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INTERSECTION HOMOLOGY AND TORUS ACTIONS

FRANCES KIRWAN

It is well known that the homology of a nonsingular complex projective variety X with a $\mathbf{C}^*$ action is determined by the homology of the connected components $F_1, \dots, F_l$ of the fixed point set $X^{\mathbf{C}^*}$ and the action of $\mathbf{C}^*$ near $X^{\mathbf{C}^*}$. In fact there are isomorphisms

$$H_i(X;\mathbf{Z}) \cong \bigoplus_{1\le\gamma\le l} H_{i-2m_\gamma}(F_\gamma;\mathbf{Z}), \tag{0.1}$$

where m_γ is the complex codimension of the stratum

$$S_\gamma = \{x \in X \mid \lim_{\lambda\to 0} \lambda\cdot x \in F_\gamma\}$$

in the Bialynicki-Birula decomposition of X (see [5, 6, 10, 13]). The same formula holds for appropriate integers m_γ when $\mathbf{C}^*$ is replaced by a torus $T = (\mathbf{C}^*)^r$.

In [9] it is shown that the formula (0.1) is valid even when X is singular, provided that the Bialynicki-Birula decomposition is "good". The aim of this paper is to generalize (0.1) to the case when X is singular in a different way, which involves replacing ordinary homology by intersection homology (with respect to the middle perversity). However only rational coefficients are considered. When X is nonsingular its intersection homology and ordinary homology coincide, but when X is singular its intersection homology behaves better in many respects than its ordinary cohomology.

It is shown that when X is singular, just as when X is nonsingular, the rational intersection homology of X is determined by the action of T on an arbitrarily small neighborhood of the fixed point set X^T. As might be expected, the formula (0.1) does not carry over directly when intersection homology replaces ordinary homology. The terms $H_{i-2m_\gamma}(F_\gamma;\mathbf{Z})$ appearing in the right-hand side are replaced by hypercohomology groups of certain complexes of sheaves over the F_γ which depend upon how the F_γ meet the singularities of X (see Theorem 2.3).

§1 of this paper contains a review of a proof of (0.1) for rational coefficients which uses equivariant Morse theory. In §2 it is shown how this proof can be extended to apply to singular varieties when intersection homology is used. The

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argument depends on the existence of a T-equivariant resolution of X, which is a consequence of Hironaka's equivariant resolution of singularities theorem announced in [24].

1.

In this section let X be a nonsingular complex projective variety with an algebraic action of a torus $T = (\mathbf{C}^*)^r$. Let $\{F_\gamma | 1 \leq \gamma \leq l\}$ be the connected components of the fixed point set X^T of T in X. Then each F_γ is a nonsingular closed subvariety of X.

Consider first the case $T = \mathbf{C}^*$. For each $x \in X$ the morphism $\mathbf{C}^* \to X$ given by $\lambda \to \lambda \cdot x$ extends uniquely to a morphism $\mathbf{C} \to X$ [21, II, 4.7]. Denote by $\lim_{\lambda \to 0} \lambda \cdot x$ the image of 0 under this morphism: it is always a fixed point of the action. The Bialynicki-Birula decomposition of X is a decomposition of X as a disjoint union of nonsingular locally closed subvarieties $S_1, \ldots, S_l$ where

$$S_\gamma = \{x \in X | \lim_{\lambda \to 0} \lambda \cdot x \in F_\gamma\}. \tag{1.1}$$

Each S_γ retracts onto the corresponding F_γ so the homology of S_γ is isomorphic to the homology of F_γ.

The Fubini-Study form ω is a Kähler form on X. By averaging we may assume that ω is invariant under the action of the maximal compact subgroup S^1 of $\mathbf{C}^*$. Thus the action of S^1 preserves the symplectic structure on X defined by ω. There exists a momentum map $\mu: X \to (\mathrm{Lie}\, S^1)^*$ for this action. That is, μ satisfies $\mu(\lambda \cdot x) = \mu(x)$ and

$$d\mu(x)(\xi) \cdot \alpha = \omega_x(\xi, \alpha_x) \tag{1.2}$$

for all $\lambda \in S^1$, $x \in X$, $\xi \in T_x X$ and $\alpha \in \mathrm{Lie}\, S^1$, where $x \mapsto \alpha_x$ is the vector field on X defined by the infinitesimal action of α (see [10, 13, 26]).

The Bialynicki-Birula decomposition of X can also be defined as the Morse stratification associated to the function $\mu: X \to (\mathrm{Lie}\, S^1)^* = \mathbf{R}$, which is a nondegenerate Morse function in the sense of [7] (see [2]). That is, a point $x \in X$ lies in S_γ if and only if the limit of its forward trajectory under the gradient flow of μ, with respect to the Kähler metric, lies in F_γ (see [10]). In particular the F_γ may be indexed in such a way that

$$\overline{S}_\gamma \subseteq \bigcup_{\beta \geq \gamma} S_\beta \tag{1.3}$$

for each γ. This means that each union

$$U_\gamma = \bigcup_{\beta \leq \gamma} S_\beta \tag{1.4}$$

is an open subset of X which contains S_γ as a closed complex submanifold, and satisfies $U_\gamma - S_\gamma = U_{\gamma-1}$. Thus there is a long exact sequence (the Thom-Gysin sequence)

(1.5)
$$\cdots \to H^{i-2m_\gamma}(S_\gamma;\mathbf{Q}) \to H^i(U_\gamma;\mathbf{Q}) \to H^i(U_{\gamma-1};\mathbf{Q}) \to H^{i+1-2m_\gamma}(S_\gamma;\mathbf{Q}) \to \cdots$$

where m_γ is the complex codimension of S_γ. The stratification $\{S_\gamma | 1 \le \gamma \le l\}$ of X is called perfect over $\mathbf{Q}$ if these long exact sequences all break up into short exact sequences

$$0 \to H^{i-2m_\gamma}(S_\gamma;\mathbf{Q}) \to H^i(U_\gamma;\mathbf{Q}) \to H^i(U_{\gamma-1};\mathbf{Q}) \to 0.$$

This happens if and only if there are isomorphisms

$$H^i(X;\mathbf{Q}) \cong \bigoplus_\gamma H^{i-2m_\gamma}(S_\gamma;\mathbf{Q}) \cong \bigoplus_\gamma H^{i-2m_\gamma}(F_\gamma;\mathbf{Q})$$

for each i.

Thus to prove that the formula (0.1) is valid with rational coefficients it suffices to show that the stratification $\{S_\gamma | 1 \le \gamma \le l\}$ is perfect. Before describing how this can be done using equivariant cohomology, let us consider what happens when $\mathbf{C}^*$ is replaced by a torus $T = (\mathbf{C}^*)^r$. Let $T_0 = (S^1)^r$ be the maximal compact subgroup of T. There exists a T_0-invariant Kähler structure on X with a momentum map $\mu: X \to (\operatorname{Lie} T_0)^*$. If α is a generic element of Lie T_0 (to be precise, if $\exp \mathbf{R}\alpha$ is dense in T_0) then the function $f: X \to \mathbf{R}$ defined by $f(x) = \mu(x)\cdot\alpha$ is a nondegenerate Morse function in the sense of [7], and its critical points are precisely the fixed points of the action (see, e.g., [2]). So there is a stratification $\{S_\gamma | 1 \le \gamma \le l\}$ of X such that a point $x \in X$ lies in S_γ if and only if the limit of its forward trajectory under the gradient flow of f, with respect to the Kähler metric, lies in F_γ. Equivalently

(1.6)
$$S_\gamma = \{x \in X | \lim_{t\to\infty} \exp(it\alpha)\cdot x \in F_\gamma\}.$$

(The limit of $\exp(it\alpha)\cdot x$ as $t \in \mathbf{R}$ tends to infinity exists because $i\alpha_x = \operatorname{grad} f(x)$ for every $x \in X$ and f is a nondegenerate Morse function.) If this stratification is perfect over $\mathbf{Q}$ then we have

(1.7)
$$H^i(X;\mathbf{Q}) \cong \bigoplus_\gamma H^{i-2m_\gamma}(S_\gamma;\mathbf{Q}) \cong \bigoplus_\gamma H^{i-2m_\gamma}(F_\gamma;\mathbf{Q})$$

where m_γ is the complex codimension of S_γ. Note that different choices of α may give different values for m_γ.

The rational T_0-equivariant cohomology of a space Y on which T_0 acts is by definition

$$H^*_{T_0}(Y;\mathbf{Q}) = H^*(Y \times_{T_0} ET_0;\mathbf{Q}),$$

where $ET_0 \to BT_0$ is a universal classifying bundle for T_0. Because X is a nonsingular projective variety it follows from [11] or [26, 5.8] that

(1.8)
$$H^*_{T_0}(X;\mathbf{Q}) \cong H^*(X;\mathbf{Q}) \otimes H^*(BT_0;\mathbf{Q})$$

and also

$$H^*_{T_0}(F_\gamma;\mathbf{Q}) \cong H^*(F_\gamma;\mathbf{Q}) \otimes H^*(BT_0;\mathbf{Q}) \tag{1.9}$$

for each γ.

Since the strata S_γ defined at (1.6) are T-invariant there exist equivariant Thom-Gysin sequences

$$\cdots \to H^{i-2m_\gamma}_{T_0}(S_\gamma;\mathbf{Q}) \to H^i_{T_0}(U_\gamma;\mathbf{Q}) \to H^i_{T_0}(U_{\gamma-1};\mathbf{Q}) \to \cdots. \tag{1.10}$$

The stratification is called equivariantly perfect over $\mathbf{Q}$ if these long exact sequences break up into short exact sequences so that

$$H^i_{T_0}(X;\mathbf{Q}) \cong \bigoplus_\gamma H^{i-2m_\gamma}_{T_0}(S_\gamma;\mathbf{Q}) \cong \bigoplus_\gamma H^{i-2m_\gamma}_{T_0}(F_\gamma;\mathbf{Q}) \tag{1.11}$$

for all i. It follows from (1.8) and (1.9) that the stratification is perfect over $\mathbf{Q}$ if and only if it is equivariantly perfect over $\mathbf{Q}$.

Atiyah and Bott have given a criterion for a stratification such as $\{S_\gamma | 1 \le \gamma \le l\}$ to be equivariantly perfect. By [3, Corollary 1.8 and Proposition 13.4] it suffices that for each γ the induced action of T_0 on the normal to S_γ at any point $x \in F_\gamma$ should have no nonzero fixed vectors. This condition is satisfied because near x the action is diffeomorphic to a linear action (see, e.g., [2, 2.2]). Thus we can conclude that the stratification $\{S_\gamma | 1 \le \gamma \le l\}$ is both perfect and equivariantly perfect over $\mathbf{Q}$, and hence that (1.7) holds.

Note that in particular we have proved the following lemma.

(1.12) **Lemma.** *The restriction maps*

$$H^*(U_\gamma;\mathbf{Q}) \to H^*(U_{\gamma-1};\mathbf{Q})$$

and

$$H^*_{T_0}(U_\gamma;\mathbf{Q}) \to H^*_{T_0}(U_{\gamma-1};\mathbf{Q})$$

are surjective for all γ.

In the next section we shall adapt this argument to the case where X may be singular and cohomology is replaced by intersection cohomology. Let us finish this section with a few remarks about the definition of the stratification $\{S_\gamma | 1 \le \gamma \le l\}$ when X is singular.

If X is a normal variety, then by [29, Corollary 1.6] there is an embedding of X in some complex projective space $\mathbf{P}_m$ such that the action of T on X extends to a linear action on $\mathbf{P}_m$. Let $\{\Phi_c : 1 \le c \le \lambda\}$ be the connected components of the fixed point set $\mathbf{P}_m^T$ of T on $\mathbf{P}_m$. Choose $\alpha \in \operatorname{Lie} T_0$ such that $\exp \mathbf{R}\alpha$ is dense in T_0, and define a stratification $\{\Sigma_c | 1 \le c \le \lambda\}$ of $\mathbf{P}_m$ by

$$\Sigma_c = \{x \in \mathbf{P}_m | \lim_{t\to\infty} \exp(it\alpha)\cdot x \in \Phi_c\}$$

as above. As before we may assume that

$$\overline{\Sigma}_c \subset \bigcup_{b \ge c} \Sigma_b \tag{1.13}$$

for each c. If $x \in \Sigma_c \cap X$ then $\exp(it\alpha)\cdot x$ lies in X for all $t \in \mathbf{R}$ because X is T-invariant and hence $\lim_{t\to\infty} \exp(it\alpha)\cdot x \in \Phi_c \cap X$ because X is closed in $\mathbf{P}_m$. Thus $\Sigma_c \cap X$ retracts onto $\Phi_c \cap X$. In particular each connected component of $\Sigma_c \cap X$ contains a unique connected component of $\Phi_c \cap X$. Let $\{F_\gamma | 1 \le \gamma \le l\}$ be the connected components of the fixed point set $X^T = X \cap \mathbf{P}_m^T$. Then for each γ there is a unique $c = c(\gamma)$ such that F_γ is a connected component of $\Phi_c \cap X$. Let

$$S_\gamma = \{x \in X | \lim_{t\to\infty} \exp(it\alpha)\cdot x \in F_\gamma\}. \tag{1.14}$$

Then S_γ is a connected component of $\Sigma_{c(\gamma)} \cap X$ and hence is a locally closed subvariety of X which retracts onto F_γ. Moreover we may assume that $\beta < \gamma$ if $c(\beta) < c(\gamma)$, so that (1.13) implies

$$\overline{S}_\gamma \subseteq \bigcup_{\beta \ge \gamma} S_\beta \tag{1.15}$$

for all γ.

Definition (1.15) of the strata S_γ depends on the choice of $\alpha \in \operatorname{Lie} T_0$ but not on the embedding of X in projective space. It will be assumed for the rest of this paper that an appropriate choice has been made of α.

2.

For any quasi-projective variety Y let $IH_i(Y;\mathbf{Q})$ and $IH^i(Y;\mathbf{Q})$ denote the ith rational intersection homology and cohomology groups of Y with respect to the middle perversity, as defined by Goresky and MacPherson in [16, 17, 28]. $IH_i(Y;\mathbf{Q})$ is the ith homology group of a subcomplex $IC_\cdot(Y;\mathbf{Q})$ of the complex of ordinary locally finite chains on Y. The intersection chains are those chains ξ such that ξ and its boundary $\partial\xi$ intersect the strata of a Whitney stratification of Y in sets of suitably small dimension. Because these conditions on ξ are local, there is a sheaf of cochain complexes $\mathbf{IC}_Y^\cdot$ on Y, satisfying $\mathbf{IC}_Y^i(U) = IC_{-i}(U;\mathbf{Q})$ for U open in Y, whose $(-i)$th hypercohomology group $\mathscr{H}^{-i}(Y;\mathbf{IC}_Y^\cdot)$ is $IH_i(Y;\mathbf{Q})$ (see [17, 2.1]).

A more sophisticated definition of intersection homology which does not depend on choosing a Whitney stratification involves giving criteria which uniquely characterize the complex of sheaves $\mathbf{IC}_Y^\cdot$ up to quasi-isomorphism [17, 4.1]. It is also possible to define intersection cohomology with coefficients in a local system on the nonsingular part of Y.

The intersection cohomology of a singular projective variety satisfies many of the properties satisfied by the ordinary cohomology of nonsingular projective varieties, such as Poincaré duality, the hard Lefschetz theorem, the Lefschetz hyperplane theorem and Hodge decomposition. One important result concerning intersection cohomology which will be needed in this paper is the decomposition theorem of Beilinson, Bertstein, Deligne and Gabber, which was conjectured in [14] and proved in [4, 6.25] (see also [20, 28]). This theorem tells us that if

$f: A \to B$ is a proper projective map of complex varieties then there exist closed subvarieties V_α of B and local systems L_α on $(V_\alpha)_{\text{nonsing}}$ such that

$$IH^i(A;\mathbf{Q}) = \bigoplus_\alpha IH^{i-l(\alpha)}(V_\alpha, L_\alpha) \tag{2.1}$$

for suitable integers $l(\alpha)$. When f is birational one of the summands $IH^{i-l(\alpha)}(V_\alpha, L_\alpha)$ is $IH^i(B;\mathbf{Q})$, so that $IH^i(B;\mathbf{Q})$ is a direct summand of $IH^i(A;\mathbf{Q})$. This will be important later. The decomposition of $IH^i(A;\mathbf{Q})$ given by the theorem is not a priori canonical, but there is a natural choice of decomposition associated to any factorization of $f: A \to B$ as an embedding of A in $B \times \mathbf{P}_m$ for some m, followed by projection onto B (see [28, §12] and [4, 5.4]).

In fact (2.1) is a consequence of applying hypercohomology to a stronger result on complexes of sheaves on B. There is a quasi-isomorphism

$$f_*\mathbf{IC}^{\cdot}_A \cong \bigoplus (i_\alpha)_*\mathbf{IC}^{\cdot}_{(V_\alpha, L_\alpha)}[-l(\alpha)], \tag{2.2}$$

where $i_\alpha: V_\alpha \to B_\alpha$ is the inclusion.

If a compact group K acts on a quasi-projective variety Y then there is a natural way of defining the equivariant intersection cohomology $IH^i_K(Y;\mathbf{Q})$ of Y (see [8, 25]; also (2.12) below).

Our aim is to prove the following result.

(2.3) **Theorem.** *Let X be a normal projective variety and let $T = (\mathbf{C}^*)^r$ be a torus acting algebraically on X. Then the rational intersection homology groups of X are isomorphic to the hypercohomology groups of a complex of sheaves on the fixed point set X^T. This complex of sheaves is determined by the action of the maximal compact subgroup T_0 of T in any neighborhood of X^T in X. More precisely let $\{S_\gamma | 1 \le \gamma \le l\}$ be the stratification of X defined at the end of* §1, *let $\{F_\gamma | 1 \le \gamma \le l\}$ be the corresponding components of X^T, and let $j_\gamma: F_\gamma \to S_\gamma$ and $i_\gamma: S_\gamma \to X$ be the inclusions. Then*

$$IH_i(X;\mathbf{Q}) = \bigoplus_\gamma \mathscr{H}^{-i}(F_\gamma, j_\gamma^* i_\gamma^! \mathbf{IC}_X)$$

for each i.

(2.4) *Remark.* The assumption that X is normal is not important here, since the intersection homology of any variety is isomorphic to the intersection homology of its normalization [16, 4.2].

(2.5) *Remark.* Suppose that X^T meets the singularities of X transversely, so that F_γ and S_γ have tubular neighborhoods in X and the inclusions $j_\gamma: F_\gamma \to S_\gamma$ and $i_\gamma: S_\gamma \to X$ are normally nonsingular [17, 5.4.1]. Under these conditions there are quasi-isomorphisms

$$i_\gamma^! \mathbf{IC}^{\cdot}_X = \mathbf{IC}^{\cdot}_{S_\gamma} \quad \text{and} \quad j_\gamma^* \mathbf{IC}^{\cdot}_{S_\gamma} \cong \mathbf{IC}_{F_\gamma}[2c_\gamma],$$

where c_γ is the codimension of F_γ in S_γ. Thus Theorem (2.3) tells us that

$$IH_i(X;\mathbf{Q}) \cong \bigoplus_\gamma IH_{i-2c_\gamma}(F_\gamma;\mathbf{Q}) .$$

Applying Poincaré duality to both sides we also get

$$IH_i(X;\mathbf{Q}) \cong \oplus_\gamma IH_{i-2m_\gamma}(F_\gamma;\mathbf{Q}) ,$$

where m_γ is the codimension of S_γ. In particular when X is nonsingular we recover (0.1).

The first step in the proof of (2.3) is to consider the stratification $\{S_\gamma | 1 \le \gamma \le l\}$ of X defined at the end of §1. For each γ, $U_\gamma = \bigcup_{\beta \le \gamma} S_\beta$ is an open T-invariant subset of X containing S_γ as a closed subvariety with $U_\gamma - S_\gamma = U_{\gamma-1}$. There is a long exact sequence of intersection cohomology

$$\cdots \to IH^i(U_\gamma, U_{\gamma-1};\mathbf{Q}) \to IH^i(U_\gamma;\mathbf{Q}) \to IH^i(U_{\gamma-1};\mathbf{Q}) \to \cdots$$

[18, 1.3].

(2.6) **Lemma.** *This long exact sequence breaks up into short exact sequences*

$$0 \to IH^i(U_\gamma;U_{\gamma-1};\mathbf{Q}) \to IH^i(U_\gamma;\mathbf{Q}) \to IH^i(U_{\gamma-1};\mathbf{Q}) \to 0.$$

Since $U_\gamma = X$ and $U_0 = \varnothing$ this lemma has the following immediate corollary.

(2.7) **Corollary.** $IH_i(X;\mathbf{Q}) \cong \bigoplus_\gamma IH_i(U_\gamma, U_{\gamma-1};\mathbf{Q})$ *for all* i.

Proof of (2.6). It is enough to prove that the restriction maps

$$IH^i(U_\gamma;\mathbf{Q}) \to IH^i(U_{\gamma-1};\mathbf{Q})$$

are all surjective.

Let $\pi: Y \to X$ be a resolution of singularities of X. Then for each γ the restriction

$$\pi: \pi^{-1}(U_\gamma) \to U_\gamma$$

of π to $\pi^{-1}(U_\gamma)$ is a resolution of singularities of U_γ. Since $\pi^{-1}(U_\gamma)$ is nonsingular its intersection cohomology coincides with its ordinary cohomology. Hence it follows from the decomposition theorem (see (2.1) above) that $IH^i(U_\gamma;\mathbf{Q})$ is a direct summand of $H^i(\pi^{-1}(U_\gamma);\mathbf{Q})$. Moreover we may choose the decompositions in such a way that the corresponding projections from $H^i(\pi^{-1}(U_\gamma);\mathbf{Q})$ onto $IH^i(U;\mathbf{Q})$ fit into commutative diagrams

$$\begin{array}{ccc} H^i(\pi^{-1}(U_\gamma);\mathbf{Q}) & \to & H^i(\pi^{-1}(U_{\gamma-1});\mathbf{Q}) \\ \downarrow & & \downarrow \\ IH^i(U_\gamma;\mathbf{Q}) & \to & IH^i(U_{\gamma-1};\mathbf{Q}) \end{array}$$

where the horizontal maps are induced by the inclusions of $U_{\gamma-1}$ in U_γ and $\pi^{-1}(U_{\gamma-1})$ in $\pi^{-1}(U_\gamma)$. Since the vertical maps are surjective, in order to prove that the restriction map $IH^i(U_\gamma;\mathbf{Q}) \to IH^i(U_{\gamma-1};\mathbf{Q})$ is surjective it suffices to prove the following lemma.

(2.8) **Lemma.** *The resolution $\pi: Y \to X$ can be chosen so that the restriction maps*

$$H^i(\pi^{-1}(U_\gamma);\mathbf{Q}) \to H^i(\pi^{-1}(U_{\gamma-1};\mathbf{Q})$$

are surjective for all γ and i.

The proof of this lemma will depend on Hironaka's equivariant resolution of singularities theorem, announced by Hironaka in 1976 [24, 9, Remark 8], which tells us that it is possible to resolve the singularities of X by a finite sequence of blow-ups along nonsingular T-invariant closed subvarieties. There does not seem to be a complete published proof of this theorem, but it is noted in [24, Bibliography with comments, 24] that a proof follows from the results of [1, 22, 23, 24].

Proof of (2.8). By Hironaka's equivariant resolution of singularities theorem there exists a resolution $\pi: Y \to X$ of X which factorizes as

$$Y = Y_0 \xrightarrow{\pi_1} Y_1 \to \cdots \xrightarrow{\pi_s} Y_s = X,$$

where $\pi_j: Y_{j-1} \to Y_j$ is the blow-up of Y_j along a nonsingular T-invariant closed subvariety V_j. In particular the action of T on X lifts to an action of T on Y. The fixed point set Y^T of this action is contained in $\pi^{-1}(X^T)$. Let the connected components of Y^T be $\{\Phi_c | 1 \le c \le \lambda\}$. Then for each c there exists a unique $\gamma = \gamma(c)$ such that

$$\Phi_c \subseteq \pi^{-1}(F_{\gamma(c)}).$$

As in §1, Y decomposes as the disjoint union of strata

$$\Sigma_c = \{x \in Y | \lim_{t\to\infty} \exp(it\alpha)\cdot x \in \Phi_c\},$$

where α is the same element of Lie T_0 as was used to define the stratification $\{S_\gamma | 1 \le \gamma \le l\}$ of X. Then

$$\pi^{-1}(S_\gamma) = \bigcup_{\gamma(c)=\gamma} \Sigma_c \quad \text{and} \quad \pi^{-1}(U_\gamma) = \bigcup_{\gamma(c)\le\gamma} \Sigma_c$$

for each γ. Moreover for each c

$$\overline{\Sigma}_c \subseteq \bigcup_{\gamma(b)\ge\gamma(c)} \Sigma_b.$$

Indeed we can reorder the indexing of the Σ_c in such a way that

$$\overline{\Sigma}_c \subseteq \bigcup_{b\ge c} \Sigma_b$$

and $\gamma(b) \geq \gamma(c)$ if $b \geq c$. In particular for each γ there is some $c_\gamma \in \{1, \dots, \lambda\}$ such that

$$\pi^{-1}(U_\gamma) = \bigcup_{\gamma(c)\leq\gamma} \Sigma_c = \bigcup_{c\leq c_\gamma} \Sigma_c .$$

We have to prove that for each i and γ the restriction map

$$H^i(\pi^{-1}(U_\gamma); \mathbf{Q}) \to H^i(\pi^{-1}(U_{\gamma-1}); \mathbf{Q})$$

is surjective. This is the composition of the restriction maps

$$H^*\left(\bigcup_{b\leq c} \Sigma_b ; \mathbf{Q}\right) \to H^*\left(\bigcup_{b<c} \Sigma_b ; \mathbf{Q}\right)$$

for $c_\gamma \geq c > c_{\gamma-1}$. But these restriction maps are all surjective by (1.12). This completes the proof of (2.8), and thus also of (2.6).

The next step is to study the relative intersection cohomology groups $IH^i(U_\gamma, U_{\gamma-1}; \mathbf{Q})$. Recall that F_γ is a closed subset of S_γ, which in turn is a closed subset of U_γ and $U_\gamma - S_\gamma = U_{\gamma-1}$. Let $i_\gamma: S_\gamma \to X$, $j_\gamma: F_\gamma \to S_\gamma$, $i_\gamma^U: U_\gamma \to X$ and $l_\gamma: S_\gamma \to U_\gamma$ be the inclusion maps. Then

$$IH_i(U_\gamma, U_{\gamma-1}; \mathbf{Q}) \cong \mathscr{H}^{-i}(S_\gamma; l_\gamma^! \mathbf{IC}^{\cdot}_{U_\gamma})$$

[17, 1.11]. Moreover

$$\mathbf{IC}^{\cdot}_{U_\gamma} \cong (i_\gamma^U)^! \mathbf{IC}^{\cdot}_X$$

because U_γ is open in X [17, 1.13 (12)]. Thus

$$\text{(2.9)} \qquad IH_i(U_\gamma, U_{\gamma-1}; \mathbf{Q}) \cong \mathscr{H}^{-i}(S_\gamma; i_\gamma^! \mathbf{IC}^{\cdot}_X).$$

(2.10) **Lemma.** *$\mathscr{H}^i(S_\gamma; i_\gamma^! \mathbf{IC}_X) \cong \mathscr{H}^i(F_\gamma; j_\gamma^* i_\gamma^! \mathbf{IC}^{\cdot}_X)$ for all i.*

Proof. For each $x \in S_\gamma$ the trajectory

$$\{\exp(it\alpha)\cdot x | t \geq 0\}$$

is contained in S_γ and it has a unique limit point which belongs to F_γ. By pushing along these trajectories we can construct a homeomorphism of an open neighborhood of S_γ in X onto an arbitrarily small open neighborhood W of F_γ in X, which restricts to a homeomorphism of S_γ onto $S_\gamma \cap W$. Since intersection homology is invariant under homeomorphism [17, 4.3] and satisfies excision [18, 1.5], this means that

$$IH_i(U_\gamma, U_{\gamma-1}; \mathbf{Q}) \cong IH_i(W, W - S_\gamma; \mathbf{Q})$$

for arbitrarily small neighborhoods W of F_γ in X. Thus

$$\mathscr{H}^i(S_\gamma; i_\gamma^! \mathbf{IC}_X) \cong \mathscr{H}^i(W \cap S_\gamma; j_W^* i_\gamma^! \mathbf{IC}_X)$$

for arbitrarily small neighborhoods W of F_γ in X, where $j_W: W \cap S_\gamma \to S_\gamma$ is the inclusion. But it follows from [21, III, 2.9 and 2.10; 30, 4.5.7] and the existence of the spectral sequence for hypercohomology [15, II, §4.6], that if $\mathscr{F}^{\cdot}$ is any complex of sheaves on S_γ then

$$\mathscr{H}^i(F_\gamma; j_\gamma^* \mathscr{F}^{\cdot}) = \varinjlim \mathscr{H}^i(W \cap S_\gamma; j_W^* \mathscr{F}^{\cdot}),$$

where the limit is over any directed set of open neighborhoods W of F_γ in X whose intersection is F_γ. Thus we can conclude that

$$\mathscr{H}^i(S_\gamma; i_\gamma^! \mathbf{IC}_X) \cong \mathscr{H}^i(F_\gamma; j_\gamma^* i_\gamma^! \mathbf{IC}_X)$$

as required.

This completes the proof of Theorem (2.3).

If we replace intersection cohomology IH^* by equivariant intersection cohomology $IH^*_{T_0}$ throughout the proof of (2.6), we find that there exist short exact sequences

$$(2.11) \qquad 0 \to IH^i_{T_0}(U_\gamma, U_{\gamma-1}; \mathbf{Q}) \to IH^i_{T_0}(U_\gamma; \mathbf{Q}) \to IH^i_{T_0}(U_{\gamma-1}; \mathbf{Q}) \to 0$$

for all i and γ. The only point to check is that the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber is valid for equivariant intersection cohomology, so that $IH^*_{T_0}(U_\gamma; \mathbf{Q})$ is a direct summand of $IH^*_{T_0}(\pi^{-1}(U_\gamma); \mathbf{Q})$ for each γ. This can be deduced from the ordinary decomposition theorem as follows. Let $T = (\mathbf{C}^*)^r$ act on $E_q = (\mathbf{C}^{q+1} - \{0\})$ for each $q \geq 0$ via

$$(t_1, \ldots, t_r) \cdot (x_1, \ldots, x_r) = (t_1 x_1, \ldots, t_r x_r).$$

This action is free and proper, and the quotient is $(\mathbf{P}_q)^r$. Suppose that Y is any quasi-projective variety and T acts linearly on Y. Then the diagonal action of T on $Y \times E_q$ is free and proper for each $q \geq 0$, and the quotient $Y \times_T E_q$ is a complex analytic variety which fibers over $(\mathbf{P}_q)^r$ with fiber Y. The quotient can also be regarded as an algebraic scheme in a natural way [29, 1.9 and 1.8 (4)], and the fibration $Y \times_T E_q \to (\mathbf{P}_q)^r$ is algebraic. The comparison theorem for spectral sequences [31] implies that the intersection cohomology group $IH^i(Y \times_T E_q; \mathbf{Q})$ is independent of q provided that $2q \geq i$. Since T is homotopy equivalent to T_0 and its classifying space is r copies of infinite projective space $(\mathbf{P}_\infty)^r$ it is natural to make the following definition:

$$(2.12) \qquad IH^i_{T_0}(Y; \mathbf{Q}) = IH^i_T(Y; \mathbf{Q}) = IH^i(Y \times_T E_q; \mathbf{Q})$$

for any $q > \frac{1}{2}i$. Alternatively one can use the more sophisticated definition of equivariant intersection cohomology given in [8, §2] (see also [25]): then it is easy to check that (2.12) holds. Finally note that if $f: A \to B$ is a proper projective map of complex varieties then so is the induced map

$$f_T: A \times_T E_q \to B \times_T E_q$$

for any q. Thus the decomposition theorem for equivariant intersection cohomology follows immediately from the ordinary decomposition theorem (2.1).

From (2.11) we obtain

(2.13) **Lemma.** *The stratification* $\{S_\gamma | 1 \le \gamma \le l\}$ *is equivariantly perfect for intersection cohomology, i.e.*

$$IH^i_{T_0}(X;\mathbf{Q}) \cong \bigoplus_\gamma IH^i_{T_0}(U_\gamma, U_{\gamma-1};\mathbf{Q}),$$

for each γ *and* i.

Since

$$IH^*_{T_0}(X;\mathbf{Q}) \cong IH^*(X;\mathbf{Q}) \otimes H^*(BT_0;\mathbf{Q}) \tag{2.14}$$

(by [8, 4.2.2; 29, Corollary 1.6; 16, 4.2]) this gives us an alternative way to obtain a formula for the intersection Betti numbers of X. Moreover if we compare this formula with the formula of Corollary (2.7) above, we get the following result.

(2.15) **Corollary.** $IH^*_{T_0}(U_\gamma, U_{\gamma-1};\mathbf{Q}) \cong IH^*(U_\gamma, U_{\gamma-1};\mathbf{Q}) \otimes H^*_{\cdot}(BT_0;\mathbf{Q})$ *for each* γ.

Proof. For each γ there is a spectral sequence abutting to $IH^*_{T_0}(U_\gamma, U_{\gamma-1};\mathbf{Q})$ with the $E_2^{p,q}$ term given by

$$H^p(BT_0;\mathbf{Q}) \otimes IH^q(U_\gamma, U_{\gamma-1};\mathbf{Q}),$$

and hence there are inequalities

$$\dim IH^i_{T_0}(U_\gamma, U_{\gamma-1};\mathbf{Q}) \le \sum_{p+q=i} \dim H^p(BT_0;\mathbf{Q}) \otimes IH^q(U_\gamma, U_{\gamma-1};\mathbf{Q})$$

(see [8, 4.2]). It follows from (2.7), (2.13) and (2.14) that these inequalities are all equalities, and hence that the spectral sequence degenerates. The result follows.

(2.16) **Corollary.** *Let* α *be any element of* $\operatorname{Lie} T_0$ *and let* F *be a connected component of the fixed point set of the subtorus* T_0^α *of* T_0 *which is the closure of* $\exp \mathbf{R}\alpha$. *Let*

$$S = \{x \in X | \lim_{t\to\infty} \exp(it\alpha)\cdot x \in F\}.$$

Then

$$IH^*_{T_0}(U, U-S) \cong IH^*(U, U-S) \otimes H^*(BT_0)$$

for any T_0*-invariant open neighborhood* U *of* S *in* X *such that* S *is closed in* U.

Proof. By excision [18, 1.5], both $IH^*_{T_0}(U, U-S)$ and $IH^*(U, U-S)$ are independent of U provided that U is a T_0-invariant open neighborhood of S in X and S is closed in U. Therefore when α is generic in the sense that $\exp \mathbf{R}\alpha$ is dense in T_0 then the result follows immediately from (2.15). Exactly the same proof gives the result in general, if one makes the following two observations. Definition (1.6) of the stratification $\{S_\gamma | 1 \leq \gamma \leq l\}$ of X makes sense when α is not generic, although now the indices correspond to the connected components of the fixed point set of T_0^α. Furthermore the criterion of Atiyah and Bott [3, Corollary 1.8 and Proposition 13.4] shows that when X is nonsingular this stratification is equivariantly perfect for the actions of both T_0 and T_0^α.

(2.17) *Remark.* The original proof in [6] of the identity

$$\dim H_i(X;\mathbf{Q}) = \sum_{1\leq\gamma\leq l} \dim H_{i-2m_\gamma}(F_\gamma;\mathbf{Q})$$

when X is nonsingular was based on the Weil conjectures which were proved by Deligne. These enable one to compute the Betti numbers of a nonsingular projective variety by counting the number of points in associated varieties defined over finite fields. This method generalizes to give the intersection Betti numbers of singular projective varieties, although now it is necessary to count points in a more sophisticated way which takes account of the singularities. The basic idea goes as follows.

We may assume that X is defined over the ring of integers $\mathscr{O}$ of an algebraic number field [4, 6.16]. For suitable primes π in $\mathscr{O}$ one has

$$\dim_{\mathbf{Q}} IH_i(X;\mathbf{Q}) = \dim_{\mathbf{Q}_l} IH_i(\overline{X}_\pi;\mathbf{Q}_l), \tag{2.18}$$

where X_π is the reduction of X modulo π and

$$\overline{X}_\pi = X_\pi \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$$

where $\overline{\mathbf{F}}_q$ is the algebraic closure of $\mathbf{F}_q = \mathscr{O}/\pi$. The right-hand side of (2.18) is l-adic intersection cohomology for suitable l (see [4, 6.1.2 and 6.1.9]). Then one applies the Lefschetz fixed point theorem to the Frobenius morphism $f: \overline{X}_\pi \to \overline{X}_\pi$ which sends a point with coordinates $(a_0:\cdots:a_m)$ to $((a_0)^q:\cdots:(a_m)^q)$. For each $r \geq 1$ the Lefschetz number $L(f^r;X) = L(f^r)$ of f^r satisfies

$$L(f^r) = \sum_i (-1)^i \mathrm{Tr}((f^r)^*: IH^i(\overline{X}_\pi;\mathbf{Q}_l) \to IH^i(\overline{X}_\pi;\mathbf{Q}_l)).$$

When $\overline{X}_\pi$ is smooth, $L(f^r)$ is just the number of fixed points of f^r, or equivalently the number of points of $\overline{X}_\pi$ with coordinates in the finite field $\mathbf{F}_{q^r}$. In general $L(f^r)$ is the sum over the fixed points $x \in X_\pi(\mathbf{F}_{q^r})$ of numbers determined by the local intersection homology of X_π at x (see [12, 19 and 17, 2.4]).

Let

$$Z(X,t) = \exp\left(\sum_{r\geq 1} L(f^r)\frac{t^r}{r}\right).$$

Then

$$Z(X, t) = \prod_i P_i(t)^{(-1)^i},$$

where $P_i(t)$ is the polynomial

$$P_i(t) = \det(1 - tf^*: IH^i(\overline{X}_\pi; \mathbf{Q}) \to IH^i(\overline{X}_\pi; \mathbf{Q}))$$

(see [21, Appendix C, 4.1]). Thus

$$P_i(t) = \prod_j (1 - \alpha_{ij} t),$$

where the α_{ij} are the eigenvalues of the endomorphism f^* of $IH^i(\overline{X}_\pi; \mathbf{Q})$, and

$$\deg P_i(t) = \dim IH_i(X; \mathbf{Q}).$$

Since $|\alpha_{ij}| = q^{i/2}$ for all j (see [4, 5.4.1]) the polynomials $P_i(t)$ and therefore the intersection Betti numbers of X are uniquely determined by $Z(X, t)$, and hence by the Lefschetz numbers $L(f^r)$.

Let us assume that the action of T on X is defined over the ring of integers $\mathscr{O}$. There is an induced torus action on $\overline{X}_\pi$ and an induced stratification of $\overline{X}_\pi$. The Lefschetz number $L(f^r)$ is then the sum of contributions L_γ coming from the strata S_γ. Each L_γ is in turn the sum of a contribution from the fixed point set F_γ and a contribution from $S_\gamma - F_\gamma$. Assume for simplicity that $T = \mathbf{C}^*$. Then T acts properly and with finite stabilizers on $S_\gamma - F_\gamma$ and the quotient is a projective variety V_γ [29, p. 40 and 2.1]. In this way one finds that $L(f^r)$ is a sum of contributions coming from the projective varieties F_γ and V_γ, and hence one could hope to use the Weil conjectures to show that

$$\text{(2.19)}\quad \begin{aligned} \dim IH^i_{T_0}(X; \mathbf{Q}) &= \sum_{j \geq 0} \dim IH^{i-2j}(X; \mathbf{Q}) \\ &= \sum_\gamma \dim(\mathscr{H}^i_{T_0}(F_\gamma; i^!_\gamma \mathbf{IC}^\cdot_X) \oplus \mathscr{H}^{-i}_{T_0}(S_\gamma - F_\gamma; k^!_\gamma \mathbf{IC}_X)), \end{aligned}$$

where $k_\gamma: S_\gamma - F_\gamma \to X$ is the inclusion. However one needs to know that the complexes $i^!\mathbf{IC}^\cdot_X$ and $k^!\mathbf{IC}_X$ are pure.

Equality (2.19) is equivalent to Theorem (2.3) combined with the breaking up of the long exact sequence

$$\cdots \to IH^i_{T_0}(U_\gamma, U_\gamma - F_\gamma; \mathbf{Q}) \to IH^i_{T_0}(U_\gamma; \mathbf{Q}) \to IH^i_{T_0}(U_\gamma - F_\gamma; \mathbf{Q}) \to \cdots$$

into short exact sequences for each γ. This is always the case when X is nonsingular by the Atiyah-Bott criterion [3, 1.8 and 2.14]. More generally it can be shown to be true by the argument of (2.6) when there is a T-equivariant resolution of singularities $\pi: Y \to X$ such that $\pi^{-1}(X^T)$ is nonsingular.

(2.20) **Example.** Let X be the cubic hypersurface in $\mathbf{P}_4$ defined by the equation

$$x^3 + y^3 + z^3 = u^2 v.$$

Let $T = \mathbf{C}^*$ act on X via

$$(t, (x: y: z: u: v)) \to (t^{-1}x: t^{-1}y: t^{-1}x: t^{-3}u: t^3 v).$$

Then X^T has connected components

$$F_1 = \{(0: 0: 0: 1: 0)\},$$
$$F_2 = \{(x: y: z: 0: 0) \in \mathbf{P}_5 | x^3 + y^3 + z^3 = 0\},$$
$$F_3 = \{(0: 0: 0: 0: 1)\}.$$

If $S_j = \{x \in X | \lim_{t \to 0} t \cdot x \in F_j\}$, then

$$S_1 = \{(x: y: z: u: v) \in X | u \neq 0\},$$
$$S_2 = \{(x: y: z: u: v) \in X | u = 0, (x, y, z) \neq 0\},$$
$$S_3 = \{(0: 0: 0: 0: 1)\}.$$

The only singular point of X is the point $P = (0: 0: 0: 0: 1)$. Thus since S_1 has codimension 0 and S_2 has codimension 1

$$\mathscr{H}^{-i}(F_1; j_1^* i_1^! \mathbf{IC}_X^\cdot) \cong IH^i(F_1; \mathbf{Q}) \cong H^i(F_1; \mathbf{Q})$$

and

$$\mathscr{H}^{-i}(F_2; j_2^* i_2^! \mathbf{IC}_X^\cdot) \cong IH^{i-2}(F_2; \mathbf{Q}) \cong H^{i-2}(F_2; \mathbf{Q}),$$

where F_1 is a single point and F_2 is a nonsingular cubic curve, hence a curve of genus 1, in $\mathbf{P}_2$. Finally

$$\mathscr{H}^i(F_3; j_3^* i_3^! \mathbf{IC}_X^\cdot) = IH_i(X, X - \{P\})$$

which is 0 when $i \leq \dim X = 3$ by the local intersection homology formula [17, 2.4]. Poincaré duality means that we do not have to calculate $IH_i(X, X - \{P\})$ when $i > 4$. Let

$$IP_t(Y) = \sum_i t^i \dim IH_i(Y; \mathbf{Q})$$

for any variety Y. Then from (2.3) we have

$$IP_t(X) = 1 + t^2(1 + 2t + t^2) + t^4 p(t),$$

where $p(t)$ is a polynomial defined by

$$t^4 p(t) = IP_t(X, X - \{P\}).$$

By Poincaré duality we must have

$$IP_t(X) = 1 + t^2 + 2t^3 + t^4 + t^6$$

and $t^4 p(t) = t^6$.

Note that if we count points over a finite field $\mathbf{F}_q$ as in Remark (2.17) then S_1 contributes q^{3r} to $L(f^r; X)$ while S_2 contributes $q^r L(f^r; F_2)$ and the contribution of S_3 is 1.

(2.21) *Remark.* The proof given of Theorem (2.3) depends on the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber. At present the only published proof of this theorem depends on the relationship between the rational intersection cohomology of complex projective varieties and the l-adic intersection cohomology of varieties defined over fields of finite characteristic (and thus is closely related to the Weil conjectures). Because of this the proof is only valid for algebraic varieties and rational coefficients. It is expected that the decomposition theorem should be true for complex analytic varieties: if so the proof of Theorem (2.3) will apply when X is any compact analytic subvariety of a Kähler manifold.

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Mathematical Institute, 24-29 St. Giles, Oxford, England