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Intersection homology of toric varieties and a conjecture of Kalai

Tom Braden and Robert MacPherson

Abstract. We prove an inequality, conjectured by Kalai, relating the g-polynomials of a polytope P, a face F, and the quotient polytope P/F, in the case where P is rational. We introduce a new family of polynomials $g(P,F)$, which measures the complexity of the part of P "far away" from the face F; Kalai's conjecture follows from the nonnegativity of these polynomials. This nonnegativity comes from showing that the restriction of the intersection cohomology sheaf on a toric variety to the closure of an orbit is a direct sum of intersection homology sheaves.

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Suppose that a d-dimensional convex polytope $P \subset \mathbb{R}^d$ is *rational*, i.e. its vertices have all coordinates rational. Then P gives rise to a polynomial $g(P) = 1+g_1(P)q+g_2(P)q^2+\cdots$ with non-negative coefficients as follows. Let X_P be the *associated toric variety* (see §6 – our variety X_P is $d+1$-dimensional and affine). The coefficient $g_i(P)$ is the rank of the $2i$-th intersection cohomology group of X_P.

The polynomial $g(P)$ turns out to depend only on the face lattice of P, (see §1). It can be thought of as a measure of the complexity of P; for example, $g(P) = 1$ if and only if P is a simplex.

Suppose that $F \subset P$ is a face of dimension $k < d$. We construct an associated polytope P/F as follows: choose an $(d-k-1)$-plane L whose intersection with P is a single point p of the interior of F. Let L' be a small parallel displacement of L that intersects the interior of P. The quotient P/F is the intersection of P with L'; it is only well-defined up to a projective transformation, but its combinatorial type is well-defined (Formally we put P/P to be the empty polytope). Faces of P/F are in one-to-one correspondence with faces of P which contain F.

In Corollary 6, we show that

$$g(P) \geq g(F)g(P/F)$$

holds, coefficient by coefficient. This was conjectured by Kalai in [11], where some of its applications were discussed. The special case of the linear and quadratic

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terms was proved in [12]. Roughly, this inequality means that the complexity of P is bounded from below by the complexity of the face F and the normal complexity $g(P/F)$ to the face F.

The principal idea is to introduce *relative g-polynomials* $g(P, F)$ for any face F of P (§2). These generalize the ordinary g-polynomials since $g(P, P) = g(P)$. They are also combinatorially determined by the face lattice. They measure the complexity of P relative to the complexity of F. For example, if P is the join of F with another polytope, then $g(P, F) = 1$ (the converse, however, does not hold).

Our main result gives an interpretation of the coefficients $g_i(P, F)$ of the relative g-polynomials as dimensions of vector spaces arising from the topology of the toric variety X_P. This shows that the coefficients are positive. Kalai's conjecture is a corollary.

The combinatorial definition of the relative g-polynomials $g(P, F)$ makes sense whether or not the polytope P is rational. We conjecture that $g(P, F) \geq 0$ for any polytope P; this would imply Kalai's conjecture for general polytopes.

This paper is organized as follows: The first three sections are entirely about the combinatorics of polyhedra. They develop the properties of relative g-polynomials as combinatorial objects, with the application to Kalai's conjecture. The last three sections concern algebraic geometry. A separate guide to their contents is included in the introduction to §§4 - 6.

1. g-numbers of polytopes

Let $P \subset \mathbb{R}^d$ be a d-dimensional convex polytope, i.e. the convex hull of a finite collection of points affinely spanning $\mathbb{R}^d$. The set of faces of P, ordered by inclusion, forms a poset which we will denote by $\mathcal{F}(P)$. We include the empty face $\emptyset = \emptyset_P$ and P itself as members of $\mathcal{F}(P)$. It is a graded poset, with the grading given by the dimension of faces. By convention we set $\dim \emptyset = -1$. Faces of P of dimension 0, 1, and $d-1$ will be referred to as vertices, edges, and facets, respectively.

Given a face F of P, the poset $\mathcal{F}(F)$ is isomorphic to the interval $[\emptyset, F] \subset \mathcal{F}(P)$. The interval $[F, P]$ is the face poset of the polytope P/F defined in the introduction.

Given the polytope P, there are associated polynomials (first introduced in [14]) $g(P) = \sum g_i(P)q^i$ and $h(P) = \sum h_i(P)q^i$, defined recursively as follows:

- $g(\emptyset) = 1$
- $h(P) = \Sigma_{\emptyset \leq F < P}(q-1)^{\dim P - \dim F - 1} g(F)$, and
- $g_0(P) = h_0(P)$, $g_i(P) = h_i(P) - h_{i-1}(P)$ for $0 < i \leq \dim P/2$, and $g_i(P) = 0$ for all other i.

The coefficients of these polynomials will be referred to as the g-numbers and h-numbers of P, respectively. We do not discuss the h-polynomial further in this paper.

These numbers depend only on the poset $\mathcal{F}(P)$. In fact, as Bayer and Billera

[1] showed, they depend only on the flag numbers of P: given a sequence of integers $I = (i_1, \dots, i_n)$ with $0 \le i_1 < i_2 < \cdots < i_n \le d$, an I-flag is an n-tuple $F_1 < F_2 < \cdots < F_n$ of faces of P with $\dim F_k = i_k$ for all k. The I-th *flag number* $f_I(P)$ is the number of I-flags. Letting P vary over all polytopes of a given dimension d, the numbers $g_i(P)$ and $h_i(P)$ can be expressed as a $\mathbb{Z}$-linear combination of the $f_I(P)$.

Conjecturally all the $g_i(P)$ should be nonnegative for all P. This is known to be true for $i = 1, 2$ [10]. For higher values of i, it can be proved for rational polytopes using the interpretation of $g_i(P)$ as an intersection cohomology Betti number of an associated toric variety.

Proposition 1. *If P is a rational polytope, then $g_i(P) \ge 0$ for all i.*

2. Relative g-polynomials

The following proposition defines a relative version of the classical g-polynomials.

Proposition 2. *There is a unique family of polynomials $g(P, F)$ associated to a polytope P and a face F of P, satisfying the following relation: for all P, F, we have*

$$\sum_{F \le E \le P} g(E, F) g(P/E) = g(P). \tag{1}$$

Proof. The equation (1) can be used inductively to compute $g(P, F)$, since the left hand side gives $g(P, F) \cdot 1$ plus terms involving $g(E, F)$ where $\dim E < \dim P$. The induction starts when $P = F$, which gives $g(F, F) = g(F)$. □

As an example, if F is a facet of P, then $g(P, F) = g(P) - g(F)$. Just as before we will denote the coefficient of q^i in $g(P, F)$ by $g_i(P, F)$.

We have the following notion of relative flag numbers. Let P be a d-polytope, and F a face of dimension e. Given a sequence of integers $I = (i_1, \dots, i_n)$ with $0 \le i_1 < i_2 < \cdots < i_n \le d$ and a number $1 \le k \le n$ with $i_k \ge e$, define the relative flag number $f_{I,k}(P, F)$ to be the number of I-flags $(F_1, \dots, F_n)$ with $F \le F_k$. Note that letting $k = n$ and $i_n = d$ gives the ordinary flag numbers of P as a special case. Also note that the numbers $f_{I,k}$ where $i_k = e$ give products of the form $f_J(F) f_{J'}(P/F)$, and all such products can be expressed this way.

Proposition 3. *Fixing $\dim P$ and $\dim F$, the relative g-number $g_i(P, F)$ is a $\mathbb{Z}$-linear combination of the $f_{I,k}(P, F)$.*

Proof. Use induction on $\dim P/F$. If $P = F$, then we have $g(P, P) = g(P)$ and the result is just the corresponding result for the ordinary flag numbers. If $P \ne F$, the equation (1) gives

$$g(P,F) = g(P) - \sum_{e=\dim F}^{\dim P-1} \sum_{\substack{\dim E=e \\ F\leq E<P}} g(E,F)g(P/E).$$

For every e the coefficients of the inner summation on the right hand side are $\mathbb{Z}$-linear combinations of the $f_{I,k}(P,F)$, using the inductive hypothesis. □

The following theorem is the main result of this paper. It will be a consequence of Theorem 11.

Theorem 4. *If P is a rational polytope and F is any face, then $g_i(P,F) \geq 0$ for all i.*

Corollary 5. (Kalai's conjecture) *If P is a rational polytope and F is any face, then*

$$g(P) \geq g(F)g(P/F),$$

where the inequality is taken coefficient by coefficient.

Proof. For any face E of P the polytope P/E is rational, so we have $g(P) = g(F,F)g(P/F)+$ other nonnegative terms. □

3. Some examples and formulas

This section contains further combinatorial results on the relative g-polynomials. They are not used in the remainder of the paper.

First, we give an interpretation of $g_1(P,F)$ and $g_2(P,F)$ analogous to the ones Kalai gave for the usual g_1 and g_2 in [10]. We begin by recalling those results from [10].

Given a finite set of points $V \subset \mathbb{R}^d$ define the space $\mathcal{A}\!f\!f(V)$ of affine dependencies of V to be

$$\{\, a \in \mathbb{R}^V \mid \Sigma_{v\in V} a_v = 0,\, \Sigma_{v\in V} a_v \cdot v = 0 \,\}.$$

If V_P is the set of vertices of a polytope $P \subset \mathbb{R}^d$, then $\mathcal{A}\!f\!f(V_P)$ is a vector space of dimension $g_1(P)$.

To describe $g_2(P)$ we need the notion of stress on a framework. A framework $\Phi = (V,E)$ is a finite collection V of points in $\mathbb{R}^d$ together with a finite collection E of straight line segments (edges) joining them. Given a finite set S, we denote the standard basis elements of $\mathbb{R}^S$ by $1_s, s \in S$. The space of stresses $\mathcal{S}(\Phi)$ is the kernel of the linear map

$$\alpha\colon \mathbb{R}^E \to \mathbb{R}^V \otimes \mathbb{R}^d,$$

defined by

$$\alpha(1_e) = 1_{v_1} \otimes (v_1 - v_2) + 1_{v_2} \otimes (v_2 - v_1),$$

where v_1 and v_2 are the endpoints of the edge e. A stress can be described physically as an assignment of a contracting or expanding force to each edge, such that the total force resulting at each vertex is zero.

To a polytope P we can associate a framework Φ_P by taking as vertices the vertices of P, and as edges the edges of P together with enough extra edges to triangulate all the 2-faces of P. Then $g_2(P)$ is the dimension of $\mathcal{S}(\Phi_P)$.

Given a polytope P and a face F, define the closed union of faces $N(P,F)$ to be the union of all facets of P containing F. Note that $N(P,\emptyset) = \partial P$, and $N(P,P) = \emptyset$. Let V_N be the set of vertices of P in $N(P,F)$, and define a framework Φ_N by taking all edges and vertices of Φ_P contained in $N(P,F)$.

Theorem 6. *We have*

$$g_1(P,F) = \dim_{\mathbb{R}} \mathcal{A}\mathit{ff}(V_P)/\,\mathcal{A}\mathit{ff}(V_N), \text{ and}$$

$$g_2(P,F) = \dim_{\mathbb{R}} \mathcal{S}(\Phi_P)/\mathcal{S}(\Phi_N),$$

using the obvious inclusions of $\mathcal{A}\mathit{ff}(V_N)$ in $\mathcal{A}\mathit{ff}(V_P)$ and $\mathcal{S}(\Phi_N)$ in $\mathcal{S}(\Phi_P)$.

The proof for g_1 is an easy exercise; the proof for g_2 will appear in a forthcoming paper [3].

Next, we have a formula which shows that $g(P,F)$ can be decomposed in the same way $g(P)$ was in Proposition 2. Given two faces E, F of a polytope P, let $E \vee F$ be the unique smallest face containing both E and F.

Proposition 7. *For any polytope P and faces $F' \leq F$ of P, we have*

$$g(P,F) = \sum_{F' \leq E} g(E,F')g(P/E,(E \vee F)/E).$$

Proof. Again, we show that this formula for $g(P,F)$ satisfies the defining relation of Proposition 2. Fix $F' \leq F$, and define $\hat{g}(P,F)$ to be the above sum. Then we have

$$\begin{aligned}\sum_{F \leq D} \hat{g}(D,F)g(P/D) &= \sum_{\substack{F' \leq E \\ F \vee E \leq D}} g(P/D)g(E,F')g(D/E,(E \vee F)/E) \\ &= \sum_{F' \leq E} g(E,F')g(P/E) \\ &= g(P).\end{aligned}$$

Since the computation of $g(P,F)$ from Proposition 2 only involves computation of $g(E,F)$ for other faces E of P, this proves that $\hat{g}(P,F) = g(P,F)$, as required. $\square$

Finally, we can carry out the inversion implicit in Proposition 2 explicitly. First we need the notion of polar polytopes. Given a polytope $P \subset \mathbb{R}^d$, we can assume that the origin lies in the interior of P by moving P by an affine motion. The polar polytope P^* is defined by

$$P^* = \{\, x \in (\mathbb{R}^*)^d \mid \langle x, y\rangle \le 1 \text{ for all } y \in P \,\}.$$

The face poset $\mathcal{F}(P^*)$ is canonically the opposite poset to $\mathcal{F}(P)$. Define $\bar{g}(P) = g(P^*)$.

Proposition 8. *We have*

$$g(P,F) = \sum_{F \le F' \le P} (-1)^{\dim P - \dim F'} g(F')\bar{g}(P/F'). \tag{2}$$

Proof. We use the following formula, due to Stanley [15]: For any polytope $P \neq \emptyset$, we have

$$\sum_{\emptyset \le F \le P} (-1)^{\dim F} \bar{g}(F) g(P/F) = 0. \tag{3}$$

Now define $\hat{g}(P,F)$ to be the right hand side of (2). We will show that the defining property (1) of Proposition 2 holds.

Pick a face F of P. We have, using (3),

$$\sum_{F \le E \le P} \hat{g}(E,F) g(P/E) = \sum_{F \le F' \le E \le P} (-1)^{\dim E - \dim F'} g(F')\bar{g}(E/F')g(P/E)$$

$$\begin{aligned} &= \sum_{F \le F' \le P} g(F') \sum_{F' \le E \le P} (-1)^{\dim E - \dim F'} \bar{g}(E/F')g(P/E) \\ &= g(P), \end{aligned}$$

as required. $\square$

Introduction to §§4 - 6

The remainder of the paper uses the topology of toric varieties to describe the polynomial $g(P,F)$ when P is rational. Given P, there is an associated affine toric variety X_P, and $g(P)$ gives the local intersection cohomology betti numbers of X_P at the unique torus fixed point p.

The main topological result is the following (Theorem 10). Let $Y \subset X_P$ be the closure of one of the torus orbits. Then the restriction of the intersection cohomology sheaf $\mathbf{IC}^{\cdot}(X)$ to Y is a direct sum of intersection cohomology sheaves, with shifts, supported on subvarieties of Y (a related result is given by Victor Ginzburg in [8], Lemma 3.5). The polynomial $g_i(P,F)$ measures the number of copies of the intersection cohomology sheaf $\mathbf{IC}^{\cdot}(\{p\})$ that appear with shift $2i$ in the restriction of the intersection cohomology sheaf of X_P to Y_F, where Y_F is the closure of the orbit corresponding to the face F.

To prove Theorem 10 we construct a certain resolution (the Seifert resolution, §5) $p\colon \widetilde{X} \to X$ of X. Its key property is that the inclusion of $\widetilde{Y} = p^{-1}(Y)$ in $\widetilde{X}$ is "$\mathbb{Q}$-homology normally nonsingular" - the restriction of the intersection cohomology sheaf of $\widetilde{X}$ to $\widetilde{Y}$ is an intersection cohomology sheaf (Proposition 14).

This construction, and hence Theorem 10, work in situations other than toric varieties; essentially any variety X with a $\mathbb{C}^*$ action contracting X onto the fixed point set Y will satisfy Theorem 10. The proof we give, while easier than the general result, only works for toric varieties.

4. Toric varieties

We will only sketch the properties of toric varieties that we will need. For a more complete presentation, see [7]. Throughout this section let P be a d-dimensional rational polytope in $\mathbb{R}^d$.

Define a toric variety X_P as follows. Embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ by

$$(x_1,\ldots,x_d) \mapsto (x_1,\ldots,x_d,1),$$

and let $\sigma = \sigma_P$ be the cone over the image of P with apex at the origin in $\mathbb{R}^{d+1}$. It is a rational polyhedral cone with respect to the standard lattice $N = \mathbb{Z}^{d+1}$. More generally, if F is a face of P, let σ_F be the cone over the image of F; set $\sigma_\emptyset = \{0\}$.

Define $X = X_P$ to be the affine toric variety X_σ corresponding to σ. It is the variety $\operatorname{Spec}\mathbb{C}[M \cap \sigma^\vee]$, where

$$\sigma^\vee = \{\mathbf{x} \in (\mathbb{R}^{d+1})^* \mid \langle \mathbf{x},\mathbf{y}\rangle \geq 0 \quad \text{for all} \quad \mathbf{y} \in \sigma\}$$

is the dual cone to σ, M is the dual lattice to N, and $\mathbb{C}[M \cap \sigma^\vee]$ is the semigroup algebra of $M \cap \sigma^\vee$. It is a $(d+1)$-dimensional normal affine algebraic variety, on which the torus $T = \mathrm{Hom}(M, \mathbb{C}^*)$ acts. Let $f_{\mathbf{v}}: X_P \to \mathbb{C}$ be the regular function corresponding to the point $\mathbf{v} \in M \cap \sigma^\vee$.

The orbits of the action of T on X are parametrized by the faces of P. Let F be any face of P, including the empty face, and let

$$\sigma_F^\perp = \{\, \mathbf{x} \in \sigma^\vee \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \text{for all} \quad \mathbf{y} \in \sigma_F \,\}$$

be the face of $\sigma^\vee$ dual to σ_F. Then the variety

$$O_F := \{\, x \in X \mid f_{\mathbf{v}}(x) \neq 0 \iff \mathbf{v} \in M \cap \sigma_F^\perp \,\}$$

is a T-orbit, isomorphic to the torus $(\mathbb{C}^*)^{d-e}$, where $e = \dim F$. Furthermore, all T-orbits arise this way. In particular, X_P has a unique T-fixed point $\{p\} = O_P$.

Given a face E, the union

$$U_E = \bigcup_{F \leq E} O_F$$

is a T-invariant open neighborhood of O_E. There is a non-canonical isomorphism $U_E \cong O_E \times X_E$ where X_E is the affine toric variety defined by the cone σ_E, considered as a subset of the affine it spans, with the lattice given by restricting N. If O_F^E denotes the orbit of X_E corresponding to a face $F \leq E$, then O_F sits in $U_E \cong O_E \times X_E$ as $O_E \times O_F^E$.

The closure of the orbit O_E is given by

$$\overline{O_E} = \bigcup_{F \geq E} O_F;$$

it is isomorphic to the affine toric variety $X_{P/E}$. More precisely, it is the affine toric variety corresponding to the cone $\tau = \sigma/\sigma_E$, the image of σ projected into $\mathbb{R}^{d+1}/\operatorname{span}\sigma_E$, with the lattice given by the projection of N; τ is a cone over a polytope projectively equivalent to P/E.

The connection between toric varieties and g-numbers of polytopes is given by the following result. Proofs appear in [5, 6]. We consider the intersection cohomology sheaf $\mathbf{IC}^{\cdot}(X)$ of a variety X as an object in the bounded derived category $D^b(X)$ of sheaves of $\mathbb{Q}$-vector spaces on X. We will take the convention that $\mathbf{IC}^{\cdot}(X)$ restricts to a constant local system placed in degree zero on an smooth open subset of X.

Proposition 9. *The local intersection cohomology groups of X_P are described as follows. Take $x \in O_F$, and let j_x be the inclusion. Then*

$$\dim \mathbb{H}^{2i} j_x^* \mathbf{IC}^{\cdot}(X_P) = g_i(F),$$

and $\mathbb{H}^k j_x^* \mathbf{IC}^{\cdot}(X_P)$ vanishes for odd k.

Definition. Call an object $\mathbf{A}$ in $D^b(X)$ *pure* if it is a direct sum of shifted intersection cohomology sheaves

$$\bigoplus_\alpha \mathbf{IC}^{\cdot}(Z_\alpha;\mathcal{L}_\alpha)[n_\alpha], \tag{4}$$

where each Z_α is an irreducible subvariety of X, $\mathcal{L}_\alpha$ is a simple local system on a Zariski open subset U_α of the smooth locus of Z_α, and n_α is an integer.

Now fix a face F of P. The following theorem is the main result of this paper. It will be proved in the following two sections.

Theorem 10. *Let $j\colon \overline{O_F}\to X_P$ be the inclusion. Then the pullback $\mathbf{A}=j^*\mathbf{IC}^{\cdot}(X_P)$ of the intersection cohomology sheaf on X_P is pure.*

As a result, since the local intersection cohomology exists only in even degrees and gives trivial local systems on the orbits O_Y, we get

$$\mathbf{A} = \bigoplus_{E\geq F}\bigoplus_{i\geq 0}\mathbf{IC}^{\cdot}(\overline{O_E})[-2i]\otimes V_E^i, \tag{5}$$

for some finite dimensional $\mathbb{Q}$-vector spaces V_E^i.

Now we can give an interpretation of the combinatorially defined polynomials $g(P,F)$ for rational polytopes which implies nonnegativity, and hence Theorem 4. Let $\{p\}=O_P$ be the unique T-fixed point of X_P.

Theorem 11. *The relative g-number $g_i(P,F)$ is given by*

$$g_i(P,F)=\dim_{\mathbb{Q}} V_P^i.$$

Proof. Taking this for the moment as a definition of $g(P,F)$, we will show that the defining relation of Proposition 2 holds. It will be enough to show that $\dim_{\mathbb{Q}} V_E^i = g(E,F)$ for a face $F\leq E\neq P$, since then taking the dimensions of the stalk cohomology groups on both sides of (5) gives exactly the desired relation (1).

Consider the commutative diagram of inclusions

$$\begin{tikzcd} \overline{O_F^E} \arrow[r, "j'"] \arrow[d, "k'"'] & X_E \arrow[d, "k"] \\ \overline{O_F} \arrow[r, "j"] & X_P \end{tikzcd}$$

where k maps $X_E\cong\{x\}\times X_E$ into $O_E\times X_E\cong U_E\subset X_P$, and k' is the restriction of k.

Then k is a normally nonsingular inclusion, so we have

$$(j')^*k^*\mathbf{IC}^{\cdot}(X_P) = (j')^*\mathbf{IC}^{\cdot}(X_E) =$$
$$\bigoplus_{F\le F'\le E}\bigoplus_{i\ge 0}\mathbf{IC}^{\cdot}(\overline{O^E_{F'}})[-2i]\otimes W^i_{F'}$$

for some vector spaces $W^i_{F'}$. On the other hand, since k' is a normally nonsingular inclusion, it is also equal to

$$(k')^*j^*\mathbf{IC}^{\cdot}(X_P) = \bigoplus_{F\le F'\le E}\bigoplus_{i\ge 0}\mathbf{IC}^{\cdot}(\overline{O^E_{F'}})[-2i]\otimes V^i_{F'}.$$

Where the $V^i_{F'}$ are as in (5).

Comparing terms, we see that $W^i_E \cong V^i_E$, so we have

$$\dim_{\mathbb{Q}} V^i_E = \dim_{\mathbb{Q}} W^i_E = g_i(E,F),$$

as required. □

5. The Seifert resolution

Fix a face F of the polytope P, and let $\tau = \sigma_F$, $X = X_P$, $Y = Y_F$. Our proof of Theorem 10 involves constructing a certain resolution $\widetilde{X}$ of X, which we call a *Seifert resolution* of the pair (X, Y). First we need to choose an action of $\mathbb{C}^*$ on X for which Y is the fixed-point set.

Let $\mathbf{a}$ be any lattice point in the relative interior of the cone τ. Define the rank-one subtorus $T_{\mathbf{a}} \subset T \cong \mathrm{Hom}(M, \mathbb{C}^*)$ to be the kernel of the restriction

$$\mathrm{Hom}(M, \mathbb{C}^*) \to \mathrm{Hom}(M \cap \mathbf{a}^{\perp}, \mathbb{C}^*).$$

The map $M \to \mathbb{Z}$ given by pairing with $\mathbf{a}$ defines a homomorphism $\mathbb{C}^* = \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) \to T = \mathrm{Hom}(M, \mathbb{C}^*)$ with image contained in $T_{\mathbf{a}}$, thus defining an action of $\mathbb{C}^*$ on X.

Proposition 12. *Y is the fixed-point set of this action, and for any $x \in X$ we have*

$$\lim_{t\to 0} t\cdot x \in Y.$$

We say that Y is an *attractor* for the $\mathbb{C}^*$ action.

Let $X^\circ = X \setminus Y$. By the proposition above, the map $X^\circ \times \mathbb{C}^* \to X^\circ$ defined by our $\mathbb{C}^*$ action extends to a map $X^\circ \times \mathbb{C} \to X$. Let $\widetilde{X}$ be the quotient $X^\circ \times \mathbb{C}/\sim$, where the equivalence relation is given by

$$(x, s) \sim (t\cdot x, t^{-1}s)$$

for $t \in \mathbb{C}^*$. There is an induced map $p\colon \widetilde{X} \to X$. We can let T act on $X^\circ \times \mathbb{C}$ by acting on the first factor; this action passes to $\widetilde{X}$, and p is an equivariant map. Let $\widetilde{Y} = p^{-1}(Y)$, $\widetilde{X}^\circ = p^{-1}(X^\circ)$.

Proposition 13. *The map p is proper, and restricts to an isomorphism $\widetilde{X}^\circ \cong X^\circ$. Furthermore, $\widetilde{Y} \cong (X^\circ \times \{0\})/\mathbb{C}^*$ is a divisor in $\widetilde{X}$, and is an attractor for the the $\mathbb{C}^*$ action on $\widetilde{X}$ defined by the lattice point a.*

We call the pair $(\widetilde{X}, \widetilde{Y})$ a Seifert resolution of (X, Y). The action of T makes $\widetilde{X}$ into a toric variety. An explicit description of its fan will be useful. Take a fan consisting of all cones of the form ρ and $\rho_{\mathbf{a}} = \rho + \mathbb{R}_{\geq 0}\mathbf{a}$, where ρ runs over all faces of σ which do not contain τ. Then $\widetilde{X}$ is the toric variety defined by this fan, and $\widetilde{Y}$ is the union of the orbits corresponding to the cones $\rho_{\mathbf{a}}$.

The inclusion $\tilde{\jmath}\colon \widetilde{Y} \to \widetilde{X}$ looks almost like the inclusion of the zero section of a line bundle; for instance, if X is conical, $Y = \{p\}$ is the cone point and a is chosen to give the conical $\mathbb{C}^*$ action, then $\widetilde{X}$ is just the blow-up of X along Y.

Proposition 14. *There is an isomorphism*

$$\tilde{\jmath}^* \mathbf{IC}^{\cdot}(\widetilde{X}) \cong \mathbf{IC}^{\cdot}(\widetilde{Y}).$$

We will prove this in the next section; first, we show how it implies Theorem 10. Consider the fiber square

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\tilde{\jmath}} & \widetilde{X} \\ \Big\downarrow q & & \Big\downarrow p \\ Y & \xrightarrow{j} & X \end{array}$$

where $q = p|_{\widetilde{Y}}$. Because p and q are proper we have

$$Rq_* \tilde{\jmath}^* \mathbf{IC}^{\cdot}(\widetilde{X}) \cong j^* Rp_* \mathbf{IC}^{\cdot}(\widetilde{X}).$$

The left hand side is $Rq_* \mathbf{IC}^{\cdot}(\widetilde{Y})$ by Proposition 14, which is pure by the decomposition theorem [2]. The decomposition theorem also implies that $\mathbf{A} = Rp_* \mathbf{IC}^{\cdot}(\widetilde{X})$ is pure, and because $\widetilde{X} \to X$ is an isomorphism on a Zariski dense subset, the intersection cohomology sheaf of X must occur in $\mathbf{A}$ with zero shift. Thus the right hand side becomes

$$j^*(\mathbf{IC}^{\cdot}(X)) \oplus j^* \mathbf{A}',$$

where $\mathbf{A}'$ is pure. Theorem 10 now follows from the following lemma.

Lemma 15. *If $\mathbf{A}, \mathbf{B}$ are objects in $D^b(X)$ and $\mathbf{A} \oplus \mathbf{B}$ is pure, then so is $\mathbf{A}$.*

Proof. Denote $\mathbf{A} \oplus \mathbf{B}$ by $\mathbf{C}$. Since $\mathbf{C}$ is pure, it is isomorphic to the direct sum

$$\bigoplus_{i \in \mathbb{Z}} {}^pH^i(\mathbf{C})[-i]$$

of its perverse homology sheaves. Each ${}^pH^i(\mathbf{C}) = {}^pH^i(\mathbf{A}) \oplus {}^pH^i(\mathbf{B})$ is a pure perverse sheaf, and since the category of perverse sheaves is abelian, ${}^pH^i(\mathbf{A})$ is pure. Then the composition

$$\bigoplus {}^pH^i(\mathbf{A})[-i] \to \bigoplus {}^pH^i(\mathbf{C})[-i] \cong \mathbf{C} \to \mathbf{A}$$

induces an isomorphism on all the perverse homology sheaves, and hence is an isomorphism (see [2], §1.3). □

6. Proof of Proposition 14

Let $\mathbf{A} = \tilde{\jmath}^* \mathbf{IC}^{\cdot}(\widetilde{X})$. We will show that $\mathbf{A}$ satisfies the vanishing conditions for intersection cohomology on the stalk and costalk cohomology groups [9], and thus must be isomorphic to $\mathbf{IC}^{\cdot}(\widetilde{Y})$.

If $\widetilde{X}$ is a line bundle over $\widetilde{Y}$, the result is immediate. In general, we can take a quotient by a finite group which acts trivially on $\widetilde{Y}$ and get a line bundle. This works for more general varieties than toric varieties, but for our purposes a combinatorial proof will suffice.

We continue the notation of the previous section. For each face ρ not containing τ, let n_ρ be the index of the lattice $(N \cap \operatorname{span}(\rho)) + \mathbf{a}\mathbb{Z}$ in N. If $n = \operatorname{lcm} n_\rho$, then we can define a lattice $N' = N + (\mathbf{a}/n)\mathbb{Z}$ containing N. We get a corresponding map of tori $T \to T'$; the kernel G is a finite cyclic group inside $T_{\mathbf{a}}$.

Proposition 16. *The quotient $\widetilde{X}/G$ is a line bundle over $\widetilde{Y}/G \cong \widetilde{Y}$.*

Using this, we prove Proposition 14. We can retract $\widetilde{X}$ onto $\widetilde{Y}$ using the $\mathbb{C}^*$ action; we get an isomorphism

$$\mathbf{A} \cong R\pi_* \mathbf{IC}^{\cdot}(\widetilde{X}),$$

where $\pi\colon \widetilde{X} \to \widetilde{Y}$ is the projection defined by the action.

For a point $y \in \widetilde{Y}$, we can find a neighborhood $N \subset \widetilde{Y}$ of y so that the stalk and costalk cohomology groups of $\mathbf{A}$ are given by

$$\mathbb{H}^i i_y^* \mathbf{A} = IH_{n-i}(\pi^{-1}(N), \pi^{-1}(\partial N)),$$
$$\mathbb{H}^i i_y^! \mathbf{A} = IH_{n-i}(\pi^{-1}(N)).$$

Since $G \subset T_{\mathbf{a}}$, elements of G preserve the fibers of π and act by transformations which are isotopic to the identity. Thus G acts trivially on the stalks and costalks of $\mathbf{A}$. The following lemma then shows that they are isomorphic to $IH_{n-i}(\pi^{-1}(N)/G, \pi^{-1}(\partial N)/G)$ and $IH_{n-i}(\pi^{-1}(N)/G)$, respectively, and hence to $IH_{n-i}(N, \partial N)$ and $IH_{n-i}(N)$, since $\widetilde{X}/G$ is a line bundle over $\widetilde{Y}$. The required vanishing follows immediately.

Lemma 17. *Let X be a pseudomanifold, acted on by a finite group G, and let Y be a G-invariant subspace. Then there is an isomorphism*

$$IH_*(X/G, Y/G; \mathbb{Q}) \cong IH_*(X, Y; \mathbb{Q})^G$$

between the intersection homology of the pair $(X/G, Y/G)$ and the G-stable part of the intersection homology of (X, Y).

Proof. Give X a G-invariant triangulation. Then the intersection homology of X can be expressed by means of simplicial chains of the barycentric subdivision, see [13, Appendix]. Now the standard argument in [4, p. 120] can be applied. □

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