Tohoku Math. J.
56 (2004), 531–551

INTERSECTION NUMBERS FOR LOADED CYCLES ASSOCIATED WITH SELBERG-TYPE INTEGRALS

Dedicated to Professor Atsushi Inoue on his sixtieth birthday

KATSUHISA MIMACHI, KATSUYOSHI OHARA AND MASAAKI YOSHIDA

(Received March 27, 2003)

Abstract. We evaluate the intersection numbers of loaded cycles associated with an n-fold Selberg-type integral. We proceed inductively using high-dimensional local systems.

Introduction. We evaluate the intersection numbers of loaded cycles associated with the n-fold Selberg-type integral

$$\int u(t)\, dt_1 \wedge \cdots \wedge dt_n, \quad u = \prod_{i=1}^{n} \prod_{(i<)\ j} (t_i - t_j)^{\alpha_{ij}}\,,$$

where t_j $(n < j)$ are mutually distinct parameters. Let $\mathcal{L} = \mathcal{L}^+$ be the local system defined by the integrand u on

$$X^n = \{t = (t_1, \ldots, t_n) \in \boldsymbol{C}^n \mid t_i \neq t_j,\ \ 1 \le i \le n,\ 1 \le j \neq i\}\,,$$

and $\mathcal{L}^-$ its dual, i.e., the local system on X defined by u^{-1}. When t_j $(n < j)$ are real, the real locus $X^n_{\boldsymbol{R}}$ of X^n breaks into disjoint n-cells. We load each cell, say $\varDelta$, with $u^{\pm 1}$ in the standard way (see §1.4) and make it loaded cycles $\varDelta_\pm$, that is, elements of the *locally finite* n-th homology group $H^{\mathrm{lf}}_n(X, \mathcal{L}^\pm)$ with coefficients in $\mathcal{L}^\pm$. There is a natural dual pairing

$$H^{\mathrm{lf}}_n(X, \mathcal{L}) \times H_n(X, \mathcal{L}^-) \to \boldsymbol{C}$$

called the *intersection pairing.* Throughout this paper we assume that the exponents α_{ij} are sufficiently generic so that the natural map

$$H_n(X, \mathcal{L}) \to H^{\mathrm{lf}}_n(X, \mathcal{L})$$

is an isomorphism; the inverse map is called the *regularization* and is denoted by reg. For general background, refer to [Yo].

The main purpose of this paper is the evaluation of the intersection numbers of these cycles. The so-called half-turn formula for these cycles plays a crucial role; this formula is important for its own sake. To help the reader's understanding, we present the context in $n = 1, 2, 3$ before stating in general n.

2000 *Mathematics Subject Classification.* Primary 34M35; Secondary 33C70.
Key words and phrases. Twisted (co)homology, loaded cycles, intersection numbers, hypergeometric integrals, Selberg-type integrals, fibre bundle.

We proceed inductively by utilizing a fibre bundle structure of the space X^n. In order to fit the notation of the coordinates and the points to induction, we fix them as follows. For each positive integer n and a sequence of mutually distinct numbers

$$t_0,\ t_{n+1},\ t_{n+2},\ \dots,$$

we consider a domain in $\boldsymbol{C}^n$ defined by

$$X^{1\cdots n}_{t_0,t_{n+1},\dots} := \{(t_1,\dots,t_n)\in \boldsymbol{C}^n \mid t_i\neq t_j,\ 1\le i\le n,\ 0\le j\neq i\}.$$

The local system $\mathcal{L}$ (of rank 1) on $X^{1\cdots n}_{t_0,t_{n+1},\dots}$ is defined by the function

$$u := \prod_{i=1}^{n}(t_i-t_0)^{\alpha_{i0}} \prod_{(i<)\ j}(t_i-t_j)^{a_{ij}}$$

on $X^{1\cdots n}_{t_0,t_{n+1},\dots}$. Via the projection

$$\pi : X^{1\cdots n}_{t_0,t_{n+1},\dots} \ni (t_1,\dots,t_n) \longmapsto t_n \in X^{n}_{t_0,t_{n+1},\dots} := \{t_n \mid t_n\neq t_0, t_{n+1},\dots\},$$

the space $X^{1\cdots n}_{t_0,t_{n+1}}$ can be regarded as a fibre bundle over the 1-dimensional space $X^{n}_{t_0,t_{n+1},\dots}$ with fibre

$$\pi^{-1}(t_n) = X^{1\cdots(n-1)}_{t_0,t_n,\dots} = \{(t_1,\dots,t_{n-1}) \mid t_i\neq t_j,\ 1\le i\le n-1,\ 0\le j\}.$$

Making use of this fibre bundle structure, we express the intersection numbers of n-dimensional cycles on $X^{1\cdots n}_{t_0,t_{n+1},\dots}$ in terms of those of k-dimensional cycles on $X^{1\cdots k}_{t_0,t_{k+1},\dots}$ for $1\le k\le n-1$.

We name the cycles on $X^{1\cdots(n-1)}_{t_0,t_n,\dots}$ in §1, and the half-turn formula, which is the key of our method, is given in §2. Finally in §3, we give formulae for the intersection numbers. An application of these formulae is given in §4. §5 recalls a fundamental structure of twisted (co)homology groups of fibre bundles stated in [OST].

1. Coding the cycles. For indeterminates $t_0, t_1, \dots$, we consider domains in real t-spaces.

1.1 1D case. On the real t_1-space, we fix $t_0<t_2<t_3<\cdots$ and name the intervals (see Figure 1)

$$D_{0\tilde{1}2\cdots} = D_{0\tilde{1}2} := \{t_1\in \boldsymbol{R} \mid t_0<t_1<t_2\},\quad D_{02\tilde{1}3\cdots} = D_{2\tilde{1}3} := \{t_1 \mid t_2<t_1<t_3\},\ \dots.$$

($\tilde{1}$ indicates that t_1 is a variable, i.e., not fixed.) We array these intervals as

$$D^1_{023\cdots} := {}^t(D_{0\tilde{1}2}, D_{2\tilde{1}3}, D_{3\tilde{1}4}, \dots).$$

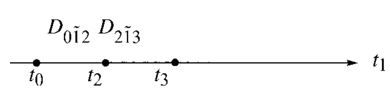

FIGURE 1. Coding the intervals.

1.2 2D case. On the real (t_1, t_2)-space, we fix $t_0 < t_3 < t_4 < \cdots$ and name the domains (see Figure 2)

$$D_{0\tilde{1}\tilde{2}3} := \{(t_1, t_2) \in \boldsymbol{R}^2 \mid t_0 < t_1 < t_2 < t_3\},\ D_{0\tilde{2}\tilde{1}3} := \{(t_1, t_2) \mid t_0 < t_2 < t_1 < t_3\},$$

$$D_{0\tilde{2}3\tilde{1}4} := \{(t_1, t_2) \mid t_0 < t_2 < t_3 < t_1 < t_4\},\ D_{0\tilde{1}3\tilde{2}4} := \{(t_1, t_2) \mid t_0 < t_1 < t_3 < t_2 < t_4\}, \ldots .$$

($\tilde{1}$ and $\tilde{2}$ indicate that t_1 and t_2 are variables, i.e., not fixed.) We array these domains as

$$D^{12}_{034\cdots} := {}^t(D^1_{0\tilde{2}34\cdots}, D^1_{03\tilde{2}4\cdots}, D^1_{034\tilde{2}5\cdots}, \ldots),$$

where

$$D^1_{0\tilde{2}34\cdots} := {}^t(D_{0\tilde{1}\tilde{2}3}, D_{0\tilde{2}\tilde{1}3}, D_{0\tilde{2}3\tilde{1}4}, D_{0\tilde{2}34\tilde{1}5}, \ldots), \ldots .$$

1.3 3D case. On the real (t_1, t_2, t_3)-space, we fix $t_0 < t_4 < t_5 < \cdots$ and name the domains

$$D_{0\tilde{1}\tilde{2}\tilde{3}4\cdots} := \{(t_1, t_2, t_3) \in \boldsymbol{R}^2 \mid t_0 < t_1 < t_2 < t_3 < t_4\}, \ldots .$$

($\tilde{1}, \tilde{2}, \tilde{3}$ indicate that t_1, t_2, t_3 are variables, i.e., not fixed.) We array these intervals as

$$D^{123}_{045\cdots} := {}^t(D^{12}_{0\tilde{3}4\cdots}, D^{12}_{04\tilde{3}5\cdots}, D^{12}_{045\tilde{3}6\cdots}, \ldots), \ldots,$$

where

$$D^{12}_{0\tilde{3}4\cdots} := {}^t(D^1_{0\tilde{2}\tilde{3}4\cdots}, D^1_{0\tilde{3}\tilde{2}4\cdots}, D^1_{0\tilde{3}4\tilde{2}5\cdots}, \cdots), \ldots$$

and

$$D^1_{0\tilde{2}\tilde{3}4\cdots} := {}^t(D_{0\tilde{1}\tilde{2}\tilde{3}4}, D_{0\tilde{2}\tilde{1}\tilde{3}4}, D_{0\tilde{2}\tilde{3}\tilde{1}4}, D_{0\tilde{2}\tilde{3}4\tilde{1}5}, \cdots), \ldots .$$

1.4 nD case. Now the reader can easily imagine what the authors would like to define. For notational simplicity, we put

$$''n := n-2, \quad 'n := n-1, \quad n' := n+1, \quad n'' := n+2.$$

On the real $(t_1, \ldots, t_n)$-space, we fix $t_0 < t_{n'} < t_{n''} < \cdots$ and name the domains

$$D_{0\tilde{1}\cdots\tilde{n}n'\cdots} := \{(t_1, \ldots, t_n) \in \boldsymbol{R}^n \mid t_0 < t_1 < \cdots < t_n < t_{n'}\}, \ldots .$$

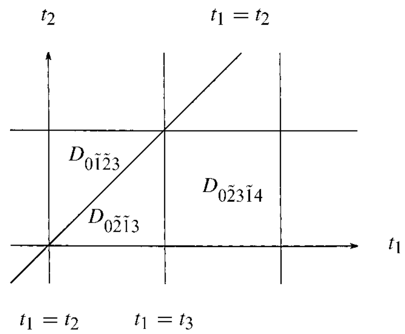

FIGURE 2. Coding the 2-domains.

($\tilde{1}, \dots, \tilde{n}$ indicate that $t_1, \dots, t_n$ are variables, i.e., not fixed.) We array these intervals inductively as

$$D^{1\cdots n}_{0n'n''\cdots} := {}^t(D^{1\cdots' n}_{0\tilde{n}n'\cdots}, D^{1\cdots' n}_{0n'\tilde{n}n''\cdots}, D^{1\cdots' n}_{0n'n''\tilde{n}\cdots}, \dots)\,, \quad \dots$$

$$\vdots$$

$$D^{1}_{0\tilde{2}\cdots\tilde{n}n'\cdots} := {}^t(D_{0\tilde{1}\tilde{2}\cdots\tilde{n}n'}, D_{0\tilde{2}\tilde{1}\cdots\tilde{n}n'}, D_{0\tilde{2}\tilde{3}\tilde{1}\cdots\tilde{n}n'}, \dots, D_{0\tilde{2}\tilde{3}\cdots\tilde{n}\tilde{1}n'}, \dots)\,, \quad \dots.$$

In this subsection, the numerals with tildes among the sub-indices of D indicate that the corresponding points are free (not fixed). When it is clear which points are fixed and which points are free, we often omit the tildes, later.

1.5 Standard loading. For complex constants $a_{ij} \in \boldsymbol{C} - \boldsymbol{Z}$, we consider the multi-valued fuction

$$u := \prod_{i=1}^{n} (t_i - t_0)^{\alpha_{i0}} \prod_{(i<)\ j} (t_i - t_j)^{a_{ij}}\,,$$

and the local system $\mathcal{L}$ defined by u. We load each domain D with the function

$$u := \prod_{i=1}^{n} \{\varepsilon^D_{i0}(t_i - t_0)\}^{\alpha_{i0}} \prod_{(i<)\ j} \{\varepsilon^D_{ij}(t_i - t_j)\}^{a_{ij}}\,,$$

where $\varepsilon^D_{i0}, \varepsilon^D_{ij} = \pm 1$ are so chosen that $\varepsilon^D_{i0}(t_i - t_0)$ and $\varepsilon^D_{ij}(t_i - t_j)$ are positive on the domain D (arguments of positve numbers are 0), and regard the domain as a *loaded cycle*. This loading is said to be *standard*. A standardly loaded cycle will often be called just by the name of the domain which supports the cycle.

1.6 Notation of the exponents.

$$a_{ij} = a_{ji}\,, \quad r_{ij} := \exp \pi\sqrt{-1}a_{ij}\,,$$

$$r_{ijk} := r_{ij} \cdot r_{jk} \cdot r_{ki}\,, \quad r_{ij\cdots k} := \prod_{\{p,q\} \subset \{i,j,\dots,k\}} r_{pq}\,,$$

$$c_{ij} := r^2_{ij}\,, \quad c_{ij\cdots k} := r^2_{ij\cdots k}\,,$$

$$d_{ij} := c_{ij} - 1\,, \quad d_{ijk} := c_{ijk} - 1, \dots, \quad d_{ij\cdot pq} := c_{ij} \cdot c_{pq} - 1, \dots.$$

2. The half-turn formulae. Let $t_a < t_b$ be adjacent points in the sequence $t_0 < t_{n+1} < \cdots$. The two points t_a and t_b were fixed when we defined the n-dimensional domains in §1 such as $D_{0\tilde{1}\tilde{2}\cdots\tilde{n}n'}, D_{0\tilde{2}\tilde{1}\cdots\tilde{n}n'}, \dots$. In this section we move t_b in the complex plane in the counterclockwise direction around t_a. Accordingly the domains deform, and so do the loaded cycles. We describe the happening when the travel of t_b is halfway completed, and t_a and t_b have exchanged positions. We assume that the resulting interval $[t_b, t_a]$ contains no other points in $\{t_0, t_{n+1}, \dots\}$. Each of the (standardly loaded) cycles with respect to the sequence

$$t_0 < t_{n+1} < \cdots < t_a < t_b < \cdots$$

is transformed into a linear combination of the (standardly loaded) cycles with respect to the sequence

$$t_0 < t_{n+1} < \cdots < t_b < t_a < \cdots .$$

This transformation is called the (positive) *half-turn* $H(ab)$ with respect to the points $t_a < t_b$. If t_b continues traveling back to the original position, by applying the half-turn transformation (do not forget to exchange the corresponding exponents) again, we get the *full-turn* transformation $F(ab)$.

2.1 1D case. Let $t_a < t_b$ be adjacent points in the sequence $t_0 < t_2 < t_3 < \cdots$. The half-turn $H(ab)$ with respect to the points $t_a < t_b$ transforms the cycles as

$$\begin{aligned}
&D_{1ab} \to r_{ab}(D_{1ba} + r_{1b}D_{b1a})\,,\\
&D_{a1b} \to -r_{a1b}D_{b1a}\,,\\
&D_{ab1} \to r_{ab}(D_{ba1} + r_{a1}D_{b1a})\,,
\end{aligned}$$

other intervals $D_{\dots ab\dots}$ being sent to $r_{ab}D_{\dots ba\dots}$. (See Figure 3.) In this subsection, the tilde on the letter 1 is omitted.

2.2 2D case. Let $t_a < t_b$ be adjacent points in the sequence $t_0 < t_3 < t_4 < \cdots$. The half-turn $H(ab)$ with respect to the points $t_a < t_b$ transforms the cycles as

$$\begin{aligned}
&D_{12ab} \to r_{ab}(D_{12ba} + r_{2b}D_{1b2a} + r_{2b}r_{1b}D_{b12a})\,,\\
&D_{1a2b} \to -r_{a2b}(D_{1b2a} + r_{1b}D_{b12a} + r_{1b}r_{12}D_{b21a})\,,\\
&D_{1ab2} \to r_{ab}(D_{1ba2} + r_{1b}D_{b1a2} + r_{a2}D_{1b2a} + r_{1b}r_{a2}D_{b12a} + r_{1b}r_{a2}r_{12}D_{b21a})\,,\\
&D_{a12b} \to r_{a12b}D_{b21a}\,,\\
&D_{a1b2} \to -r_{a1b}(D_{b1a2} + r_{a2}D_{b12a} + r_{a2}r_{12}D_{b21a})\,,\\
&D_{ab12} \to r_{ab}(D_{ba12} + r_{a1}D_{b1a2} + r_{a1}r_{a2}D_{b12}a)\,.
\end{aligned}$$

(See Figure 4.) If 1 or 2 is away from a and b, then the move reduces to the $1D$-case. In this subsection, the tildes on the letters 1 and 2 are omitted.

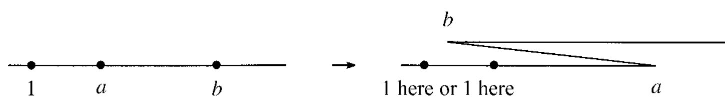

FIGURE 3. Half-turn of D_{1ab}.

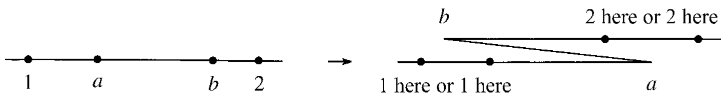

FIGURE 4. Half-turn of D_{1ab2}.

2.3 3D case. Let $t_a < t_b$ be adjacent points in the sequence $t_0 < t_4 < t_5 < \cdots$. The half-turn $H(ab)$ with respect to the points $t_a < t_b$ transforms the cycles as

$$
\begin{aligned}
D_{123ab} \to\ & r_{ab}(D_{123ba} + r_{3b}D_{12b3a} + r_{3b}r_{2b}D_{1b23a} + r_{3b}r_{2b}r_{1b}D_{b123a})\,,\\
D_{12a3b} \to\ & -r_{a3b}(D_{12b3a} + r_{2b}D_{1b23a} + r_{2b}r_{23}D_{1b32a} + r_{2b}r_{1b}D_{b123a} + r_{2b}r_{1b}r_{23}D_{b132a}\\
& + r_{2b}r_{1b}r_{23}r_{13}D_{b312a})\,,\\
D_{12ab3} \to\ & r_{ab}(D_{12ba3} + r_{2b}D_{1b2a3} + r_{2b}r_{3a}D_{1b23a} + r_{2b}r_{3a}r_{32}D_{1b32a} + r_{2b}r_{1b}D_{b12a3}\\
& + r_{2b}r_{1b}r_{3a}D_{b123a} + r_{2b}r_{1b}r_{3a}r_{32}D_{b132a} + r_{2b}r_{1b}r_{3a}r_{32}r_{31}D_{b312a})\,,\\
D_{1a23b} \to\ & r_{a23b}(D_{1b23a} + r_{1b}D_{b123a} + r_{1b}r_{12}D_{b213a} + r_{1b}r_{12}r_{31}D_{b231a})\,,\\
D_{1a2b3} \to\ & -r_{a2b}(D_{1b2a3} + r_{1b}D_{b12a3} + r_{a3}D_{1b23a}\\
& + r_{1b}r_{a3}\textstyle\sum_\sigma D_{b1^\sigma 2^\sigma 3^\sigma a}\ (\sigma : \text{permutations of } 123)\,,\\
D_{a123b} \to\ & -r_{a123b}D_{b321a}\,,
\end{aligned}
$$

$D_{a12b3}\,,\quad D_{a1b23}\,,\quad D_{ab123}$: as above.

If 1, 2 or 3 is away from a, b and c, then the move reduces to the 2D-case. In this subsection, the tildes on the letters 1, 2 and 3 are omitted.

2.4 nD case. By the examples above, we hope that the reader can imagine what will happen in high-dimensional cases. Since the situation is fairly complicated, we state the action of half-turn properly.

A shuffle of the three ordered sets

$$X = \{\dots, x_i, x_{i+1}, \dots\}\,, \quad Y = \{\dots, y_j, y_{j+1}, \dots\}\,, \quad Z = \{\dots, z_k, z_{k+1}, \dots\}$$

is a permutaion of the elements of $X \cup Y \cup Z$ such that the order of the elements of X, the order of the elements of Y, and the order of the elements of Z are preserved. The set of such shuffles is denoted by $S(X, Y, Z)$. For example, if

$$X = \{x_1, x_2\}\,, \quad Y = \{y_1, y_2\}\,, \quad Z = \emptyset\,,$$

then

$$S(X, Y, Z) = \{x_1x_2y_1y_2,\ x_1y_1x_2y_2,\ x_1y_1y_2x_2,\ y_1x_1x_2y_2,\ y_1x_1y_2x_2,\ y_1y_2x_1x_2\}\,.$$

For two members x and y of a shuffle s, if x is on the left side of y, we denote '$x \prec y$ in s'. Let Y^{-1} be the order reversed set of Y, that is,

$$Y^{-1} = \{\dots, y_{j+1}, y_j, \dots\}\,.$$

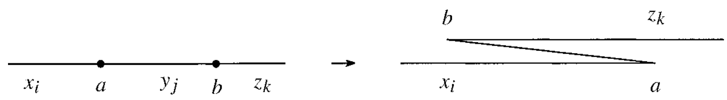

FIGURE 5. Half-turn $H(a, b)$.

For a shuffle $s \in S(X, Y^{-1}, Z)$, define $\varepsilon_s(\cdot, \cdot)$ as

$$\varepsilon_s(x_i, y_j) := \begin{cases} 0 & \text{if } x_i \prec y_j \text{ in } s\,, \\ 1 & \text{if } x_i \succ y_j \text{ in } s\,, \end{cases} \qquad \varepsilon_s(y_j, z_k) := \begin{cases} 0 & \text{if } y_j \prec z_k \text{ in } s\,, \\ 1 & \text{if } y_j \succ z_k \text{ in } s\,, \end{cases}$$

and

$$\varepsilon_s(x_i, z_k) := \begin{cases} 0 & \text{if } x_i \prec z_k \text{ in } s\,, \\ 1 & \text{if } x_i \succ z_k \text{ in } s\,. \end{cases}$$

Consider an n-dimensional domain D_{***} defined in §1.4. Recall that it is coded by an arrangement of n moving points $t_1, \dots, t_n$ in the real line divided by fixed ones $t_0 < t_{n+1} < \cdots$. Let $t_a < t_b$ be adjacent points in the sequence $t_0 < t_{n+1} < \cdots$, and $t_{a'}$ be the left-adjacent point (if any) to t_a, and $t_{b'}$ the right-adjacent point to t_b. Let t_X, $t_{Y'}$ and t_Z be the ordered subsets of $\{t_0, \dots, t_n\}$ situated in the interval $[t_{a'}, t_a]$, $[t_a, t_b]$ and $[t_b, t_{b'}]$, respectively. The indices of the points in t_X, $t_{Y'}$ and t_Z define ordered subsets X, Y' and Z of $\{0, \dots, n\}$; put $Y = \{a, Y', b\}$. Thus the domain D_{***} has the coding $D_{\dots XYZ\dots}$. Now we are ready to state the move of the half-turn.

PROPOSITION 1. *Let a be the left extreme element of Y, and b the right extreme. Suppose that $X \cup (Y - \{a, b\}) \cup Z \subset \{1, \dots, n\}$. Then the half-turn $H(ab)$ with respect to the points $t_a < t_b$ transforms the cycles*

$$D_{\dots XYZ\dots} = D_{\dots x_i x_{i+1}\dots a \dots y_j y_{j+1} \dots b \dots z_k z_{k+1} \dots}$$

into

$$(-)^{\#Y} r_Y \sum_{s \in S(X, Y^{-1}, Z)} r(s) D_{\dots s \dots}\,,$$

where

$$r(s) = \prod_{i,j,k} r_{x_i y_j}^{\varepsilon_s(x_i, y_j)} r_{y_j z_k}^{\varepsilon_s(y_j, z_k)} r_{x_i z_k}^{\varepsilon_s(x_i, z_k)}, \qquad r_Y = \prod_{\{p,q\} \subset Y} r_{pq}\,.$$

PROOF. See Figure 5. Remember that x_i move on the left-side of a, y_j between a and b, and z_k on the right-side of b. After the half-turn of b around a, let us regard

$$\dots x_i x_{i+1} \dots b \dots y_{j+1} y_j \dots a \dots z_k z_{k+1} \dots$$

as the ground state. Starting from this ground state, the very right one among $\{x_i\}$ can pass b and the y_j's, the very left one among $\{z_k\}$ can pass a and the y_j's; then the next right one among $\{x_i\}$ and so on, and we get shuffles of X, Y^{-1} and Z. Each time when x_i passes y_j, the argument of $t_{x_i} - t_{y_j}$ increases by π, so $r_{x_i y_j}$ is multiplied. This is also the case when x_i passes z_k, and z_k passes y_j.

Since the moves of y_j are reversed, for the orientation reason, we have the overall factor $(-)^{\#Y}$. The other overall factor r_Y comes from the factors

$$\prod_{p<q}(t_{y_p}-t_{y_q})^{\alpha_{y_p y_q}}$$

of u; note that the half-turn increases the argument of every factor by π. □

3. Intersection numbers. Consider for each positive integer n and a sequence of mutually distinct numbers

$$t_0\,,\quad t_{n+1}\,,\quad t_{n+2},\ldots,$$

a domain in $\boldsymbol{C}^n$ defined by

$$X^{1\cdots n}_{t_0,t_{n+1},\ldots} := \{(t_1,\ldots,t_n)\in \boldsymbol{C}^n \mid t_i\neq t_j,\ 1\le i\le n,\ 0\le j\neq i\}\,.$$

The local system $\mathcal{L}$ (of rank 1) is defined by the function

$$u := \prod_{i=1}^{n}(t_i-t_0)^{\alpha_{i0}}\prod_{(i<)\ j}(t_i-t_j)^{a_{ij}}$$

on $X^{1\cdots n}_{t_0,t_{n+1}}$. Via the projection

$$\pi : X^{1\cdots n}_{t_0,t_{n+1},\ldots}\ni (t_1,\ldots,t_n)\longmapsto t_n\in X^n_{t_0,t_{n+1},\ldots} := \{t_n \mid t_n\neq t_0, t_{n+1},\ldots\}\,,$$

the space $X^{1\cdots n}_{t_0,t_{n+1}}$ can be regarded as a fibre bundle over the 1-dimensional space $X^n_{t_0,t_{n+1},\ldots}$ with fibre

$$\pi^{-1}(t_n)=X^{1\cdots(n-1)}_{t_0,t_n,\ldots}\,.$$

As explained in §5, we have the isomorphism

$$H_n(X^{1\cdots n}_{t_0,t_{n+1},\ldots},\mathcal{L})\cong H_1(X^n_{t_0,t_{n+1},\ldots},\mathcal{H}_{n-1})\,,$$

where $\mathcal{H}_{n-1}$ is the locally constant sheaf of germs of locally flat sections of the bundle

$$\bigcup_{t_n\in X^n_{t_0,t_{n+1},\ldots}} H_{n-1}(X^{1\cdots(n-1)}_{t_0,t_n,\ldots},\iota^*\mathcal{L})\,,$$

where $\iota : X^{1\cdots(n-1)}_{t_0,t_n,\ldots}\to X^{1\cdots n}_{t_0,t_{n+1},\ldots}$ is the inclusion.

When $t_0<t_{n+1}<t_{n+2}<\cdots$, the set $D^{1\cdots n}_{0(n+1)\cdots}$ of loaded cycles form a basis of $H^{\mathrm{lf}}_n(X^{1\cdots n}_{t_0,t_{n+1},\ldots},\mathcal{L})$. Assume that the $(n-1)$-dimensional intersection numbers of the cycles in $D^{1\cdots(n-1)}_{0n\cdots}$ and their duals are already known. Then the intersection numbers of the cycles

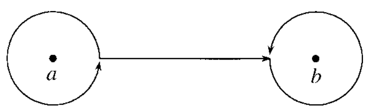

FIGURE 6. reg I_{ab}.

in $D^{1\dots n}_{0(n+1)\cdots}$ and their duals can be evaluated as follows. For adjacent points $t_a < t_b$ $(a, b \in \{0, n+1, \dots\})$ in t_n-space, consider the interval $I_{ab} := (t_a, t_b) \subset X^n_{t_0,t_{n+1},\cdots}$ and load it with a section σ of $\mathcal{H}^{\mathrm{lf}}_{n-1}$ to be an element $I_{ab} \otimes \sigma$ of $H^{\mathrm{lf}}_1(X^n_{t_0,t_{n+1},\dots}, \mathcal{H}^{\mathrm{lf}}_{n-1})$. We regularize the cycle $I_{ab} \otimes \sigma$ as

$$\mathrm{reg} I_{ab} \otimes \sigma := S(a; a+\varepsilon) \otimes (F(an) - \mathrm{id})^{-1}\sigma + (a+\varepsilon, b-\varepsilon) \otimes \sigma$$
$$- S(b; b-\varepsilon) \otimes (F(nb) - \mathrm{id})^{-1}\sigma\,,$$

where $S(a; a+\varepsilon)$ is a positively oriented circle with center a and radius ε starting and ending at $a+\varepsilon$ (see Figure 6). Also,

$$F(nb) : H^{\mathrm{lf}}_{n-1}(X^{1\cdots(n-1)}_{t_0,\dots,t_a,t_n,t_b,\dots}, \iota^*\mathcal{L}) \to H^{\mathrm{lf}}_{n-1}(X^{1\cdots(n-1)}_{t_0,\dots,t_a,t_n,t_b,\dots}, \iota^*\mathcal{L})$$

is the full-turn operator with respect to the move of the point t_n along the circle $S(b; b-\varepsilon)$.

For adjacent points $t_c < t_d$ in the t_n-space, consider the loaded cycle $I_{cd} \otimes \check{\sigma}$ with support on the interval (t_c, t_d) and loaded with a section $\check{\sigma}$ of $\check{\mathcal{H}}^{\mathrm{lf}}_{n-1}$ to be an element of $H^{\mathrm{lf}}_1(X^n_{t_0,t_{n+1},\dots}, \check{\mathcal{H}}^{\mathrm{lf}}_{n-1})$. If $(a, b) = (c, d)$, then the intersection number $(I_{ab} \otimes \sigma) \bullet (I_{cd} \otimes \check{\sigma})$ is given (§5) by

$$(\mathrm{reg}\ I_{ab} \otimes \sigma) \bullet (I_{cd} \otimes \check{\sigma}) = -\{(F(an) - \mathrm{id})^{-1}\sigma\} \bullet \check{\sigma}|_{t_n=a+\varepsilon}$$
$$-\sigma \bullet \check{\sigma}|_{t_n=(a+b)/2} - \{(F(nb) - \mathrm{id})^{-1}\sigma\} \bullet \check{\sigma}|_{t_n=b-\varepsilon}\,;$$

recall that we assumed that the $(n-1)$-dimensional intersection numbers are already known. If $b = c$, then $(I_{ab} \otimes \sigma) \bullet (I_{cd} \otimes \check{\sigma})$ is given by

$$\{(F(nb) - \mathrm{id})^{-1} H(nb)\sigma\} \bullet \check{\sigma}|_{t_n=b+\varepsilon}\,,$$

where

$$H(nb) : H^{\mathrm{lf}}_{n-1}(X^{1\cdots(n-1)}_{t_0,\dots,t_a,t_n,t_b,\dots}, \iota^*\mathcal{L}) \to H^{\mathrm{lf}}_{n-1}(X^{1\cdots(n-1)}_{t_0,\dots,t_a,t_b,t_n,\dots}, \iota^*\mathcal{L})$$

is the half-turn operator with respect to the move of the point t_n along the half of the circle $S(b, b-\varepsilon)$ (note that $F(nb) = H(nb)H(bn)$); if $d = a$, then $(I_{ab} \otimes \sigma) \bullet (I_{cd} \otimes \check{\sigma})$ is given by

$$\{(F(an) - \mathrm{id})^{-1} H(an)\}\sigma \bullet \check{\sigma}|_{t_n=a-\varepsilon}\,,$$

otherwise 0.

The following examples show the actual process. We represent the half-turn and full-turn operators $H(\cdot\cdot)$ and $F(\cdot\cdot)$ by matrices, which will be denoted by roman letters $\mathrm{H}(\cdot\cdot)$ and $\mathrm{F}(\cdot\cdot)$, respectively.

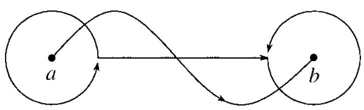

FIGURE 7. $(\mathrm{reg}\ I_{ab} \otimes \sigma) \bullet (I_{ab} \otimes \check{\sigma})$.

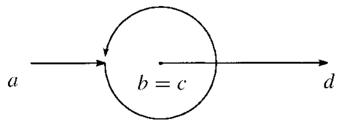

FIGURE 8. (reg $I_{ab} \otimes \sigma) \bullet (I_{bd} \otimes \check{\sigma})$.

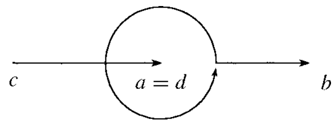

FIGURE 9. (reg $I_{ab} \otimes \sigma) \bullet (I_{ca} \otimes \check{\sigma})$.

3.1 1D case. The half-turn $H(1a)$ of t_1 around t_a is represented by the scalar $\mathrm{H}(1a) = r_{1a}$, and the full-turn $F(1a)$ by the scalar $\mathrm{F}(1a) = c_{1a} = r_{1a}^2$. For adjacent $t_a < t_b$, we have

$$-(c_{a1}-1)^{-1} - 1 - (c_{1b}-1)^{-1} = -\frac{d_{a1\cdot 1j}}{d_{a1}d_{1b}}, \quad (c_{1b}-1)^{-1} r_{1b} = \frac{r_{1b}}{d_{1b}}.$$

The intersection matix is given as

$$I^1_{02\cdots} := D^1_{02\cdots} \bullet \check{D}^1_{02\cdots} = \begin{pmatrix} -\dfrac{d_{01\cdot 12}}{d_{01}d_{12}} & \dfrac{r_{12}}{d_{12}} & 0 & \cdots \\ \dfrac{r_{21}}{d_{21}} & -\dfrac{d_{21\cdot 13}}{d_{21}d_{13}} & \dfrac{r_{13}}{d_{13}} & 0 \\ 0 & \dfrac{r_{31}}{d_{31}} & -\dfrac{d_{31\cdot 14}}{d_{31}d_{14}} & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix}$$

3.2 2D case. For adjacent $t_a < t_2 < t_b$, let us represent the half-turns and the full-turns of t_2 around t_a and t_b by matrices with respect to the column vectors $D^1_{\cdots a2b\cdots}$ and $D^1_{\cdots 2ab\cdots}$ as

$$\begin{aligned} H(a2)D^1_{\cdots a2b\cdots} &= \mathrm{H}(a2)D^1_{\cdots 2ab\cdots}, \quad & F(a2)D^1_{\cdots a2b\cdots} &= \mathrm{F}(a2)D^1_{\cdots a2b\cdots}, \\ H(2b)D^1_{\cdots a2b\cdots} &= \mathrm{H}(2b)D^1_{\cdots ab2\cdots}, \quad & F(2b)D^1_{\cdots a2b\cdots} &= \mathrm{F}(2b)D^1_{\cdots a2b\cdots}. \end{aligned}$$

Our convention on the product of operators is from the left to the right; for example, $F(a2) = H(a2)H(2a)$ operates $H(a2)$ first and then $H(2a)$; so we have $\mathrm{F}(a2) = \mathrm{H}(a2)\mathrm{H}(2a)$. Put

$$\begin{aligned} \mathrm{F}(a2b) &:= -(\mathrm{F}(a2)-\mathrm{id})^{-1} - \mathrm{id} - (\mathrm{F}(2b)-\mathrm{id})^{-1}, \\ \mathrm{G}(2b) &:= (\mathrm{F}(2b)-\mathrm{id})^{-1}\mathrm{H}(2b). \end{aligned}$$

The 2D-intersection matrix is given as

$$I^{12}_{03\cdots} := D^{12}_{03\cdots} \bullet \check{D}^{12}_{03\cdots} = \begin{pmatrix} D^{1}_{0\tilde{2}3\cdots} \\ D^{1}_{03\tilde{2}4\cdots} \\ \vdots \end{pmatrix} \bullet ({}^t\check{D}^{1}_{0\tilde{2}3\cdots}, {}^t\check{D}^{1}_{03\tilde{2}4\cdots}, \ldots)$$

$$= \begin{pmatrix} \mathrm{F}(023) & \mathrm{G}(23) & 0 & \cdots \\ \mathrm{G}(32) & \mathrm{F}(324) & \mathrm{G}(24) & 0 \\ 0 & \mathrm{G}(42) & \mathrm{F}(425) & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I^{1}_{023\cdots} & 0 & 0 & \cdots \\ 0 & I^{1}_{0324\cdots} & 0 & 0 \\ 0 & 0 & I^{1}_{03425\cdots} & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix},$$

where I^1_{**} are already evaluated 1D-intersection matrices:

$$I^{1}_{023\cdots} = \begin{pmatrix} D_{0\tilde{1}2\cdots} \\ D_{02\tilde{1}3\cdots} \\ \vdots \end{pmatrix} \bullet ({}^t\check{D}_{0\tilde{1}2\cdots}, \ldots), \quad I^{1}_{0324\cdots} = \begin{pmatrix} D_{0\tilde{1}324\cdots} \\ D_{03\tilde{1}24\cdots} \\ \vdots \end{pmatrix} \bullet ({}^t\check{D}_{0\tilde{1}324\cdots}, \ldots), \ldots .$$

3.3 3D case. For adjacent $t_a < t_3 < t_b$, let us represent the half-turns and the full-turns of t_3 around t_a and t_b by matrices with respect to the column vectors $D^{12}_{\cdots a3b\cdots}$ and $D^{12}_{\cdots 3ab\cdots}$:

$$H(a3)D^{12}_{\cdots a3b\cdots} = \mathrm{H}(a3)D^{12}_{\cdots 3ab\cdots}, \quad F(a3)D^{12}_{\cdots a3b\cdots} = \mathrm{F}(a3)D^{12}_{\cdots a3b\cdots},$$
$$H(3b)D^{12}_{\cdots a3b\cdots} = \mathrm{H}(3b)D^{12}_{\cdots ab3\cdots}, \quad F(3b)D^{12}_{\cdots a3b\cdots} = \mathrm{F}(3b)D^{12}_{\cdots a3b\cdots}.$$

Put

$$\mathrm{F}(a3b) := -(\mathrm{F}(a3) - \mathrm{id})^{-1} - \mathrm{id} - (\mathrm{F}(3b) - \mathrm{id})^{-1},$$
$$\mathrm{G}(3b) := (\mathrm{F}(3b) - \mathrm{id})^{-1} H(3b).$$

The 3D-intersection matrix is given as

$$I^{123}_{04\cdots} := D^{123}_{04\cdots} \bullet \check{D}^{123}_{04\cdots} = \begin{pmatrix} D^{12}_{0\tilde{3}4\cdots} \\ D^{12}_{04\tilde{3}5\cdots} \\ \vdots \end{pmatrix} \bullet ({}^t\check{D}^{12}_{0\tilde{3}4\cdots}, {}^t\check{D}^{12}_{04\tilde{3}5\cdots}, \ldots)$$

$$= \begin{pmatrix} \mathrm{F}(034) & \mathrm{G}(34) & 0 & \cdots \\ \mathrm{G}(43) & \mathrm{F}(435) & \mathrm{G}(35) & 0 \\ 0 & \mathrm{G}(53) & \mathrm{F}(536) & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I^{12}_{034\cdots} & 0 & 0 & \cdots \\ 0 & I^{12}_{0435\cdots} & 0 & 0 \\ 0 & 0 & I^{12}_{04536\cdots} & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix},$$

where I^{12}_{**} are already evaluated 2D-intersection matrices.

3.4 nD case. Now the reader can easily imagine what will happen in high-dimensional cases. For adjacent $t_a < t_n < t_b$, let us represent the half-turns and the full-turns of t_n around t_a and t_b by matrices with respect to the column vectors $D^{1\cdots n'}_{\cdots anb\cdots}$ and $D^{1\cdots n'}_{\cdots nab\cdots}$:

$$H(an)D^{1\cdots n'}_{\cdots anb\cdots} = \mathrm{H}(an)D^{1\cdots n'}_{\cdots nab\cdots}\,, \quad F(an)D^{1\cdots n'}_{\cdots anb\cdots} = \mathrm{F}(an)D^{1\cdots n'}_{\cdots anb\cdots}\,,$$

$$H(nb)D^{1\cdots n'}_{\cdots anb\cdots} = \mathrm{H}(nb)D^{1\cdots n'}_{\cdots abn\cdots}\,, \quad F(nb)D^{1\cdots n'}_{\cdots anb\cdots} = \mathrm{F}(nb)D^{1\cdots n'}_{\cdots anb\cdots}\,.$$

Put

$$\mathrm{F}(anb) := -(\mathrm{F}(an) - \mathrm{id})^{-1} - \mathrm{id} - (\mathrm{F}(nb) - \mathrm{id})^{-1}\,,$$

$$\mathrm{G}(4b) := (\mathrm{F}(4b) - \mathrm{id})^{-1}\mathrm{H}(4b)\,.$$

PROPOSITION 2. *The nD-intersection matrix is given as*

$$I^{12\cdots n}_{0n'\cdots} := D^{12\cdots n}_{0n'\cdots} \bullet \check{D}^{12\cdots n}_{0n'\cdots} = \begin{pmatrix} D^{1\cdots' n}_{0\tilde{n}n'\cdots} \\ D^{1\cdots' n}_{0n'\tilde{n}n''\cdots} \\ \vdots \end{pmatrix} \bullet ({}^t\check{D}^{1\cdots' n}_{0\tilde{n}n'\cdots}, {}^t\check{D}^{1\cdots' n}_{0n'\tilde{n}n''\cdots}, \dots)$$

$$= \begin{pmatrix} \mathrm{F}(0nn') & \mathrm{G}(nn') & 0 & \cdots \\ \mathrm{G}(n'n) & \mathrm{F}(n'nn'') & \mathrm{G}(nn'') & 0 \\ 0 & \mathrm{G}(n''n) & \mathrm{F}(n''nn''') & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I^{1\cdots' n}_{0nn'\cdots} & 0 & 0 & \cdots \\ 0 & I^{1\cdots' n}_{0n'nn''\cdots} & 0 & 0 \\ 0 & 0 & I^{1\cdots' n}_{0n'n''nn'''\cdots} & \ddots \\ \vdots & 0 & \ddots & \ddots \end{pmatrix},$$

where $I^{1\cdots' n}_{**}$ are already evaluated $(n-1)$D-intersection matrices.

4. An application to the Selberg integral. To show that our inductive method can serve as practical use, we derive the result obtained in [MiY1] and [MiY2] as applications of our method in 1,2 and 3-dimensional cases.

So far, we considered infinitely many t_i's, and the exponents α_{ij} were independent. Now we work in $(t_1, \dots, t_n)$-space with the three kinds of hyperplanes

$$(t_0 =)\ 0 = t_j\,, \quad t_j = 1\ (= t_{n+1})\,, \quad t_i = t_j \quad (1 \le i, j \le n)$$

with respective exponents

$$c_{01} = \cdots = c_{0n} = a\,, \quad c_{1(n+1)} = \cdots = c_{n(n+1)} = b\,, \quad r_{ij} = g \quad (1 \le i, j \le n)\,.$$

The integral in question is the Selberg integral

$$\int u\, \frac{dt_1 \wedge \cdots \wedge dt_n}{\prod_{i=1}^n t_i(1-t_i)}\,, \quad u = \prod_{i=1}^n t_i^a (1-t_i)^b \prod_{1 \le i < j \le n} |t_i - t_j|^{2g}\,.$$

The symmetric group acts on the coordinates $t_1, \dots, t_n$ of

$$X^n = \{(t_1, \dots, t_n) \in \boldsymbol{C}^n \mid t_i \neq 0, 1, t_j\ (i \neq j)\}\,.$$

So it acts also on the homology group $H_n^{\mathrm{lf}}(X^n, \mathcal{L})$. Its S_n-invariant subspace is 1-dimensional, and is generated by

$$C_n := \sum_{\sigma \in S_n} D_{0\widetilde{1^\sigma}\widetilde{2^\sigma}\cdots\widetilde{n^\sigma}(n+1)} .$$

Note that $D_{0\widetilde{1^\sigma}\widetilde{2^\sigma}\cdots\widetilde{n^\sigma}(n+1)}$ is a standardly loaded cycle with support on the simplex

$$\{(t_1, \dots, t_n) \in \boldsymbol{R}^n \mid 0 < t_{1^\sigma} < \cdots < t_{n^\sigma} < 1\} .$$

In this section we evaluate the nD-intersection number $J_n := C_n \bullet \check{C}_n$ for $n = 1, 2, 3$.

4.1 1D case.

$$J_1 = D_{0\tilde{1}2} \bullet \check{D}_{0\tilde{1}2} = -\frac{d_{01\cdot 12}}{d_{01}d_{12}} = \frac{1-ab}{(1-a)(1-b)} .$$

4.2 2D case. We evaluate $J_2 = (D_{0\tilde{1}\tilde{2}3} + D_{0\tilde{2}\tilde{1}3}) \bullet (\check{D}_{0\tilde{1}\tilde{2}3} + \check{D}_{0\tilde{2}\tilde{1}3})$. Recall that these intersection numbers are computed as

$$\begin{aligned} I_{03}^{12} &:= D_{03}^{12} \bullet \check{D}_{03}^{12} = D_{0\tilde{2}3}^{1} \bullet {}^t\check{D}_{0\tilde{2}3}^{1} = \begin{pmatrix} D_{0\tilde{1}\tilde{2}3} \\ D_{0\tilde{2}\tilde{1}3} \end{pmatrix} \bullet (\check{D}_{0\tilde{1}\tilde{2}3}, \check{D}_{0\tilde{2}\tilde{1}3}) \\ &= \mathrm{F}(023) I_{023}^{1} , \end{aligned}$$

where

$$I_{023}^{1} := D_{023}^{1} \bullet {}^t\check{D}_{023}^{1} = \begin{pmatrix} D_{0\tilde{1}23} \\ D_{02\tilde{1}3} \end{pmatrix} \bullet (\check{D}_{0\tilde{1}23}, \check{D}_{02\tilde{1}3}) = -\begin{pmatrix} \dfrac{d_{01\cdot 12}}{d_{01}d_{12}} & -\dfrac{r_{12}}{d_{12}} \\ -\dfrac{r_{21}}{d_{21}} & \dfrac{d_{21\cdot 13}}{d_{21}d_{13}} \end{pmatrix} ,$$

and

$$\mathrm{F}(02) := \mathrm{H}(02)\mathrm{H}(20) = \begin{pmatrix} -r_{012} & 0 \\ r_{02}r_{01} & r_{02} \end{pmatrix} \begin{pmatrix} -r_{012} & 0 \\ r_{20}r_{12} & r_{20} \end{pmatrix} ,$$

$$\begin{aligned} \mathrm{F}(023) :&= -(\mathrm{F}(02) - \mathrm{id})^{-1} - \mathrm{id} - (\mathrm{F}(23) - \mathrm{id})^{-1} \\ &= -\begin{pmatrix} c_{012} - 1 & 0 \\ -c_{02}r_{12}d_{01} & c_{02} - 1 \end{pmatrix}^{-1} - \mathrm{id} - \begin{pmatrix} c_{23} - 1 & -c_{23}r_{12}d_{13} \\ 0 & c_{123} - 1 \end{pmatrix}^{-1} \\ &= -\begin{pmatrix} \dfrac{1}{d_{012}} + 1 + \dfrac{1}{d_{23}} & \dfrac{c_{23}r_{12}d_{13}}{d_{23}d_{123}} \\ \dfrac{c_{02}r_{12}d_{01}}{d_{012}d_{02}} & \dfrac{1}{d_{02}} + 1 + \dfrac{1}{d_{123}} \end{pmatrix} = -\begin{pmatrix} \dfrac{d_{012\cdot 23}}{d_{012}d_{23}} & \dfrac{c_{23}r_{12}d_{13}}{d_{23}d_{123}} \\ \dfrac{c_{02}r_{12}d_{01}}{d_{012}d_{02}} & \dfrac{d_{02\cdot 123}}{d_{02}d_{123}} \end{pmatrix} . \end{aligned}$$

Now put

$$c_{01} = c_{02} = a , \quad c_{13} = c_{23} = b , \quad r_{12} = g ,$$

and add the entries of

$$\mathrm{F}(023) = \begin{pmatrix} \dfrac{1 - a^2g^2b}{(1-a^2g^2)(1-b)} & \dfrac{gb}{1-g^2b^2} \\ \dfrac{ag}{1-a^2g^2} & \dfrac{1-ag^2b^2}{(1-a)(1-g^2b^2)} \end{pmatrix}$$

vertically:

$$(f_1, f_2) := \left(\frac{1-agb}{(1-ag)(1-b)}, \frac{1-agb}{(1-a)(1-gb)}\right),$$

and add the entries of

$$I^1_{023} = \begin{pmatrix} \dfrac{1-ag^2}{(1-a)(1-g^2)} & -\dfrac{g}{1-g^2} \\ -\dfrac{g}{1-g^2} & \dfrac{1-g^2b}{(1-g^2)(1-b)} \end{pmatrix}$$

horizontally:

$$^t(g_1, g_2) := {}^t\!\left(\frac{ag+1}{(1-a)(g+1)}, \frac{gb+1}{(g+1)(1-b)}\right).$$

We thus have

$$\begin{aligned} J_2 = f_1 g_1 + f_2 g_2 &= \frac{1-agb}{(1-a)(1-b)(g+1)}\left(\frac{ag+1}{ag-1} + \frac{gb+1}{1-gb}\right) \\ &= \frac{2\,(1-agb)(1-ag^2b)}{(1-a)(1-ag)(1-b)(1-gb)(g+1)}, \end{aligned}$$

which agrees with the result in [MiY1].

4.3 3D case. We evaluate $J_3 := C_3 \bullet \check{C}_3$, where $C_3 = \sum_{\sigma \in S_3} D_{0\widetilde{1^\sigma}\widetilde{2^\sigma}\widetilde{3^\sigma}4}$. Recall that these intersection numbers are computed as

$$\begin{aligned} I^{123}_{04} &:= D^{123}_{04} \bullet \check{D}^{123}_{04} = D^{12}_{0\tilde{3}4} \bullet {}^t\check{D}^{12}_{0\tilde{3}4} = \begin{pmatrix} D^1_{0\tilde{2}\tilde{3}4} \\ D^1_{0\tilde{3}\tilde{2}4} \end{pmatrix} \bullet ({}^t\check{D}^1_{0\tilde{2}\tilde{3}4}, {}^t\check{D}^1_{0\tilde{3}\tilde{2}4}) \\ &= \begin{pmatrix} D_{0\tilde{1}\tilde{2}\tilde{3}4} \\ D_{0\tilde{2}\tilde{1}\tilde{3}4} \\ D_{0\tilde{2}\tilde{3}\tilde{1}4} \\ D_{0\tilde{1}\tilde{3}\tilde{2}4} \\ D_{0\tilde{3}\tilde{1}\tilde{2}4} \\ D_{0\tilde{3}\tilde{2}\tilde{1}4} \end{pmatrix} \bullet (\check{D}_{0\tilde{1}\tilde{2}\tilde{3}4}, \check{D}_{0\tilde{2}\tilde{1}\tilde{3}4}, \check{D}_{0\tilde{2}\tilde{3}\tilde{1}4}, \check{D}_{0\tilde{1}\tilde{3}\tilde{2}4}, \check{D}_{0\tilde{3}\tilde{1}\tilde{2}4}, \check{D}_{0\tilde{3}\tilde{2}\tilde{1}4}) \\ &= \mathrm{F}(034)\, I^{12}_{034}. \end{aligned}$$

All data needed are evaluated already; we repeat them as follows. The action of the half-turn $H(03) : D^{12}_{034} \to D^{12}_{304}$ is given by

$$
\begin{aligned}
&D_{0\tilde{1}\tilde{2}34} \to r_{0123} D_{3\tilde{2}\tilde{1}04}\,,\\
&D_{0\tilde{2}\tilde{1}34} \to r_{0213} D_{3\tilde{1}\tilde{2}04}\,,\\
&D_{0\tilde{2}3\tilde{1}4} \to -r_{023}(D_{3\tilde{2}0\tilde{1}4} + r_{01} D_{3\tilde{2}\tilde{1}04} + r_{01}r_{21} D_{3\tilde{1}\tilde{2}04})\,,\\
&D_{0\tilde{1}3\tilde{2}4} \to -r_{013}(D_{3\tilde{1}0\tilde{2}4} + r_{02} D_{3\tilde{1}\tilde{2}04} + r_{02}r_{12} D_{3\tilde{2}\tilde{1}04})\,,\\
&D_{03\tilde{1}\tilde{2}4} \to r_{03}(D_{30\tilde{1}\tilde{2}4} + r_{01} D_{3\tilde{1}0\tilde{2}4} + r_{01}r_{02} D_{3\tilde{1}\tilde{2}04})\,,\\
&D_{03\tilde{2}\tilde{1}4} \to r_{03}(D_{30\tilde{2}\tilde{1}4} + r_{02} D_{3\tilde{2}0\tilde{1}4} + r_{02}r_{01} D_{3\tilde{2}\tilde{3}04})\,,
\end{aligned}
$$

the full-turn $F(03) : D^{12}_{034} \to D^{12}_{034}$ by the matrix $\mathrm{F}(03) = \mathrm{H}(03)\mathrm{H}(30)$:

$$
\begin{pmatrix}
c_{0123} & 0 & 0 & 0 & 0 & 0\\
0 & c_{0213} & 0 & 0 & 0 & 0\\
-c_{023}r_{31}r_{21}d_{01} & -c_{023}r_{31}d_{01\cdot 21} & c_{023} & 0 & 0 & 0\\
-c_{013}r_{32}d_{02\cdot 12} & -c_{013}r_{32}r_{12}d_{02} & 0 & c_{013} & 0 & 0\\
-c_{03}r_{31}r_{32}d_{01} & c_{03}c_{01}r_{123}d_{02} & 0 & -c_{03}r_{31}d_{01} & c_{03} & 0\\
c_{03}c_{02}r_{213}d_{01} & -c_{03}r_{32}r_{31}d_{02} & -c_{03}r_{32}d_{02} & 0 & 0 & c_{03}
\end{pmatrix}.
$$

The full-turn matrix F(34) can be expressed in a similar way:

$$
\begin{pmatrix}
c_{34} & 0 & 0 & -c_{34}r_{23}d_{24} & -c_{34}r_{23}r_{13}d_{24} & c_{34}c_{24}r_{123}d_{14}\\
0 & c_{34} & -c_{34}r_{13}d_{14} & 0 & c_{34}c_{14}r_{123}d_{24} & -c_{34}r_{13}r_{23}d_{14}\\
0 & 0 & c_{314} & 0 & -c_{314}r_{23}r_{21}d_{24} & -c_{314}r_{23}d_{24\cdot 21}\\
0 & 0 & 0 & c_{324} & -c_{324}r_{13}d_{14\cdot 12} & -c_{324}r_{13}r_{12}d_{14}\\
0 & 0 & 0 & 0 & c_{3124} & 0\\
0 & 0 & 0 & 0 & 0 & c_{3214}
\end{pmatrix}.
$$

Since they are triangular matrices,

$$\mathrm{F}(034) = -(\mathrm{F}(03) - \mathrm{id})^{-1} - \mathrm{id} - (\mathrm{F}(34) - \mathrm{id})^{-1}$$

can be computed without much difficulty as

$$
-\left(\begin{matrix}
\frac{d_{0123\cdot 34}}{d_{0123}d_{34}} & 0 & 0 \\
0 & \frac{d_{0123\cdot 34}}{d_{0123}d_{34}} & \frac{c_{34}r_{13}d_{14}}{d_{34}d_{134}} \\
\frac{c_{023}r_{13}r_{12}d_{01}}{d_{023}d_{0123}} & \frac{c_{023}r_{13}d_{01\cdot 12}}{d_{023}d_{0123}} & \frac{d_{023\cdot 314}}{d_{023}d_{314}} \\
\frac{c_{013}r_{23}d_{02\cdot 12}}{d_{013}d_{0123}} & \frac{c_{013}r_{23}r_{12}d_{02}}{d_{013}d_{0123}} & 0 \\
\frac{d_{01}d_{013\cdot 02\cdot 12}c_{03}r_{13}r_{23}}{d_{03}d_{013}d_{0123}} & -\frac{r_{123}d_{02}d_{03\cdot 13}c_{01}c_{03}}{d_{03}d_{013}d_{0123}} & 0 \\
-\frac{r_{123}d_{01}d_{23\cdot 03}c_{02}c_{03}}{d_{03}d_{023}d_{0123}} & \frac{d_{02}d_{023\cdot 01\cdot 12}c_{03}r_{13}r_{23}}{d_{03}d_{023}d_{0123}} & \frac{c_{03}r_{23}d_{02}}{d_{03}d_{023}}
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
\frac{c_{34}r_{23}d_{24}}{d_{34}d_{234}} & \frac{d_{24}d_{234\cdot 12\cdot 14}c_{34}r_{23}r_{13}}{d_{34}d_{234}d_{1234}} & -\frac{r_{123}d_{14}d_{23\cdot 34}c_{24}c_{34}}{d_{34}d_{234}d_{1234}} \\
0 & -\frac{r_{123}d_{24}d_{13\cdot 34}c_{34}c_{14}}{d_{34}d_{134}d_{1234}} & \frac{d_{14}d_{134\cdot 12\cdot 24}c_{34}r_{23}r_{13}}{d_{34}d_{134}d_{1234}} \\
0 & \frac{c_{134}r_{23}r_{12}d_{24}}{d_{134}d_{1234}} & \frac{c_{134}r_{23}d_{24\cdot 12}}{d_{134}d_{1234}} \\
\frac{d_{013\cdot 324}}{d_{013}d_{324}} & \frac{c_{234}r_{13}d_{14\cdot 12}}{d_{234}d_{1234}} & \frac{c_{234}r_{13}r_{12}d_{14}}{d_{234}d_{1234}} \\
\frac{c_{03}r_{13}d_{01}}{d_{03}d_{013}} & \frac{d_{03\cdot 3124}}{d_{03}d_{3124}} & 0 \\
0 & 0 & \frac{d_{03\cdot 3214}}{d_{03}d_{3214}}
\end{matrix}\right).
$$

The intersection matrix I_{034}^{12} can be computed by

$$
I_{034}^{12} = \begin{pmatrix} \mathrm{F}(023) & \mathrm{G}(23) \\ \mathrm{G}(32) & \mathrm{F}(324) \end{pmatrix} \begin{pmatrix} I_{0234}^{1} & 0 \\ 0 & I_{0324}^{1} \end{pmatrix},
$$

where

$$I^1_{0234} = \begin{pmatrix} -\frac{d_{01\cdot 12}}{d_{01}d_{12}} & \frac{r_{12}}{d_{12}} & 0 \\ \frac{r_{21}}{d_{21}} & -\frac{d_{21\cdot 13}}{d_{21}d_{13}} & \frac{r_{13}}{d_{13}} \\ 0 & \frac{r_{31}}{d_{31}} & -\frac{d_{31\cdot 14}}{d_{31}d_{14}} \end{pmatrix}, \quad I^1_{0324} : \text{exchange 2 and 3},$$

and the matrix-representation of the operators

$$F(023) : D^1_{0234} \to D^1_{0234}, \quad G(23) = (F(23) - \mathrm{id})^{-1} H(23) : D^1_{0234} \to D^1_{0324}$$

are given as follows.

$$\begin{array}{lclcl} & H(02) & & H(20) & \\ D_{0\tilde{1}234} & \to & -r_{012} D_{2\tilde{1}034} & \to & c_{012} D_{0\tilde{1}234}, \\ D_{02\tilde{1}34} & \to & r_{02}(D_{20\tilde{1}34} + r_{01} D_{0\tilde{1}234}) & \to & c_{02} D_{02\tilde{1}34} - c_{02} r_{21} d_{01} D_{0\tilde{1}234}, \\ D_{023\tilde{1}4} & \to & r_{02} D_{203\tilde{1}4} & \to & c_{02} D_{023\tilde{1}4}, \end{array}$$

$$\begin{array}{lclcl} & H(23) & & H(32) & \\ D_{0\tilde{1}234} & \to & r_{23}(D_{0\tilde{1}324} + r_{13} D_{03\tilde{1}24}) & \to & c_{23} D_{0\tilde{1}234} - c_{23} r_{12} d_{13} D_{02\tilde{1}34}, \\ D_{02\tilde{1}34} & \to & -r_{213} D_{03\tilde{1}24} & \to & c_{213} D_{02\tilde{1}34}, \\ D_{023\tilde{1}4} & \to & r_{23}(D_{032\tilde{1}4} + r_{12} D_{03\tilde{1}24}) & \to & c_{23} D_{023\tilde{1}4} - c_{23} r_{31} d_{12} D_{02\tilde{1}34}, \end{array}$$

$$\begin{aligned} \mathrm{F}(023) &= \begin{pmatrix} d_{012} & 0 & 0 \\ -c_{02} r_{12} d_{01} & d_{02} & 0 \\ 0 & 0 & d_{02} \end{pmatrix}^{-1} + \mathrm{id} + \begin{pmatrix} d_{23} & -c_{23} r_{12} d_{13} & 0 \\ 0 & d_{123} & 0 \\ 0 & -c_{23} r_{31} d_{12} & d_{23} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{d_{012\cdot 23}}{d_{012} d_{23}} & \frac{c_{23} r_{12} d_{13}}{d_{23} d_{123}} & 0 \\ \frac{c_{02} r_{21} d_{01}}{d_{012} d_{02}} & \frac{d_{02\cdot 123}}{d_{02} d_{123}} & 0 \\ 0 & \frac{c_{23} r_{31} d_{12}}{d_{123} d_{23}} & \frac{d_{02\cdot 23}}{d_{02} d_{23}} \end{pmatrix}, \end{aligned}$$

$$-G(23) = (\mathrm{H}(23)\mathrm{H}(32) - \mathrm{id})^{-1}\mathrm{H}(23) = \{\mathrm{H}(32) - \mathrm{H}(23)^{-1}\}^{-1}$$

$$= \begin{pmatrix} r_{32} - \dfrac{1}{r_{23}} & r_{32}r_{12} - \dfrac{1}{r_{23}r_{21}} & 0 \\ 0 & -r_{32}r_{31}r_{12} + \dfrac{1}{r_{23}r_{21}r_{13}} & 0 \\ 0 & r_{32}r_{13} - \dfrac{1}{r_{23}r_{13}} & r_{32} - \dfrac{1}{r_{23}} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \dfrac{r_{23}}{d_{23}} & \dfrac{d_{23\cdot 21}r_{123}}{r_{12}d_{23}d_{123}} & 0 \\ 0 & -\dfrac{r_{123}}{d_{123}} & 0 \\ 0 & \dfrac{d_{23\cdot 13}r_{123}}{r_{13}d_{23}d_{123}} & \dfrac{r_{23}}{d_{23}} \end{pmatrix}.$$

The matrix-representation of the operators

$$F(324) : D^1_{0324} \to D^1_{0324}\,, \quad G(32) = (F(32) - \mathrm{id})^{-1}H(32) : D^1_{0324} \to D^1_{0234}$$

are given as follows.

$$\begin{array}{lclcl} & H(32) & & H(23) & \\ D_{0\tilde{1}324} & \to & r_{32}(D_{0\tilde{1}234} + r_{12}D_{02\tilde{1}34}) & \to & c_{32}D_{0\tilde{1}324} - c_{32}r_{13}d_{12}D_{03\tilde{1}24}\,, \\ D_{03\tilde{1}24} & \to & -r_{312}D_{02\tilde{1}34} & \to & c_{312}D_{03\tilde{1}24}\,, \\ D_{032\tilde{1}4} & \to & r_{32}(D_{023\tilde{1}4} + r_{13}D_{02\tilde{1}34}) & \to & c_{32}D_{032\tilde{1}4} - c_{32}r_{21}d_{31}D_{03\tilde{1}24}\,, \end{array}$$

$$\begin{array}{lclcl} & H(24) & & H(42) & \\ D_{0\tilde{1}324} & \to & r_{24}D_{0\tilde{1}342} & \to & c_{24}D_{0\tilde{1}324}, \\ D_{03\tilde{1}24} & \to & r_{24}(D_{03\tilde{1}42} + r_{14}D_{034\tilde{1}2}) & \to & c_{24}D_{03\tilde{1}24} - c_{24}r_{12}d_{14}D_{032\tilde{1}4}, \\ D_{032\tilde{1}4} & \to & -r_{214}D_{034\tilde{1}2} & \to & c_{214}D_{032\tilde{1}4}, \end{array}$$

$$\mathrm{F}(324) = \begin{pmatrix} d_{23} & -c_{23}r_{13}d_{12} & 0 \\ 0 & d_{312} & 0 \\ 0 & -c_{32}r_{21}d_{31} & d_{23} \end{pmatrix}^{-1} + \mathrm{id} + \begin{pmatrix} d_{24} & 0 & 0 \\ 0 & d_{24} & -c_{24}r_{12}d_{14} \\ 0 & 0 & d_{214} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \dfrac{d_{23\cdot 24}}{d_{23}d_{24}} & \dfrac{c_{23}r_{13}d_{12}}{d_{23}d_{312}} & 0 \\ 0 & \dfrac{d_{312\cdot 24}}{d_{312}d_{24}} & \dfrac{c_{24}r_{12}d_{14}}{d_{24}d_{214}} \\ 0 & \dfrac{c_{32}r_{21}d_{31}}{d_{312}d_{23}} & \dfrac{d_{23\cdot 214}}{d_{23}d_{214}} \end{pmatrix},$$

G(32) being obtained from G(23) by exchanging 2 and 3.

Now we put

$$c_{01} = c_{02} = c_{03} = a\,, \quad c_{14} = c_{24} = c_{34} = c\,, \quad r_{12} = r_{23} = r_{31} = g\,.$$

The vertical sums of the entries of F(034) are $(f_1, f_1, f_2, f_2, f_3, f_3)$, where

$$(f_1, f_2, f_3) := (1 - ag^2b)\left(\frac{1}{(1-ag^2)(1-b)}, \frac{1}{(1-ag)(1-gb)}, \frac{1}{(1-a)(1-g^2b)}\right).$$

Let us honestly compute

$$(h_1, \ldots, h_6) := (f_1, f_1, \ldots, f_3)\begin{pmatrix} \mathrm{F}(023) & \mathrm{G}(23) \\ \mathrm{G}(32) & \mathrm{F}(324) \end{pmatrix}.$$

Then we have

$$h_1 = -\frac{(1-ag^2b)(1-ag^3b)}{(1-ag)(1-ag^2)(1-b)(1-gb)(g+1)},$$

$$h_2 = -\frac{(1-ag^2b)(1-ag^3b)(abg^4-(a+b)g^3+2abg^3-(a+b)g^2-(a+b)g+2g+1)}{(1-a)(1-ag)(1-ag^2)(1-b)(1-gb)(1-g^2b)(g+1)(g^2+g+1)},$$

$$h_3 = -\frac{(1-ag^2b)(1-ag^3b)}{(1-a)(1-ag)(1-gb)(1-g^2b)(g+1)},$$

$$h_4 = h_1, \quad h_5 = h_2, \quad h_6 = h_3.$$

On the other hand, the horizontal sums of the entries of I^1_{0234} are

$$(I_1, I_2, I_3) = \frac{1}{g+1}\left(\frac{ag+1}{1-a}, 1-g, \frac{gb+1}{1-b}\right).$$

We have

$$\begin{aligned}&(1-g^2b)(g^2+g+1)(ag+1)\\ &\quad + (acg^4-(a+b)g^3+2abg^3-(a+b)g^2-(a+b)g+2g+1)(1-g)\\ &\quad + (1-ag^2)(g^2+g+1)(gb+1) = 3(1-ag^4b)(g+1),\end{aligned}$$

and so the sum J_3 can be computed, and factors as

$$J_3 = -\frac{3!(1-g^4ab)(1-g^3ab)(1-g^2ab)}{(1-a)(1-ag)(1-ag^2)(1-b)(1-gb)(1-g^2b)(g+1)(g^2+g+1)},$$

which agrees with the result obtained geometrically in [MiY2].

5. General theory on the twisted (co)homology groups on fibred spaces ([OST]). Let $\pi : X \to B$ be a fibre bundle, and $\mathcal{L}$ a local system on X. Assume pure (co)dimensionality of the total space and the fibres:

$$H_i(X, \mathcal{L}) = 0, \qquad H^i(X, \mathcal{L}) = 0, \qquad \text{if} \quad i \neq n := \dim_{\mathbf{C}} X,$$

$$H_i(\pi^{-1}(b), \iota_b^*\mathcal{L}) = 0, \quad H^i(\pi^{-1}(b), \iota_b^*\mathcal{L}) = 0, \quad \text{if} \quad i \neq f := \dim_{\mathbf{C}} \pi^{-1}(b),$$

where $\iota_b : \pi^{-1}(b) \to X$ is the inclusion map. Then we have the natural isomorphisms

$$H_n(X, \mathcal{L}) \cong H_{n-f}(B, \mathcal{H}_f), \quad H^n(X, \mathcal{L}) \cong H^{n-f}(B, \mathcal{H}^f),$$

where $\mathcal{H}_f$ and $\mathcal{H}^f$ are local systems on B defined as the locally constant sheaves of germs of locally flat sections of the bundles

$$\bigcup_{b\in B} H_f(\pi^{-1}(b), \iota_b^*\mathcal{L}) \quad \text{and} \quad \bigcup_{b\in B} H^f(\pi^{-1}(b), \iota_b^*\mathcal{L}),$$

respectively.

Let $\gamma \in H_n(X, \mathcal{L})$ be represented by the finite sum

$$\sum a_i \delta_i \otimes u_i, \quad a_i \in \boldsymbol{C},$$

where $\delta_i \in H_{n-f}(B, \boldsymbol{Z})$, and u_i is a section of $\mathcal{H}_f$ along the support $|\delta_i|$ of δ_i. Let $\gamma' \in H_n^{\mathrm{lf}}(X, \check{\mathcal{L}})$ be represented by the locally finite sum $\sum a_i' \delta_i' \otimes u_i'$, where $\delta_i' \in H_{n-f}^{\mathrm{lf}}(B, \boldsymbol{Z})$, and u_i' is a section of $\check{\mathcal{H}}_f^{\mathrm{lf}}$, which is defined as the locally constant sheaf of germs of horizonatal sections of the bundle

$$\bigcup_{b\in B} H_f^{\mathrm{lf}}(\pi^{-1}(b), \iota_b^*\check{\mathcal{L}})$$

along the support $|\delta_i'|$ of δ_i'. Then the intersection number $\gamma \cdot \gamma'$ is equal to

$$\sum_{\{b\}=|\delta_i|\cap|\delta_j|} a_i a_j' (\delta_i \cdot \delta')(b)(u_i \cdot u_j')(b),$$

where $(\delta_i \cdot \delta_j')(b)$ is the topological intersection number at p, and $(u_i \cdot u_j')(b)$ is defined by the intersection pairing between $H_f(\pi^{-1}(b), \iota_b^*\mathcal{L})$ and $H_f^{\mathrm{lf}}(\pi^{-1}(b), \iota_b^*\check{\mathcal{L}})$.

Let $f \in H_c^n(X, \mathcal{L})$ be represented by the finite sum

$$\sum a_i g_i \otimes v_i, \quad a_i \in \boldsymbol{C},$$

where g_i is a compactly supported $(n-f)$-form on B and v_i is a section of $\mathcal{H}_c^f$, that is, v_i is a compactly supported f-form with values in $\mathcal{L}$ and with parameter b on the generic fibre. Let $f' \in H^n(X, \check{\mathcal{L}})$ be represented by the finite sum $\sum a_i' g_i' \otimes v_i'$, where g_i' is an $(n-f)$-form on B and v_i is a section of $\check{\mathcal{H}}^f$, that is, v_i is an f-form with values in $\check{\mathcal{L}}$ and with parameter b on the generic fibre. The intersection number $f \cdot f'$ is equal to

$$\sum a_i a_j' \int (v_i \cdot v_j')(b) g_i \wedge g_j',$$

where $(v_i \cdot v_j)(b)$ is defined by the intersection pairing between $H^f(\pi^{-1}(b), \iota_b^*\mathcal{L})$ and $H^f(\pi^{-1}(b), \iota_b^*\check{\mathcal{L}})$.

The de Rham theorem and the Fubini theorem imply the assersion for cohomology groups. The intersection form for homology groups is defined ([KY]) through that of cohomology groups (this is the compatibility of the two intersection theories). So the assersion for the cohomology groups leads to that for the homology groups.

REFERENCES

[KY] M. KITA AND M. YOSHIDA, Intersection theory for twisted cycles, Math. Nach. 166 (1994), 287–304.

[MiY1] K. MIMACHI AND M. YOSHIDA, Intersection numbers of twisted cycles and the correlation functions of the conformal field theory, Comm. Math. Phys. 234 (2003), 339–358.

[MiY2] K. MIMACHI AND M. YOSHIDA, Intersection numbers of twisted cycles with the Selberg integral and an application to the conformal field theory, Comm. Math. Phys. 250 (2004), 23–25.

[Oh] K. OHARA, Intersection forms on twisted cohomology groups associated with Selberg-type integrals, preprint 2002.

[OST] K. OHARA, Y. SUGIKI AND N. TAKAYAMA, Quadratic relations for generalized hypergeometric functions ${}_pF_{p-1}$, Funk. Ekvac. 46 (2003), 213–252.

[Yo] M. YOSHIDA, Hypergeometric Functions, My Love, Vieweg Verlag, Wiesbaden, 1997.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
TOKYO 152–8551
JAPAN

E-mail address: mimachi@math.titech.ac.jp

DEPARTMENT OF COMPUTATIONAL SCIENCE
KANAZAWA UNIVERSITY
KANAZAWA 920–1192
JAPAN

E-mail address: ohara@air.s.kanazawa-u.ac.jp

DEPARTMENT OF MATHEMATICS
KYUSHU UNIVERSITY
FUKUOKA 810–8560
JAPAN
E-mail address: myoshida@math.kyushu-u.ac.jp