], [27], a classification of *convex-body-valued* valuations $\mathrm{Z} : \mathcal{P} \to \mathcal{K}$ was obtained where $\mathcal{P}$ is the set of convex polytopes in $\mathbb{R}^n$ containing the origin and addition in $\mathcal{K}$ is Minkowski addition of convex bodies (defined by $K + L = \{x + y : x \in K, y \in L\}$). A valuation Z is called $\mathrm{GL}(n)$ *covariant*, if there exists a $q \in \mathbb{R}$ such that for all $\phi \in \mathrm{GL}(n)$ and all bodies K,

$$\mathrm{Z}(\phi K) = |\det \phi|^q \,\phi\, \mathrm{Z}\,K,$$

where $\det \phi$ is the determinant of ϕ. It is called $\mathrm{GL}(n)$ *contravariant*, if there exists a $q \in \mathbb{R}$ such that $\phi \in \mathrm{GL}(n)$ and all bodies K

$$\mathrm{Z}(\phi K) = |\det \phi|^q \,\phi^{-t}\, \mathrm{Z}\,K,$$

where ϕ^{-t} is the transpose of the inverse of ϕ. Since each body $K \in \mathcal{K}$ is determined by its support function, $h(K, \cdot) : S^{n-1} \to \mathbb{R}$, where $h(K, u) = \max\{u \cdot x : x \in K\}$ and where $u \cdot x$ denotes the standard inner product of u and x, these valuations can be defined via support functions. For $n > 2$, the classification theorems [27] are the following. An

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operator $\mathrm{Z} : \mathcal{P} \to \mathcal{K}$ is a $\mathrm{GL}(n)$ contravariant valuation if and only if there is a constant $c \geq 0$ such that

$$\mathrm{Z}\, P = c\, \Pi\, P$$

for every $P \in \mathcal{P}$. Here $\Pi\, P$ is the projection body of P, that is, $h(\Pi\, P, u) = \mathrm{vol}(P|u^\perp)$ for $u \in S^{n-1}$, where vol is the $(n-1)$-dimensional volume, $u^\perp$ is the subspace orthogonal to u, and $P|u^\perp$ is the image of the orthogonal projection of P onto $u^\perp$. An operator $\mathrm{Z} : \mathcal{P} \to \mathcal{K}$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there are constants $c_0 \in \mathbb{R}$ and $c_1 \geq 0$ such that

$$\mathrm{Z}\, P = c_0\, m(P) + c_1\, \mathrm{M}\, P$$

for every $P \in \mathcal{P}$. Here an operator is called *trivial*, if it is a linear combination of the identity and central reflection, while $m(P)$ is the moment vector and $\mathrm{M}\, P$ is the moment body of P, defined by,

$$m(P) = \int_P x\, dx \quad \text{and} \quad h(\mathrm{M}\, P, u) = \int_P |x \cdot u|\, dx,$$

for $u \in S^{n-1}$.

These results establish a classification of $\mathrm{GL}(n)$ covariant and contravariant valuations within the Brunn-Minkowski theory. In this paper we ask the corresponding question in the dual Brunn-Minkowski theory. In the dual theory convex bodies are replaced by star bodies and Minkowski addition of convex bodies is replaced by radial addition of star bodies (see next section for definitions). The natural question to ask is for a classification of *star-body-valued* valuations.

Let $\mathcal{S}$ denote the set of star bodies in $\mathbb{R}^n$, where a set $K \subset \mathbb{R}^n$ is a star body, if it is sharshaped with respect to the origin and has a continuous radial function $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$ (defined by $\rho(K, u)\, u \in \partial K$). Let $\mathcal{P}_0$ denote the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors and let $P^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1$ for every $y \in P\}$ denote the polar body of $P \in \mathcal{P}_0$.

Theorem. *An operator $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\, \mathrm{I}\, P^*$$

for every $P \in \mathcal{P}_0$.

Here $\mathrm{I}\, P^*$ is the *intersection body* of $P^* \in \mathcal{P}_0$, that is, the star body whose radial function is given for $u \in S^{n-1}$ by

$$\rho(\mathrm{I}\, P^*, u) = \mathrm{vol}(P^* \cap u^\perp).$$

In recent years, these intersections bodies have attracted increased interest within different subjects. They first appear in Busemann's [5] theory of area in Finsler spaces and were first explicitly defined and named by Lutwak [29]. Intersection bodies turned out to be critical for the solution of the *Busemann-Petty problem*: If the central hyperplane sections of an origin-symmetric convex body in $\mathbb{R}^n$ are always smaller in volume than those of another such body, is its volume also smaller? Lutwak [29] showed that the answer to the Busemann-Petty problem is affirmative if the body with the smaller sections is an intersection body of a star body. This led to the final solution that the answer is affirmative if $n \le 4$ and negative otherwise (see [7], [8], [10], [18], [19], [20], [35], [38], [39]). For further applications of intersection bodies, see [6], [11], [12], [14], [21], [34], and the books and surveys [9], [22], [23], [31], [36], [37].

The next section lists some basics regarding convex bodies, star bodies and valuations. Section 2 contains the proof of the theorem.

1 Notation and background material

General references on convex bodies and star bodies are the books by Gardner [9], Leichtweiß [23], Schneider [36], and Thompson [37]. We work in Euclidean n-space, $\mathbb{R}^n$, and write $x = (x_1, \dots, x_n)$ for $x \in \mathbb{R}^n$. Let $e_1, \dots, e_n$ denote the vectors of the standard basis of $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, let $x \cdot y = x_1\, y_1 + \dots + x_n\, y_n$ denote the inner product of x and y and let $|x|$ denote the length of x.

Let $K \in \mathcal{S}$. Then its radial function can extended to $v \in \mathbb{R}^n$, $v \neq 0$, by

$$\rho(K, v) = \max\{\lambda \ge 0 : \lambda\, v \in K\}.$$

It follows immediately that for $s > 0$ and $\phi \in \mathrm{GL}(n)$,

$$\rho(K, s\,v) = \frac{1}{s}\,\rho(K, v) \quad \text{and} \quad \rho(\phi K, v) = \rho(K, \phi^{-1} v). \tag{1}$$

The radial sum $K_1 \,\tilde{+}\, K_2$ of $K_1, K_2 \in \mathcal{S}$ is the star body whose radial function is given by

$$\rho(K_1 \,\tilde{+}\, K_2, v) = \rho(K_1, v) + \rho(K_2, v).$$

The set $\mathcal{S}$ equipped with the operation $\tilde{+}$ is an abelian semigroup and $\{0\}$ is its neutral element.

Let $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ be a valuation which is $\mathrm{GL}(n)$ *contravariant of weight* $q \in \mathbb{R}$, that is, for all $\phi \in \mathrm{GL}(n)$ and all $P \in \mathcal{P}_0$,

$$\mathrm{Z}\,\phi P = |\det \phi|^q\, \phi^{-t}\, \mathrm{Z}\, P.$$

We associate with Z an operator $\mathrm{Z}^* : \mathcal{P}_0 \to \mathcal{S}$ by setting $\mathrm{Z}^*\, P = \mathrm{Z}\, P^*$ for $P \in \mathcal{P}_0$. Let $P_1, P_2, P_1 \cup P_2 \in \mathcal{P}_0$. Since

$$(P_1 \cup P_2)^* = P_1^* \cap P_2^* \quad \text{and} \quad (P_1 \cap P_2)^* = P_1^* \cup P_2^*,$$

we obtain

$$\begin{aligned} \mathrm{Z}^*\, P_1 \,\tilde{+}\, \mathrm{Z}^*\, P_2 &= \mathrm{Z}\, P_1^* \,\tilde{+}\, \mathrm{Z}\, P_2^* &&= \mathrm{Z}(P_1^* \cup P_2^*) \,\tilde{+}\, \mathrm{Z}(P_1^* \cap P_2^*) \\ &= \mathrm{Z}(P_1 \cap P_2)^* \,\tilde{+}\, \mathrm{Z}(P_1 \cup P_2)^* &&= \mathrm{Z}^*(P_1 \cap P_2) \,\tilde{+}\, \mathrm{Z}^*(P_1 \cup P_2). \end{aligned}$$

Thus Z^* is a valuation on $\mathcal{P}_0$. Let $P \in \mathcal{P}_0$ and $\phi \in \mathrm{GL}(n)$. Since

$$(\phi P)^* = \phi^{-t} P^* \tag{2}$$

and since Z is $\mathrm{GL}(n)$ contravariant of weight q, we obtain

$$\mathrm{Z}^*(\phi P) = \mathrm{Z}(\phi P)^* = \mathrm{Z}(\phi^{-t} P^*) = |\det \phi|^{-q} \phi\, \mathrm{Z}^*\, P. \tag{3}$$

Thus Z^* is $\mathrm{GL}(n)$ covariant of weight $-q$.

2 Proof of the Theorem

Lutwak [30] showed that for all $\phi \in \mathrm{GL}(n)$ and all $K \in \mathcal{S}$

$$\mathrm{I}(\phi K) = |\det \phi|\, \phi^{-t}\, \mathrm{I}\, K.$$

By (3), $P \mapsto \mathrm{I}\, P^*$ is a $\mathrm{GL}(n)$ covariant valuation on $\mathcal{P}_0$ and we prove that up to multiplication with a constant this is the unique non-trivial such valuation. The proof consists of three steps. First, we extend $\mathrm{GL}(n)$ covariant valuations defined on $\mathcal{P}_0$ to valuations defined on a larger set of polytopes. Next, we derive a classification of valuations which are $\mathrm{GL}(n)$ covariant of weight $q \geq 0$. Then we derive a classification of valuations which are $\mathrm{GL}(n)$ contravariant of weight $q > 0$. This classification of contravariant valuations and (3) provide a classification of valuations which are $\mathrm{GL}(n)$ covariant of weight $q < 0$. Combined these results prove the theorem.

2.1 Extension

Let $\overline{\mathcal{P}}_0$ denote the set of convex polytopes P which are either in $\mathcal{P}_0$ or are the intersection of a polytope $P_0 \in \mathcal{P}_0$ and a polyhedral cone with apex at the origin and at most n facets. As a first step, we extend valuations $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ to simple valuations on $\overline{\mathcal{P}}_0$. Here a valuation is called *simple* if $\mathrm{Z}(P) = \{0\}$ for every $P \in \overline{\mathcal{P}}_0$ with dimension less than n.

We need the following definitions. For $A, A_1, \dots, A_k \subset \mathbb{R}^n$, let $[A_1, \dots, A_k]$ denote the convex hull of $A_1, \dots, A_k$ and let

$$A^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \quad \text{for every} \ \ y \in A\}.$$

For a central hyperplane H (that is, a hyperplane containing the origin), let H^+ and H^- denote the complementary closed halfspaces bounded by H. Let $\mathcal{P}_0(H)$ denote the set of convex polytopes in H that contain the origin in their interiors relative to H.

Let $\mathcal{C}_+(S^{n-1})$ denote the set of non-negative continuous functions on the unit sphere S^{n-1} and let $\overline{\mathcal{C}}_+(S^{n-1})$ denote the set of non-negative functions that are continuous almost everywhere on S^{n-1}. Note that if $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ is a valuation then the operator Y defined by $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)$ is a valuation taking values in $\mathcal{C}_+(S^{n-1})$. Let H be a central hyperplane and let $A \subset S^{n-1}$. For $P \in \mathcal{P}_0(H)$, we say that $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ *is vanishing on A at P* if

$$\lim_{u,v \to 0} \mathrm{Y}[P, u, v] = 0 \ \text{ locally uniformly on } A$$

for $u \in H^- \backslash H, v \in H^+ \backslash H$. For $P \in \mathcal{P}_0(H)$, we say that Y is *bounded at P* if there exists a constant $c \in \mathbb{R}$ such that

$$\mathrm{Y}[P, u, v](x) \leq c$$

for every $x \in S^{n-1}$ and $u \in H^- \backslash H, v \in H^+ \backslash H$ if $|u|, |v| \leq 1$ and $[P, u, v] = [P, u] \cup [P, v]$.

Lemma 1. *Let $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ be a valuation such that for every central hyperplane H and $P \in \mathcal{P}_0(H)$, Y is bounded and vanishing on $S^{n-1} \backslash H$ at P. Then Y can be extended to a simple valuation $\mathrm{Y} : \overline{\mathcal{P}}_0 \to \overline{\mathcal{C}}_+(S^{n-1})$ and for $P \in \overline{\mathcal{P}}_0$ bounded by central hyperplanes $H_1, \dots, H_n$, $\mathrm{Y}\, P$ is continuous and bounded on $S^{n-1} \backslash (H_1 \cup \dots \cup H_n)$.*

Proof. Let $\mathcal{P}_j$, $j = 1, \dots, n$, be the set of convex polytopes P such that there exist $P_0 \in \mathcal{P}_0$ and hyperplanes $H_1, \dots, H_j$ containing the origin with linearly independent normal vectors and

$$P = P_0 \cap H_1^+ \cap \cdots \cap H_j^+. \tag{4}$$

Define Y on $\mathcal{P}_j$, $j = 1, \dots, n$, inductively, starting with $j = 1$, in the following way. For $P \in \mathcal{P}_j$ and $u \in H_1 \cap \cdots \cap H_{j-1}$, $u \in H_j^- \backslash H_j$, set

$$\mathrm{Y}\, P = \lim_{u \to 0} \mathrm{Y}[P, u] \quad \text{on} \quad S^{n-1} \backslash (H_1 \cup \cdots \cup H_j). \tag{5}$$

We show that Y is well defined (that is, the limit in (5) exists and does not depend on the choice of H_j), that $\mathrm{Y}\, P$ is continuous and bounded on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_j)$, that for every hyperplane H and $P \in \mathcal{P}_j(H)$, we have for $u, v \in H_1 \cap \cdots \cap H_j$, $u \in H^+ \backslash H$, $v \in H^- \backslash H$,

$$\lim_{u,v \to 0} \mathrm{Y}[P, u, v] = 0 \text{ locally uniformly on } S^{n-1} \backslash H \tag{6}$$

and

$$\mathrm{Y}[P, u, v] \text{ is uniformly bounded on } S^{n-1} \text{ for } |u|, |v| \le 1, \tag{7}$$

and that Y has the following additivity properties:
If $P \in \mathcal{P}_{j-1}$ and H is a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_j$, then

$$\mathrm{Y}\, P = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-) \quad \text{on} \quad S^{n-1} \backslash (H_1 \cup \cdots \cup H_{j-1} \cup H). \tag{8}$$

If $P, Q, P \cap Q, P \cup Q \in \mathcal{P}_j$ are defined by (4) with halfspaces $H_1^+, \dots, H_j^+$, then

$$\mathrm{Y}\, P + \mathrm{Y}\, Q = \mathrm{Y}(P \cup Q) + \mathrm{Y}(P \cap Q) \quad \text{on} \quad S^{n-1} \backslash (H_1 \cup \cdots \cup H_j). \tag{9}$$

The operator Y is well defined and a valuation on $\mathcal{P}_0$. Suppose that Y is well defined by (5) on $\mathcal{P}_{k-1}$, that $\mathrm{Y}\, P$ is continuous and bounded on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_{k-1})$ for $P \in \mathcal{P}_{k-1}$ and that (6), (7), (8) (if $k > 1$) and (9) hold for $j < k$.

First, we show that the limit in (5) exists and that for $P \in \mathcal{P}_k$ bounded by hyperplanes $H_1, \dots, H_k$, $\mathrm{Y}\, P$ is bounded and continuous on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$. Let $u' \in H_1 \cap \cdots \cap H_{k-1}$, $u' \in H_k^- \backslash H_k$ be chosen such that $[P, u] \subseteq [P, u']$ and $-u' \in P$. Then applying (9) with $j = k - 1$ gives on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H_k, u', -u'] = \mathrm{Y}[P, u'] + \mathrm{Y}[P \cap H_k, u, -u']. \tag{10}$$

Consequently, for $x \in S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$|\, \mathrm{Y}[P, u](x) - \mathrm{Y}[P, u'](x)| \le \mathrm{Y}[P \cap H_k, u', -u'](x) + \mathrm{Y}[P \cap H_k, u, -u'](x). \tag{11}$$

Combined with (6) for $j = k - 1$, this implies that the limit in (5) exists locally uniformly and that $\mathrm{Y}\, P$ is continuous on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$. By (10) we have

$$\mathrm{Y}[P, u'](x) \le \mathrm{Y}[P, u](x) + \mathrm{Y}[P \cap H_k, u', -u'](x).$$

For u fixed, $\mathrm{Y}[P, u]$ is bounded by the induction assumption. Let $u' \to 0$. Then (7) implies that $\mathrm{Y}\, P$ is bounded on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$. For $k > 1$ we show that $\mathrm{Y}\, P$ as defined by (5) does not depend on the choice of the hyperplane H_k. Let $u \in H_1 \cap \cdots \cap H_{k-1}$,

$u \in H_k^- \backslash H_k$. Choose $w \in H_2 \cap \cdots \cap H_k$, $w \in H_1^- \backslash H_1$. Then applying (8) for $j = k-2$ gives on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$\Upsilon[P, u, w] = \Upsilon([P, u, w] \cap H_k^+) + \Upsilon([P, u, w] \cap H_k^-). \tag{12}$$

We have $[P, u, w] \cap H_k^- = [P \cap H_k, u, w]$ and $w \in H_k$. By (5) and (6) for $j = k-2$, we get on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$\lim_{u \to 0} \Upsilon[P \cap H_k, u, w] = 0.$$

Combined with $[P, u, w] \cap H_k^+ = [P, w]$, this implies that on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$\lim_{u \to 0} \Upsilon[P, u, w] = \Upsilon[P, w]. \tag{13}$$

Similarly, we get on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$\lim_{w \to 0} \Upsilon[P, u, w] = \Upsilon[P, u]. \tag{14}$$

Note that by an argument similar to (11) $\lim_{u,w \to 0} \Upsilon[P, u, w]$ exists. Thus (12) combined with (13) and (14) implies that

$$\lim_{u,w \to 0} \Upsilon[P, u, w] = \lim_{u \to 0} \Upsilon[P, u] = \lim_{w \to 0} \Upsilon[P, w] = \Upsilon P. \tag{15}$$

Thus Υ is well defined on $\mathcal{P}_k$.

Next, we show that (6) and (7) hold for $j = k < n$. Let $\varepsilon > 0$ be chosen. Let $P \in \mathcal{P}_k(H)$ be bounded by $H_1, \ldots, H_k$. Since $P \subset H$, ΥP is defined on $S^{n-1} \backslash H$. Let $x \in S^{n-1} \backslash H$. Choose $z \in H \cap H_1 \cap \cdots \cap H_{k-1}$ and $z \in H_k^- \backslash H_k$. Then $[P, z] \in \mathcal{P}_{k-1}(H)$ and by (6) for $j = k-1$,

$$\Upsilon[P, z, u, v] < \varepsilon \tag{16}$$

locally around x for $u, v \in H_1 \cap \cdots \cap H_k$, $u \in H^- \backslash H$ and $v \in H^+ \backslash H$ with $|u|, |v|$ sufficiently small. Since $[P \cap H_k, u, v] \in \mathcal{P}_{k-1}(H_k)$, (6) for $j = k-1$ implies that

$$\lim_{w \to 0} \Upsilon[P \cap H_k, u, v, -w, w] = 0 \quad \text{locally uniformly} \tag{17}$$

for $w \in H \cap H_1 \cap \cdots \cap H_{k-1}$ and $w \in H_k^- \backslash H_k$. Since Υ is a valuation,

$$\Upsilon[P, z, u, v] + \Upsilon[P \cap H_k, -w, w, u, v] = \Upsilon[P, w, u, v] + \Upsilon[P \cap H_k, -w, z, u, v].$$

Let $w \to 0$, then by (5) and (17)

$$\Upsilon[P, z, u, v] = \Upsilon[P, u, v] + \Upsilon[P \cap H_k, z, u, v]. \tag{18}$$

Since $\Upsilon \geq 0$, (18) combined with (16) implies that

$$\Upsilon[P, u, v] \leq \varepsilon$$

locally around x for $|u|, |v|$ sufficiently small. Thus (6) holds for $j = k$. It follows from (7) that $\Upsilon[P, z, u, v]$ is uniformly bounded for $|u|, |v| \leq 1$. Thus (18) implies that (7) holds for $j = k$.

Next, we show that (8) holds for $j = k$. Let $P \in \mathcal{P}_{k-1}$, that is, there exist $P_0 \in \mathcal{P}_0$ and hyperplanes $H_1, \ldots, H_{k-1}$ such that $P = P_0 \cap H_1^+ \cap \cdots \cap H_{k-1}^+$. Choose $u \in H_1 \cap \cdots \cap H_{k-1}$, such that $u \in P \cap H^+ \backslash H$ and $-u \in P \cap H^-$. Then P, $[P \cap H, u, -u]$, $[P \cap H^+, -u]$, $[P \cap H^-, u]$ have the hyperplanes $H_1, \ldots, H_{k-1}$ in common. Applying (9) for $j = k-1$ gives on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_{k-1} \cup H)$

$$\mathrm{Y}\, P + \mathrm{Y}[P \cap H, u, -u] = \mathrm{Y}[P \cap H^+, -u] + \mathrm{Y}[P \cap H^-, u].$$

By (6) and definition (5), this implies that (8) holds for $j = k$.

Finally, we show that (9) holds for $j = k$. Choose $u \in H_1 \cap \cdots \cap H_{k-1}$, $u \notin H_k$ such that $-u \in P \cap Q$. Applying (9) for $j = k-1$ shows that on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_k)$

$$\mathrm{Y}[P, u] + \mathrm{Y}[Q, u] = \mathrm{Y}[P \cup Q, u] + \mathrm{Y}[P \cap Q, u].$$

Because of definition (5) this implies that (9) holds for $j = k$.

The induction is now complete and Y is extended to $\overline{\mathcal{P}}_0$. As last step, we show that Y is a valuation on $\overline{\mathcal{P}}_0$. In addition to (8) and (9) it suffices to prove that if $P \in \mathcal{P}_n$ and H is a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_n$, then

$$\mathrm{Y}\, P = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-) \tag{19}$$

on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_n \cup H)$.

First, let $n = 2$. Let P be bounded by H_1, H_2, and let $P \cap H^+$ and $P \cap H^-$ be bounded by H_1, H and H, H_2, respectively. For $u \in H \cap (H_1^- \backslash H_1) \cap (H_2^- \backslash H_2)$, it follows from (8) that $\mathrm{Y}[P, u] = \mathrm{Y}[P \cap H^+, u] + \mathrm{Y}[P \cap H^-, u]$. By (5), this implies that

$$\lim_{u \to 0} \mathrm{Y}[P, u] = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-). \tag{20}$$

On the other hand, it follows from (8) that

$$\begin{aligned} \mathrm{Y}[P, u] &= \mathrm{Y}([P, u] \cap H_1^+) + \mathrm{Y}([P, u] \cap H_1^-) \\ &= \mathrm{Y}[P, w] + \mathrm{Y}([P, u] \cap H_1^- \cap H^-) + \mathrm{Y}([P, u] \cap H_1^- \cap H^+) \\ &= \mathrm{Y}[P, w] + \mathrm{Y}[P \cap H_1, u] + \mathrm{Y}[0, u, w]. \end{aligned} \tag{21}$$

where $w \in H_1$ depends on u. Because of (5), we have $\lim_{u \to 0} \mathrm{Y}[P, w] = \mathrm{Y}\, P$ and because of (6), we have $\lim_{u \to 0} \mathrm{Y}[P \cap H_1, u] = 0$. By (8), $\mathrm{Y}[P \cap H_2, u] = \mathrm{Y}[P \cap H_2, w] + \mathrm{Y}[0, u, w]$. Since by (6) $\lim_{u \to 0} \mathrm{Y}[P \cap H_2, u] = 0$ and $\lim_{u \to 0} \mathrm{Y}[P \cap H_2, w] = 0$, this implies that $\lim_{u \to 0} \mathrm{Y}[0, u, w] = 0$. Thus it follows from (21) that $\lim_{u \to 0} \mathrm{Y}[P, u] = \mathrm{Y}\, P$. Combined with (20) this implies (19).

Second, let $n \geq 3$. Let $P = P_0 \cap H_1^+ \cap \cdots \cap H_n^+$, $P_0 \in \mathcal{P}_0$. Since $P \cap H^+$, $P \cap H^- \in \mathcal{P}_n$, we can say that $P \cap H^+$ is bounded by $H_1, H, H_3, \ldots, H_n$ and that $P \cap H^-$ is bounded by $H, H_2, H_3, \ldots, H_n$, where $H_1 \cap H_2 \cap \cdots \cap H_{n-1} \subseteq H$. Therefore on $S^{n-1} \backslash (H_1 \cup \cdots \cup H_n \cup H)$

$$\mathrm{Y}\, P = \lim_{u \to 0} \mathrm{Y}[P, u]$$

and

$$\mathrm{Y}(P \cap H^+) = \lim_{u \to 0} \mathrm{Y}[P \cap H^+, u], \;\; \mathrm{Y}(P \cap H^-) = \lim_{u \to 0} \mathrm{Y}[P \cap H^-, u]$$

where $u \in H_1 \cap H_2 \cap \cdots \cap H_{n-1}, u \in H_n^- \backslash H_n$. Applying (8) for $j = n$ shows that

$$\mathrm{Y}[P, u] = \mathrm{Y}[P \cap H^+, u] + \mathrm{Y}[P \cap H^-, u].$$

Because of definition (5) this implies (19). This completes the proof of the lemma. $\square$

We also require the following lemmas. The proofs are similar to that of Lemma 1 and are omitted.

Lemma 2. *Let* $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ *be a valuation such that for every central hyperplane* H*,* Y *is vanishing on* S^{n-1} *at* $\mathcal{P}_0(H)$*. Then* Y *can be extended to a simple valuation* $\mathrm{Y} : \overline{\mathcal{P}}_0 \to \mathcal{C}_+(S^{n-1})$*.*

Lemma 3. *Let* $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ *be a valuation such that for every central hyperplane* H*,* Y *is vanishing and bounded on* $S^{n-1}\backslash H^\perp$ *at* $\mathcal{P}_0(H)$*. Then* Y *can be extended to a simple valuation* $\mathrm{Y} : \overline{\mathcal{P}}_0 \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0$ *bounded by hyperplanes* $H_1, \dots, H_n$*,* $\mathrm{Y}\,P$ *is continuous and bounded on* $S^{n-1}\backslash(H_1^\perp \cup \dots \cup H_n^\perp)$*.*

Lemma 4. *Let* $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ *be a valuation such that for every central hyperplane* H*,* Y *is vanishing on* $S^{n-1}\backslash H^\perp$ *at* $\mathcal{P}_0(H)$*. Then* Y *can be extended to a simple valuation* $\mathrm{Y} : \overline{\mathcal{P}}_0 \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0$ *bounded by hyperplanes* $H_1, \dots, H_n$*,* $\mathrm{Y}\,P$ *is continuous on* $S^{n-1}\backslash(H_1^\perp \cup \dots \cup H_n^\perp)$*.*

Note that if $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ is $\mathrm{GL}(n)$ covariant (contravariant), then the extended operator Z is $\mathrm{GL}(n)$ covariant (contravariant) on $\overline{\mathcal{P}}_0$.

2.2 Covariant valuations

We prove the following result.

Proposition 1. *Let* $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* $q \geq 0$*. Then there are constants* $c_1, c_2 \geq 0$ *such that*

$$\mathrm{Z}\,P = c_1\,P \,\tilde{+}\, c_2(-P)$$

for every $P \in \mathcal{P}_0$*.*

To extend Z to $\overline{\mathcal{P}}_0$, we apply Lemmas 1 and 2 and need the following result.

Lemma 5. *Let* $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* q *and let* $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ *be defined by* $\mathrm{Y}\,P(\cdot) = \rho(\mathrm{Z}\,P, \cdot)$*. Then for every central hyperplane* H *and* $P \in \mathcal{P}_0(H)$*, the following holds: If* $q > -1$*, then* Y *is vanishing on* $S^{n-1}\backslash H$ *at* P*. If* $q > 0$*, then* Y *is vanishing on* S^{n-1} *at* P*. If* $q \geq 0$*, then* Y *is bounded at* P*.*

Proof. Since Z is rotation covariant, it suffices to prove the statements for $H = e_n^\perp$. Let $P \in \mathcal{P}_0(H)$. Let $u \in H^-\backslash H$ and $v \in H^+\backslash H$ be chosen such that $[P, u, v] = [P, u] \cup [P, v]$ and let $r > 0$ be suitably small. Since Z is a valuation, we have

$$\mathrm{Z}[P, u, v] + \mathrm{Z}[P, -r\,u, -r\,v] = \mathrm{Z}[P, u, -r\,u] + \mathrm{Z}[P, v, -r\,v]$$

and

$$\mathrm{Z}[P, u, -r\,u] + \mathrm{Z}[P, r\,u, -u] = \mathrm{Z}[P, u, -u] + \mathrm{Z}[P, r\,u, -r\,u].$$

Thus to prove the lemma it suffices to show that $\mathrm{Y}[P, u, -u]$ is bounded on S^{n-1} for $|u| \leq 1$ for $q \geq 0$ and that $\lim_{u\to 0} \mathrm{Y}[P, u, -u] = 0$ locally uniformly on $S^{n-1}\backslash H$ for $q > -1$ or uniformly on S^{n-1} for $q > 0$.

Define $\phi_u \in \mathrm{GL}(n)$ by $\phi_u\, e_j = e_j$, $j = 1, \dots, n-1$, and $\phi_u\, e_n = u$. Then

$$\phi_u^{-1} x = (x_1 - \frac{u_1}{u_n} x_n, \dots, x_{n-1} - \frac{u_{n-1}}{u_n} x_n, \frac{x_n}{u_n}). \tag{22}$$

Since Z is $\mathrm{GL}(n)$ covariant of weight q, we obtain by (1)

$$\mathrm{Z}[P, u, -u](x) = \mathrm{Z}(\phi_u [P, e_n, -e_n])(x) = u_n^q\, \mathrm{Z}[P, e_n, -\, e_n](\phi_u^{-1} x).$$

and

$$\mathrm{Y}[P, u, -u](x) = u_n^{q+1}\ \mathrm{Y}[P, e_n, -e_n](u_n\, x_1 - u_1\, x_n, \dots, u_n\, x_{n-1} - u_{n-1}\, x_n, x_n). \tag{23}$$

For $q > -1$, this implies that $\lim_{u\to 0} \mathrm{Y}[P, u, -u] = 0$ locally uniformly on $S^{n-1}\backslash H$. If $x \in S^{n-1} \cap H$, it follows from (23) and (1) that

$$\mathrm{Y}[P, u, -u](x) = u_n^q\ \mathrm{Y}[P, e_n, -e_n](x_1, \dots, x_{n-1}, 0).$$

Thus, we obtain that $\lim_{u\to 0} \mathrm{Y}[P, u, -u] = 0$ uniformly on S^{n-1} for $q > 0$. Let $x \in S^{n-1}$. It follows from (23) and (1) that

$$\mathrm{Y}[P, u, -u](x) \le \frac{u_n^{q+1}}{|\phi_u^{-1} x|} \max_{w \in S^{n-1}} \mathrm{Y}[P, e_n, -e_n](w). \tag{24}$$

If $|u_n| \ge 4\, |x_n|$, then by (22)

$$|\phi_u^{-1} x| = |x + \frac{x_n}{u_n}(e_n - u)| \ge 1 - \frac{1}{4} |e_n - u| \ge \frac{1}{2}.$$

If $|u_n| \le 4\, |x_n|$, then by (22), $|\phi_u^{-1} x| \ge |(\phi_u^{-1} x) \cdot e_n| \ge \frac{1}{4}$. Thus (24) implies that Y is bounded at P for $q \ge 0$. $\square$

We also write Z for the extended operator. Let T be the simplex with vertices $0, e_1, \dots, e_n$. We determine $\mathrm{Z}\, T$. Since Z is $\mathrm{GL}(n)$ covariant,

$$\rho(\mathrm{Z}\, T, (x_1, \dots, x_n)) = \rho(\mathrm{Z}\, T, (x_{i_1}, \dots, x_{i_n})) \tag{25}$$

for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$. Let Z' be a simple and $\mathrm{GL}(n)$ covariant valuation on $\overline{\mathcal{P}}_0$. Note that it suffices to show that $\mathrm{Z}\, T = \mathrm{Z}'\, T$ to show that $\mathrm{Z}\, P = \mathrm{Z}'\, P$ for every $P \in \mathcal{P}_0$. This implies that Proposition 1 is a consequence of the following lemmas.

First, let $q > 0$. Note that in this case it follows from Lemma 2 and Lemma 5 that $\rho(\mathrm{Z}\, T, \cdot)$ is continuous on S^{n-1}.

Lemma 6. *Let* $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* $q > 0$. *Then* $\mathrm{Z}\, T = \{0\}$.

Proof. For $0 < \lambda_j < 1$, $j = 2, \dots, n$, let H_j be the central hyperplane with normal vector $\lambda_j\, e_1 - (1 - \lambda_j)\, e_j$. Then H_j dissects T into two simplices $T \cap H_j^+$ and $T \cap H_j^-$. Since Z is a simple valuation, we have

$$\mathrm{Z}\, T = \mathrm{Z}(T \cap H_j^+) \tilde{+} \mathrm{Z}(T \cap H_j^-). \tag{26}$$

For $j = 2, \dots, n$, define ϕ_j, ψ_j by

$$\begin{aligned} \phi_j e_j &= (1-\lambda_j)\, e_1 + \lambda_j\, e_j \quad &\text{and}& \quad \phi_j e_i = e_i \ \text{ for } \ i \neq j, \\ \psi_j e_1 &= (1-\lambda_j)\, e_1 + \lambda_j\, e_j \quad &\text{and}& \quad \psi_j e_i = e_i \ \text{ for } \ i \neq 1. \end{aligned}$$

Then $T \cap H_j^+ = \phi_j T$ and $T \cap H_j^- = \psi_j T$. Set $f(x) = \rho(\mathrm{Z}\, T, x)$ and let $x \neq 0$. Since Z is $\mathrm{GL}(n)$ covariant of weight q, (26) and (1) imply that

$$f(x) = \lambda_j^q\, f(\phi_j^{-1} x) + (1-\lambda_j)^q\, f(\psi_j^{-1} x). \tag{27}$$

Note that

$$\begin{aligned} \phi_j^{-1} e_j &= -\frac{1-\lambda_j}{\lambda_j}\, e_1 + \frac{1}{\lambda_j}\, e_j \quad &\text{and}& \quad \phi_j^{-1} e_i = e_i \ \text{ for } \ i \neq j, \\ \psi_j^{-1} e_1 &= \frac{1}{1-\lambda_j}\, e_1 - \frac{\lambda_j}{1-\lambda_j}\, e_j \quad &\text{and}& \quad \psi_j^{-1} e_i = e_i \ \text{ for } \ i \neq 1. \end{aligned}$$

From (27) with $x = e_1$, we obtain

$$f(e_1 - \lambda_j\, e_j) = \frac{1-\lambda_j^q}{(1-\lambda_j)^{q+1}}\, f(e_1). \tag{28}$$

Since $(1-\lambda_j^q)/(1-\lambda_j)^{q+1} \to \infty$ as $\lambda_j \to 1$, we obtain that $f(e_1) = 0$ and by (25) that $f(e_i) = 0$, $i = 1, \dots, n$. Similarly, we obtain that $f(-e_i) = 0$, $i = 1, \dots, n$. From (27), we obtain

$$f((1-\lambda_j)\, e_1 + \lambda_j\, e_j) = \lambda_j^q\, f(e_j) + (1-\lambda_j)^q\, f(e_1).$$

It follows from this and (28) that $f(x_1\, e_1 + x_j\, e_j) = 0$ for every $x_1, x_j \in \mathbb{R}$, $(x_1, x_j) \neq (0,0)$. Let $x' = x_2\, e_2 + \dots + x_{j-1}\, e_{j-1}$. Then by (27)

$$f((1-\lambda_j)\, e_1 + \lambda_j\, e_j + x') = \lambda_j^q\, f(e_j + x') + (1-\lambda_j)^q\, f(e_1 + x'). \tag{29}$$

By (25), $f(e_1 + x') = f(e_j + x')$. Thus by using induction on the number of vanishing coordinates, we obtain from (29) that $f(x) = 0$ for every $x \neq 0$. $\square$

Next, we consider the case $q = 0$. Note that in this case it follows from Lemma 1 and Lemma 5 that $\rho(\mathrm{Z}\, T, \cdot)$ is continuous and uniformly bounded on $S^{n-1} \backslash (e_1^\perp \cup \dots \cup e_n^\perp)$.

Lemma 7. *Let* $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* $q = 0$. *Then there are constants* $c_1, c_2 \geq 0$ *such that*

$$\rho(\mathrm{Z}\, T, x) = c_1\, \rho(T, x) + c_2\, \rho(-T, x)$$

for $x \notin e_1^\perp \cup \dots \cup e_n^\perp$.

Proof. We define H_j, ϕ_j, ψ_j, and f as in the proof of Lemma 6. Let $x \notin e_1^\perp \cup \dots \cup e_n^\perp$. Note that for given x, there is a dense set of λ_2 such that the subsequent expressions are well defined, that is, for example, $\phi_2^{-1} x \notin e_1^\perp \cup \dots \cup e_n^\perp$. As in (27), we have

$$f(x) = f(\phi_2^{-1} x) + f(\psi_2^{-1} x). \tag{30}$$

Using this repeatedly, we obtain

$$f(\psi_2^k\, x) = \sum_{j=1}^{k} f(\phi_2^{-1}\psi_2^j\, x) + f(x). \tag{31}$$

Let $x' = x_2\, e_2 + \cdots + x_n\, e_n$. Note that for ψ_2 the vectors e_i, $i = 2, \ldots, n$, are eigenvectors with eigenvalue 1 and the vector $e_1 - e_2$ is an eigenvector with eigenvalue $(1-\lambda_2)$. Setting $x = (e_1 - e_2) + x'$ in (31) gives

$$f((1-\lambda_2)^k\,(e_1-e_2) + x') = \sum_{j=1}^{k} f(\phi_2^{-1}((1-\lambda_2)^j\,(e_1-e_2) + x')) + f(x).$$

Let $k \to \infty$. Since f is uniformly bounded, continuous and non-negative, we obtain that $f(\phi_2^{-1} x') = 0$. Note that

$$\phi_2^{-1}\, x' = -\frac{1-\lambda_2}{\lambda_2}\, x_2\, e_1 + \frac{1}{\lambda_2}\, x_2\, e_2 + x_3\, e_3 + \cdots + x_n\, e_n.$$

Using (25) and the continuity of f, we conclude that

$$f(x_1, \ldots, x_n) = 0 \;\text{ if } x_i \neq 0 \text{ for } i = 1, \ldots, n, \text{ and not all } x_i \text{ have the same sign.} \tag{32}$$

Let $x_1, \ldots, x_n > 0$. Then by (30) and (32) we have

$$f(\phi_2\, x) = f(x) + f(\psi_2^{-1}\phi_2\, x) = f(x).$$

Thus

$$f(\phi_n \cdots \phi_2 x) = f(\phi_{n-1} \cdots \phi_2 x) = \cdots = f(x).$$

Since

$$\phi_n \cdots \phi_2\,(x_1, \ldots, x_n) = (x_1 + (1-\lambda_2)\,x_2 + \cdots + (1-\lambda_n)\,x_n, \lambda_2\,x_2, \ldots, \lambda_n\,x_n), \tag{33}$$

we obtain that

$$f(x_1, \ldots, x_n) = f(1, \ldots, 1) \quad \text{for } \; x_1 + \cdots + x_n = n,\; 0 < x_2, \ldots, x_n < 1.$$

By choosing λ_i such that $\lambda_i\, x_i < 1$, we obtain from this and (33) that

$$f(x_1, \ldots, x_n) = f(1, \ldots, 1) \quad \text{for } \; x_1 + \cdots + x_n = n,\; x_1, \ldots, x_n > 0.$$

Similarly, we obtain that

$$f(-x_1, \ldots, -x_n) = f(-1, \ldots, -1) \quad \text{for } \; x_1 + \cdots + x_n = n,\; x_1, \ldots, x_n > 0.$$

Thus $f(x) = c_1\, \rho(T, x) + c_2\, \rho(-T, x)$ with suitable constants $c_1, c_2 \geq 0$. □

2.3 Contravariant valuations

We prove the following result.

Proposition 2. *Let* $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $q > 0$. *Then there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = c\, \mathrm{I}\, P$$

for every $P \in \mathcal{P}_0$.

To extend Z to $\overline{\mathcal{P}}_0$, we apply Lemmas 2, 3 and 4 and need the following result.

Lemma 8. *Let* $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* q *and let* $\mathrm{Y} : \mathcal{P}_0 \to \mathcal{C}_+(S^{n-1})$ *be defined by* $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)$. *For every central hyperplane* H, *the following holds: If* $q > 0$, *then* Y *is vanishing on* $S^{n-1}\backslash H^\perp$ *at* $\mathcal{P}_0(H)$. *If* $q > 1$, *then* Y *is vanishing on* S^{n-1} *at* $\mathcal{P}_0(H)$. *If* $q \geq 1$, *then* Y *is bounded on* S^{n-1} *at* $\mathcal{P}_0(H)$.

Proof. Since Z is rotation contravariant, it suffices to prove the statements for $H = e_n^\perp$. Let $P \in \mathcal{P}_0(H)$. Let $u \in H^-\backslash H$, $v \in H^+\backslash H$ be chosen such that $[P, u, v] = [P, u] \cup [P, v]$ and let $r > 0$ be suitably small. Since Z is a valuation, we have

$$\mathrm{Z}[P, u, v] + \mathrm{Z}[P, -r\, u, -r\, v] = \mathrm{Z}[P, u, -r\, u] + \mathrm{Z}[P, v, -r\, v]$$

and

$$\mathrm{Z}[P, u, -r\, u] + \mathrm{Z}[P, r\, u, -u] = \mathrm{Z}[P, u, -u] + \mathrm{Z}[P, r\, u, -r\, u].$$

Thus to prove the lemma it suffices to show that $\mathrm{Y}[P, u, -u]$ is bounded on S^{n-1} for $|u| \leq 1$ for $q \geq 1$ and that $\lim_{u\to 0} \mathrm{Y}[P, u, -u] = 0$ locally uniformly on $S^{n-1}\backslash H^\perp$ for $q > 0$ or uniformly on S^{n-1} for $q > 1$.

Define $\phi_u \in \mathrm{GL}(n)$ by $\phi_u\, e_j = e_j$, $j = 1, \dots, n-1$, and $\phi_u\, e_n = u$. Then

$$\phi_u^t x = (x_1, \dots, x_{n-1}, x_1\, u_1 + \cdots + x_n\, u_n). \tag{34}$$

Since Z is $\mathrm{GL}(n)$ contravariant of weight q, we obtain by (1)

$$\mathrm{Z}[P, u, -u](x) = \mathrm{Z}(\phi_u[P, e_n, -e_n])(x) = u_n^q\, \mathrm{Z}[P, e_n, -\, e_n](\phi_u^t x).$$

and

$$\mathrm{Y}[P, u, -u](x) = u_n^q\; \mathrm{Y}[P, e_n, -e_n](x_1, \dots, x_{n-1}, x_1\, u_1 + \cdots + x_n\, u_n). \tag{35}$$

For $q > 0$, this implies that $\lim_{u\to 0} \mathrm{Y}[P, u, -u] = 0$ locally uniformly on $S^{n-1}\backslash H^\perp$. If $x \in S^{n-1} \cap H^\perp$, it follows from (35) and (1) that

$$\mathrm{Y}[P, u, -u](x) = u_n^{q-1}\; \mathrm{Y}[P, e_n, -e_n](0, \dots, 0, x_n).$$

Thus, we obtain that $\lim_{u\to 0} \mathrm{Y}[P, u, -u] = 0$ uniformly on S^{n-1} for $q > 1$. Let $x \in S^{n-1}$. It follows from (35) and (1) that

$$\mathrm{Y}[P, u, -u](x) \leq \frac{u_n^q}{|\phi_u^t x|} \max_{w\in S^{n-1}} \mathrm{Y}[P, e_n, -e_n](w). \tag{36}$$

Since $|u_n| \leq 1$, by (34)

$$\frac{|\phi_u^t x|^2}{u_n^2} = \frac{x_1^2}{u_n^2} + \cdots + \frac{x_{n-1}^2}{u_n^2} + \frac{(x \cdot u)^2}{u_n^2} \geq 1.$$

Thus (36) implies that Y is bounded at P for $q \geq 0$. $\square$

We also write Z for the extended operator. Let T be the simplex with vertices $0, e_1, \ldots, e_n$. We determine $\operatorname{Z} T$. Since Z is $\mathrm{GL}(n)$ contravariant,

$$\rho(\operatorname{Z} T, (x_1, \ldots, x_n)) = \rho(\operatorname{Z} T, (x_{i_1}, \ldots, x_{i_n})) \tag{37}$$

for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$. Let Z' be a simple and $\mathrm{GL}(n)$ contravariant valuation on $\overline{\mathcal{P}}_0$. Note that it suffices to show that $\operatorname{Z} T = \operatorname{Z}' T$ to show that $\operatorname{Z} P = \operatorname{Z}' P$ for every $P \in \mathcal{P}_0$. This implies that Proposition 2 is a consequence of the following lemmas.

First, let $q > 1$. Note that in this case it follows from Lemma 2 and Lemma 8 that $\rho(\operatorname{Z} T, \cdot)$ is continuous on S^{n-1}.

Lemma 9. *Let* $\operatorname{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $q > 1$. *Then* $\operatorname{Z} T = \{0\}$.

Proof. We define H_j, ϕ_j, ψ_j, and f as in the proof of Lemma 6. Let $x \neq 0$. Since Z is $\mathrm{GL}(n)$ contravariant of weight q, (26) and (1) imply that

$$f(x) = \lambda_j^q\, f(\phi_j^t\, x) + (1 - \lambda_j)^q\, f(\psi_j^t\, x). \tag{38}$$

From this with $x = e_1$, we obtain

$$f(e_1 + (1 - \lambda_j)\, e_j) = \frac{1 - (1 - \lambda_j)^{q-1}}{\lambda_j^q}\, f(e_1). \tag{39}$$

Since $(1 - (1 - \lambda_j)^{q-1})/\lambda_j^q \to \infty$ as $\lambda_j \to 0$, we obtain that $f(e_1) = 0$ and by (37) that $f(e_i) = 0$, $i = 1, \ldots, n$. Similarly, we obtain that $f(-e_i) = 0$, $i = 1, \ldots, n$. From (38), we obtain

$$f(\lambda_j\, e_1 - (1 - \lambda_j)\, e_j) = \lambda_j^{q-1}\, f(e_1) + (1 - \lambda_j)^{q-1}\, f(-e_j).$$

It follows from this and (39) that $f(x_1\, e_1 + x_j\, e_j) = 0$ for every $x_1, x_j \in \mathbb{R}$, $(x_1, x_j) \neq (0, 0)$. Let $x' = x_2\, e_2 + \cdots + x_{j-1}\, e_{j-1}$. Then by (38)

$$f(\lambda_j\, e_1 - (1 - \lambda_j)\, e_j + x') = \lambda_j^q\, f(\lambda_j\, e_1 + x') + (1 - \lambda_j)^q\, f(-(1 - \lambda_j)\, e_j + x'). \tag{40}$$

By (37), $f(-(1 - \lambda_j)\, e_j + x') = f(-(1 - \lambda_j)\, e_1 + x')$. Thus by using induction on the number of vanishing coordinates, we obtain from (40) that $f(x) = 0$ for every $x \neq 0$. $\square$

Next, we consider the case $0 < q < 1$. Let $\mathbb{R}_a^n$ be the set of $x \in \mathbb{R}^n$ not on the coordinate axes, that is, $x \in \mathbb{R}^n$, $x \neq \lambda\, e_j$, $\lambda \in \mathbb{R}$, $j = 1, \ldots, n$. Note that for $0 < q < 1$ it follows from Lemma 4 and Lemma 8 that $\rho(\operatorname{Z} T, \cdot)$ is continuous on $\mathbb{R}_a^n$.

Lemma 10. *Let* $\operatorname{Z} : \mathcal{P}_0 \to \mathcal{S}$ *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $0 < q < 1$. *Then*

$$\rho(\operatorname{Z} T, x) = 0$$

for $x \in \mathbb{R}_a^n$.

Proof. We define H_j, ϕ_j, ψ_j, and f as in the proof of Lemma 6. We consider $x \in \mathbb{R}_a^n$ for which the subsequent expressions are well defined, that is, for example, $\phi_2^t\, x \in \mathbb{R}_a^n$. Since Z is $\mathrm{GL}(n)$ contravariant of weight q, (26) and (1) imply that

$$f(x) = \lambda_2^q\, f(\phi_2^t\, x) + (1 - \lambda_2)^q\, f(\psi_2^t\, x). \tag{41}$$

For $x_2 = 1 - \lambda_2$, it follows that

$$\begin{aligned} f(x_1, 1-\lambda_2, x_3, \dots, x_n) &= \lambda_2^q\, f(x_1, (x_1+\lambda_2)(1-\lambda_2), x_3, \dots, x_n) \\ &\quad + (1-\lambda_2)^q\, f((x_1+\lambda_2)(1-\lambda_2), (1-\lambda_2), x_3, \dots, x_n). \end{aligned}$$

Let $x_1 \to -\lambda_2, x_3, \dots, x_n \to 0$. Since the left hand side is well defined and f is non-negative, this implies that f is uniformly bounded on $S^{n-1} \cap \mathbb{R}^n_a$. By (41),

$$f(\phi_2^{-t} x) = \lambda_2^q\, f(x) + (1-\lambda_2)^q\, f(\psi_2^t \phi_2^{-t}\, x).$$

Setting $x_1 = -1, x_2 = 1$ and dividing by λ_2 gives

$$\begin{aligned} &f(-\lambda_2, 2-\lambda_2, \lambda_2\, x_3, \dots, \lambda_2\, x_n) \\ &\quad = \lambda_2^{q-1}\, f(-1, 1, x_3, \dots, x_n) + (1-\lambda_2)^q\, f(\lambda_2, 2-\lambda_2, \lambda_2\, x_3, \dots, \lambda_2\, x_n). \end{aligned}$$

Let $\lambda_2 \to 0$. Since f is uniformly bounded on $S^{n-1} \cap \mathbb{R}^n_a$, this shows that

$$f(-1, 1, x_3, \dots, x_n) = 0.$$

Combined with (41) for $x = (-1, 1, x_3, \dots, x_n)$ this gives

$$0 = \lambda_2^q\, f(-1, -1+2\,\lambda_2, x_3, \dots, x_n) + (1-\lambda_2)^q\, f(-1+2\lambda_2, 1, x_3, \dots, x_n).$$

Since f is non-negative, this implies that

$$f(-1, -1+2\,\lambda_2, x_3, \dots, x_n) = f(-1+2\lambda_2, 1, x_3, \dots, x_n) = 0.$$

Similarly, we obtain that $f(1, -1, x_3, \dots, x_n) = 0$ and conclude that $f(x) = 0$ for every $x \in \mathbb{R}^n_a$. □

Finally, we consider the case $q = 1$. Note that in this case it follows from Lemma 3 and Lemma 8 that $\rho(\mathrm{Z}\,T, \cdot)$ is continuous and uniformly bounded on $S^{n-1} \cap \mathbb{R}^n_a$.

Lemma 11. *Let $\mathrm{Z} : \mathcal{P}_0 \to \mathcal{S}$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q = 1$. Then there is a constant $c \geq 0$ such that*

$$\rho(\mathrm{Z}\,T, x) = c\, \rho(\mathrm{I}\,T, x)$$

for $x \in \mathbb{R}^n_a$.

Proof. We define H_j, ϕ_j, ψ_j, and f as in the proof of Lemma 6. Let $x \in \mathbb{R}^n_a$. Note that for given x, there is a dense set of λ_j such that the subsequent expressions are well defined, that is, for example, $\phi_j^t x \in \mathbb{R}^n_a$. Since Z is $\mathrm{GL}(n)$ contravariant of weight $q = 1$, (26) and (1) imply that

$$f(x) = \lambda_j\, f(\phi_j^t x) + (1-\lambda_j)\, f(\psi_j^t x). \tag{42}$$

Using this repeatedly, we obtain

$$f(x) = \lambda_2 \cdots \lambda_n\, f(\phi_n^t \cdots \phi_2^t x) + \sum_{j=3}^n \lambda_2 \cdots \lambda_{j-1} (1-\lambda_j)\, f(\psi_j^t \phi_{j-1}^t \cdots \phi_2^t x) + (1-\lambda_2)\, f(\psi_2^t x).$$

Using this repeatedly, we obtain

$$\begin{aligned} f((\psi_2^{-t})^k x) \;=\; & \lambda_2 \cdots \lambda_n \sum_{i=1}^{k} (1-\lambda_j)^{k-i} f(\phi_n^t \cdots \phi_2^t (\psi_2^{-t})^i x) \\ & + \sum_{j=3}^{n} \lambda_2 \cdots \lambda_{j-1} \sum_{i=1}^{k} (1-\lambda_j)^{k-i} f(\psi_j^t \phi_{j-1}^t \cdots \phi_2^t (\psi_2^{-t})^i x) + (1-\lambda_2)^k f(x). \end{aligned} \tag{43}$$

Let $x' = x_3 e_3 + \cdots + x_n e_n$. Note that for ψ_2^t the vectors $e_1 + e_2$ and e_i, $i = 3, \ldots, n$, are eigenvectors with eigenvalue 1 and the vector e_1 is an eigenvector with eigenvalue $(1-\lambda_2)$. Setting $x = e_1 + x'$ in (43) and dividing by $(1-\lambda_2)^k$ gives

$$\begin{aligned} f(e_1 + (1-\lambda_2)^k x') \;=\; & \lambda_2 \cdots \lambda_n \sum_{i=1}^{k} f(\phi_n^t \cdots \phi_2^t (e_1 + (1-\lambda_2)^i x')) \\ & + \sum_{j=3}^{n} \lambda_2 \cdots \lambda_{j-1} \sum_{i=1}^{k} f(\psi_j^t \phi_{j-1}^t \cdots \phi_2^t (e_1 + (1-\lambda_2)^k x')) + f(x). \end{aligned}$$

Let $k \to \infty$. Since f is uniformly bounded, continuous and non-negative, we obtain that $f(\phi_n^t \ldots \phi_2^t e_1) = 0$. Note that $\phi_n^t \cdots \phi_2^t e_1 = e_1 + (1-\lambda_2)\, e_2 + \cdots + (1-\lambda_n)\, e_n$. Thus we conclude that

$$f(x_1, \ldots, x_n) = 0 \quad \text{for } x_1, \ldots, x_n > 0. \tag{44}$$

Next, let $x_1 < 0, x_2, \ldots, x_n > 0$ and $(1-\lambda_j)\, x_1 + \lambda_j\, x_j > 0$, $j = 2, \ldots, n$. Then by (42) and (44)

$$f(x) = \lambda_2\, f(\phi_2^t x) + (1-\lambda_2)\, f(\psi_2^t x) = \lambda_2\, f(\phi_2^t\, x)$$

and

$$f(\phi_n^t \cdots \phi_2^t\, x) = \frac{1}{\lambda_2 \cdots \lambda_n}\, f(x).$$

Setting $x = (-1, 1, \ldots, 1)$ and $a = f(-1, 1, \ldots, 1)$, we obtain

$$f(-1, -1 + 2\,\lambda_2, \ldots, -1 + 2\,\lambda_n) = \frac{a}{\lambda_2 \cdots \lambda_n}.$$

We use induction on the number of negative coordinates. Suppose f is determined by a for x with at most $(k-1)$ negative coordinates. Let $x_1, x_2 < 0$ and $(1-\lambda_2)\, x_1 + \lambda_2\, x_2 > 0$. Then

$$f(x) = \lambda_2\, f(\phi_2^t x) + (1-\lambda_2)\, f(\psi_2^t x)$$

and $\phi_2^t x$ as well as $\psi_2^t x$ have at most $(k-1)$ negative coordinates. Thus $f(x)$ is determined by a. We obtain that for given a there is at most one operator Z. Since $\mathrm{Z} = c\, \mathrm{I}$ with a suitable $c \geq 0$ is such an operator, this concludes the proof of the lemma. $\square$

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