

Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface

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1. Introduction

Let n and d be coprime positive integers, and define $\mathcal{M}(n, d)$ to be the moduli space of (semi)stable holomorphic vector bundles of rank n , degree d and fixed determinant on a compact Riemann surface Σ . This moduli space is a compact Kähler manifold which has been studied from many different points of view for more than three decades (see for instance Narasimhan and Seshadri [41]). The subject of this article is the characterization of the intersection pairings in the cohomology ring¹ $H^*(\mathcal{M}(n, d))$. A set of generators of this ring was described by Atiyah and Bott in their seminal 1982 paper [2] on the Yang-Mills equations on Riemann surfaces (where in addition inductive formulas for the Betti numbers of $\mathcal{M}(n, d)$ obtained earlier using number-theoretic methods [13], [25] were rederived). By Poincaré duality, knowledge of the intersection pairings between products of these generators (or equivalently knowledge of the evaluation on the fundamental class of products of the generators) completely determines the structure of the cohomology ring.

In 1991 Donaldson [15] and Thaddeus [47] gave formulas for the intersection pairings between products of these generators in $H^*(\mathcal{M}(2, 1))$ (in terms of Bernoulli numbers). Then using physical methods, Witten [50] found formulas for generating functions from which could be extracted the intersection pairings between products of these generators in $H^*(\mathcal{M}(n, d))$ for general rank n . These generalized his (rigorously proved) formulas [49] for the symplectic volume of $\mathcal{M}(n, d)$: For instance, the symplectic volume of $\mathcal{M}(2, 1)$ is given by

$$(1.1) \quad \text{vol}(\mathcal{M}(2, 1)) = \left(1 - \frac{1}{2^{2g-3}}\right) \frac{\zeta(2g-2)}{2^{g-2}\pi^{2g-2}} = \frac{2^{g-1} - 2^{2-g}}{(2g-2)!} |B_{2g-2}|$$

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¹Throughout this paper all cohomology groups will have complex coefficients, unless specified otherwise.

where g is the genus of the Riemann surface, ζ is the Riemann zeta function and B_{2g-2} is a Bernoulli number (see [49], [47], [15]). The purpose of this paper is to obtain a mathematically rigorous proof of Witten's formulas for general rank n . Our announcement [32] sketched the arguments we shall use, concentrating mainly on the case of rank $n = 2$.

The proof involves an application of the *nonabelian localization principle* [31], [50]. Let K be a compact connected Lie group with Lie algebra \mathfrak{k} , let (M, ω) be a compact symplectic manifold equipped with a Hamiltonian action of K and suppose that 0 is a regular value of the moment map $\mu: M \rightarrow \mathfrak{k}^*$ for this action. One can use equivariant cohomology on M to study the cohomology ring of the reduced space, or symplectic quotient, $M_{\text{red}} = \mu^{-1}(0)/K$, which is an orbifold with an induced symplectic form ω_0 . In particular, it is shown in [36] that there is a natural surjective homomorphism from the equivariant cohomology $H_K^*(M)$ of M to the cohomology $H^*(M_{\text{red}})$ of the reduced space. For any cohomology class $\eta_0 \in H^*(M_{\text{red}})$ coming from $\eta \in H_K^*(M)$ via this map, we derived in [31] a formula (the residue formula, Theorem 8.1 of [31]) for the evaluation $\eta_0[M_{\text{red}}]$ of η_0 on the fundamental class of M_{red} . This formula involves the data that enter the Duistermaat-Heckman formula [17], and its generalization the abelian localization formula [3], [8], [9] for the action of a maximal torus T of K on M : that is, the set \mathcal{F} of connected components F of the fixed point set M^T of the action of T on M , and the equivariant Euler classes e_F of their normal bundles in M . Let \mathfrak{t} be the Lie algebra of T ; then the composition $\mu_T: M \rightarrow \mathfrak{t}^*$ of $\mu: M \rightarrow \mathfrak{k}^*$ with the natural map from \mathfrak{k}^* to \mathfrak{t}^* is a moment map for the action of T on M . In the case when $K = \text{SU}(2)$ and the order of the stabilizer in K of a generic point of $\mu^{-1}(0)$ is n_0 , the residue formula can be expressed in the form

$$(1.2) \quad \eta_0 e^{\omega_0}[M_{\text{red}}] = \frac{n_0}{2} \text{Res}_{X=0} \left((2X)^2 \sum_{F \in \mathcal{F}_+} h_F^\eta(X) dX \right)$$

where the subset \mathcal{F}_+ of \mathcal{F} consists of those components F of the fixed point set M^T on which the value taken by the T -moment map $\mu_T: M \rightarrow \mathfrak{t}^* \cong \mathbb{R}$ is positive, and for $F \in \mathcal{F}_+$ the inclusion of F in M is denoted by i_F and the meromorphic function h_F^η of $X \in \mathbb{C}$ is defined by

$$h_F^\eta(X) = \int_F \frac{i_F^* \eta(X) e^{\bar{\omega}(X)}}{e_F(X)} = e^{\mu_T(F)(X)} \int_F \frac{i_F^* \eta(X) e^\omega}{e_F(X)}$$

when $X \in \mathbb{C}$ has been identified with $\text{diag}(2\pi i, -2\pi i)X \in \mathfrak{t} \otimes \mathbb{C}$. Here $\bar{\omega}$ is the extension $\omega + \mu$ of the symplectic form ω on M to an equivariantly closed 2-form, while as before ω_0 denotes the induced symplectic form on M_{red} . Finally $\text{Res}_{X=0}$ denotes the ordinary residue at $X = 0$.

The moduli space $\mathcal{M}(n, d)$ was described by Atiyah and Bott [2] as the symplectic reduction of an infinite-dimensional symplectic affine space \mathcal{A} with

respect to the action of an infinite-dimensional group \mathcal{G} (the *gauge group*).² However $\mathcal{M}(n, d)$ can also be exhibited as the symplectic quotient of a finite-dimensional symplectic space $M(c)$ by the Hamiltonian action of the finite-dimensional group $K = \text{SU}(n)$. One characterization of the space $M(c)$ is that it is the symplectic reduction of the infinite-dimensional affine space \mathcal{A} by the action of the *based* gauge group \mathcal{G}_0 (which is the kernel of the evaluation map $\mathcal{G} \rightarrow K$ at a prescribed basepoint: see [28]). Now if a compact group G containing a closed normal subgroup H acts in a Hamiltonian fashion on a symplectic manifold Y , then one may “reduce in stages”: the space $\mu_H^{-1}(0)/H$ has a residual Hamiltonian action of the quotient group G/H with moment map $\mu_{G/H}: \mu_H^{-1}(0)/H \rightarrow (\mathfrak{g}/\mathfrak{h})^*$, and $\mu_G^{-1}(0)/G$ is naturally identified as a symplectic manifold with $\mu_{G/H}^{-1}(0)/(G/H)$. Similarly $M(c)$ has a Hamiltonian action of $\mathcal{G}/\mathcal{G}_0 \cong K$, and the symplectic reduction with respect to this action is identified with the symplectic reduction of \mathcal{A} with respect to the full gauge group \mathcal{G} .

Our strategy for obtaining Witten’s formulas is to apply nonabelian localization to this extended moduli space $M(c)$, which has a much more concrete (and entirely finite-dimensional) characterization described in Section 4 below. Unfortunately technical difficulties arise, because $M(c)$ is both singular and noncompact. The noncompactness of $M(c)$ causes the more serious problems, the most immediate of which is that there are infinitely many components F of the fixed point set $M(c)^T$. These, however, are easy to identify (roughly speaking they correspond to bundles which are direct sums of line bundles), and there are obvious candidates for the equivariant Euler classes of their normal bundles, if the singularities of $M(c)$ are ignored. In the case when $n = 2$, for example, a naïve application of the residue formula (1.2), with some sleight of hand, would yield

$$\begin{aligned}
 (1.3) \quad \text{vol}(\mathcal{M}(2, 1)) &= e^{\omega_0}[\mathcal{M}(2, 1)] = (-1)^g \text{Res}_{X=0} \sum_{j=0}^{\infty} \frac{e^{(2j+1)X}}{2^{g-2} X^{2g-2}} \\
 &= (-1)^g \text{Res}_{X=0} \frac{e^X}{2^{g-2} X^{2g-2} (1 - e^{2X})} \\
 &= (-1)^{g-1} \text{Res}_{X=0} \frac{1}{2^{g-1} X^{2g-2} \sinh(X)}.
 \end{aligned}$$

This does give the correct answer (it agrees with (1.1) above). However it is far from obvious how this calculation might be justified, since the infinite sum does not converge in a neighbourhood of 0, where the residue is taken, and

²To obtain his generating functionals, Witten formally applied his version of nonabelian localization to the action of the gauge group on the infinite dimensional space \mathcal{A} .

indeed the sum of the residues at 0 of the individual terms in the sum does not converge.

These difficulties can be overcome by making use of a different approach to nonabelian localization given recently by Guillemin-Kalkman [23] and independently by Martin [39]. This is made up of two steps: the first is to reduce to the case of a torus action, and the second, when $K = T$ is a torus, is to study the change in the evaluation on the fundamental class of the reduced space $\mu_T^{-1}(\xi)/T$ of the cohomology class induced by η , as ξ varies in \mathfrak{t}^* . It is in fact an immediate consequence of the residue formula that if T is a maximal torus of K and $\xi \in \mathfrak{t}^*$ is any regular value sufficiently close to 0 of the T -moment map $\mu_T: M \rightarrow \mathfrak{t}^*$, then the evaluation $\eta_0[M_{\text{red}}]$ of $\eta_0 \in H^*(M_{\text{red}})$ on the fundamental class of $M_{\text{red}} = \mu^{-1}(0)/K$ is equal to the evaluation of a related element of $H^*(\mu_T^{-1}(\xi)/T)$ on the fundamental class of the T -reduced space $\mu_T^{-1}(\xi)/T$. This was first observed by Guillemin and Kalkman [23] and by Martin [39], who gave an independent proof which showed that $\eta_0[M_{\text{red}}]$ is also equal to an evaluation on

$$\mu^{-1}(0)/T = (M_{\mathfrak{t}} \cap \mu_T^{-1}(0))/T$$

where $M_{\mathfrak{t}} = \mu^{-1}(\mathfrak{t})$. In our situation the space $M_{\mathfrak{t}}$ turns out to be "periodic" in a way which enables us to avoid working with infinite sums except in a very trivial sense. This is done by comparing the results of relating evaluations on $(M_{\mathfrak{t}} \cap \mu_T^{-1}(\xi))/T$ for different values of ξ in two ways: using the periodicity and using Guillemin and Kalkman's arguments, which can be made to work in spite of the noncompactness of $M(c)$. The singularities can be dealt with because $M(c)$ is embedded naturally and equivariantly in a nonsingular space, and integrals over $M(c)$ can be rewritten as integrals over this nonsingular space.

In the case when $n = 2$ our approach gives expressions for the pairings in $H^*(\mathcal{M}(2, 1))$ as residues similar to those in (1.3) above. When $n > 2$ we consider the action of a suitable one-dimensional subgroup \hat{T}_1 of T , with Lie algebra $\hat{\mathfrak{t}}_1$ say, on the quotient of $\mu^{-1}(\hat{\mathfrak{t}}_1)$ by a subgroup of T whose Lie algebra is a complementary subspace to $\hat{\mathfrak{t}}_1$ in \mathfrak{t} . This leads to an inductive formula for the pairings on $H^*(\mathcal{M}(n, d))$, and thus to expressions for these pairings as iterated residues (see Theorems 8.1 and 9.12 below, which are the central results of this paper). Witten's formulas, on the other hand, express the pairings as infinite sums over those elements of the weight lattice of $SU(n)$ which lie in the interior of a fundamental Weyl chamber (see Section 2). These infinite sums are difficult to calculate in general, and there is apparently (see [50, Section 5]) no direct proof even that they are always zero when the pairings they represent vanish on dimensional grounds. However, thanks to an argument of

Szenes (see Proposition 2.2 below), Witten's formulas can be identified with the iterated residues which appear in our approach.

Over the moduli space $\mathcal{M}(n, d)$ there is a natural line bundle \mathcal{L} (the Quillen line bundle [43]) whose fibre at any point representing a semistable holomorphic bundle E is the determinant line

$$\det \bar{\partial} = \det H^1(\Sigma, E) \otimes \det H^0(\Sigma, E)^*$$

of the associated $\bar{\partial}$ -operator. Our expressions for pairings in $H^*(\mathcal{M}(n, d))$ as iterated residues, together with the Riemann-Roch formula, lead easily (cf. Section 4 of [45]) to a proof of the Verlinde formula for

$$\dim H^0(\mathcal{M}(n, d), \mathcal{L}^k)$$

for positive integers k (proved by Beauville and Laszlo in [6], by Faltings in [20], by Kumar, Narasimhan and Ramanathan in [38] and by Tsuchiya, Ueno and Yamada in [48]).

This paper is organized as follows. In Section 2 we describe the generators for the cohomology ring $H^*(\mathcal{M}(n, d))$ and Witten's formulas for the intersection pairings among products of these generators. In Section 3 we outline tools from the Cartan model of equivariant cohomology, which will be used in later sections, and the different versions of localization which will be relevant. In Section 4 we recall properties of the extended moduli space $M(c)$, and in Section 5 we construct the equivariant differential forms representing equivariant Poincaré duals which enable us to rewrite integrals over singular spaces as integrals over ambient nonsingular spaces. Then Section 6 begins the application of nonabelian localization to the extended moduli space, and Section 7 analyses the fixed point sets which arise in this application. Section 8 uses induction to complete the proof of Witten's formulas when the pairings are between cohomology classes of a particular form, Section 9 extends the inductive argument to give formulas for all pairings, and in Section 10 it is shown that these agree with Witten's formulas. Finally as an application, Section 11 gives a proof of the Verlinde formula for $\mathcal{M}(n, d)$.

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2. The cohomology of the moduli space $\mathcal{M}(n, d)$ and Witten's formulas for intersection pairings

In order to avoid exceptional cases, we shall assume throughout that the Riemann surface Σ has genus $g \geq 2$.

A set of generators for the cohomology³ $H^*(\mathcal{M}(n, d))$ of the moduli space $\mathcal{M}(n, d)$ of stable holomorphic vector bundles of coprime rank n and degree d and fixed determinant on a compact Riemann surface Σ of genus $g \geq 2$ is given in [2] by Atiyah and Bott. It may be described as follows. There is a universal rank n vector bundle

$$\mathbb{U} \rightarrow \Sigma \times \mathcal{M}(n, d)$$

which is unique up to tensor product with the pullback of any holomorphic line bundle on $\mathcal{M}(n, d)$; for definiteness Atiyah and Bott impose an extra normalizing condition which determines the universal bundle up to isomorphism, but this is not crucial to their argument (see [2, p. 582]). Then by [2, Prop. 2.20] the following elements of $H^*(\mathcal{M}(n, d))$ for $2 \leq r \leq n$ make up a set of generators:

$$\begin{aligned} f_r &= ([\Sigma], c_r(\mathbb{U})), \\ b_r^j &= (\alpha_j, c_r(\mathbb{U})), \\ a_r &= (1, c_r(\mathbb{U})). \end{aligned}$$

Here, $[\Sigma] \in H_2(\Sigma)$ and $\alpha_j \in H_1(\Sigma)$ ($j = 1, \dots, 2g$) form standard bases of $H_2(\Sigma, \mathbb{Z})$ and $H_1(\Sigma, \mathbb{Z})$, and the bracket represents the slant product $H^N(\Sigma \times \mathcal{M}(n, d)) \otimes H_j(\Sigma) \rightarrow H^{N-j}(\mathcal{M}(n, d))$. More generally, if $K = \mathrm{SU}(n)$ and Q is an invariant polynomial of degree s on its Lie algebra $\mathfrak{k} = \mathfrak{su}(n)$ then there is an associated element of $H^*(BSU(n))$ and hence an associated element of $H^*(\Sigma \times \mathcal{M}(n, d))$ which is a characteristic class $Q(\mathbb{U})$ of the universal bundle \mathbb{U} . Hence the slant product gives rise to classes

$$\begin{aligned} ([\Sigma], Q(\mathbb{U})) &\in H^{2s-2}(\mathcal{M}(n, d)), \\ (\alpha_j, Q(\mathbb{U})) &\in H^{2s-1}(\mathcal{M}(n, d)) \end{aligned}$$

and

$$(1, Q(\mathbb{U})) \in H^{2s}(\mathcal{M}(n, d)).$$

In particular, letting $\tau_r \in S^r(\mathfrak{k}^*)^K$ denote the invariant polynomial associated to the r^{th} Chern class, we recover

$$(2.1) \quad \begin{aligned} f_r &= ([\Sigma], \tau_r(\mathbb{U})), \\ b_r^j &= (\alpha_j, \tau_r(\mathbb{U})) \end{aligned}$$

³In this paper, all cohomology groups are assumed to be with complex coefficients.

and

$$a_r = (1, \tau_r(\mathbb{U})).$$

A special role is played by the invariant polynomial $\tau_2 = -\langle \cdot, \cdot \rangle / 2$ on \mathbf{k} given by the Killing form or invariant inner product. We normalize the inner product as follows for $K = \text{SU}(n)$:

$$(2.2) \quad \langle X, X \rangle = -\text{Trace}(X^2) / (4\pi^2).$$

The class f_2 associated to $-\langle \cdot, \cdot \rangle / 2$ is the cohomology class of the symplectic form on $\mathcal{M}(n, d)$.

As was noted in the introduction, Atiyah and Bott identify $\mathcal{M}(n, d)$ with the symplectic reduction of an infinite dimensional affine space \mathcal{A} of connections by the action of an infinite dimensional Lie group \mathcal{G} (the gauge group). They show that associated to this identification there is a natural surjective homomorphism of rings from the equivariant cohomology ring $H_{\mathcal{G}}^*(\mathcal{A})$ to $H^*(\mathcal{M}(n, d))$, where $\bar{\mathcal{G}}$ is the quotient of \mathcal{G} by its central subgroup S^1 . There is a canonical \mathcal{G} -equivariant universal bundle over $\Sigma \times \mathcal{A}$, and the slant products of its Chern classes with $1 \in H_0(\Sigma)$, $\alpha_j \in H_1(\Sigma)$ for $1 \leq j \leq 2g$ and $[\Sigma] \in H_2(\Sigma)$ give generators of $H_{\mathcal{G}}^*(\mathcal{A})$ which by abuse of notation we shall also call a_r, b_r^j and f_r . (In fact $H_{\mathcal{G}}^*(\mathcal{A})$ is freely generated by $a_1, \dots, a_n, f_2, \dots, f_n$ and b_r^j for $1 < r \leq n$ and $1 \leq j \leq 2g$, subject only to the usual commutation relations.) The surjection from \mathcal{G} to $\bar{\mathcal{G}}$ induces an inclusion from $H_{\bar{\mathcal{G}}}^*(\mathcal{A})$ to $H_{\mathcal{G}}^*(\mathcal{A})$ such that

$$H_{\mathcal{G}}^*(\mathcal{A}) \cong H_{\bar{\mathcal{G}}}^*(\mathcal{A}) \otimes H^*(BS^1)$$

if we identify $H^*(BS^1)$ with the polynomial subalgebra of $H_{\bar{\mathcal{G}}}^*(\mathcal{A})$ generated by a_1 , and then the generators a_r, f_r and b_r^j for $1 < r \leq n$ determine generators of $H_{\bar{\mathcal{G}}}^*(\mathcal{A})$ and thus of $H^*(\mathcal{M}(n, d))$. These are the generators we shall use in this paper. The normalization condition imposed by Atiyah and Bott corresponds to use of the isomorphism

$$H_{\bar{\mathcal{G}}}^*(\mathcal{A}) \cong H_{\mathcal{G}}^*(\mathcal{A}) \otimes H^*(BS^1)$$

obtained by identifying $H^*(BS^1)$ with the polynomial subalgebra of $H_{\mathcal{G}}^*(\mathcal{A})$ generated by $2(g - 1)a_1 + f_2$; they choose this condition because it has a nice geometrical interpretation in terms of a universal bundle over $\Sigma \times \mathcal{M}(n, d)$.

In Sections 4 and 5 of [50], Witten obtained formulas for generating functionals from which one may extract all intersection pairings

$$\prod_{r=2}^n a_r^{m_r} f_r^{n_r} \prod_{k_r=1}^{2g} (b_r^{k_r})^{p_{r,k_r}} [\mathcal{M}(n, d)].$$

Let us begin with pairings of the form

$$(2.3) \quad \prod_{r=2}^n a_r^{m_r} \exp f_2 [\mathcal{M}(n, d)].$$

When m_r is sufficiently small to ensure convergence of the sum, Witten obtains⁴

$$(2.4) \quad \prod_{r=2}^n a_r^{m_r} \exp f_2[\mathcal{M}(n, d)] = c^\rho \Gamma(-1)^{n_+(g-1)} \left(\sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+} \frac{c^{-\lambda} \prod_{r=2}^n \tau_r(2\pi i \lambda)^{m_r}}{\mathcal{D}^{2g-2}(2\pi i \lambda)} \right),$$

where

$$(2.5) \quad \Gamma = \frac{n^{2g}}{\#\Pi_1(K')} \left(\frac{\text{vol}(K')}{(2\pi)^{\dim K'}} \right)^{2g-2} ((2\pi)^{n_+} \mathcal{D}(\rho))^{2g-2} = n^g$$

is a universal constant for $K = \text{SU}(n)$ and $K' = K/Z(K)$, and the Weyl odd polynomial \mathcal{D} on \mathfrak{t}^* is defined by

$$\mathcal{D}(X) = \prod_{\gamma > 0} \gamma(X)$$

where γ runs over the positive roots. Here, ρ is half the sum of the positive roots, and $n_+ = n(n-1)/2$ is the number of positive roots. The sum over λ in (2.4) runs over those elements of the weight lattice Λ^w that are in the interior of the fundamental Weyl chamber.⁵ The element

$$(2.6) \quad c = e^{2\pi i d/n} \text{diag}(1, \dots, 1)$$

is a generator of the centre $Z(K)$ of K ; so since $\lambda \in \mathfrak{t}^*$ is in $\text{Hom}(T, U(1))$, we may evaluate λ on c as in (2.4); c^λ is defined as $\exp \lambda(\tilde{c})$ where \tilde{c} is any element of the Lie algebra of T such that $\exp \tilde{c} = c$. Note that in fact when d is coprime to n (so that when n is even d is odd) we have $c^\rho = (-1)^{n-1}$.

Witten's formula [50, (5.21)] covers pairings involving the f_r for $r > 2$ and the b_r^j as well as f_2 and the a_r . He obtains it by reducing to the special case of pairings of the form (2.3) above (see [50, §5], in particular the calculations (5.11)–(5.20)) and then applying [50, (4.74)] to this special case. In the special case of pairings of the form (2.3), Witten's formula [50, (5.21)] follows from our Theorem 8.1 using Proposition 2.2 below. Moreover our formula (Theorem 9.12) for pairings involving all the generators a_r , b_r^j and f_r reduces to the special case just as Witten's does (see Propositions 10.2 and 10.3). Thus Witten's formulas are equivalent to ours, although they look very different (being expressed in terms of infinite sums indexed by dominant weights instead of in terms of iterated residues).

For the sake of concreteness it is worth examining the special case when rank $n = 2$ so that the degree d is odd. In fact, since tensoring by a fixed line bundle of degree e induces a homeomorphism between $\mathcal{M}(n, d)$ and

⁴In fact $\mathcal{M}(n, d)$ is an n^{2g} -fold cover of the space for which Witten computes pairings; this accounts for the factor n^{2g} in our formula (2.5). Taking this into account, (2.4) follows from a special case of Witten's formula [50, (5.21)].

⁵The weight lattice $\Lambda^w \subset \mathfrak{t}^*$ is the dual lattice of the integer lattice $\Lambda^I = \text{Ker}(\exp)$ in \mathfrak{t} .

$M(n, d + ne)$, we may assume that $d = 1$. In this case the dominant weights λ are just the positive integers. The relevant generators of $H^*(\mathcal{M}(2, 1))$ are

$$(2.7) \quad f_2 \in H^2(\mathcal{M}(2, 1))$$

(which is the cohomology class of the symplectic form on $\mathcal{M}(2, 1)$) and

$$(2.8) \quad a_2 \in H^4(\mathcal{M}(2, 1));$$

these arise from the invariant polynomial $\tau_2 = -\langle \cdot, \cdot \rangle / 2$ by $a_2 = \tau_2(1)$, $f_2 = \tau_2([\Sigma])$ (see (2.1)). We find then that formula (2.4) reduces for $m \leq g - 2$ to⁶ ([50, (4.44)])

$$(2.9) \quad a_2^j \exp(f_2)[\mathcal{M}(2, 1)] = \frac{2^{2g}}{2(8\pi^2)^{g-1}} \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^{2j}}{n^{2g-2-2j}} \right).$$

Thus one obtains the formulas found by Thaddeus in Section 5 of [47] for the intersection pairings $a_2^m f_2^n [\mathcal{M}(2, 1)]$; these intersection pairings are given by Bernoulli numbers, or equivalently are given in terms of the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} 1/n^s$. As Thaddeus shows in Section 4 of [47], this is enough to determine all the intersection pairings in the case when rank n is two, because all the pairings

$$a_2^m f_2^n \prod_{k=1}^{2g} (b_2^k)^{p_k} [\mathcal{M}(2, 1)]$$

are zero except those of the form

$$a_2^m f_2^n b_2^{2i_1-1} b_2^{2i_1} \dots b_2^{2i_q-1} b_2^{2i_q} [\mathcal{M}(2, 1)]$$

where $m + 2n + 3q = 3g - 3$ and $1 \leq i_1 < \dots < i_q \leq g$. This expression equals the evaluation of $a_2^m f_2^n$ on the corresponding moduli space of rank 2 and degree 1 bundles over a Riemann surface of genus $g - q$ if $q \leq g - 2$, and equals 4 if $q = g - 1$.

Szenes [45] has proved that the expression on the right-hand side of (2.4) may be rewritten in a particular form. To state the result we must introduce some notation. The Lie algebra $\mathfrak{t} = \mathfrak{t}_n$ of the maximal torus T of $SU(n)$ is

$$\mathfrak{t} = \{(X_1, \dots, X_n) \in \mathbb{R}^n: X_1 + \dots + X_n = 0\}.$$

Define coordinates $Y_j = e_j(X) = X_j - X_{j+1}$ on \mathfrak{t} for $j = 1, \dots, n - 1$. The positive roots of $SU(n)$ are then $\gamma_{jk}(X) = X_j - X_k = Y_j + \dots + Y_{k-1}$ for $1 \leq j < k \leq n$. The integer lattice Λ^I of $SU(n)$ is generated by the simple roots $e_j, j = 1, \dots, n - 1$. The dual lattice to Λ^I with respect to the inner product $\langle \cdot, \cdot \rangle$ introduced at (2.2) is the weight lattice $\Lambda^w \subset \mathfrak{t}$; in terms of the inner

⁶Here, we have identified $-a_2$ with Witten's class Θ and f_2 with Witten's class ω .

product $\langle \cdot, \cdot \rangle$, it is given by $\Lambda^w = \{X \in \mathfrak{t}: Y_j \in \mathbb{Z} \text{ for } j = 1, \dots, n-1\}$. We define also $\Lambda_{\text{reg}}^w(\mathfrak{t}_n) = \{X \in \Lambda^w: Y_j \neq 0 \text{ for } j = 1, \dots, n-1 \text{ and } \gamma_{jk}(X) \neq 0 \text{ for any } j \neq k\}$.

Definition 2.1. Let $f: \mathfrak{t} \otimes \mathbb{C} \rightarrow \mathbb{C}$ be a meromorphic function of the form

$$(2.10) \quad f(X) = g(X)e^{-\gamma(X)}$$

where $\gamma(X) = \gamma_1 Y_1 + \dots + \gamma_{n-1} Y_{n-1}$ for $(\gamma_1, \dots, \gamma_{n-1}) \in \mathbb{R}^{n-1}$. We define

$$[[\gamma]] = ([[\gamma]]_1, \dots, [[\gamma]]_{n-1})$$

to be the element of \mathbb{R}^{n-1} for which $0 \leq [[\gamma]]_j < 1$ for all $j = 1, \dots, n-1$ and $[[\gamma]] = \gamma \bmod \mathbb{Z}^{n-1}$. (In other words, $[[\gamma]] = \sum_{j=1}^{n-1} [[\gamma]]_j e_j$ is the unique element of $\mathfrak{t} \cong \mathbb{R}^{n-1}$ which is in the fundamental domain defined by the simple roots for the translation action on \mathfrak{t}_n of the integer lattice, and which is equivalent to γ under translation by the integer lattice.)

We also define the meromorphic function $[[f]]: \mathfrak{t} \otimes \mathbb{C} \rightarrow \mathbb{C}$ by

$$[[f]](X) = g(X)e^{-[[\gamma]](X)}.$$

PROPOSITION 2.2 (Szenes). *Let $f: \mathfrak{t} \otimes \mathbb{C} \rightarrow \mathbb{C}$ be defined by*

$$f(X) = \frac{\prod_{r=2}^n \tau_r(X)^{m_r} e^{-\tilde{c}(X)}}{\mathcal{D}(X)^{2g-2}}.$$

Provided that the m_r are sufficiently small to ensure convergence of the sum,

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n) \cap \mathfrak{t}_+} f(2\pi i \lambda) = \frac{1}{n!} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{\sum_{w \in W_{n-1}} [[w(f)]](X)}{(e^{-Y_{n-1}} - 1) \dots (e^{-Y_1} - 1)} \right),$$

where $W_{n-1} \cong S_{n-1}$ is the Weyl group of $SU(n-1)$ embedded in $SU(n)$ in the standard way using the first $n-1$ coordinates X_1, \dots, X_{n-1} .

Remark 2.3. Here, we have introduced coordinates $Y_j = e_j(X)$ on \mathfrak{t} using the simple roots

$$\{e_j: j = 1, \dots, n-1\}$$

of \mathfrak{t} , and Λ_{reg}^w denotes the regular part of the weight lattice Λ^w (see below). Also, we have introduced the unique element \tilde{c} of \mathfrak{t} which satisfies $e^{2\pi i \tilde{c}} = c$ and which belongs to the fundamental domain defined by the simple roots for the translation action on T_n of the integer lattice Λ^I . This simply means that $\langle \tilde{c}, X \rangle = \gamma_1 Y_1 + \dots + \gamma_{n-1} Y_{n-1}$ where $0 \leq \gamma_j < 1$ for $1 \leq j \leq n-1$. (In the notation introduced in Definition 2.1 this says that $\tilde{c} = [[(d/n, d/n, \dots, -(n-1)d/n)]]$.) Also, \mathfrak{t}_+ denotes the fundamental Weyl chamber, which is a fundamental domain for the action of the Weyl group on \mathfrak{t} .

If $g(Y_k, \dots, Y_{n-1})$ is a meromorphic function of Y_k, \dots, Y_{n-1} , we interpret $\text{Res}_{Y_k=0} g(Y_k, \dots, Y_{n-1})$ as the ordinary one-variable residue of g regarded as a function of Y_k with Y_{k+1}, \dots, Y_{n-1} held constant.

The rest of this section will be devoted to a proof of Proposition 2.2. We shall prove the following theorem:

THEOREM 2.4. *Let $f: \mathfrak{t}_n \otimes \mathbb{C} \rightarrow \mathbb{C}$ be a meromorphic function of the form $f(X) = g(X)e^{-\gamma(X)}$ where $\gamma(X) = \gamma_1 Y_1 + \dots + \gamma_{n-1} Y_{n-1}$ with $0 \leq \gamma_{n-1} < 1$, and $g(X)$ is a rational function of X with poles only on the zeros of the roots γ_{jk} and decaying sufficiently fast at infinity. Then*

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n)} f(2\pi i \lambda) = \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{\sum_{w \in W_{n-1}} [[w(f)]](X)}{(e^{-Y_{n-1}} - 1) \dots (e^{-Y_1} - 1)} \right)$$

where W_{n-1} is the Weyl group of $\text{SU}(n-1)$ embedded in $\text{SU}(n)$ using the first $n-1$ coordinates.

Remark 2.5. Notice that if f is as in the hypothesis of the theorem (but here one may omit the hypothesis that $0 \leq \gamma_{n-1} < 1$) then

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n)} f(2\pi i \lambda) = \sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n)} [[f]](2\pi i \lambda),$$

where $[[f]]$ is as in Definition 2.1.

Proof of Proposition 2.2 given Theorem 2.4. The function

$$(2.11) \quad f(X) = \frac{\prod_{r=2}^n \tau_r(X)^{m_r} e^{-\tilde{c}(X)}}{\mathcal{D}(X)^{2g-2}}$$

satisfies the hypotheses of the theorem, provided that the m_r are small enough to ensure convergence of the sum. Notice that if $\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n)$ then $e^{-2\pi i \tilde{c}(\lambda)} = c^{-\lambda}$ satisfies $c^{-\lambda} = c^{-w\lambda}$ for all elements w of the Weyl group W . Thus for this particular f we have that

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n)} f(2\pi i \lambda) = n! \sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n) \cap \mathfrak{t}_+} f(2\pi i \lambda).$$

Thus,

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n) \cap \mathfrak{t}_+} f(2\pi i \lambda) = \frac{1}{n!} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{\sum_{w \in W_{n-1}} [[w(f)]](X)}{(e^{-Y_{n-1}} - 1) \dots (e^{-Y_1} - 1)} \right)$$

which is the statement of Proposition 2.2. □

It remains to prove Theorem 2.4. By induction on n it suffices to prove:

LEMMA 2.6. *Let $f = f_{(n)}: \mathfrak{t}_n \rightarrow \mathbb{C}$ be as in the statement of Theorem 2.4. Define $f_{(n-1)}: \mathfrak{t}_{n-1} \rightarrow \mathbb{C}$ by*

$$f_{(n-1)}(Y_1, \dots, Y_{n-2}) = \text{Res}_{Y_{n-1}=0} \frac{f(Y_1, \dots, Y_{n-1})}{e^{-Y_{n-1}} - 1}.$$

Then

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_n)} f_{(n)}(2\pi i \lambda) = \sum_{\lambda \in \Lambda_{\text{reg}}^w(\mathfrak{t}_{n-1})} \sum_{j=1}^{n-1} (q_j f)_{(n-1)}(2\pi i \lambda),$$

where q_j is the element of the Weyl group $W_{n-1} \cong S_{n-1}$ represented by swapping the coordinates X_j and X_{n-1} .

Remark 2.7. Note that by Remark 2.5, the sum $\sum_{j=1}^{n-1} (q_j f)_{(n-1)}(2\pi i \lambda)$ is equal to

$$\sum_{j=1}^{n-1} [(q_j f)_{(n-1)}](2\pi i \lambda).$$

Remark 2.8. Note that the function $[(q_j f)_{(n-1)}]$ satisfies the hypotheses of Theorem 2.4.

Proof of Lemma 2.6. Let l_j ($j = 1, \dots, n - 2$) be integers such that

$$(2.12) \quad l_j + l_{j+1} + \dots + l_k \neq 0 \text{ for any } 1 \leq j \leq k \leq n - 2.$$

Define $L_{(l_1, \dots, l_{n-2})}$ to be the line $\{(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1}): Y_{n-1} \in \mathbb{C}\}$. The condition (2.12) states that all the roots γ_{jk} for $1 \leq j < k \leq n - 1$ are nonzero on $L_{(l_1, \dots, l_{n-2})}$.

Let $f: \mathfrak{t} \otimes \mathbb{C} \rightarrow \mathbb{C}$ be a meromorphic function as in the statement of Theorem 2.4, having poles only at the zeros of the roots γ_{jk} . We shall think of f as a function $f(Y_1, \dots, Y_{n-1})$ of the coordinates Y_1, \dots, Y_{n-1} . Define $\Lambda_{(l_1, \dots, l_{n-2})}^{\text{reg}}$ to be

$$\Lambda_{(l_1, \dots, l_{n-2})}^{\text{reg}} = \{X \in L_{(l_1, \dots, l_{n-2})}: Y_j(X) = X_j - X_{j+1} \in 2\pi i \mathbb{Z}, \\ \gamma_{jk}(X) \neq 0 \text{ for any } j \neq k\}.$$

The sum of all residues of the function $g_{(l_1, \dots, l_{n-2})}$ on \mathbb{C} given by

$$g_{(l_1, \dots, l_{n-2})}(Y_{n-1}) = \frac{f(2\pi i l_1, 2\pi i l_2, \dots, Y_{n-1})}{e^{-Y_{n-1}} - 1}$$

is zero and these residues occur when $Y_{n-1} \in 2\pi i \mathbb{Z}$. Therefore we find that the sum

$$\sum_{p \in \Lambda_{(l_1, \dots, l_{n-2})}^{\text{reg}}} f(p)$$

is given by

$$\begin{aligned}
 (2.13) \quad & - \sum_{(2\pi i l_1, \dots, 2\pi i l_{n-2}, 2\pi i l_{n-1}) \in \Lambda_{(l_1, \dots, l_{n-2})}^{\text{reg}}} \text{Res}_{Y_{n-1}=2\pi i l_{n-1}} \\
 & \frac{f(2\pi i l_1, 2\pi i l_2, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1}} - 1} \\
 & = \sum_{j=1}^{n-2} \text{Res}_{Y_{n-1}=-2\pi i(l_j+\dots+l_{n-2})} \frac{f(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1}} - 1} \\
 & \quad + \text{Res}_{Y_{n-1}=0} \frac{f(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1}} - 1}.
 \end{aligned}$$

PROPOSITION 2.9. *Let p_j be the point $X \in L_{(l_1, \dots, l_{n-2})}$ for which $Y_{n-1} = -2\pi i(l_j + \dots + l_{n-2})$, or equivalently $X_n = X_j$. Then*

$$\begin{aligned}
 & \text{Res}_{Y_{n-1}=-2\pi i(l_j+\dots+l_{n-2})} \left(\frac{f(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1}} - 1} \right) \\
 & = \text{Res}_{Y_{n-1}=0} \left(\frac{q_j(f) \left(2\pi i l_1^{(j)}, \dots, 2\pi i l_{n-2}^{(j)}, Y_{n-1} \right)}{e^{-Y_{n-1}} - 1} \right).
 \end{aligned}$$

Here, we define an involution $q_j: \mathbf{t} \rightarrow \mathbf{t}$ (for $j = 1, \dots, n-1$) by

$$q_j(X_1, \dots, X_j, \dots, X_{n-1}, X_n) = (X_1, \dots, X_{j-1}, X_{n-1}, X_{j+1}, \dots, X_{n-2}, X_j, X_n).$$

The integers $l_1^{(j)}, \dots, l_{n-1}^{(j)}$ are defined by the equation

$$(2.14) \quad q_j(X)|_{X=(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})} = (2\pi i l_1^{(j)}, \dots, 2\pi i l_{n-2}^{(j)}, 2\pi i l_{n-1}^{(j)} + Y_{n-1}).$$

Proof. For $j \leq n-2$, the involution q_j is given in the coordinates (Y_1, \dots, Y_{n-1}) by $q_j: (Y_1, \dots, Y_{n-1}) \mapsto (Y'_1, \dots, Y'_{n-1})$ where $Y'_k = Y_k$ for $k \neq j-1, j, n-2, n-1$ and

$$(2.15) \quad Y'_{j-1} = Y_{j-1} + \dots + Y_{n-2},$$

$$(2.16) \quad Y'_j = - \sum_{j \leq k \leq n-2} Y_k,$$

$$(2.17) \quad Y'_{n-2} = - \sum_{j \leq k \leq n-3} Y_k,$$

$$(2.18) \quad Y'_{n-1} = Y_j + \dots + Y_{n-1}.$$

For $j = n-1$, q_j is the identity map. Notice that Y'_{n-1} is the only one of the transformed coordinates that involves Y_{n-1} . Notice also that q_j takes p_j to a point where $Y'_{n-1} = 0$.

We now examine the image of $L_{(l_1, \dots, l_{n-2})}$ under q_j . The integers $l_1^{(j)}, \dots, l_{n-1}^{(j)}$ were defined by the equation (2.14): in fact $l_k^{(j)} = l_k$ for $k \neq$

$j - 1, j, n - 2, n - 1$ and

$$(2.19) \quad l_{j-1}^{(j)} = l_{j-1} + \dots + l_{n-2},$$

$$(2.20) \quad l_j^{(j)} = - \sum_{j \leq k \leq n-2} l_k,$$

$$(2.21) \quad l_{n-2}^{(j)} = - \sum_{j \leq k \leq n-3} l_k,$$

$$(2.22) \quad l_{n-1}^{(j)} = l_j + \dots + l_{n-2}.$$

We have that

$$\begin{aligned} & \text{Res}_{Y_{n-1} = -2\pi i(l_j + \dots + l_{n-2})} \left(\frac{f(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1}} - 1} \right) \\ &= \text{Res}_{Y_{n-1} = -2\pi i(l_j + \dots + l_{n-2})} \left(\frac{f(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1} - 2\pi i(l_j + \dots + l_{n-2})} - 1} \right) \end{aligned}$$

(because $e^{2\pi i l_k} = 1$ for all $k = j, \dots, n - 2$)

$$= \text{Res}_{Y_{n-1}^{(j)} = 0} \left(\frac{q_j(f)(2\pi i l_1^{(j)}, \dots, 2\pi i l_{j-1}^{(j)}, 2\pi i l_j^{(j)}, 2\pi i l_{j+1}^{(j)}, \dots, 2\pi i l_{n-2}^{(j)}, Y_{n-1}^{(j)})}{e^{-Y_{n-1}^{(j)}} - 1} \right)$$

by the formulas (2.15–2.18) where we have defined $Y_{n-1}^{(j)} = Y_{n-1} + 2\pi i(l_j + \dots + l_{n-2})$ so that $dY_{n-1}^{(j)} = dY_{n-1}$. This completes the proof. \square

COROLLARY 2.10.

$$(2.23) \quad \sum_{p \in \Lambda_{(l_1, \dots, l_{n-2})}^{\text{reg}}} f(p) = \sum_{j=1}^{n-2} \text{Res}_{Y_{n-1} = 0} \left(\frac{q_j(f)(2\pi i l_1^{(j)}, \dots, 2\pi i l_{n-2}^{(j)}, Y_{n-1})}{e^{-Y_{n-1}} - 1} \right) + \text{Res}_{Y_{n-1} = 0} \frac{f(2\pi i l_1, \dots, 2\pi i l_{n-2}, Y_{n-1})}{e^{-Y_{n-1}} - 1},$$

where the integers $l_1^{(j)}, \dots, l_{n-2}^{(j)}$ are as defined in (2.19–2.21).

Proof. This follows by addition of the results of Proposition 2.9 over all $j = 1, \dots, n - 1$; on one side this yields the sum on the right-hand side of (2.13) (which according to (2.13) is equal to $\sum_{p \in \Lambda_{(l_1, \dots, l_{n-2})}^{\text{reg}}} f(p)$), and on the other side yields the sum on the right-hand side of (2.23). \square

We shall complete the proof of Lemma 2.6 by summing the equality given in Corollary 2.10 over all possible (l_1, \dots, l_{n-2}) satisfying (2.12); the proof reduces to the following lemma.

LEMMA 2.11. *In the notation of Proposition 2.9, $(l_1^{(j)}, \dots, l_{n-2}^{(j)}) \in \Lambda_{\text{reg}}^w(\mathfrak{t}_{n-1})$. Moreover for any $(l'_1, \dots, l'_{n-2}) \in \Lambda_{\text{reg}}^w(\mathfrak{t}_{n-1})$ there is exactly one sequence of integers (l_1, \dots, l_{n-2}) satisfying (2.12) such that*

$$(l_1^{(j)}, \dots, l_{n-2}^{(j)}) = (l'_1, \dots, l'_{n-2}).$$

Proof. This follows immediately from the proof of Proposition 2.9 and the fact that the restriction of q_j to \mathfrak{t}_{n-1} is given by the action of an element of the Weyl group W_{n-1} and hence maps $\Lambda_{\text{reg}}^w(\mathfrak{t}_{n-1})$ to itself bijectively. \square

This completes the proof of Lemma 2.6 and hence of Theorem 2.4 and Proposition 2.2.

3. Residue formulas and nonabelian localization

Let (M, ω) be a compact symplectic manifold with a Hamiltonian action of a compact connected Lie group K with Lie algebra \mathfrak{k} . Let $\mu: M \rightarrow \mathfrak{k}^*$ be a moment map for this action.

The K -equivariant cohomology with complex coefficients $H_K^*(M)$ of M may be identified with the cohomology of the chain complex

$$(3.1) \quad \Omega_K^*(M) = (S(\mathfrak{k}^*) \otimes \Omega^*(M))^K$$

of equivariant differential forms on M , equipped with the differential⁷

$$(3.2) \quad (D\eta)(X) = d(\eta(X)) - \iota_{X^\#}(\eta(X))$$

where $X^\#$ is the vector field on M generated by the action of X (see Chapter 7 of [7]). Here $(\Omega^*(M), d)$ is the de Rham complex of differential forms on M (with complex coefficients), and $S(\mathfrak{k}^*)$ denotes the algebra of polynomial functions on the Lie algebra \mathfrak{k} of K . An element $\eta \in \Omega_K^*(M)$ may be thought of as a K -equivariant polynomial function from \mathfrak{k} to $\Omega^*(M)$, or alternatively as a family of differential forms on M parametrized by $X \in \mathfrak{k}$. The standard definition of degree is used on $\Omega^*(M)$ and degree two is assigned to elements of \mathfrak{k}^* .

In fact as a vector space, though not in general as a ring, when M is a compact symplectic manifold with a Hamiltonian action of K then $H_K^*(M)$ is isomorphic to $H^*(M) \otimes H_K^*$ where $H_K^* = \Omega_K^*(\text{pt}) = S(\mathfrak{k}^*)^K$ is the equivariant cohomology of a point (see [36, Prop. 5.8]).

The map $\Omega_K^*(M) \rightarrow \Omega_K^*(\text{pt}) = S(\mathfrak{k}^*)^K$ given by integration over M passes to $H_K^*(M)$. Thus for any D -closed element $\eta \in \Omega_K^*(M)$ representing a cohomology class $[\eta]$, there is a corresponding element $\int_M \eta \in \Omega_K^*(\text{pt})$ which depends only on $[\eta]$. The same is true for any D -closed element $\eta = \sum_j \eta_j$ which is a formal series of elements η_j in $\Omega_K^j(M)$ without polynomial dependence on X : we shall in particular consider terms of the form

$$\eta(X)e^{(\bar{\omega}(X))}$$

⁷This definition of the equivariant cohomology differential differs by a factor of i from that used in [50] but is consistent with that used in [31].

where $\eta \in \Omega_K^*(M)$ and

$$\bar{\omega}(X) = \omega + \mu(X) \in \Omega_K^2(M).$$

Here $\mu: M \rightarrow \mathfrak{k}^*$ is identified in the natural way with a linear function on \mathfrak{k} with values in $\Omega^0(M)$. It follows directly from the definition of a moment map⁸ that $D\bar{\omega} = 0$.

If X lies in \mathfrak{t} , the Lie algebra of a chosen maximal torus T of K , then there is a formula for $\int_M \eta(X)$ (the *abelian localization formula* [2], [7], [8], [9]) which depends only on the fixed point set of T in M . It tells us that

$$(3.3) \quad \int_M \eta(X) = \sum_{F \in \mathcal{F}} \int \frac{i_F^* \eta(X)}{e_F(X)}$$

where \mathcal{F} indexes the components F of the fixed point set of T in M , the inclusion of F in M is denoted by i_F and $e_F \in H_T^*(M)$ is the equivariant Euler class of the normal bundle to F in M . In particular, applying (3.3) with η replaced by the formal equivariant cohomology class $\eta e^{\bar{\omega}}$ we have

$$(3.4) \quad h^\eta(X) \stackrel{\text{def}}{=} \int_M \eta(X) e^{\bar{\omega}(X)} = \sum_{F \in \mathcal{F}} h_F^\eta(X),$$

where

$$(3.5) \quad h_F^\eta(X) = e^{\mu(F)(X)} \int_F \frac{i_F^* \eta(X) e^\omega}{e_F(X)}.$$

Note that the moment map μ takes a constant value $\mu(F) \in \mathfrak{t}^*$ on each $F \in \mathcal{F}$, and that the integral in (3.5) is a rational function of X .

We shall assume throughout that 0 is a regular value of the moment map $\mu: M \rightarrow \mathfrak{k}^*$; equivalently the action of K on $\mu^{-1}(0)$ has only finite isotropy groups. The reduced space

$$M_{\text{red}} = \mu^{-1}(0)/K$$

is then a compact symplectic orbifold. The cohomology (with complex coefficients, as always in this paper) $H^*(M_{\text{red}})$ of this reduced space is naturally isomorphic to the equivariant cohomology $H_K^*(\mu^{-1}(0))$ of $\mu^{-1}(0)$, and by Theorem 5.4 of [36] the inclusion of $\mu^{-1}(0)$ in M induces a surjection on equivariant cohomology

$$H_K^*(M) \rightarrow H_K^*(\mu^{-1}(0)).$$

Composing we obtain a natural surjection

$$\Phi: H_K^*(M) \rightarrow H^*(M_{\text{red}})$$

which we shall denote by

$$\eta \mapsto \eta_0.$$

⁸We follow the convention that $d\mu(X) = \iota_{X\#}\omega$; some authors have $d\mu(X) = -\iota_{X\#}\omega$.

When there is no danger of confusion we shall use the same symbol for $\eta \in H_K^*(M)$ and any equivariantly closed differential form in $\Omega_K^*(M)$ which represents it. Note that $(\bar{\omega})_0 \in H^*(M_{\text{red}})$ is represented by the symplectic form ω_0 induced on M_{red} by ω .

Remark. Later we shall be working with not only the reduced space $M_{\text{red}} = \mu^{-1}(0)/K$ with respect to the action of the nonabelian group K , but also $\mu^{-1}(0)/T$ and $M_{\text{red}}^T(\xi) = \mu_T^{-1}(\xi)/T$ for regular values ξ of the T -moment map μ_T which is the composition of μ with restriction from \mathfrak{k}^* to \mathfrak{t}^* . We shall use the same notation η_0 for the image of η under the surjective homomorphism Φ for whichever of the spaces $\mu^{-1}(0)/K$, $\mu^{-1}(0)/T$ or $\mu_T^{-1}(0)/T$ we are considering, and the notation η_ξ if we are working with $\mu_T^{-1}(\xi)/T$. It should be clear from the context which version of the map Φ is being used.

The main result (the residue formula, Theorem 8.1) of [31] gives a formula for the evaluation on the fundamental class $[M_{\text{red}}] \in H_*(M_{\text{red}})$, or equivalently (if we represent cohomology classes by differential forms) the integral over M_{red} , of the image $\eta_0 e^{\omega_0}$ in $H^*(M_{\text{red}})$ of any formal equivariant cohomology class on M of the type $\eta e^{\bar{\omega}}$ where $\eta \in H_K^*(M)$.

THEOREM 3.1 (Residue formula, [31, Th. 8.1]). *Let $\eta \in H_K^*(M)$ induce $\eta_0 \in H^*(M_{\text{red}})$. Then*

$$(3.6) \quad \eta_0 e^{\omega_0} [M_{\text{red}}] = n_0 C_K \text{Res} \left(\mathcal{D}^2(X) \sum_{F \in \mathcal{F}} h_F^\eta(X) [dX] \right),$$

where the constant⁹ C_K is defined by

$$(3.7) \quad C_K = \frac{(-1)^{s+n_+}}{|W| \text{vol}(T)},$$

and n_0 is the order of the stabilizer in K of a generic point¹⁰ of $\mu^{-1}(0)$.

⁹This constant differs by a factor of $(-1)^s (2\pi)^{s-l}$ from that of [31, Th. 8.1]. The reason for the factor of $(2\pi)^{s-l}$ is that in this paper we shall adopt the convention that weights $\beta \in \mathfrak{t}^*$ send the integer lattice $\Lambda^l = \text{Ker}(\exp: \mathfrak{t} \rightarrow T)$ to \mathbb{Z} rather than to $2\pi\mathbb{Z}$, and that the roots of K are the nonzero weights of its complexified adjoint action. In [31] the roots send Λ^l to $2\pi\mathbb{Z}$. The reason for the factor of $(-1)^s$ is an error in Section 5 of [31]. In the last paragraph of p.307 of [31] the appropriate form to consider is $\prod_{j=1}^s (\theta^j dz'_j)$, and since 1-forms anticommute this is $(-1)^s / i^s$ times the term in $\exp(idz'(\theta))$ which contributes to the integral (5.4) of [31]. The constant also differs by a factor of i^s from that of [33, Th. 3.1], because in that paper the convention adopted on the equivariant cohomology differential is that of [50], not that of [31].

¹⁰Note that in [31] and [32] n_0 is stated incorrectly to be the order of the subgroup of K which acts trivially on $\mu^{-1}(0)$ (i.e. the kernel of the action of K on $\mu^{-1}(0)$); see the correction in Section 3 of [33]. When $K = T$ is abelian, however, the stabilizer in K of a generic point of $\mu^{-1}(0)$ is equal to the kernel of the action of K on $\mu^{-1}(0)$. Moreover since the coadjoint action of T on \mathfrak{t}^* is trivial, when $K = T$ this subgroup acts trivially on the normal bundle to $\mu^{-1}(0)$ in M and hence is the kernel of the action of K on M .

In this formula $|W|$ is the order of the Weyl group W of K , and we have introduced $s = \dim K$ and $l = \dim T$, while $n_+ = (s - l)/2$ is the number of positive roots. The measure $[dX]$ on \mathfrak{t} and volume $\text{vol}(T)$ of T are obtained from the restriction of a fixed invariant inner product on \mathfrak{k} , which is used to identify \mathfrak{k}^* with \mathfrak{k} throughout. Also, \mathcal{F} denotes the set of components of the fixed point set of T , and if F is one of these components then the meromorphic function h_F^η on $\mathfrak{t} \otimes \mathbb{C}$ is defined by (3.5). The polynomial $\mathcal{D}: \mathfrak{t} \rightarrow \mathbb{R}$ is defined by

$$\mathcal{D}(X) = \prod_{\gamma > 0} \gamma(X),$$

where γ runs over the positive roots of K . Note that it would perhaps be more natural to combine $(-1)^{n_+}$ from the constant C_K with $\mathcal{D}^2(X)$ and replace them by the product

$$\prod_{\gamma} \gamma(X)$$

of all the positive and negative roots of K .

The formula (3.6) was called a residue formula in [31] because the quantity Res (whose general definition was given in Section 8 of [31]) can be expressed as a multivariable residue¹¹, whose domain is a class of meromorphic differential forms on $\mathfrak{t} \otimes \mathbb{C}$. It is a linear map, but in order to apply it to individual terms in the residue formula some choices have to be made which do not affect the residue of the whole sum. Once the choices have been made one finds that many of the terms in the sum contribute zero, and the formula can be rewritten as a sum over a certain subset \mathcal{F}_+ of the set \mathcal{F} of components of the fixed point set M^T . When the rank of K is one and \mathfrak{t} is identified with \mathbb{R} , we can take

$$\mathcal{F}_+ = \{F \in \mathcal{F}: \mu_T(F) > 0\}.$$

In this paper we shall be particularly interested in the case where K has rank one, for which the results are as follows.

COROLLARY 3.2 [35], [51], [31]. *In the situation of Theorem 3.1, let $K = U(1)$. Then*

$$\eta_0 e^{\omega_0} [M_{\text{red}}] = -n_0 \text{Res}_{X=0} \left(\sum_{F \in \mathcal{F}_+} h_F^\eta(X) \right)$$

where n_0 is the order of the stabilizer in K of a generic point in $\mu^{-1}(0)$. Here, the meromorphic function h_F^η on \mathbb{C} is as defined by (3.5), and $\text{Res}_{X=0}$ denotes the coefficient of $1/X$, where $X \in \mathbb{R}$ has been identified with $2\pi i X \in \mathfrak{k}$.

¹¹An alternative definition in terms of iterated 1-variable residues is given in Section 3 of [33].

Remark. The notation $\text{Res}_{X=0}$ is intended to indicate the variable X with respect to which the residue is calculated, as well as the point 0 at which the residue is taken, so that, for example, $\text{Res}_{-X=0}f(X) = -\text{Res}_{X=0}f(X)$. It would perhaps be more natural to use the notation $\text{Res}_{X=0}f(X)dX$, but we shall have numerous formulas involving iterated residues of this type which would then become too long and unwieldy.

COROLLARY 3.3 (cf. [31, Cor. 8.2]). *In the situation of Theorem 3.1, let $K = \text{SU}(2)$. Then*

$$\eta_0 e^{\omega_0} [M_{\text{red}}] = \frac{n_0}{2} \text{Res}_{X=0} \left((2X)^2 \sum_{F \in \mathcal{F}_+} h_F^\eta(X) \right).$$

Here, n_0 , $\text{Res}_{X=0}$, h_F^η and \mathcal{F}_+ are as in Corollary 3.2, and $X \in \mathbb{R}$ has been identified with $\text{diag}(2\pi i, -2\pi i)X \in \mathfrak{t}$.

Remark 3.4. Note that if the degree of η is equal to the dimension of M_{red} then

$$\eta_0 e^{\omega_0} [M_{\text{red}}] = \eta_0 [M_{\text{red}}].$$

Alternatively for $K = U(1)$ or $K = \text{SU}(2)$, if we multiply ω and μ by a real scalar $\varepsilon > 0$ and let ε tend to 0 we obtain

$$\eta_0 [M_{\text{red}}] = -n_0 \text{Res}_{X=0} \left(\sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^* \eta(X)}{e_F(X)} \right)$$

or

$$\eta_0 [M_{\text{red}}] = \frac{n_0}{2} \text{Res}_{X=0} \left((2X)^2 \sum_{F \in \mathcal{F}_+} \int_F \frac{i_F^* \eta(X)}{e_F(X)} \right).$$

The results we have stated so far require the symplectic manifold M to be compact, and this condition is not satisfied in the situation in which we would like to apply these results (in order to obtain formulas for the intersection pairings in the cohomology of moduli spaces of bundles over compact Riemann surfaces). Luckily there are other related results due to Guillemin and Kalkman [23], and independently, Martin [39], which as we shall see can be generalized to noncompact symplectic manifolds.

Guillemin and Kalkman and Martin have approached the problem of finding a formula for

$$\eta_0 [M_{\text{red}}] = \int_{M_{\text{red}}} \eta_0$$

in terms of data on M localised near M^T in a slightly different way from that described above. As Guillemin and Kalkman observe, it follows immediately

from the residue formula that if $\xi \in \mathfrak{t}^*$ is a regular value of the T -moment map $\mu_T: M \rightarrow \mathfrak{t}^*$ which is sufficiently close to 0 then

$$(3.8) \quad \eta_0[M_{\text{red}}] = \frac{(-1)^{n_+} n_0(\mathcal{D}^2\eta)_\xi [\mu_T^{-1}(\xi)/T]}{n_0^T |W|}$$

where n_0 (respectively n_0^T) is the order of the stabilizer in K (respectively T) of a generic point of $\mu^{-1}(0)$ (respectively $\mu_T^{-1}(0)$) and $\mu_T^{-1}(\xi)/T$ is the reduced space for the action of T on M with respect to the shifted moment map $\mu_T - \xi$. Also $(\mathcal{D}^2\eta)_\xi \in H^*(\mu_T^{-1}(\xi)/T)$ is the image of $\mathcal{D}^2\eta$ under the surjection $\Phi: H_T^*(M) \rightarrow H^*(M_{\text{red}})$. Here $\eta \in H_K^*(M)$ and $\mathcal{D} \in S(\mathfrak{t}^*) = H_T^*$ are regarded as elements of $H_T^*(M)$ via the natural identification of $H_K^*(M)$ with the Weyl invariant part $(H_T^*(M))^W$ of $H_T^*(M)$ and the natural inclusion $H_T^* \rightarrow H_T^*(M)$. Martin gives a direct proof of (3.8) without appealing to the residue formula, which shows also that for any ξ sufficiently close to 0

$$(3.9) \quad \eta_0[M_{\text{red}}] = \frac{n_0(\mathcal{D}\eta)_\xi [\mu^{-1}(\xi)/T]}{n'_0 |W|}$$

where n'_0 is the order of the stabilizer in T of a generic point in $\mu^{-1}(0)$, provided that $\mu^{-1}(\xi)/T$ is oriented appropriately.

Remark 3.5. The symplectic form ω induces an orientation on M , and the induced symplectic forms on $M_{\text{red}} = \mu^{-1}(0)/K$ and on $\mu_T^{-1}(\xi)/T$ induce orientations on these quotients. We have made a choice of positive Weyl chamber for K in \mathfrak{t} ; this determines a Borel subgroup B (containing T) of the complexification G of K , such that the weights of the adjoint action of T on the quotient $\mathfrak{g}/\mathfrak{b}$ of the Lie algebra \mathfrak{g} of G by the Lie algebra \mathfrak{b} of B are the positive roots of K . We then get an orientation of the flag manifold K/T by identifying it with the complex space G/B . Modulo the action of finite isotropy groups we have a fibration

$$\mu^{-1}(0)/T \rightarrow \mu^{-1}(0)/K$$

with fibre K/T ; thus the symplectic orientation of $\mu^{-1}(0)/K$ and the orientation of K/T determined by the choice of Weyl chamber induce an orientation of $\mu^{-1}(0)/T$. Since 0 is a regular value of μ , if ξ is sufficiently close to 0 there is a homeomorphism from $\mu^{-1}(0)/T$ to $\mu^{-1}(\xi)/T$ induced by a T -equivariant isotopy of M ; so we get an induced orientation of $\mu^{-1}(\xi)/T$ to be used below.

Note that given a positive Weyl chamber we have another choice of orientation on $\mu^{-1}(\xi)/T$ which is compatible with the symplectic orientation on $\mu_T^{-1}(\xi)/T$ and the orientation of the normal bundle to $\mu^{-1}(\xi)/T$ in $\mu_T^{-1}(\xi)/T$ induced by identifying it in the natural way with the kernel of the restriction map $\mathfrak{k}^* \rightarrow \mathfrak{t}^*$, thence via the fixed invariant inner product on \mathfrak{k} with $\mathfrak{k}/\mathfrak{t}$ and thus finally with the complex vector space $\mathfrak{g}/\mathfrak{b}$ as above. Because we have

used the inner product to identify $\mathfrak{k}/\mathfrak{t}$ with its dual here, this orientation differs from the one chosen above by a factor of $(-1)^{n_+}$ where n_+ is the number of positive roots.

PROPOSITION 3.6 (Reduction to the abelian case, [S. Martin], [39]). *If T is a maximal torus of K and K acts effectively on M , then for any regular value ξ of μ_T sufficiently close to 0, we have that*

$$\begin{aligned} \int_{\mu^{-1}(0)/K} (\eta e^{\bar{\omega}})_0 &= \frac{n_0}{n'_0 |W|} \int_{\mu^{-1}(0)/T} (\mathcal{D}\eta e^{\bar{\omega}})_0 = \frac{n_0}{n'_0 |W|} \int_{\mu^{-1}(\xi)/T} (\mathcal{D}\eta e^{\bar{\omega}})_\xi \\ &= \frac{(-1)^{n_+} n_0}{n_0^T |W|} \int_{\mu_T^{-1}(\xi)/T} (\mathcal{D}^2 \eta e^{\bar{\omega}})_\xi \end{aligned}$$

where n_0 is the order of the stabilizer in K of a generic point of $\mu^{-1}(0)$ and n_0^T (respectively n'_0) is the order of the stabilizer in T of a generic point of $\mu_T^{-1}(0)$ (respectively $\mu^{-1}(0)$).

Remark 3.7. Note that $(-1)^{n+\mathcal{D}^2}$ is the product of all the roots of K , both positive and negative.

Martin proves this result by considering the diagram

$$\begin{array}{ccc} \mu^{-1}(0)/T & \cong & \mu^{-1}(\xi)/T \hookrightarrow \mu_T^{-1}(\xi)/T \\ \downarrow & & \\ M_{\text{red}} & = & \mu^{-1}(0)/K \end{array}$$

where the homeomorphism from $\mu^{-1}(0)/T$ to $\mu^{-1}(\xi)/T$ is induced by a T -equivariant isotopy of M (for ξ sufficiently close to 0). For simplicity we shall consider the case when $n_0 = n'_0 = n_0^T = 1$. As before we use a fixed invariant inner product on \mathfrak{k} to identify \mathfrak{k}^* with \mathfrak{k} , which splits T -equivariantly as the direct sum of \mathfrak{t} and its orthogonal complement \mathfrak{t}^\perp . The projection of $\mu: M \rightarrow \mathfrak{k}^* \cong \mathfrak{k}$ onto \mathfrak{t}^\perp then defines a T -equivariant section of the bundle $M \times \mathfrak{t}^\perp$ on M , which has equivariant Euler class $(-1)^{n+\mathcal{D}}$ if we orient $\mathfrak{t}^\perp \cong \mathfrak{k}/\mathfrak{t}$ by identifying it with the dual of the complex vector space $\mathfrak{g}/\mathfrak{b}$ as in Remark 3.5. Hence if ξ is a regular value of μ_T then $\mu^{-1}(\xi)/T$ is a zero-section of the induced orbifold bundle $\mu_T^{-1}(\xi) \times_T \mathfrak{t}^\perp$ on $\mu_T^{-1}(\xi)/T$, whose Euler class is $(-1)^{n+\mathcal{D}_\xi}$. Thus under the conventions for orientations described in Remark 3.5, evaluating the restriction to $\mu^{-1}(\xi)/T$ of an element of $H^*(\mu_T^{-1}(\xi))/T$ on the fundamental class $[\mu^{-1}(\xi)/T]$ gives the same result as multiplying by $(-1)^{n+\mathcal{D}_\xi}$ and evaluating on the fundamental class $[\mu_T^{-1}(\xi)/T]$.

Now Martin observes that since the natural map

$$\Pi: \mu^{-1}(0)/T \rightarrow \mu^{-1}(0)/K = M_{\text{red}}$$

is a fibration with fibre K/T , modulo the action of finite isotropy groups which act trivially on cohomology with complex coefficients, and since the Euler

characteristic of K/T is nonzero (in fact it is the order $|W|$ of the Weyl group of K), the evaluation of a cohomology class $\eta_0 \in H^*(M_{\text{red}})$ on $[M_{\text{red}}]$ is given by the evaluation of an associated cohomology class on $[\mu^{-1}(0)/T]$. More precisely we have

$$(3.10) \quad \eta_0[M_{\text{red}}] = \frac{e(V)}{|W|} \Pi^*(\eta_0)[\mu^{-1}(0)/T]$$

where $e(V)$ is the Euler class of the vertical subbundle of the tangent bundle to $\mu^{-1}(0)/T$ with respect to the fibration Π . As this Euler class is induced by \mathcal{D} under the orientation conventions of Remark 3.5, this completes the proof. \square

Remark 3.8. In this proof we saw that \mathcal{D}_ξ is the cohomology class in $H^*(\mu_T^{-1}(\xi)/T)$ which is Poincaré dual to the homology class represented by $\mu^{-1}(\xi)/T$. Thus \mathcal{D}_ξ may be represented by a closed differential form on $\mu_T^{-1}(\xi)/T$ with support in an arbitrarily small neighbourhood of $\mu^{-1}(\xi)/T$. If we interpret \mathcal{D}_ξ in this way, Martin's proof of Proposition 3.6 is valid even when M is noncompact and has singularities, provided that for ξ near 0 the subset $\mu^{-1}(\xi)$ is compact and does not meet the singularities of M .

Note also that K and hence T act with at most finite isotropy groups on a neighbourhood of $\mu^{-1}(0)$ in $\mu_T^{-1}(0)$, and so $\mu_T^{-1}(0)/T$ has at worst orbifold singularities in a neighbourhood of $\mu^{-1}(0)/T$. This means that in Proposition 3.6 we do not need to perturb the value of the T -moment map μ_T from 0 to a nearby regular value ξ if, as above, we represent \mathcal{D}_0 by a differential form on $\mu_T^{-1}(0)/T$ with support in a sufficiently small neighbourhood of $\mu^{-1}(0)/T$.

This result reduces the problem of finding a formula for $\eta_0[M_{\text{red}}]$ in terms of data on M localized near M^T to the case when $K = T$ is itself a torus. Guillemin and Kalkman, and independently Martin, then follow essentially the same line. This is to consider the change in

$$\eta_\xi[\mu_T^{-1}(\xi)/T],$$

for fixed $\eta \in H_T^*(M)$, as ξ varies through the regular values of μ_T . This is sufficient, if M is a compact symplectic manifold, because the image $\mu_T(M)$ is bounded; so if ξ is far enough from 0 then $\mu_T^{-1}(\xi)/T$ is empty and thus $\eta_\xi[\mu_T^{-1}(\xi)/T] = 0$.

More precisely, the convexity theorem of Atiyah [1] and Guillemin and Sternberg [24] tells us that the image $\mu_T(M)$ is a convex polytope; it is the convex hull in \mathfrak{t}^* of the set

$$\{\mu_T(F) : F \in \mathcal{F}\}$$

of the images $\mu_T(F)$ (each a single point of \mathfrak{t}^*) of the connected components F of the fixed point set M^T . This convex polytope is divided by codimension-one "walls" into subpolytopes, themselves convex hulls of subsets of

$\{\mu_T(F): F \in \mathcal{F}\}$, whose interiors consist entirely of regular values of μ_T . When ξ varies in the interior of one of these subpolytopes there is no change in $\eta_\xi[\mu_T^{-1}(\xi)/T]$, so it suffices to understand what happens as ξ crosses a codimension-one wall.

Any such wall is the image $\mu_T(M_1)$ of a connected component M_1 of the fixed point set of a circle subgroup T_1 of T . The quotient group T/T_1 acts on M_1 , which is a symplectic submanifold of M , and the restriction of the moment map μ_T to M_1 has an orthogonal decomposition

$$\mu_T|_{M_1} = \mu_{T/T_1} \oplus \mu_{T_1}$$

where $\mu_{T/T_1}: M_1 \rightarrow (\mathfrak{t}/\mathfrak{t}_1)^*$ is a moment map for the action of T/T_1 on M_1 and $\mu_{T_1}: M_1 \rightarrow \mathfrak{t}_1^*$ is constant (because T_1 acts trivially on M_1). If ξ_1 is a regular value of μ_{T/T_1} then there is a reduced space

$$(M_1)_{\text{red}} = \mu_{T/T_1}^{-1}(\xi_1)/(T/T_1).$$

Guillemin and Kalkman show that if T acts effectively on M (or equivalently if $n_0^T = 1$; see Footnote 9) then, for an appropriate choice of ξ_1 , the change in $\eta_\xi[\mu_T^{-1}(\xi)/T]$ as ξ crosses the wall $\mu_T(M_1)$ can be expressed as

$$(\text{res}_{M_1}(\eta))_{\xi_1}[(M_1)_{\text{red}}]$$

for a certain residue operation (see Footnote 11 below)

$$\text{res}_{M_1}: H_T^*(M) \rightarrow H_{T/T_1}^{*-d_1}(M_1)$$

where $d_1 = \text{codim}M_1 - 2$. (Of course care is needed here about the direction in which the wall is crossed; this can be resolved by a careful analysis of orientations.) By induction on the dimension of T this gives a method for calculating $\eta_\xi[\mu_T^{-1}(\xi)/T]$ in terms of data on M localized near M^T .

It is easiest to see how this version of localization is related to the residue formula of [31] in the special case when $K = T = U(1)$. In this case

$$\Omega_T^*(M) \cong \mathbb{C}[X] \otimes \Omega^*(M)^T$$

is the tensor product of a polynomial ring in one variable X (representing a coordinate function on the Lie algebra \mathfrak{t}) with the algebra of T -invariant de Rham forms on M . The Guillemin-Kalkman residue operation

$$\text{res}_{M_1}: H_T^*(M) \rightarrow H_{T/T_1}^{*-d_1}(M_1)$$

is then given in terms of the ordinary residue on \mathbb{C} by

$$\text{res}_{M_1}(\eta) = \text{Res}_{X=0} \frac{\eta|_{M_1}(X)}{e_{M_1}(X)}$$

where $\eta|_{M_1}(X)$ and the equivariant Euler class $e_{M_1}(X)$ of the normal bundle to M_1 in M are regarded as polynomials in X with coefficients in $H^*(M_1)$.

More precisely we formally decompose this normal bundle (using the splitting principle if necessary) as a sum of complex line bundles ν_j on which T acts with nonzero weights $\beta_j \in \mathfrak{t}^* \cong \mathbb{R}$, and because $c_1(\nu_j) \in H^*(M_1)$ is nilpotent we can express

$$\frac{\eta|_{M_1}(X)}{e_{M_1}(X)} = \frac{\eta|_{M_1}(X)}{\prod_j (c_1(\nu_j) + \beta_j X)} = \frac{\eta|_{M_1}(X)}{\prod_j (\beta_j X)} \prod_j \left(1 + \frac{c_1(\nu_j)}{\beta_j X} \right)^{-1}$$

as a finite Laurent series in X with coefficients in $H^*(M_1)$. Then $\text{res}_{M_1}(\eta)$ is simply the coefficient of $1/X$ in this expression.¹² Since $T_1 = T$ acts trivially on M_1 , we have $M_{1,\text{red}} = M_1$ and M_1 is a connected component of the fixed point set M^T , i.e. $M_1 \in \mathcal{F}$. Therefore

$$(\text{res}_{M_1}(\eta))_{\xi_1} [(M_1)_{\text{red}}] = \text{Res}_{X=0} \int_{M_1} \frac{\eta|_{M_1}(X)}{e_{M_1}(X)}.$$

Of course as $K = T = U(1)$ the convex polytope $\mu_T(M)$ in $\mathfrak{t}^* \cong \mathbb{R}$ is a closed interval, divided into subintervals by the points $\{\mu_T(F) : F \in \mathcal{F}\}$. Thus the argument of Guillemin and Kalkman just described, amplified by some careful consideration of orientations, tells us that if $\xi > 0$ is a regular value of μ_T and $n_0^T = 1$ then the difference

$$\eta_\xi[\mu_T^{-1}(\xi)/T] - \eta_0[\mu_T^{-1}(0)/T]$$

can be expressed as

$$(3.11) \quad \sum_{M_1 \in \mathcal{F} : 0 < \mu_T(M_1) < \xi} \text{res}_{M_1}(\eta)[M_1] = \text{Res}_{X=0} \sum_{F \in \mathcal{F} : 0 < \mu_T(F) < \xi} \int_F \frac{i_F^* \eta(X)}{e_F(X)}.$$

If we take $\xi > \sup(\mu_T(M))$ then this gives the same result as Corollary 3.2 (cf. Remark 3.4).

PROPOSITION 3.9 (dependence of symplectic quotients on parameters (Guillemin and Kalkman [23], S. Martin [39])). *If $K = T = U(1)$ and n_0^T is*

¹²When the dimension l of T is greater than one the Guillemin-Kalkman residue operation

$$\text{res}_{M_1} : H_T^*(M) \rightarrow H_{T/T_1}^{*-d_1}(M_1)$$

is defined in almost exactly the same way, by choosing a coordinate system $X = (X_1, \dots, X_l)$ on \mathfrak{t} where X_1 is a coordinate on \mathfrak{t}_1 , and taking the coefficient of $1/X_1$ in $\frac{\eta|_{M_1}(X)}{e_{M_1}(X)}$ expanded formally as a Laurent series in X_1 with coefficients in $\mathbb{C}[X_2, \dots, X_l] \otimes \Omega^*(M)^T$.

the order of the stabilizer in T of a generic point of $\mu_T^{-1}(0)$ then¹³

$$\int_{\mu_T^{-1}(\xi_1)/T} (\eta e^{\bar{\omega}})_{\xi_1} - \int_{\mu_T^{-1}(\xi_0)/T} (\eta e^{\bar{\omega}})_{\xi_0} = n_0^T \sum_{F \in \mathcal{F}: \xi_0 < \mu_T(F) < \xi_1} \text{Res}_{X=0} e^{\mu_T(F)X} \int_F \frac{\eta(X) e^\omega}{e_F(X)},$$

where $X \in \mathbb{C}$ has been identified with $2\pi iX \in \mathfrak{t} \otimes \mathbb{C}$ and $\xi_0 < \xi_1$ are two regular values of the moment map.

Remark 3.10. As we have already noted these results can be deduced easily from the residue formula of [31] when M is a compact symplectic manifold. However the proof of Proposition 3.9, just like that of Proposition 3.6 (see Remark 3.8), can be adapted to apply in circumstances when M is not compact and the residue formula of [31] is not valid. Indeed, as Guillemin and Kalkman observe, in the case when $K = T = U(1)$ the basis of their argument applies to any compact oriented $U(1)$ -manifold Y with boundary such that the action of $T = U(1)$ on the boundary ∂Y is locally free. Let us suppose for simplicity that T acts effectively on M (i.e. that $n_0^T = 1$; see Footnote 9) and let ζ be a $U(1)$ -invariant de Rham one-form on $Y - Y^T$ with the property that $\iota_v(\zeta) = 1$, where the vector field v is the infinitesimal generator of the $U(1)$ -action. Guillemin and Kalkman showed that, at the level of forms, the map $\Phi: H_T^*(Y) \rightarrow H^*(\partial Y/T)$ which is the composition of the restriction map from $H_T^*(Y)$ to $H_T^*(\partial Y)$ with the inverse of the canonical isomorphism $H_T^*(\partial Y) \rightarrow H^*(\partial Y/T)$ is given by

$$\Phi(\eta) = \text{Res}_{X=0} \iota_v \left(\frac{\zeta \eta}{X - d\zeta} \right)$$

(see (1.18) of [23], noting that Guillemin and Kalkman have a different convention for the equivariant cohomology differential, which accounts for the minus sign). If tubular neighbourhoods U_1, \dots, U_N of the components F_1, \dots, F_N of the fixed point set Y^T are removed from Y , then Stokes' theorem can be applied to the manifold with boundary $Y - \bigcup_{j=1}^N U_j$ using the formal identity

$$D \left(\frac{\zeta \eta}{X - d\zeta} \right) = \eta$$

on $Y - \bigcup_{j=1}^N U_j$ to give, after using the fact that $\int_{\partial Y} \alpha = \int_{\partial Y/T} \iota_v(\alpha)$ and taking

¹³The convention of Guillemin and Kalkman for the sign of the moment map differs from ours (see Footnote 8). This accounts for a difference in sign between their formula and ours.

residues at $X = 0$, the formula

$$\int_{\partial Y/T} \Phi(\eta) = \text{Res}_{X=0} \sum_{j=1}^N \int_{F_j} \frac{\eta|_{F_j}(X)}{e_{F_j}(X)}$$

where e_{F_j} is the equivariant Euler class of the normal bundle to F_j in Y .

The formula of Proposition 3.9 comes directly from this when the manifold with boundary Y is $\mu_T^{-1}[\xi_0, \xi_1]$ for a moment map $\mu_T: M \rightarrow \mathfrak{t}^* \cong \mathbb{R}$ with regular values $\xi_0 < \xi_1$, but there is no need for μ_T to be a moment map or for M to have a symplectic structure for the formula to be valid. It is enough for $\mu_T: M \rightarrow \mathbb{R}$ to be a smooth T -invariant map with regular values $\xi_0 < \xi_1$ such that T acts freely on the intersections of $\mu_T^{-1}(\xi_0)$ and $\mu_T^{-1}(\xi_1)$ with the support of the equivariant differential form η . There is also no need to assume that M is compact; it suffices to suppose that $\mu_T: M \rightarrow \mathbb{R}$ is a proper map. Indeed, the assumption that μ_T is proper can itself be weakened; the same proof applies provided only that the intersection of $\mu_T^{-1}[\xi_0, \xi_1]$ with the support of the equivariant differential form η is compact.

4. Extended moduli spaces

In [28] certain “extended moduli spaces” of flat connections on a compact Riemann surface with one boundary component are studied. They have natural symplectic structures, and can be used to exhibit the moduli spaces $\mathcal{M}(n, d)$ of interest to us as finite-dimensional symplectic quotients or reduced spaces. Our aim is to obtain Witten’s formulas for intersection pairings on $H^*(\mathcal{M}(n, d))$ by applying nonabelian localization to these extended moduli spaces. They have a gauge-theoretic description (cf. the introduction to this paper), but we shall use a more concrete (and entirely finite-dimensional) characterization given in [28].

The space with which we want to work is defined by

$$(4.1) \quad M(c) = (\varepsilon_K \times e_c)^{-1}(\Delta) \subset \text{Hom}(\mathbb{F}, K) \times \mathfrak{k},$$

where \mathbb{F} is the free group on $2g$ generators $\{x_1, \dots, x_{2g}\}$; we identify \mathbb{F} with the fundamental group of the surface Σ with one point removed, in such a way that x_1, \dots, x_{2g} correspond to the generators $\alpha_1, \dots, \alpha_{2g}$ of $H_1(\Sigma, \mathbb{Z})$ chosen in Section 2. Then $\varepsilon_K: \text{Hom}(\mathbb{F}, K) \rightarrow K$ is the evaluation map on the relator $r = \prod_{j=1}^g [x_j, x_{j+g}]$ and

$$(4.2) \quad \varepsilon_K(h_1, \dots, h_{2g}) = \prod_{j=1}^g [h_j, h_{j+g}].$$

The map $e_c: \mathfrak{k} \rightarrow K$ is defined by

$$(4.3) \quad e_c(Y) = c \exp(Y),$$

where the generator c of the centre of K is as defined in (2.6) above. The diagonal in $K \times K$ is denoted Δ . The space $M(c)$ then has canonical projection maps pr_1, pr_2 which make the following diagram commute:

$$(4.4) \quad \begin{array}{ccc} M(c) & \xrightarrow{\text{pr}_2} & \mathfrak{k} \\ \text{pr}_1 \downarrow & & \downarrow e_c \\ \text{Hom}(\mathbb{F}, K) & \xrightarrow{\varepsilon_K} & K. \end{array}$$

In other words, $M(c)$ is the fibre product of $\text{Hom}(\mathbb{F}, K)$ and \mathfrak{k} under the maps ε_K and e_c . The action of K on $M(c)$ is given by the adjoint actions on K and \mathfrak{k} . The space $M(c)$ has the following properties (see [28] and [29]):

PROPOSITION 4.1. (a) *The space $M(c)$ is smooth near all $(h, \Lambda) \in \text{Hom}(\mathbb{F}, K) \times \mathfrak{k}$ for which the linear space $z(h) \cap \ker(d \exp)_\Lambda \neq \{0\}$. Here, $z(h)$ is the Lie algebra of the stabilizer $Z(h)$ of h .*

(b) *There is a K -invariant 2-form ω on $\text{Hom}(\mathbb{F}, K) \times \mathfrak{k}$ whose restriction to $M(c)$ is closed and which defines a nondegenerate bilinear form on the Zariski tangent space to $M(c)$ at every (h, Λ) in an open dense subset of $M(c)$ containing $M(c) \cap (K^{2g} \times \{0\})$. Thus the form ω gives rise to a symplectic structure on this open subset of $M(c)$.*

(c) *With respect to the symplectic structure given by the 2-form ω , a moment map $\mu: M(c) \rightarrow \mathfrak{k}^*$ for the action of K on $M(c)$ is given by the restriction to $M(c)$ of $-\text{pr}_2$, where $\text{pr}_2: M(c) \rightarrow \mathfrak{k}$ is the projection map to \mathfrak{k} (composed with the canonical isomorphism $\mathfrak{k} \rightarrow \mathfrak{k}^*$ given by the invariant inner product on \mathfrak{k}).*

(d) *The space $M(c)$ is smooth in a neighbourhood of $\mu^{-1}(0)$.*

(e) *The symplectic quotient $M_{\text{red}} = M(c) \cap \mu^{-1}(0)/K$ can be naturally identified with $\varepsilon_K^{-1}(c)/K = \mathcal{M}(n, d)$.*

Remark 4.2. We shall also use μ to denote the map

$$\mu: K^{2g} \times \mathfrak{k} \rightarrow \mathfrak{k}$$

defined by

$$\mu(h, \Lambda) = -\Lambda,$$

even though it is only its restriction to $M(c)$ which is a moment map in any obvious sense. That is why we write $M(c) \cap \mu^{-1}(0)/K$ instead of $\mu^{-1}(0)/K$ in (e) above.

Remark 4.3. Using our description (4.4) of $M(c)$ as a fibre product, it is easy to identify the components F of the fixed point set of the action of T . We examine the fixed point sets of the action of T on $\text{Hom}(\mathbb{F}, K)$ and \mathfrak{k} and

find

$$(4.5) \quad \begin{array}{ccc} M(c)^T & \xrightarrow{\text{pr}_2} & \mathfrak{t} \\ \text{pr}_1 \downarrow & & \downarrow e_c \\ \text{Hom}(\mathbb{F}, T) & \xrightarrow{\varepsilon_K} & 1 \in T. \end{array}$$

(Notice that ε_K sends $\text{Hom}(\mathbb{F}, T)$ to 1 because T is abelian.) Thus

$$(4.6) \quad M(c)^T = \text{Hom}(\mathbb{F}, T) \times e_c^{-1}(1) = T^{2g} \times \{\delta - \tilde{c} : \delta \in \Lambda^I \subset \mathfrak{t}\}$$

where \tilde{c} is a fixed element of \mathfrak{t} for which $\exp \tilde{c} = c$. (Here, Λ^I denotes the integer lattice $\text{Ker}(\exp) \subset \mathfrak{t}$.) If we ignore the singularities of $M(c)$, this description also enables us to find a plausible candidate for the equivariant Euler class e_{F_δ} of the normal bundle of each component $T^{2g} \times (\delta - \tilde{c})$ in $M(c)^T$ (indexed by $\delta \in \Lambda^I$). This should be simply the equivariant Euler class of the normal bundle to T^{2g} in K^{2g} , implying that e_{F_δ} is in fact independent of δ and is given by

$$(4.7) \quad e_{F_\delta}(X) = \left(\prod_\gamma \gamma \right)^g = ((-1)^{n+} \mathcal{D}(X)^2)^g.$$

The symplectic volume of the component F_δ is independent of δ (indeed these components are all identified symplectically with T^{2g}); we denote the volume of F_δ by $\int_F e^\omega$. The constant value taken by the moment map μ_T on the component $F = F_\delta$ is given by $\tilde{c} - \delta$.

We shall need also the following property (proved in [30]):

PROPOSITION 4.4. *The generating classes a_r, b_r^j and f_r ($r = 2, \dots, n, j = 1, \dots, 2g$) extend to classes $\tilde{a}_r(X), \tilde{b}_r^j(X)$ and $\tilde{f}_r(X) \in H_K^*(M(c))$.*

Indeed, because of our conventions on the equivariant differential, the construction of [30] (which will be described at the beginning of Section 9) tells us that the equivariant differential form $\tilde{a}_r(X) \in \Omega_K^*(M(c))$ whose restriction represents the cohomology class $a_r \in H^*(\mathcal{M}(n, d))$ is $\tau_r(-X)$, where as above $\tau_r \in S^r(\mathfrak{k}^*)^K \cong H_K^*(\text{pt})$ is the invariant polynomial which is associated to the r^{th} Chern class (see [30]). Moreover \tilde{f}_2 is the extension $\tilde{\omega} = \omega + \mu$ of the symplectic form ω to an equivariantly closed differential form (see [30] again).

Finally we shall need to work with the symplectic subspace $M_{\mathfrak{t}}(c) = M(c) \cap \mu^{-1}(\mathfrak{t})$ of $M(c)$, which is no longer acted on by K but is acted on by T . The space $M_{\mathfrak{t}}(c)$ has an important periodicity property:

LEMMA 4.5. *Suppose Λ_0 lies in the integer lattice $\Lambda^I = \text{Ker}(\exp)$ in \mathfrak{t} . Then there is a homeomorphism $s_{\Lambda_0}: K^{2g} \times \mathfrak{k} \rightarrow K^{2g} \times \mathfrak{k}$ defined by*

$$s_{\Lambda_0}: (h, \Lambda) \mapsto (h, \Lambda + \Lambda_0)$$

which restricts to a homeomorphism $s_{\Lambda_0}: M_{\mathfrak{t}}(c) \rightarrow M_{\mathfrak{t}}(c)$.

Proof. This is an immediate consequence of the definition of $M_{\mathfrak{t}}(c)$ and the fact that $\exp(\Lambda + \Lambda_0) = \exp(\Lambda) \exp(\Lambda_0)$ when Λ and Λ_0 commute. \square

Let us examine the behaviour of the images in $H_T^*(M_{\mathfrak{t}}(c))$ of these extensions $\tilde{a}_r(X), \tilde{b}_r^j(X), \tilde{f}_r(X) \in H_K^*(M(c))$ of the generating classes a_r, b_r^j, f_r (see Proposition 4.4) under pullback under these homeomorphisms $s_{\Lambda_0}: M_{\mathfrak{t}}(c) \rightarrow M_{\mathfrak{t}}(c)$. By abuse of language, we shall refer to these images also as $\tilde{a}_r(X), \tilde{b}_r^j(X)$ and $\tilde{f}_r(X)$. We noted above that the classes $\tilde{a}_r(X)$ are the images in $H_K^*(M(c))$ of the polynomials $\tau_r(-X) \in H_K^* = S(\mathfrak{k}^*)^K$ (cf. (2.1)). Moreover (by [30, (8.18)]) the classes $\tilde{b}_r^j(X) \in H_K^*(M(c))$ are of the form $\tilde{b}_r^j(X) = \text{pr}_1^*(\tilde{b}_r^j(X))_1$ where $(\tilde{b}_r^j(X))_1 \in H_K^*(K^{2g})$ and $\text{pr}_1: M(c) \rightarrow K^{2g}$ is the projection in (4.4). It follows that

$$s_{\Lambda_0}^* \tilde{b}_r^j(X) = \tilde{b}_r^j(X)$$

and

$$s_{\Lambda_0}^* \tilde{a}_r(X) = \tilde{a}_r(X).$$

Furthermore we see from (8.30) of [30] that $\tilde{f}_2(X)$ is of the form

$$(4.8) \quad \tilde{f}_2(X) = \text{pr}_1^* f_2^1 + \langle \mu, X \rangle$$

where $f_2^1 \in H_K^*(K^{2g})$ and $\mu: M(c) \rightarrow \mathfrak{k}$ is the moment map (which is the restriction to $M(c)$ of minus the projection $K^{2g} \times \mathfrak{k} \rightarrow \mathfrak{k}$; see Proposition 4.1). It follows from this that for any Λ_0 in the integer lattice Λ^I of \mathfrak{t} (the kernel of the exponential map),

$$(4.9) \quad s_{\Lambda_0}^* \tilde{f}_2(X) = \tilde{f}_2(X) - \langle \Lambda_0, X \rangle.$$

5. Equivariant Poincaré duals

We are aiming to apply nonabelian localization to the extended moduli space $M(c)$ defined in the previous section. In order to overcome the problem that $M(c)$ is singular, instead of working with integrals over $M(c)$ of equivariant differential forms, we shall integrate over $K^{2g} \times \mathfrak{k}$ after first multiplying by a suitable equivariantly closed differential form on $K^{2g} \times \mathfrak{k}$ with support near $M(c)$ which can be thought of as representing the equivariant Poincaré dual to $M(c)$ in $K^{2g} \times \mathfrak{k}$. So we need to construct such an equivariantly closed differential form.

Remark 5.1. In our earlier article [32] covering the case when the bundles have rank $n = 2$, we overcame the problem of the singularities of $M(c)$ in a slightly different way, by perturbing the central constant $c \in \text{SU}(n)$ to a nearby element of the maximal torus T . This method can be generalized to cover the cases when $n > 2$, but it seems a little more straightforward to use equivariant Poincaré duals, so we adopt the latter approach here.

Remark 5.2. Related constructions of equivariantly closed differential forms representing the Poincaré dual to a submanifold appear in the literature.¹⁴ In Kalkman's paper [34] and Mathai-Quillen's paper [40], an equivariantly closed differential form which is rapidly decreasing away from a submanifold and represents the Poincaré dual to the submanifold is given; such a form is often referred to as the *Thom form*, as the cohomology class it represents is the Thom class of the normal bundle to the submanifold. The forms constructed in [34] and [40] are not compactly supported; a construction of a compactly supported equivariantly closed form representing the Poincaré dual of a submanifold is given in Section 2.3 of [19]. For completeness, in this section we provide a construction of an equivariantly closed form representing the Poincaré dual.

First we consider the simpler problem of constructing an equivariant Poincaré dual to the origin in a one-dimensional representation χ of a circle. If we did not need to find a form with support near the origin we could represent the equivariant Poincaré dual by χ itself, regarded as an equivariant differential form. However compact support will be important later, so we need to be a little more careful.

LEMMA 5.3. *Let $T = U(1)$ act on \mathbb{C} via a weight $\chi: T \rightarrow U(1)$. Then we can find an equivariantly closed differential form $\alpha_\chi \in \Omega_T^2(\mathbb{C})$ on \mathbb{C} with compact support arbitrarily close to 0, such that*

$$\int_{\mathbb{C}} \eta \alpha_\chi = \eta|_0 \in H_T^*$$

for all equivariantly closed forms $\eta \in \Omega_T^*(\mathbb{C})$. Moreover $\alpha_\chi \in \chi + D(\Omega_T^*(\mathbb{C}))$, so that α_χ represents the same equivariant cohomology class on \mathbb{C} as χ .

Proof. Let X^\sharp denote the vector field on \mathbb{C} given by the infinitesimal action of $X \in \mathfrak{t}$. There is a T -invariant closed differential 1-form on $\mathbb{C} - \{0\}$, given in polar coordinates (r, θ) by $\frac{d\theta}{2\pi}$, such that $\iota_{X^\sharp}(\frac{d\theta}{2\pi})$ is identically equal to $\chi(X)$ for every $X \in \mathfrak{t}$. We can choose a smooth T -invariant function $b: \mathbb{C} \rightarrow [0, \infty)$ with support in an arbitrarily small neighbourhood of 0 which is identically equal to 1 on some smaller neighbourhood of 0, and let

$$\alpha_\chi(X) = \chi(X) + D\left((1-b)\frac{d\theta}{2\pi}\right) = \chi(X) + d\left((1-b)\frac{d\theta}{2\pi}\right) + (b-1)\chi(X)$$

where D is the equivariant differential defined at (3.2) and d is the ordinary differential. Then α_χ is equivariantly closed and is zero outside the support of b .

¹⁴We thank P. Paradan for pointing out that the references cited below contain such constructions.

Suppose that $\eta \in \Omega_T^*(\mathbb{C})$ is equivariantly closed. We wish to show that

$$\int_{\mathbb{C}} \eta \alpha_X = \eta|_0.$$

First we shall show that the integral

$$\int_{\mathbb{C}} \eta \alpha_X$$

is independent of the choice of the function b .

If $\rho > 0$ is sufficiently small and $R > 0$ is sufficiently large, then b is identically equal to 1 on the disc D_ρ centre 0 and radius ρ , and b is identically equal to 0 outside the disc D_R centre 0 and radius R . Then

$$\int_{\mathbb{C}} \eta \alpha_X = \chi \int_{D_\rho} \eta + \int_{D_R - D_\rho} \eta \alpha_X.$$

Now η is a polynomial function from \mathfrak{t} to the ordinary de Rham complex $\Omega^*(\mathbb{C})$, so we can write

$$\eta = \eta^{(0)} + \eta^{(1)} + \eta^{(2)}$$

where $\eta^{(j)}$ is a polynomial function from \mathfrak{t} to $\Omega^j(\mathbb{C})$ for $j = 0, 1, 2$. Similarly

$$\alpha_X = \alpha_X^{(0)} + \alpha_X^{(1)} + \alpha_X^{(2)}$$

where $\alpha_X^{(0)} = b\chi$, $\alpha_X^{(1)} = 0$ and $\alpha_X^{(2)} = d((1 - b)\frac{d\theta}{2\pi})$. Since $D\eta = d\eta - \iota_{X\#}\eta$ is zero, we have $d\eta^{(0)} = \iota_{X\#}\eta^{(2)}$. As any 2-form on \mathbb{C} is a C^∞ function on \mathbb{C} multiplied by the nowhere vanishing 2-form given in polar coordinates by $\frac{r d\theta dr}{2\pi}$, and since $\iota_{X\#}(\frac{r d\theta dr}{2\pi}) = \chi(X)r dr$, it follows that

$$\chi(X)\eta^{(2)}(X) = \frac{d\theta}{2\pi}d\eta^{(0)}(X)$$

on $\mathbb{C} - \{0\}$ where $d\theta$ is defined. Hence

$$\begin{aligned} \int_{D_R - D_\rho} \eta \alpha_X &= \int_{D_R - D_\rho} \eta^{(2)} \alpha_X^{(0)} + \eta^{(0)} \alpha_X^{(2)} \\ &= \int_{D_R - D_\rho} b \frac{d\theta}{2\pi} d\eta^{(0)} + \eta^{(0)} d((1 - b)\frac{d\theta}{2\pi}) \\ &= - \int_{D_R - D_\rho} d(b\eta^{(0)} \frac{d\theta}{2\pi}) \\ &= \int_{\partial D_\rho} b\eta^{(0)} \frac{d\theta}{2\pi} - \int_{\partial D_R} b\eta^{(0)} \frac{d\theta}{2\pi} \\ &= \int_{\partial D_\rho} \eta^{(0)} \frac{d\theta}{2\pi} \end{aligned}$$

by Stokes' theorem, since b is identically one on ∂D_ρ and identically zero on

∂D_R . It follows that

$$\int_{\mathbb{C}} \eta \alpha_\chi = \chi \int_{D_\rho} \eta + \int_{\partial D_\rho} \eta^{(0)} \frac{d\theta}{2\pi}$$

is independent of the choice of b .

Now ρ can be taken arbitrarily small, and $\chi \int_{D_\rho} \eta \rightarrow 0$ as $\rho \rightarrow 0$. Moreover by continuity, for fixed $X \in \mathfrak{t}$ and any $\varepsilon > 0$ we can choose ρ so that $\eta^{(0)}$ differs from $\eta^{(0)}|_0 = \eta|_0$ by at most ε on D_ρ . Then

$$\left| \int_{\partial D_\rho} \eta^{(0)} \frac{d\theta}{2\pi} - \eta^{(0)}|_0 \right| = \left| \int_{\partial D_\rho} (\eta^{(0)} - \eta^{(0)}|_0) \frac{d\theta}{2\pi} \right| \leq \varepsilon.$$

Thus $\int_{\mathbb{C}} \eta \alpha_\chi - \eta|_0$ tends to zero as ρ tends to 0. Since $\int_{\mathbb{C}} \eta \alpha_\chi$ and $\eta|_0$ are independent of ρ we deduce that

$$\int_{\mathbb{C}} \eta \alpha_\chi = \eta|_0$$

as required. □

LEMMA 5.4. *Let T be a torus acting trivially on \mathbb{R} . Then we can find an equivariantly closed differential form $\alpha_0 \in \Omega_T^*(\mathbb{R})$ on \mathbb{R} with compact support arbitrarily close to 0, such that*

$$\int_{\mathbb{R}} \eta \alpha_0 = \eta|_0 \in H_T^*$$

for all equivariantly closed forms $\eta \in \Omega_T^*(\mathbb{R})$.

Proof. We have $\Omega_T^*(\mathbb{R}) = S(\mathfrak{t}^*) \otimes \Omega^*(\mathbb{R})$ and $\eta \in S(\mathfrak{t}^*) \otimes \Omega^0(\mathbb{R})$ is equivariantly closed if and only if it is constant on \mathbb{R} , so we can take α_0 to be the standard volume form on \mathbb{R} multiplied by any bump function compactly supported near 0 with unit integral. □

COROLLARY 5.5. *Let T be a torus acting linearly on \mathbb{C}^n with weights χ_1, \dots, χ_n and trivially on \mathbb{R}^m . Then we can find an equivariantly closed differential form $\alpha \in \Omega_T^{2n}(\mathbb{C}^n \times \mathbb{R}^m)$ on $\mathbb{C}^n \times \mathbb{R}^m$ with compact support arbitrarily close to 0, such that*

$$\int_{\mathbb{C}^n \times \mathbb{R}^m} \eta \alpha = \eta_0 \in H_T^*$$

for all equivariantly closed forms $\eta \in \Omega_T^*(\mathbb{C}^n \times \mathbb{R}^m)$. Moreover if $m = 0$ then $\alpha \in \chi_1 \dots \chi_n + D(\Omega_T^*(\mathbb{C}^n))$.

Proof. The action of T on the copy of \mathbb{C} in \mathbb{C}^n on which it acts via the weight χ_j factors through an action of $T/\ker \chi_j \cong U(1)$ (unless $\chi_j = 0$ in which case we can replace $\ker \chi_j$ by any subtorus of T of codimension one). We can construct $\alpha_{\chi_j} \in \Omega_{U(1)}^*(\mathbb{C})$ as in Lemma 5.3 and m copies of α_0 as in

Lemma 5.4, and then define α to be the wedge product of the pullbacks of the α_{χ_j} and α_0 to $\Omega_T^*(\mathbb{C}^n \times \mathbb{R}^m)$ via the projections of $\mathbb{C}^n \times \mathbb{R}^m$ to \mathbb{C} and \mathbb{R} and the homomorphisms $T \rightarrow U(1)$ induced by the weights χ_j . \square

Now we shall relax our assumption that c is a central element of K , and assume only that $c \in T$. This will be important later when we apply induction on n (see Remark 6.4 below).

COROLLARY 5.6. *Let T be the maximal torus of $K = \text{SU}(n)$ acting on K by conjugation. If $c \in T$ then we can find a T -equivariantly closed differential form $\alpha \in \Omega_T^*(K)$ on K with support arbitrarily close to c such that*

$$\int_K \eta \alpha = \eta|_c \in H_T^*$$

for all T -equivariantly closed differential forms $\eta \in \Omega_T^*(K)$.

Proof. There is a T -equivariant diffeomorphism ϕ from a T -invariant neighbourhood U of 0 in the Lie algebra \mathfrak{k} of K to a T -invariant neighbourhood V of c in K given by

$$\phi(X) = c \exp(X).$$

By Corollary 5.5 we can find $\tilde{\alpha} \in \Omega_T^*(\mathfrak{k})$ with arbitrarily small compact support contained in U , such that

$$\int_{\mathfrak{k}} \eta \tilde{\alpha} = \eta|_0 \in H_T^*$$

for all equivariantly closed forms $\eta \in \Omega_T^*(\mathfrak{k})$. Then we can define α to be $(\phi^{-1})^*(\tilde{\alpha})$. \square

Note that

$$M(c) = \left\{ (h_1, \dots, h_{2g}, \Lambda) \in K^{2g} \times \mathfrak{k} : \prod_{j=1}^g h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} = c \exp(\Lambda) \right\}$$

can be expressed as $M(c) = P^{-1}(c)$ where $P: K^{2g} \times \mathfrak{k} \rightarrow K$ is defined by

$$P(h_1, \dots, h_{2g}, \Lambda) = \prod_{j=1}^g h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} \exp(-\Lambda).$$

PROPOSITION 5.7. *If T is the maximal torus of $K = \text{SU}(n)$ and $c \in T$ then there is a T -equivariantly closed differential form $\alpha \in \Omega^*(K^{2g} \times \mathfrak{k})$ of degree $n^2 - 1$ on $K^{2g} \times \mathfrak{k}$ with support contained in a neighbourhood of $M(c)$ of the form $P^{-1}(V)$ where V is an arbitrarily small neighbourhood of c in K , such that*

$$\int_{K^{2g} \times \mathfrak{k}} \eta \alpha = \int_{M(c)} \eta|_{M(c)} \in H_T^*$$

for any T -equivariantly closed form $\eta \in \Omega_T^*(K^{2g} \times \mathfrak{k})$ for which the intersection of $P^{-1}(\bar{V})$ with the support of η is compact.

Proof. By Corollary 5.6 we can find a T -equivariantly closed differential form $\hat{\alpha} \in \Omega_T^*(K)$ on K with support in V such that

$$\int_K \eta \hat{\alpha} = \eta|_c \in H_T^*$$

for all T -equivariantly closed forms $\eta \in \Omega_T^*(K)$. Let $\alpha = P^*(\hat{\alpha})$; by the functoriality of the equivariant pushforward map (cf. Section 3 of [3]) this has the properties we want. \square

Remark 5.8. In fact if V' is any neighbourhood of c in K containing \bar{V} then we have

$$\int_{P^{-1}(V')} \eta \alpha = \int_{M(c)} \eta|_{M(c)} \in H_T^*$$

for any T -equivariantly closed form $\eta \in \Omega_T^*(P^{-1}(V'))$ on $P^{-1}(V')$ for which the intersection of $P^{-1}(\bar{V})$ with the support of η is compact.

Remark 5.9. As we are going to use Proposition 5.7 to convert integrals over $M(c)$ into integrals over $K^{2g} \times \mathbf{k}$ (or at least over neighbourhoods of $M(c)$ in $K^{2g} \times \mathbf{k}$ of the form $P^{-1}(V)$ for arbitrarily small neighbourhoods V of c in K), we shall need to be able to extend T -equivariant cohomology classes η on $M(c)$ to T -equivariant cohomology classes on neighbourhoods of $M(c)$ in $K^{2g} \times \mathbf{k}$ of this form $P^{-1}(V)$. This will always be possible by the continuity properties of cohomology (see e.g. [14, VIII 6.18]) because η will always have compact support in $M(c)$; more precisely we will in fact be converting integrals over $M(c) \cap (K^{2g} \times B)$ for compact subsets B of \mathbf{k} into integrals over $P^{-1}(V) \cap (K^{2g} \times B)$.

Note that the centre Z_n of $K = \mathrm{SU}(n)$ is a finite group of order n which acts trivially on $K^{2g} \times \mathbf{k}$.

LEMMA 5.10. *Suppose that $c = \mathrm{diag}(c_1, \dots, c_n) \in T$ is such that the product of no proper subsequence of c_1, \dots, c_n is 1. Then the quotient T/Z_n of T by the centre Z_n of $K = \mathrm{SU}(n)$ acts freely on $P^{-1}(V) \cap \mu^{-1}(0)$ for any sufficiently small T -invariant neighbourhood V of c in K .*

Proof. Suppose that T/Z_n does not act freely on $P^{-1}(V) \cap \mu^{-1}(0)$. Then there exist $t_1, \dots, t_n \in \mathbb{C}$, not all equal, such that $t_1 \dots t_n = 1$, and some element $(h, 0) = (h_1, \dots, h_{2g}, 0)$ of $P^{-1}(V) \cap \mu^{-1}(0)$ fixed by $\mathrm{diag}(t_1, \dots, t_n)$. Then each h_j is block diagonal with respect to the decomposition of $\{1, \dots, n\}$ as the union of $\{i: t_i = t_1\}$ and $\{i: t_i \neq t_1\}$, which implies that

$$P(h, 0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where A and B are products of commutators and hence satisfy $\det A = 1 = \det B$. The result follows. \square

Remark 5.11. It follows from this lemma that we can extend the definition of the composition

$$\Phi: H_T^*(M(c)) \rightarrow H_T^*(M(c) \cap \mu^{-1}(0)) \cong H^*(M(c) \cap \mu^{-1}(0)/T)$$

to

$$\Phi: H_T^*(P^{-1}(V)) \rightarrow H_T^*(P^{-1}(V) \cap \mu^{-1}(0)) \cong H^*(P^{-1}(V) \cap \mu^{-1}(0)/T).$$

By 1.18 of [23] (see Remark 3.10 above), when $T = U(1)$ is a circle then Φ is given on the level of forms by

$$\Phi(\eta) = \text{Res}_{X=0\iota_v} \left(\frac{\zeta\eta}{X - d\zeta} \right)$$

where the vector field v is the infinitesimal generator of the $U(1)$ action and ζ is a $U(1)$ -invariant differential 1-form on $P^{-1}(V) \cap \mu^{-1}(0)$ such that $\iota_v(\zeta) = 1$. (Strictly speaking the residue is an invariant form on $P^{-1}(V) \cap \mu^{-1}(0)$ which descends to a form on $(P^{-1}(V) \cap \mu^{-1}(0))/T$.) Thus when $T = U(1)$,

$$\int_{M(c) \cap \mu^{-1}(0)/T} \Phi(\eta) = \int_{M(c) \cap \mu^{-1}(0)} \text{Res}_{X=0} \frac{\zeta\eta}{X - d\zeta},$$

and it follows that if α is defined as in Proposition 5.7 for $n = 2$ and V' is any neighbourhood of c in $SU(2)$ containing \bar{V} ,

$$\begin{aligned} \int_{P^{-1}(V') \cap \mu^{-1}(0)/T} \Phi(\eta\alpha) &= \int_{P^{-1}(V') \cap \mu^{-1}(0)} \text{Res}_{X=0} \frac{\zeta\eta\alpha}{X - d\zeta} \\ &= \int_{M(c) \cap \mu^{-1}(0)} \text{Res}_{X=0} \frac{\zeta\eta}{X - d\zeta} = \int_{M(c) \cap \mu^{-1}(0)/T} \Phi(\eta) \end{aligned}$$

for any T -equivariantly closed differential form $\eta \in \Omega_T^*(P^{-1}(V'))$ such that the intersection of $P^{-1}(\bar{V})$ with the support of η is compact. Here we have used the same notation for η and its restriction to $M(c)$.

When $n > 2$, so that the maximal torus T of $K = SU(n)$ has dimension higher than one, then $\Phi(\eta)$ and $\int_{M(c) \cap \mu^{-1}(0)/T} \Phi(\eta)$ are given by similar formulas involving $n - 1$ iterated residues (see [23]). In particular the support of $\Phi(\eta)$ is contained in the image of the support of η , and

$$\int_{P^{-1}(V') \cap \mu^{-1}(0)/T} \Phi(\eta\alpha) = \int_{M(c) \cap \mu^{-1}(0)/T} \Phi(\eta)$$

for any T -equivariantly closed differential form $\eta \in \Omega_T^*(P^{-1}(V'))$ such that the intersection of $P^{-1}(\bar{V})$ with the support of η is compact.

6. Nonabelian localization applied to extended moduli spaces

Naïve application of the residue formula (Theorem 3.1) to the extended moduli space $M(c)$, using (2.1) and Remark 4.3 and ignoring the fact that $M(c)$ is noncompact and has singularities, yields

$$(6.1) \quad \prod_{r=2}^n a_r^{m_r} \exp(f_2)[\mathcal{M}(n, d)] \\ = nC_K \text{Res} \left(\mathcal{D}^2(X) \left(\int_F e^\omega \right) \sum_{\delta \in \Lambda^I} \frac{\prod_{r=2}^n \tau_r(-X)^{m_r} e^{(\tilde{\alpha}-\delta)(X)}}{((-1)^{n+} \mathcal{D}^2(X))^g} \right)$$

where the constant C_K is as defined at (3.7). The main problem with (6.1) (related to the noncompactness of $M(c)$, which permits the fixed point set $M(c)^T$ to be the union of infinitely many components F_δ) is that the sum over δ does not converge for $X \in \mathfrak{t}$. In this section we shall instead apply the version of nonabelian localization due to Guillemin-Kalkman and Martin (Propositions 3.6 and 3.9) to $M(c)$, using Remarks 3.8 and 3.10; this will lead to a proof that (6.1) is true if interpreted appropriately (see Remark 8.6). First we use Proposition 3.6.

LEMMA 6.1. *Let $|W| = n!$ be the order of the Weyl group W of $K = \text{SU}(n)$, and let $c = \text{diag}(e^{2\pi id/n}, \dots, e^{2\pi id/n})$ where d is coprime to n . If V is a sufficiently small neighbourhood of c in K so that the quotient T/Z_n of T by the centre Z_n of $K = \text{SU}(n)$ acts freely on $P^{-1}(V) \cap \mu^{-1}(0)$ (see Lemma 5.10), then for any $\eta \in H_K^*(X)$,*

$$\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\bar{\omega}}) = \frac{1}{|W|} \int_{N(c)} \Phi(\mathcal{D}\eta e^{\bar{\omega}}) = \frac{1}{|W|} \int_{N(V)} \Phi(\mathcal{D}\eta e^{\bar{\omega}} \alpha)$$

where

$$N(c) = M(c) \cap \mu^{-1}(0)/T$$

for $\mu: K^{2g} \times \mathbf{k} \rightarrow \mathbf{k}$ given by minus the projection onto \mathbf{k} and

$$N(V) = P^{-1}(V) \cap \mu^{-1}(0)/T.$$

Also α is a T -equivariantly closed form on $K^{2g} \times \mathbf{k}$ representing the T -equivariant Poincaré dual to $M(c)$, which is chosen as in Proposition 5.7 so that the support of α is contained in $P^{-1}(V)$ and has compact intersection with $\mu^{-1}(0)$.

Proof. Since $\mathcal{M}(n, d) = M(c) \cap \mu^{-1}(0)/K$, we can first identify $\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\bar{\omega}})$ with

$$\frac{1}{|W|} \int_{N(c)} \Phi(\mathcal{D}\eta e^{\bar{\omega}})$$

via Proposition 3.6, whose proof works in this situation even though $M(c)$ is noncompact and singular, because μ is proper and $M(c)$ is nonsingular in a neighbourhood of $\mu^{-1}(0)$ (see Remark 3.8). Then we use Remark 5.11. \square

Next we need to summarize some conventions on the roots and weights of $SU(n)$. The simple roots $\{e_j: j = 1, \dots, n - 1\}$ of $SU(n)$ are elements of \mathfrak{t}^* ; in terms of the standard identification of \mathfrak{t} with $\{(X_1, \dots, X_n) \in \mathbb{R}^n: \sum_i X_i = 0\}$ under which $(X_1, \dots, X_n) \in \mathbb{R}^n$ satisfying $\sum_i X_i = 0$ corresponds to $X = \text{diag}(2\pi i X_1, \dots, 2\pi i X_n) \in \mathfrak{t}$, they are given by

$$(6.2) \quad e_j(X) = X_j - X_{j+1}.$$

The dual basis to the basis of simple roots (with respect to the inner product $\langle \cdot, \cdot \rangle$ defined at (2.2) above, which is the usual Euclidean inner product on \mathbb{R}^n) is the set of *fundamental weights* $w_j \in \mathfrak{t}^*$ given by

$$(6.3) \quad w_j(X) = X_1 + \dots + X_j.$$

If we use this same inner product to identify \mathfrak{t}^* with \mathfrak{t} , the simple roots become identified with a set of generators

$$\hat{e}_j = (0, \dots, 0, 1, -1, 0, \dots, 0)$$

for the integer lattice Λ^I of \mathfrak{t} , and the fundamental weights correspond to elements $\hat{w}_j \in \mathfrak{t}$ given by

$$\hat{w}_j = (1, \dots, 1, 0, \dots, 0) - \frac{j}{n}(1, \dots, 1).$$

In particular we have

$$(6.4) \quad \hat{w}_{n-1} = \frac{1}{n}(1, \dots, 1, -(n - 1)).$$

Since we shall later apply induction on n , it will be convenient to label certain spaces, groups and Lie algebras by the associated value of n . In particular the space $M(c)$ will sometimes be denoted by $M_n(c)$, the maximal torus T of $SU(n)$ by T_n , its Lie algebra \mathfrak{t} by \mathfrak{t}_n , and the map Φ by Φ_n .

We define a one-dimensional torus $\hat{T}_1 \cong S^1$ in $SU(n)$ generated by \hat{e}_1 : it is identified with S^1 via

$$(6.5) \quad t \in S^1 \mapsto (t, t^{-1}, 1, \dots, 1) \in \hat{T}_1.$$

The (one-dimensional) Lie algebra $\hat{\mathfrak{t}}_1$ is spanned by \hat{e}_1 . Its orthocomplement in \mathfrak{t} is

$$(6.6) \quad \mathfrak{t}_{n-1} = \left\{ (X_1, \dots, X_n) \in \mathbb{R}^n: X_1 = X_2, \sum_{j=1}^n X_j = 0 \right\}.$$

Define T_{n-1} to be the torus given by $\exp(\mathfrak{t}_{n-1})$:

$$T_{n-1} = \left\{ (t_1, t_1, t_3, \dots, t_{n-1}, t_n) \in U(1)^n : (t_1)^2 \left(\prod_{j=3}^n t_j \right) = 1 \right\};$$

then T_{n-1} is isomorphic to the maximal torus of $SU(n-1)$ (i.e. $T_{n-1} \cong (S^1)^{n-2}$) so this does not conflict with the notation already adopted.

Remark 6.2. The multiplication map $\hat{T}_1 \times T_{n-1} \rightarrow \hat{T}_1 T_{n-1} = T_n$ is a covering map with fibre $\hat{T}_1 \cap T_{n-1} = \mathbb{Z}_2 = \{(t, t^{-1}, 1, \dots, 1) : t = t^{-1}\}$.

There is the following decomposition of the ring homomorphism Φ_n .

PROPOSITION 6.3. *For any symplectic manifold M equipped with a Hamiltonian action of T_n such that T_{n-1} acts locally freely on $\mu_{T_{n-1}}^{-1}(0)$, the symplectic quotient $\mu_{T_n}^{-1}(0)/T_n$ may be identified with the symplectic quotient of $\mu_{T_{n-1}}^{-1}(0)/T_{n-1}$ by the induced Hamiltonian action of \hat{T}_1 . Moreover if in addition T_n acts locally freely on $\mu_{T_n}^{-1}(0)$ then the ring homomorphism $\Phi_n: H_{T_n}^*(M) \rightarrow H^*(\mu_{T_n}^{-1}(0)/T_n)$ factors as*

$$\Phi_n = \hat{\Phi}_1 \circ \Phi_{n-1}$$

where

$$\begin{aligned} \Phi_{n-1}: H_{T_n}^*(M) &\rightarrow H_{T_n}^*(\mu_{T_{n-1}}^{-1}(0)) \cong H_{\hat{T}_1 \times T_{n-1}}^*(\mu_{T_{n-1}}^{-1}(0)) \\ &\cong H_{\hat{T}_1}^*(\mu_{T_{n-1}}^{-1}(0)/T_{n-1}) \end{aligned}$$

and

$$\begin{aligned} \hat{\Phi}_1: H_{\hat{T}_1}^*(\mu_{T_{n-1}}^{-1}(0)/T_{n-1}) &\rightarrow H^*((\mu_{T_{n-1}}^{-1}(0) \cap \mu_{\hat{T}_1}^{-1}(0))/T_{n-1} \times \hat{T}_1) \\ &\cong H^*(\mu_{T_n}^{-1}(0)/T_n). \end{aligned}$$

Proof. The isomorphisms

$$H_{T_n}^*(\mu_{T_{n-1}}^{-1}(0)) \cong H_{\hat{T}_1 \times T_{n-1}}^*(\mu_{T_{n-1}}^{-1}(0)) \cong H_{\hat{T}_1}^*(\mu_{T_{n-1}}^{-1}(0)/T_{n-1})$$

follow from Remark 6.2 and the fact that the cohomology with complex coefficients of the classifying space of a finite group is trivial.

Since μ_{T_n} is a T_n -invariant map, its projection $\mu_{\hat{T}_1}$ onto $\hat{\mathfrak{t}}_1$ descends to $\mu_{T_{n-1}}^{-1}(0)/T_{n-1}$ and defines a moment map for the induced \hat{T}_1 -action with respect to the induced symplectic structure on $\mu_{T_{n-1}}^{-1}(0)/T_{n-1}$. The rest then follows from Remark 6.2 and naturality (cf. [23], after (2.9)). \square

Remark 6.4. From now on, thanks to Lemma 6.1, we shall be working with quotients by T and subgroups of T , rather than quotients by K . Because of this our arguments will apply to $M(c)$ when c belongs to T but is no longer

necessarily a central element of K . This will be important later, when we apply induction on n using Proposition 6.3. The only condition we will need to impose on $c \in T$ is that $c = \text{diag}(c_1, \dots, c_n)$ where the product of no proper subsequence of (c_1, \dots, c_n) is 1; this is certainly true for our original choice of c when $c_j = e^{2\pi id/n}$ for all j with d coprime to n .

So for any $c \in T$, let us define

$$\begin{aligned} M(c) &= M_n(c) = P^{-1}(c) \\ &= \left\{ (h_1, \dots, h_{2g}, \Lambda) \in K^{2g} \times \mathfrak{k} : \prod_{j=1}^g h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} = c \exp(\Lambda) \right\} \end{aligned}$$

where $P: K^{2g} \times \mathfrak{k} \rightarrow K$ is defined by

$$P(h_1, \dots, h_{2g}, \Lambda) = \prod_{j=1}^g h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} \exp(-\Lambda).$$

Let us also define

$$(6.7) \quad N_n(c) = M(c) \cap \mu^{-1}(0)/T_n$$

and

$$(6.8) \quad N_n(V) = P^{-1}(V) \cap \mu^{-1}(0)/T_n$$

where V is a small T -invariant neighbourhood of c in K .

PROPOSITION 6.5. *Suppose $c = \text{diag}(c_1, \dots, c_n) \in T$ is such that the product of no proper subset of (c_1, \dots, c_n) is 1. Then the group T_{n-1}/\mathbb{Z}_n , where \mathbb{Z}_n consists of the identity matrix multiplied by n^{th} roots of unity, acts freely on $P^{-1}(V) \cap \mu_{T_{n-1}}^{-1}(0)$ for any sufficiently small T -invariant neighbourhood V of c in K . Hence the quotient $P^{-1}(V) \cap \mu_{T_{n-1}}^{-1}(0)/T_{n-1}$ is smooth.*

Proof. The conjugation action of $(t_1, \dots, t_n) \in U(1)^n$ on the space of $n \times n$ matrices sends

$$(A_{ij}) \mapsto (t_i t_j^{-1} A_{ij}).$$

Clearly \mathbb{Z}_n acts trivially. Let us assume that $(h, \Lambda) \in M(c) \cap \mu_{T_{n-1}}^{-1}(0)$ is fixed by the action of some element of T_{n-1} which is not in \mathbb{Z}_n . After rearranging the coordinates X_3, \dots, X_n if necessary, we may assume that there is some k between 3 and n such that this element of T_{n-1} is of the form $(t_1, t_1, t_3, \dots, t_{n-1}, t_n)$ where $t_i = t_1$ if and only if $i \leq k$. Then each h_j is block diagonal of the form

$$\begin{bmatrix} h_j^1 & 0 \\ 0 & h_j^2 \end{bmatrix}$$

where h_j^1 is a $k \times k$ matrix and h_j^2 is $(n - k) \times (n - k)$. As the determinant of any commutator is one, it follows that $\prod_{j=1}^n [h_{2j-1}, h_{2j}]$ is block diagonal of the form

$$(6.9) \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

where $\det A = \det B = 1$. But Λ is also block diagonal of the same form

$$\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix},$$

and since $(h, \Lambda) \in \mu_{T_{n-1}}^{-1}(0)$ the diagonal entries of Λ are $(2\pi i \lambda, -2\pi i \lambda, 0, \dots, 0)$ for some $\lambda \in \mathbb{R}$. Thus as $k \geq 3$ both Λ_1 and Λ_2 have trace 0, so $\det \exp \Lambda_1 = 1 = \det \exp \Lambda_2$. Since $(h, \Lambda) \in M(c)$ it follows that the matrix A must equal

$$\text{diag}(c_1, \dots, c_k) \exp \Lambda_1,$$

and hence

$$c_1 \dots c_k = \det A = 1.$$

This contradiction to the hypotheses on c shows that T_{n-1}/\mathbb{Z}_n acts freely on $M(c) \cap \mu_{T_{n-1}}^{-1}(0)$, and the same argument shows that T_{n-1}/\mathbb{Z}_n acts freely on $P^{-1}(V) \cap \mu_{T_{n-1}}^{-1}(\hat{V})$ for any sufficiently small T -invariant neighbourhood V of c in K and any sufficiently small neighbourhood \hat{V} of 0 in \mathfrak{t}_{n-1} . The result follows. □

Definition 6.6. There are coordinates

$$Y_k = e_k(X) = \langle \hat{e}_k, X \rangle$$

on \mathfrak{t} , corresponding to the simple roots $e_k \in \mathfrak{t}^*$.

We are now in a position to exploit Proposition 3.9 and Remark 3.10, by using the translation map s_{Λ_0} defined by Lemma 4.5, where $\Lambda_0 = \hat{e}_1$ lies in the integer lattice Λ^I and so satisfies $\exp(\Lambda_0) = 1$.

LEMMA 6.7. *Suppose $c = \text{diag}(c_1, \dots, c_n) \in T$ is such that the product of no proper subset of (c_1, \dots, c_n) is 1. Suppose also that η is a polynomial in the $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$, so that $s_{\hat{e}_1}^* \eta = \eta$. If V is a sufficiently small T -invariant neighbourhood of c in K so that $P^{-1}(V) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)/T_{n-1}$ is smooth (see Proposition 6.5), and if $N_n(V) = P^{-1}(V) \cap \mu^{-1}(0)/T_n$ as before, then*

$$\begin{aligned} \int_{N_n(V)} \Phi_n(\eta e^{\bar{\omega}} e^{-Y_1} \alpha) &= \int_{P^{-1}(V) \cap \mu^{-1}(-\hat{e}_1)/T_n} \Phi_n(\eta e^{\bar{\omega}} \alpha) \\ &= \int_{N_n(V)} \Phi_n(\eta e^{\bar{\omega}} \alpha) - n_0 \\ &\quad \times \sum_{F \in \mathcal{F}: -\|\hat{e}_1\|^2 < \langle \hat{e}_1, \mu(F) \rangle < 0} \text{Res}_{Y_1=0} \int_F \frac{\Phi_{n-1}(\eta e^{\bar{\omega}} \alpha)}{e_F} \end{aligned}$$

where \mathcal{F} is the set of components of the fixed point set of the action of \hat{T}_1 on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$, and e_F denotes the \hat{T}_1 -equivariant Euler class of the normal to F in $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ for any $F \in \mathcal{F}$, while n_0 is the order of the subgroup of $\hat{T}_1/\hat{T}_1 \cap T_{n-1}$ that acts trivially on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$. Also α is the T -equivariantly closed differential form on $K^{2g} \times \mathbf{k}$ given by Proposition 5.7 which represents the equivariant Poincaré dual of $M(c)$, chosen so that the support of α is contained in $P^{-1}(V)$.

Proof. Since $\mu^{-1}(\hat{\mathbf{t}}_1) = K^{2g} \times \hat{\mathbf{t}}_1$ is contained in $\mu_{T_{n-1}}^{-1}(0)$, it follows from Proposition 6.5 that if V is a sufficiently small T -invariant neighbourhood of c in K , then T_{n-1}/\mathbb{Z}_n acts freely on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)$ and so the quotient $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ is smooth.

Since the restriction of $\mu_{\hat{T}_1}$ to $\mu^{-1}(\hat{\mathbf{t}}_1)$ is proper, and the support of α is contained in $P^{-1}(V)$, by Remark 3.10, Guillemin and Kalkman’s proof of Proposition 3.9 can be applied to the \hat{T}_1 -invariant function induced by $\mu_{\hat{T}_1}$ on the smooth manifold $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ and the \hat{T}_1 -equivariant form induced by $\eta e^{\bar{\omega}} \alpha$. In fact since $\hat{T}_1 \cap T_{n-1} \cong \mathbb{Z}_2$ acts trivially we can work with the action of $\hat{T}_1/\hat{T}_1 \cap T_{n-1}$ instead of the action of \hat{T}_1 (the Lie algebra and moment map are of course the same). This fits better with the choice of coordinates Y_k defined by the simple roots \hat{e}_k because the simple root \hat{e}_1 takes $(t, t^{-1}, 1, \dots, 1) \in \hat{T}_1$ to t^2 and thus induces an isomorphism from $\hat{T}_1/\hat{T}_1 \cap T_{n-1}$ to S^1 . By combining this with Proposition 6.3 we get

$$\begin{aligned} & \int_{P^{-1}(V) \cap \mu^{-1}(0)/T_n} \Phi_n(\eta e^{\bar{\omega}} \alpha) - \int_{P^{-1}(V) \cap \mu^{-1}(-\hat{e}_1)/T_n} \Phi_n(\eta e^{\bar{\omega}} \alpha) \\ &= n_0 \text{Res}_{Y_1=0} \sum_{F \in \mathcal{F}: -\|\hat{e}_1\|^2 < \langle \hat{e}_1, \mu(F) \rangle < 0} \int_F \frac{\Phi_{n-1}(\eta e^{\bar{\omega}} \alpha)}{e_F}. \end{aligned}$$

Now note that the restriction of $P: K^{2g} \times \mathbf{k} \rightarrow K$ to $\mu^{-1}(\mathbf{t}) = K^{2g} \times \mathbf{t}$ is invariant under the translation s_{Λ_0} for $\Lambda_0 \in \Lambda^I$. Therefore by construction the restriction of α to $\mu^{-1}(\mathbf{t})$ is also invariant under this translation. Thus by (4.9) and Definition 6.6

$$\int_{P^{-1}(V) \cap \mu^{-1}(-\hat{e}_1)/T_n} \Phi_n(\eta e^{\bar{\omega}} \alpha) = \int_{N_n(V)} \Phi_n(s_{\hat{e}_1}^*(\eta e^{\bar{\omega}} \alpha)) = \int_{N_n(V)} \Phi_n(\eta e^{\bar{\omega}} e^{-Y_1} \alpha).$$

The result follows. □

Remark 6.8. It will follow from the proof of Proposition 7.1 below that $n_0 = 1$ here (see Remark 7.2).

7. Fixed point sets of the circle action

In this section we shall consider the components $F \in \mathcal{F}$ of the fixed point set of the action of \hat{T}_1 on the quotient $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ (which appeared in Lemma 6.7). Since $P^{-1}(c) = M(c)$ and V is an arbitrarily small T -invariant neighbourhood of c in K , we may assume that every $F \in \mathcal{F}$ contains a component of the fixed point set of the action of \hat{T}_1 on $M(c) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$, and each of these components is contained in a unique $F \in \mathcal{F}$. So we shall start by analysing the components of the fixed point set of the action of \hat{T}_1 on $M(c) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$. We shall find that they can be described inductively in terms of products of spaces of the form $N(c)$ (see Remark 6.4) for smaller values of n . This will enable us to use induction in the next two sections to express the intersection pairings $\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\bar{\omega}})$ on the moduli spaces $\mathcal{M}(n, d)$ as iterated residues (see Theorem 8.1 and Theorem 9.12).

PROPOSITION 7.1. *Suppose that $c = \text{diag}(c_1, \dots, c_n) \in \text{SU}(n)$ is such that the product of no proper subsequence of (c_1, \dots, c_n) is 1. Then the components of the fixed point set of the action of \hat{T}_1 on the quotient $(M(c) \cap \mu^{-1}(\hat{\mathbf{t}}_1))/T_{n-1}$ may be described as follows. For any subset I of $\{3, \dots, n\}$ let $I_1 = I \cup \{1\}$ and let $I_2 = \{1, \dots, n\} - I_1$. Let H_I be the subgroup of $\text{SU}(n)$ given by*

$$H_I = \{(a_{ij}) \in \text{SU}(n): a_{ij} = 0 \text{ if } (i,j) \in (I_1 \times I_2) \cup (I_2 \times I_1)\}.$$

Suppose that $\lambda \in \mathbb{R}$ is a solution of

$$e^{-2\pi i \lambda} = c_{i_1} \dots c_{i_r} = \prod_{j \in I_1} c_j$$

where r is the number of elements of $I_1 = \{i_1, \dots, i_r\}$, so that

$$e^{2\pi i \lambda} = \prod_{j \in I_2} c_j.$$

Then there is a component of the fixed point set given by $F_{I,\lambda} = \tilde{F}_{I,\lambda}/T_{n-1}$ where

$$\tilde{F}_{I,\lambda} = M(c) \cap (H_I^{2g} \times \{\lambda \hat{e}_1\}),$$

and every component is of this form for some subset I of $\{3, \dots, n\}$ and solution λ to the equation above.

Proof. Suppose the \hat{T}_1 orbit of a point $(h_1, \dots, h_{2g}, \Lambda) \in \text{SU}(n)^{2g} \times \hat{\mathbf{t}}_1$ is contained in its orbit under T_{n-1} . A general element of the \hat{T}_1 orbit of an $n \times n$

matrix $A = (a_{ij})$ under conjugation looks like

$$\begin{bmatrix} a_{11} & t^2 a_{12} & t a_{13} & \dots & t a_{1n} \\ t^{-2} a_{21} & a_{22} & t^{-1} a_{23} & \dots & t^{-1} a_{2n} \\ t^{-1} a_{31} & t a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ t^{-1} a_{n1} & t a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

while a general element of the T_{n-1} orbit of A looks like

$$\begin{bmatrix} a_{11} & a_{12} & t_1 t_3^{-1} a_{13} & \dots & t_1 t_n^{-1} a_{1n} \\ a_{21} & a_{22} & t_1 t_3^{-1} a_{23} & \dots & t_1 t_n^{-1} a_{2n} \\ t_3 t_1^{-1} a_{31} & t_3 t_1^{-1} a_{32} & a_{33} & \dots & t_3 t_n^{-1} a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ t_n t_1^{-1} a_{n1} & t_n t_1^{-1} a_{n2} & t_n t_3^{-1} a_{n3} & \dots & a_{nn} \end{bmatrix}$$

For each t there exist t_1, t_3, \dots, t_n such that these two matrices are equal when A is any of h_1, \dots, h_{2g} and Λ . Choose $t \neq t^{-1}$ and let I denote the set of j in $\{3, \dots, n\}$ for which $t_1 t_j^{-1} = t$. Similarly, define J to be the set of j in $\{3, \dots, n\}$ for which $t_1 t_j^{-1} = t^{-1}$, and let $K = \{3, \dots, n\} - I - J$. Reordering the coordinates one finds that all the h_j and Λ are block diagonal where the blocks correspond to $I \cup \{1\}$, $J \cup \{2\}$ and K . Conversely, if all the h_j are block diagonal of this form and $\Lambda \in \hat{\mathfrak{t}}_1$, then the \hat{T}_1 orbit of $(h_1, \dots, h_{2g}, \Lambda)$ is contained in its T_{n-1} orbit since given any $t \in U(1)$ we can find $(t_1, t_1, t_3, \dots, t_n)$ in T_{n-1} satisfying $t_1 t_j^{-1} = t$ if $j \in I$ and $t_1 t_j^{-1} = t^{-1}$ if $j \in J$.

We next prove that K is empty. Suppose otherwise; then as the determinant of any commutator is one, $\det \prod_{j=1}^g [h_{2j-1}^{[K]}, h_{2j}^{[K]}] = 1$ (where the superscript $[K]$ denotes the block of the matrix corresponding to K). Thus the K block in c also has determinant 1. This is impossible by the hypothesis on c .

Suppose now that $(h_1, \dots, h_{2g}, \Lambda) \in M(c) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)$ lies in $H_I^{2g} \times \hat{\mathfrak{t}}_1$. Then

$$\Lambda = \lambda \hat{e}_1 = 2\pi i \text{diag}(\lambda, -\lambda, 0, \dots, 0)$$

for some $\lambda \in \mathbb{R}$, so the blocks $\Lambda^{[I_1]}$ and $\Lambda^{[I_2]}$ of Λ corresponding to $I_1 = I \cup \{1\}$ and $I_2 = J \cup \{2\}$ satisfy $\det \exp \Lambda^{[I_1]} = e^{2\pi i \lambda}$ and $\det \exp \Lambda^{[I_2]} = e^{-2\pi i \lambda}$. But

$$\det \left(\prod_{j=1}^g [h_{2j-1}^{[I_1]}, h_{2j}^{[I_1]}] \right) = 1 = \det \left(\prod_{j=1}^g [h_{2j-1}^{[I_2]}, h_{2j}^{[I_2]}] \right)$$

because the determinant of any commutator is one. It therefore follows from the definition of $M(c)$ that

$$e^{-2\pi i \lambda} = \prod_{j \in I_1} c_j.$$

This is enough to complete the proof. □

Remark 7.2. The proof of this proposition shows that the elements of \hat{T}_1 which act trivially on the quotient $(M(c) \cap \mu^{-1}(\hat{t}_1))/T_{n-1}$ are precisely those represented by t satisfying $t = t^{-1}$; i.e., $t = \pm 1$, or equivalently those in $\hat{T}_1 \cap T_{n-1}$. Thus the size n_0 of the subgroup of $\hat{T}_1/\hat{T}_1 \cap T_{n-1}$ acting trivially on the quotient $M(c) \cap \mu^{-1}(\hat{t}_1)/T_{n-1}$ is 1 (cf. Lemma 6.7).

PROPOSITION 7.3. *Suppose that $c = \text{diag}(c_1, \dots, c_n) \in \text{SU}(n)$ is such that the product of no proper subsequence of (c_1, \dots, c_n) is 1. Suppose that I is a subset of $\{3, \dots, n\}$ with $r - 1$ elements where $1 \leq r \leq n - 1$, and let $I_1 = I \cup \{1\} = \{i_1, \dots, i_r\}$ and $I_2 = \{1, \dots, n\} - I_1 = \{i_{r+1}, \dots, i_n\}$. Suppose also that $\lambda \in \mathbb{R}$ is a solution of*

$$e^{-2\pi i \lambda} = \prod_{j \in I_1} c_j$$

so that $e^{2\pi i \lambda} = \prod_{j \in I_2} c_j$. Let

$$c(I_1, \lambda) = \text{diag}(c_{i_1}^{I, \lambda}, \dots, c_{i_r}^{I, \lambda})$$

and

$$c(I_2, -\lambda) = \text{diag}(c_{i_{r+1}}^{I, \lambda}, \dots, c_{i_n}^{I, \lambda})$$

where $c_j^{I, \lambda} = c_j$ if $j \geq 3$, while $c_1^{I, \lambda} = c_1 e^{2\pi i \lambda}$ and $c_2^{I, \lambda} = c_2 e^{-2\pi i \lambda}$. Let $F_{I, \lambda}$ be defined as in Proposition 7.1. Then there is a finite to one (in fact $(r(n-r))^{2g}$ to one) surjective smooth map

$$\Psi_{I, \lambda}: (S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda)) \rightarrow F_{I, \lambda}.$$

Proof. We define a homomorphism

$$\rho_I: S^1 \times \text{SU}(r) \times \text{SU}(n-r) \rightarrow H_I \subset \text{SU}(n)$$

given by

$$\rho_I: (s, A, B) \mapsto \begin{bmatrix} s^{n-r} A & 0 \\ 0 & s^{-r} B \end{bmatrix}$$

with respect to the decomposition of $\{1, \dots, n\}$ as $I_1 \cup I_2$. Note that ρ_I restricts to an $r(n-r)$ -to-one surjective homomorphism

$$\rho_I: S^1 \times T_r \times T_{n-r} \rightarrow T_n.$$

If $\varepsilon_r: \text{SU}(r)^{2g} \rightarrow \text{SU}(r)$ is defined by $\varepsilon_r(h_1, \dots, h_{2g}) = \prod_{j=1}^g [h_{2j-1}, h_{2j}]$ then

$$(7.1) \quad \begin{aligned} & \varepsilon_n(\rho_I(s_1, A_1, B_1), \dots, \rho_I(s_{2g}, A_{2g}, B_{2g})) \\ &= \begin{bmatrix} \varepsilon_r(A_1, \dots, A_{2g}) & 0 \\ 0 & \varepsilon_{n-r}(B_1, \dots, B_{2g}) \end{bmatrix}. \end{aligned}$$

Let us define a map

$$\Psi_{I,\lambda}: (S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda)) \rightarrow F_{I,\lambda}$$

as the quotient of

$$\begin{aligned} \tilde{\Psi}_{I,\lambda}: (S^1)^{2g} \times & \left(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \lambda)) \right) \\ & \times \left(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\lambda)) \right) \rightarrow \tilde{F}_{I,\lambda} \end{aligned}$$

defined by

$$\begin{aligned} \tilde{\Psi}_{I,\lambda} & \left((s_1, \dots, s_{2g}), (h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}, 0), (h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}, 0) \right) \\ & = (\rho_I(s_1, h_1^{[I_1]}, h_1^{[I_2]}), \rho_I(s_2, h_2^{[I_1]}, h_2^{[I_2]}), \dots, \rho_I(s_{2g}, h_{2g}^{[I_1]}, h_{2g}^{[I_2]}), \\ & \quad 2\pi i \text{diag}(\lambda, -\lambda, 0, \dots, 0)). \end{aligned}$$

Here, $\tilde{F}_{I,\lambda}$ is as defined in Proposition 7.1.

We must check that the image of $\tilde{\Psi}_{I,\lambda}$ is contained in $\tilde{F}_{I,\lambda}$. We have

$$\varepsilon_r(h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}) = \text{diag}(c_{i_1}^{I,\lambda}, \dots, c_{i_r}^{I,\lambda}) = c(I_1, \lambda)$$

and

$$\varepsilon_{n-r}(h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}) = \text{diag}(c_{i_{r+1}}^{I,\lambda}, \dots, c_{i_n}^{I,\lambda}) = c(I_2, -\lambda).$$

In order to show that $\tilde{\Psi}_{I,\lambda} \left((s_1, \dots, s_{2g}), (h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}, 0), (h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}, 0) \right)$ lies in $\tilde{F}_{I,\lambda}$ we need to check that if $\Lambda = 2\pi i \text{diag}(\lambda, -\lambda, 0, \dots, 0)$ then $c \exp(\Lambda)$ is block diagonal of the form

$$\begin{bmatrix} c(I_1, \lambda) & 0 \\ 0 & c(I_2, -\lambda) \end{bmatrix}$$

with respect to the decomposition of $\{1, \dots, n\}$ as $I_1 \cup I_2$. This follows by the choice of $c_1^{I,\lambda}, \dots, c_n^{I,\lambda}$.

We must also check that the map $\Psi_{I,\lambda}$ is well defined on the quotient by the action of $T_r \times T_{n-r}$; in other words we must check that for any $t = (t_1, \dots, t_n) \in U(1)^n$ satisfying $t_{i_1}, \dots, t_{i_r} = t_{i_{r+1}}, \dots, t_{i_n} = 1$ so that $t_{I_1} = (t_{i_1} \dots t_{i_r}) \in T_r$ and $t_{I_2} = (t_{i_{r+1}} \dots t_{i_n}) \in T_{n-r}$, we have

$$\begin{aligned} (7.2) \quad \tilde{\Psi}_{I,\lambda} & \left((s_1, \dots, s_{2g}), (t_{I_1} h_1^{[I_1]} (t_{I_1})^{-1}, \dots, t_{I_1} h_{2g}^{[I_1]} (t_{I_1})^{-1}, 0), \right. \\ & \left. ((t_{I_2} h_1^{[I_2]} (t_{I_2})^{-1}, \dots, t_{I_2} h_{2g}^{[I_2]} (t_{I_2})^{-1}, 0) \right) \\ & = \tilde{t} \tilde{\Psi}_{I,\lambda} \left((s_1, \dots, s_{2g}), (h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}, 0), (h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}, 0) \right) \end{aligned}$$

for some $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_n)$ satisfying $\prod_{j=1}^n \tilde{t}_j = 1$ and $\tilde{t}_1 = \tilde{t}_2$. For any $s \in U(1)$, we may conjugate all the $h_j^{[I_1]}$ by s^{n-r} and all the $h_j^{[I_2]}$ by s^{-r} without changing the image under $\tilde{\Psi}_{I,\lambda}$; choosing s so that $s^{n-r} \tilde{t}_1 = s^{-r} \tilde{t}_2$ we find that the

equation (7.2) is satisfied for $\tilde{t}_j = t_j s^{n-r}$ (when $j \in I_1$) and $\tilde{t}_j = t_j s^{-r}$ (when $j \in I_2$).

To show that $\Psi_{I,\lambda}$ is finite-to-one and surjective, suppose that $(h_1, \dots, h_{2g}, \Lambda) \in \tilde{F}_{I,\lambda}$; we must check that a finite (and nonzero) number of $T_r \times T_{n-r}$ orbits in

$$(S^1)^{2g} \times \left(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \lambda)) \right) \times \left(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\lambda)) \right)$$

map into the T_{n-1} orbit of $(h_1, \dots, h_{2g}, \Lambda)$. Now by the definition of $\tilde{F}_{I,\lambda}$ we have $\Lambda = 2\pi i \text{diag}(\lambda, -\lambda, 0, \dots, 0)$ and each h_j is block diagonal of the form

$$\begin{bmatrix} h_j^{[I_1]} & 0 \\ 0 & h_j^{[I_2]} \end{bmatrix}$$

with respect to the decomposition of $\{1, \dots, n\}$ as $I_1 \cup I_2$. So

$$\tilde{\Psi}_{I,\lambda} \left((s_1, \dots, s_{2g}), (H_1^{[I_1]}, \dots, H_{2g}^{[I_1]}, 0), (H_1^{[I_2]}, \dots, H_{2g}^{[I_2]}, 0) \right)$$

belongs to the T_{n-1} orbit of $(h_1, \dots, h_{2g}, \Lambda)$ if and only if there is some $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_n)$ satisfying $\prod_{j=1}^n \tilde{t}_j = 1$ and $\tilde{t}_1 = \tilde{t}_2$ such that

$$s_j^{n-r} H_j^{[I_1]} = \tilde{t}_{I_1} h_j^{[I_1]} (\tilde{t}_{I_1})^{-1}$$

and

$$s_j^{-r} H_j^{[I_2]} = \tilde{t}_{I_2} h_j^{[I_2]} (\tilde{t}_{I_2})^{-1}$$

where $\tilde{t}_{I_1} = (\tilde{t}_{i_1}, \dots, \tilde{t}_{i_r}) \in T_r$ and $\tilde{t}_{I_2} = (\tilde{t}_{i_{r+1}}, \dots, \tilde{t}_{i_n}) \in T_{n-r}$. Since $\det H_j^{[I_1]} = 1 = \det H_j^{[I_2]}$ and $\det h_j^{[I_1]} \det h_j^{[I_2]} = 1$, by the argument of the previous paragraph this happens if and only if $(s_j)^{r(n-r)} = \det h_j^{[I_1]}$ and $s_j^{n-r} H_j^{[I_1]}$ is conjugate to $h_j^{[I_1]}$ and $s_j^{-r} H_j^{[I_2]}$ is conjugate to $h_j^{[I_2]}$. Thus $\Psi_{I,\lambda}$ is surjective and $(r(n-r))^{2g}$ -to-one. \square

Remark 7.4. Note that by the definition of $c_j^{I,\lambda}$ (see Proposition 7.3) no proper subsequence of $(c_{i_1}^{I,\lambda}, \dots, c_{i_r}^{I,\lambda})$ or $(c_{i_{r+1}}^{I,\lambda}, \dots, c_{i_n}^{I,\lambda})$ has product equal to 1, because the same is true of (c_1, \dots, c_n) .

Remark 7.5. It follows from Proposition 7.3 that if $\Psi_{I,\lambda}$ is orientation-preserving (and we shall see below in Remark 7.13 that $\Psi_{I,\lambda}$ takes a natural symplectic orientation on $(S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda))$ to the symplectic orientation induced by ω on $F_{I,\lambda}$) then

$$(7.3) \quad \int_{F_{I,\lambda}} \Phi_{n-1}(\eta e^{\bar{\omega}}) = (r(n-r))^{-2g} \int_{(S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda))} \Psi_{I,\lambda}^* \Phi_{n-1}(\eta e^{\bar{\omega}})$$

where both sides are elements of $H_{\tilde{I}_1}^*$. To be more precise we should replace $\eta e^{\bar{\omega}}$ on each side of this equation by its restriction to $\tilde{F}_{I,\lambda}$, and as in Proposition 6.3

and Lemma 6.7 we use the double cover $\hat{T}_1 \times T_{n-1} \rightarrow T_n$ to define

$$\Phi_{n-1}: H_{T_n}^*(\tilde{F}_{I,\lambda}) \rightarrow H_{\hat{T}_1}^*(F_{I,\lambda}) \cong H_{\hat{T}_1}^* \otimes H^*(F_{I,\lambda}).$$

Recall from the proof of the last proposition that the homomorphism

$$\rho_I: S^1 \times \text{SU}(r) \times \text{SU}(n-r) \rightarrow H_I \subset \text{SU}(n)$$

given by

$$\rho_I: (s, A, B) \mapsto \begin{bmatrix} s^{n-r}A & 0 \\ 0 & s^{-r}B \end{bmatrix}$$

with respect to the decomposition of $\{1, \dots, n\}$ as $I_1 \cup I_2$ restricts to an $r(n-r)$ -to-one surjective homomorphism

$$\rho_I: S^1 \times T_r \times T_{n-r} \rightarrow T_n.$$

It is easy to check that the inclusions of \hat{T}_1 , T_r and T_{n-r} in T_n induce an isomorphism

$$\hat{\rho}_r: \hat{T}_1 \times T_r \times T_{n-r} \rightarrow T_n$$

such that ρ_I and $\hat{\rho}_r$ have the same restriction to $T_r \times T_{n-r}$. The composition of this restriction with the natural surjection from T_n to $T_n/\hat{T}_1 \cong T_{n-1}/(\hat{T}_1 \cap T_{n-1})$ gives an isomorphism

$$T_r \times T_{n-r} \rightarrow T_n/\hat{T}_1 \cong T_{n-1}/(\hat{T}_1 \cap T_{n-1}).$$

Moreover the composition of ρ_I with the inverse of $\hat{\rho}_r$ defines a finite (in fact $r(n-r)$ -to-one) cover

$$\nu_I: S^1 \times T_r \times T_{n-r} \rightarrow \hat{T}_1 \times T_r \times T_{n-r}$$

which restricts to the identity on $T_r \times T_{n-r}$ and induces a finite cover $\nu_I: S^1 \rightarrow \hat{T}_1$ and isomorphisms on Lie algebras and equivariant cohomology.

The argument in the proof of the last proposition to show that the map $\Psi_{I,\lambda}$ is well defined on the quotient by the action of $T_r \times T_{n-r}$ may be rephrased as the statement that the map

$$\begin{aligned} \tilde{\Psi}_{I,\lambda}: (S^1)^{2g} \times & \left(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \lambda)) \right) \\ & \times \left(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\lambda)) \right) \rightarrow \tilde{F}_{I,\lambda} \end{aligned}$$

defined in the proof of Proposition 7.3 satisfies

$$\begin{aligned} \tilde{\Psi}_{I,\lambda} & \left(t((s_1, \dots, s_{2g}), (h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}, 0), (h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}, 0)) \right) \\ & = \rho_I(t) \tilde{\Psi}_{I,\lambda} \left((s_1, \dots, s_{2g}), (h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}, 0), (h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}, 0) \right) \end{aligned}$$

for all $t \in S^1 \times T_r \times T_{n-r}$, where S^1 acts trivially. Thus $\tilde{\Psi}_{I,\lambda}$ and ρ_I induce $\tilde{\Psi}_{I,\lambda}^*$ from $H_{T_n}^*(\tilde{F}_{I,\lambda})$ to

$$\begin{aligned} H_{S^1}^* \otimes H^*((S^1)^{2g}) & \otimes H_{T_r}^*(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \lambda))) \\ & \otimes H_{T_{n-r}}^*(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\lambda))) \end{aligned}$$

and

$$\begin{aligned} \nu_I^* \int_{(S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda))} \Psi_{I, \lambda}^* \Phi_{n-1}(\eta e^{\bar{\omega}}) \\ = \int_{(S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda))} (1 \otimes \Phi_r \otimes \Phi_{n-r}) \tilde{\Psi}_{I, \lambda}^*(\eta e^{\bar{\omega}}), \end{aligned}$$

where both sides are elements of $H_{S^1}^*$. Hence by (7.3), if Y_1 is the coordinate on $\hat{\mathfrak{t}}_1$ given by the restriction of $Y_1 = X_1 - X_2$ on \mathfrak{t} and Y_1^I is the coordinate on the Lie algebra of S^1 obtained from Y_1 via the isomorphism on Lie algebras induced by $\nu_I: S^1 \rightarrow \hat{T}_1$, then

$$\begin{aligned} \text{Res}_{Y_1=0} \int_{F_{I, \lambda}} \Phi_{n-1}(\eta e^{\bar{\omega}}) \\ = (r(n-r))^{-2g} \text{Res}_{Y_1^I=0} \\ \int_{(S^1)^{2g} \times N_r(c(I_1, \lambda)) \times N_{n-r}(c(I_2, -\lambda))} (1 \otimes \Phi_r \otimes \Phi_{n-r}) \tilde{\Psi}_{I, \lambda}^*(\eta e^{\bar{\omega}}). \end{aligned}$$

Since S^1 acts trivially, the residue operation $\text{Res}_{Y_1^I=0}: H_{S^1}^* \rightarrow \mathbb{C}$ can be extended to map

$$\begin{aligned} H_{S^1}^* \otimes H^*((S^1)^{2g}) \otimes H_{T_r}^* \left(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \lambda)) \right) \\ \otimes H_{T_{n-r}}^* \left(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\lambda)) \right) \end{aligned}$$

to

$$\begin{aligned} H^*((S^1)^{2g}) \otimes H_{T_r}^* \left(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \lambda)) \right) \\ \otimes H_{T_{n-r}}^* \left(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\lambda)) \right) \end{aligned}$$

so that it commutes with Φ_r and Φ_{n-r} and with integration over $N_r(c(I_1, \lambda))$ and integration over $N_{n-r}(c(I_2, -\lambda))$. In particular by expressing integrals over products as iterated integrals we obtain

$$\begin{aligned} \text{Res}_{Y_1=0} \int_{F_{I, \lambda}} \Phi_{n-1}(\eta e^{\bar{\omega}}) \\ = (r(n-r))^{-2g} \int_{N_r(c(I_1, \lambda))} \Phi_r \left(\text{Res}_{Y_1^I=0} \int_{N_{n-r}(c(I_2, -\lambda))} \Phi_{n-r} \left(\int_{(S^1)^{2g}} \tilde{\Psi}_{I, \lambda}^*(\eta e^{\bar{\omega}}) \right) \right) \\ = (r(n-r))^{-2g} \int_{N_{n-r}(c(I_2, -\lambda))} \Phi_{n-r} \left(\text{Res}_{Y_1^I=0} \int_{N_r(c(I_1, \lambda))} \Phi_r \left(\int_{(S^1)^{2g}} \tilde{\Psi}_{I, \lambda}^*(\eta e^{\bar{\omega}}) \right) \right). \end{aligned}$$

This will be important when we apply induction later.

Recall from Lemma 6.7 that \mathcal{F} is the set of components of the fixed point set of the action of \hat{T}_1 on the quotient $P^{-1}(V) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)/T_{n-1}$, where V is a sufficiently small T -invariant neighbourhood of c in K . Every $F \in \mathcal{F}$ contains a component $F_{I,\lambda}$ of the fixed point set of the action of \hat{T}_1 on $M(c) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)/T_{n-1}$, and each $F_{I,\lambda}$ is contained in a unique $F \in \mathcal{F}$ (see Proposition 7.1 for the definition of $F_{I,\lambda}$). For each I and λ we now need to understand the normal bundle in $P^{-1}(V) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)/T_{n-1}$ to the component $F \in \mathcal{F}$ of the fixed point set which contains $F_{I,\lambda}$. First, we observe that there is the following decomposition:

Remark 7.6. Let I be a subset of $\{3, \dots, n\}$ with $r - 1$ elements where $1 \leq r \leq n - 1$, let $I_1 = I \cup \{1\}$ and let $I_2 = \{1, \dots, n\} - I_1$. Then

$$\mathcal{D}_n(X) = \mathcal{D}_r^{[I_1]}(X) \mathcal{D}_{n-r}^{[I_2]}(X) \tau_I(X)$$

where

$$\mathcal{D}_r^{[I_1]}(X) = \prod_{1 \leq j < k \leq r} (X_{i_j} - X_{i_k})$$

is the product of the positive roots of $SU(r)$ embedded in $SU(n)$ via the inclusion of I_1 in $\{1, \dots, n\}$,

$$\mathcal{D}_{n-r}^{[I_2]}(X) = \prod_{r+1 \leq j < k \leq n} (X_{i_j} - X_{i_k})$$

is the product of the positive roots of $SU(n - r)$ embedded in $SU(n)$ via the inclusion of I_2 in $\{1, \dots, n\}$, and

$$\tau_I(X) = \pm \prod_{1 \leq j \leq r < k \leq n} (X_{i_j} - X_{i_k}),$$

where the sign is $+$ or $-$ depending on whether the permutation

$$\begin{bmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{bmatrix}$$

is even or odd. Note also that

$$(-1)^{r(n-r)} (\tau_I(X))^2 = \prod_{(i,j) \in I_1 \times I_2 \cup I_2 \times I_1} (X_i - X_j).$$

Now we can find the \hat{T}_1 -equivariant Euler class of the normal bundle in $P^{-1}(V) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)/T_{n-1}$ to the component $F \in \mathcal{F}$ of the fixed point set which contains $F_{I,\lambda}$.

LEMMA 7.7. *Let I be a subset of $\{3, \dots, n\}$ with $r - 1$ elements where $1 \leq r \leq n - 1$, and let $\lambda \in \mathbb{R}$ be a solution of the equation*

$$e^{-2\pi i \lambda} = \prod_{j \in I_1} c_j.$$

Then the \hat{T}_1 -equivariant Euler class of the normal bundle in $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ to the component $F \in \mathcal{F}$ of the fixed point set of the action of \hat{T}_1 on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ which contains $F_{I,\lambda}$ is given by $e_F = (-1)^{r(n-r)g} \Phi_{n-1}(\tau_I^{2g})$.

Proof. The proof of Proposition 7.1 shows that the component $F \in \mathcal{F}$ of the fixed point set of the action of \hat{T}_1 on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ which contains $F_{I,\lambda}$ is

$$F = P^{-1}(V) \cap (H_I^{2g} \times \hat{\mathbf{t}}_1)/T_{n-1},$$

whereas $\mu^{-1}(\hat{\mathbf{t}}_1) = K^{2g} \times \hat{\mathbf{t}}_1$. The T -equivariant Chern roots of the normal bundle to H_I^{2g} in K^{2g} are $X_i - X_j$ for $(i, j) \in I_1 \times I_2 \cup I_2 \times I_1$ with multiplicity g . The result follows by Remark 7.6. \square

LEMMA 7.8. Let I be a subset of $\{3, \dots, n\}$ with $r - 1$ elements where $1 \leq r \leq n - 1$, let $I_1 = I \cup \{1\}$, let $I_2 = \{1, \dots, n\} - I_1$ and let $\lambda \in \mathbb{R}$ be a solution of the equation

$$e^{-2\pi i \lambda} = \prod_{j \in I_1} c_j.$$

Let F be the component of the fixed point set of the action of \hat{T}_1 on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ which contains $F_{I,\lambda}$, where $F_{I,\lambda}$ is as defined in Proposition 7.1. We then have

$$\begin{aligned} \int_F \frac{\Phi_{n-1}(\eta e^{\bar{\omega}} \alpha)}{e_F} &= (-1)^{r(n-r)(g-1)} \int_{F_{I,\lambda}} \Phi_{n-1}\left(\frac{\eta e^{\bar{\omega}}}{\tau_I^{2g-1}}\right) \\ &= (-1)^{r(n-r)(g-1)} \int_{F_{I,\lambda}} \Phi_{n-1}\left(\frac{\mathcal{D}_r^{[I_1]}(X) \mathcal{D}_{n-r}^{[I_2]}(X) \eta e^{\bar{\omega}}}{\mathcal{D}_n(X) \tau_I^{2g-2}}\right) \end{aligned}$$

where α is the T_n -equivariant differential form on $K^{2g} \times \mathbf{k}$ given by Proposition 5.7 which is supported near $M(c)$ and represents the equivariant Poincaré dual of $M(c)$ in $K^{2g} \times \mathbf{k}$.

Proof. The T_n -equivariant differential form α on $K^{2g} \times \mathbf{k}$ which represents the equivariant Poincaré dual of $M(c) = P^{-1}(c)$ in $K^{2g} \times \mathbf{k}$ was defined in Proposition 5.7 as a pullback via the map $P: K^{2g} \times \mathbf{k} \rightarrow K$. By using the restriction $P: H_I^{2g} \times \hat{\mathbf{t}}_1 \rightarrow H_I$ we can similarly define a T_n -equivariant differential form α_I on $H_I^{2g} \times \hat{\mathbf{t}}_1$ which represents the equivariant Poincaré dual of $M(c) \cap (H_I^{2g} \times \hat{\mathbf{t}}_1)$ in $H_I^{2g} \times \hat{\mathbf{t}}_1$. The restriction of $\Phi_{n-1}(\alpha_I)$ then represents the Poincaré dual to $F_{I,\lambda}$ in F , provided suitable orientations are chosen. Note that $\{1, \dots, 1\} \times \mathbf{k}$ is transverse to both $M(c) = P^{-1}(c)$ and $\mu^{-1}(0) = K^{2g} \times \{0\}$ in $K^{2g} \times \mathbf{k}$, and that if $\Lambda \in \mathbf{k}$ then

$$\mu(1, \dots, 1, \Lambda) = -\Lambda$$

while

$$P(1, \dots, 1, \Lambda) = \exp(-\Lambda).$$

From the orientation conventions of Remark 3.5 it follows that the normal to $P^{-1}(H_I)$ in $K^{2g} \times \mathbf{k}$ is T_n -equivariantly isomorphic to the kernel of the restriction map $\mathbf{k}^* \rightarrow \mathbf{h}_I^*$. Thus the restriction of $(-1)^{r(n-r)}\tau_I\alpha_I$ to $H_I^{2g} \times \hat{\mathbf{t}}_1$ has compact support near $M(c)$ and locally represents the equivariant Poincaré dual to $M(c)$ in $K^{2g} \times \mathbf{k}$, so we can substitute it for α on $H_I^{2g} \times \hat{\mathbf{t}}_1$ and we can substitute $(-1)^{r(n-r)}\Phi_{n-1}(\tau_I\alpha_I)$ for $\Phi_{n-1}(\alpha)$ on F .

We have that $e_F = (-1)^{r(n-r)g}\Phi_{n-1}(\tau_I^{2g})$ by the last lemma. We therefore get

$$\begin{aligned} \int_F \frac{\Phi_{n-1}(\eta e^{\bar{\omega}}\alpha)}{e_F} &= (-1)^{r(n-r)(g-1)} \int_F \Phi_{n-1}\left(\frac{\eta e^{\bar{\omega}}\alpha_I}{\tau_I^{2g-1}}\right) \\ &= (-1)^{r(n-r)(g-1)} \int_{F_{I,\lambda}} \Phi_{n-1}\left(\frac{\eta e^{\bar{\omega}}}{\tau_I^{2g-1}}\right), \end{aligned}$$

and Remark 7.6 completes the proof. □

Remark 7.9. The condition for $F \in \mathcal{F}$ to appear in the sum in the statement of Lemma 6.7 was that

$$(7.4) \quad -\|\hat{e}_1\|^2 < \langle \hat{e}_1, \mu(F) \rangle < 0.$$

Let I be a subset of $\{3, \dots, n\}$ with $r - 1$ elements where $1 \leq r \leq n - 1$, and let $\lambda \in \mathbb{R}$ be a solution of the equation

$$e^{-2\pi i\lambda} = \prod_{j \in I_1} c_j.$$

If $F \in \mathcal{F}$ is the component of the fixed point set of the action of \hat{T}_1 on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ which contains $F_{I,\lambda}$, then

$$\mu_{\hat{T}_1}(F) = \mu_{\hat{T}_1}(F_{I,\lambda}) = -\lambda \hat{e}_1.$$

We thus find that for each I there is precisely one solution $\lambda \in \mathbb{R}$ to the equation

$$e^{-2\pi i\lambda} = \prod_{j \in I_1} c_j$$

such that the component F of the fixed point set of the action of \hat{T}_1 on $P^{-1}(V) \cap \mu^{-1}(\hat{\mathbf{t}}_1)/T_{n-1}$ which contains $F_{I,\lambda}$ contributes to the sum in Lemma 6.7. This solution is $\lambda = \delta_I$ where δ_I is the noninteger part of

$$\frac{i}{2\pi} \log \prod_{j \in I_1} c_j,$$

and so we have

$$\mu_{\hat{T}_1}(F) = -\delta_I \hat{e}_1.$$

(Note that since $\prod_{j \in I_1} c_j$ has modulus 1 but is not equal to 1, the noninteger part of $\frac{i}{2\pi} \log \prod_{j \in I_1} c_j$ is well defined as an element of the open interval $(0, 1)$ in \mathbb{R} .) We therefore define

$$F_I = F_{I, \delta_I},$$

and also $\Psi_I = \Psi_{I, \delta_I}$ and $\tilde{\Psi}_I = \tilde{\Psi}_{I, \delta_I}$.

We can now deduce the following result:

PROPOSITION 7.10. *If $\eta(X)$ is a polynomial in $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$, so that $s_{\hat{e}_1}^* \eta = \eta$, then*

$$\begin{aligned} \int_{N_n(c)} \Phi_n(\eta e^{\tilde{\omega}}) - \int_{N_n(c)} \Phi_n(\eta e^{\tilde{\omega}} e^{-Y_1}) &= \int_{N_n(c)} \Phi_n\left((1 - e^{-Y_1})\eta e^{\tilde{\omega}}\right) \\ &= \sum_{1 \leq r \leq n-1} \sum_{I \subseteq \{3, \dots, n\}, |I|=r-1} (-1)^{r(n-r)(g-1)} \text{Res}_{Y_1=0} \int_{F_I} \Phi_{n-1}\left(\frac{\eta e^{\tilde{\omega}}}{\tau_I^{2g-1}}\right). \end{aligned}$$

Proof. Recall that the coordinates $Y_k = e_k(X) = \langle \hat{e}_k, X \rangle$ were introduced in Definition 6.6. The result then follows immediately from Lemma 6.7, Lemma 7.8 and Remark 7.4 above, together with Lemma 6.1 and Remark 7.2. \square

Remark 7.11. This proposition is also true for formal equivariant cohomology classes $\eta = \sum_{j=0}^{\infty} \eta_j$ with $\eta_j \in H_K^j(M(c))$, because all but finitely many η_j contribute zero to both sides of the equations.

COROLLARY 7.12. *Suppose η is a polynomial in $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$, so that $s_{\hat{e}_1}^* \eta = \eta$. Then*

$$\begin{aligned} \int_{N_n(c)} \Phi_n(\mathcal{D}_n \eta e^{\tilde{\omega}}) &= \sum_{1 \leq r \leq n-1} \sum_{I \subseteq \{3, \dots, n\}, |I|=r-1} (-1)^{r(n-r)(g-1)} \text{Res}_{Y_1=0} \\ &\int_{F_I} \Phi_{n-1} \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta e^{\tilde{\omega}}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right). \end{aligned}$$

Proof. This follows by application of Remark 7.6 and Proposition 7.10 with η replaced by the formal equivariant cohomology class $\eta \mathcal{D}_n / (1 - e^{-Y_1})$. This is valid by Remark 7.11 because Y_1 divides $\mathcal{D}_n(X)$ and so $\mathcal{D}_n / (1 - e^{-Y_1})$ can be expressed as a power series in Y_1 whose coefficients are polynomials in the other coordinates Y_2, \dots, Y_{n-1} . \square

Remark 7.13. Recall from the proof of Proposition 7.3 that

$$\begin{aligned} \tilde{\Psi}_I: (S^1)^{2g} \times \left(\mu_{\text{SU}(r)}^{-1}(0) \cap M_r(c(I_1, \delta_I)) \right) \\ \times \left(\mu_{\text{SU}(n-r)}^{-1}(0) \cap M_{n-r}(c(I_2, -\delta_I)) \right) \rightarrow \tilde{F}_{I, \delta_I} \end{aligned}$$

is defined for δ_I as in Remark 7.9 by

$$\begin{aligned} \tilde{\Psi}_I & \left((s_1, \dots, s_{2g}), (h_1^{[I_1]}, \dots, h_{2g}^{[I_1]}, 0), (h_1^{[I_2]}, \dots, h_{2g}^{[I_2]}, 0) \right) \\ & = ((\rho_I(s_1, h_1^{[I_1]}, h_1^{[I_2]}), \rho_I(s_2, h_2^{[I_1]}, h_2^{[I_2]}), \dots, \rho_I(s_{2g}, h_{2g}^{[I_1]}, h_{2g}^{[I_2]}), \\ & \quad 2\pi i \text{diag}(\delta_I, -\delta_I, 0, \dots, 0)) \end{aligned}$$

when we use the map

$$\rho_I: S^1 \times \text{SU}(r) \times \text{SU}(n-r) \rightarrow S(U(r) \times U(n-r)) \subset \text{SU}(n)$$

given by

$$\rho_I: (s, A, B) \mapsto \begin{bmatrix} s^{n-r} A & 0 \\ 0 & s^{-r} B \end{bmatrix}$$

with respect to the decomposition of $\{1, \dots, n\}$ as $I_1 \cup I_2$, which restricts to an $r(n-r)$ -to-one surjective homomorphism

$$\rho_I: S^1 \times T_r \times T_{n-r} \rightarrow T_n.$$

Since $\bar{\omega} = \omega + \mu$ is constructed using the inner product \langle, \rangle defined at (2.2) on the Lie algebra \mathfrak{k} of $K = \text{SU}(n)$, and since ρ_I embeds the Lie algebras of S^1 , $\text{SU}(r)$ and $\text{SU}(n-r)$ as mutually orthogonal subspaces of \mathfrak{k} ,

$$\tilde{\Psi}_I^*(\bar{\omega}) = \bar{\omega}_r + \bar{\omega}_{n-r} + \Omega - \delta_I \hat{e}_1$$

for some $\Omega \in H^2((S^1)^{2g})$, where $\bar{\omega}_r$ and $\bar{\omega}_{n-r}$ are defined like $\bar{\omega}$ but with n replaced by r and $n-r$. Thus we have

$$\tilde{\Psi}_I^* \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta e^{\bar{\omega}}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) = e^{\bar{\omega}_r + \bar{\omega}_{n-r} + \Omega - \delta_I Y_1} \tilde{\Psi}_I^* \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right).$$

Since $\tilde{\Psi}_I^*(\mathcal{D}_r^{[I_1]}) = \mathcal{D}_r$ and $\tilde{\Psi}_I^*(\mathcal{D}_{n-r}^{[I_2]}) = \mathcal{D}_{n-r}$, we can combine this with Corollary 7.12 and Remark 7.5 to obtain the result on which is based the inductive proof of Witten's formulas in the next section.

PROPOSITION 7.14. *If $c \in T$ satisfies the conditions of Remark 6.4, and if $\eta(X)$ is a polynomial in $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$ so that $s_{\hat{e}_1}^* \eta = \eta$, then*

$$\begin{aligned} \int_{N_n(c)} \Phi_n(\mathcal{D}_n \eta e^{\bar{\omega}}) & = \sum_{1 \leq r \leq n-1} \sum_{I \subseteq \{3, \dots, n\}, |I|=r-1} (-1)^{r(n-r)(g-1)} \text{Res}_{Y_1=0} \\ & \int_{F_I} \Phi_{n-1} \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta e^{\bar{\omega}}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \end{aligned}$$

where

$$\text{Res}_{Y_1=0} \int_{F_I} \Phi_{n-1} \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta e^{\bar{\omega}}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right)$$

is equal to $(r(n-r))^{-2g}$ times

$$\int_{N_r(c(I_1, \delta_I))} \Phi_r \left(\mathcal{D}_r e^{\bar{\omega}_r} \text{Res}_{Y_1^I=0} \int_{N_{n-r}(c(I_2, -\delta_I))} \right. \\ \left. \times \Phi_{n-r} \left(\mathcal{D}_{n-r} e^{\bar{\omega}_{n-r}} \int_{(S^1)^{2g}} e^{\Omega} \tilde{\Psi}_I^* \left(\frac{\eta e^{-\delta_I Y_1}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \right) \right)$$

and also to $(r(n-r))^{-2g}$ times

$$\int_{N_{n-r}(c(I_2, -\delta_I))} \Phi_{n-r} \left(\mathcal{D}_{n-r} e^{\bar{\omega}_{n-r}} \text{Res}_{Y_1^I=0} \int_{N_r(c(I_1, \delta_I))} \right. \\ \left. \times \Phi_r \left(\mathcal{D}_r e^{\bar{\omega}_r} \int_{(S^1)^{2g}} e^{\Omega} \tilde{\Psi}_I^* \left(\frac{\eta e^{-\delta_I Y_1}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \right) \right).$$

Here $c(I_1, \delta_I)$ and $c(I_2, -\delta_I)$ are defined as in Proposition 7.3 with δ_I as in Remark 7.4 and $\bar{\omega}_r, \bar{\omega}_{n-r}$ and Ω as in Remark 7.13.

Remark 7.15. For any $\gamma \in T_n$ a unique $\tilde{\gamma} \in \mathfrak{t}_n$ can be chosen so that $\exp \tilde{\gamma} = \gamma$ and $\tilde{\gamma}$ belongs to the fundamental domain defined by the simple roots for the translation action on \mathfrak{t}_n of the integer lattice Λ^I (i.e. $\tilde{\gamma} = \gamma_1 \hat{e}_1 + \dots + \gamma_{n-1} \hat{e}_{n-1}$ with $0 \leq \gamma_j < 1$ for $1 \leq j \leq n-1$). Suppose that $\tilde{c}(I_1, \delta_I) \in \mathfrak{t}_r$ and $\tilde{c}(I_2, -\delta_I) \in \mathfrak{t}_{n-r}$ are chosen in this way in the fundamental domains defined by the simple roots for the translation actions on \mathfrak{t}_r and \mathfrak{t}_{n-r} of their integer lattices, satisfying

$$\exp \tilde{c}(I_1, \delta_I) = c(I_1, \delta_I) = \text{diag}(c_{i_1}^{I, \delta_I}, \dots, c_{i_r}^{I, \delta_I})$$

and

$$\exp \tilde{c}(I_2, -\delta_I) = c(I_2, -\delta_I) = \text{diag}(c_{i_{r+1}}^{I, \delta_I}, \dots, c_{i_n}^{I, \delta_I}),$$

where (as in Proposition 7.3 and Remark 7.9) we define δ_I to be the noninteger part of $\frac{i}{2\pi} \log \prod_{j \in I_1} c_j$ and let $c_j^{I, \delta_I} = c_j$ if $j \geq 3$, and $c_1^{I, \delta_I} = c_1 e^{2\pi i \delta_I}$ and $c_2^{I, \delta_I} = c_2 e^{-2\pi i \delta_I}$.

In the proof of the main theorem (Theorem 8.1) of the next section we shall need to consider the elements w_I^1 and w_I^2 of the subgroup S_{n-1} of the Weyl group $W \cong S_n$ of $SU(n)$ given by the permutations

$$\begin{bmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & \dots & n-r & n-r+1 & \dots & n \\ i_{r+1} & i_{r+2} & \dots & i_n & i_1 & \dots & i_r \end{bmatrix},$$

in the cases when $i_r = 1$ and $i_{r+1} = 2$ and $i_n = n$ and when $i_1 = 1$ and $i_n = 2$ and $i_r = n$ respectively. We will use the fact that if $i_r = 1$ and $i_{r+1} = 2$ and $i_n = n$ then

$$w_I^1(\tilde{c}) = \begin{bmatrix} \tilde{c}(I_1, \delta_I) & 0 \\ 0 & \tilde{c}(I_2, -\delta_I) \end{bmatrix} + (1 - \delta_I)\hat{e}_1,$$

where the block diagonal form is taken with respect to the decomposition of $\{1, \dots, n\}$ as $\{1, \dots, r\} \cup \{r + 1, \dots, n\}$. To see why this is the case, note that

$$w_I^1(\tilde{c})(X) = \gamma_1(X_{i_1} - X_{i_2}) + \dots + \gamma_{n-1}(X_{i_{n-1}} - X_{i_n})$$

where γ_k is the noninteger part of $\frac{1}{2\pi i} \log \prod_{j \leq k} c_{i_j}$, so that $\gamma_r = 1 - \delta_I$ and if $k < r$ then γ_k is the noninteger part of $\frac{1}{2\pi i} \log \prod_{j \leq k} c_{i_j}^{I, \delta_I}$ whereas if $k > r$ then γ_k is the noninteger part of

$$-\delta_I + \frac{1}{2\pi i} \log \prod_{r < j \leq k} c_{i_j} = \frac{1}{2\pi i} \log \prod_{r < j \leq k} c_{i_j}^{I, \delta_I}.$$

Similarly if $i_1 = 1$ and $i_n = 2$ and $i_r = n$ then

$$w_I^2(\tilde{c}) = \begin{bmatrix} \tilde{c}(I_2, -\delta_I) & 0 \\ 0 & \tilde{c}(I_1, \delta_I) \end{bmatrix} - \delta_I \hat{e}_1$$

where the block diagonal form is taken with respect to the decomposition of $\{1, \dots, n\}$ as $\{1, \dots, n - r\} \cup \{n - r + 1, \dots, n\}$.

8. Proof of the iterated residue formula

In this section we shall use induction to prove Witten's formulas as given in Section 2 (see Proposition 2.2) involving iterated residues, for pairings of the form

$$(8.1) \quad \prod_{r=2}^n a_r^{m_r} \prod_{k_r=1}^{2g} (b_r^{k_r})^{p_{r,k_r}} \exp(f_2)[\mathcal{M}(n, d)]$$

for nonnegative integers m_r and $p_{r,k}$. The induction is based on Proposition 7.14. In the next section we shall extend the proof to give formulas for all pairings, and in the following section we shall show that these formulas are equivalent to those of Witten.

We are aiming to prove the following:

THEOREM 8.1. *Let $c = \text{diag}(e^{2\pi i d/n}, \dots, e^{2\pi i d/n})$ where $d \in \{1, \dots, n-1\}$ is coprime to n , and suppose that $\eta \in H_{\text{SU}(n)}^*(M_n(c))$ is a polynomial $Q(\tilde{a}_2, \dots, \tilde{a}_n, \tilde{b}_2^1, \dots, \tilde{b}_n^{2g})$ in the equivariant cohomology classes \tilde{a}_r and \tilde{b}_j^i*

for $2 \leq r \leq n$ and $1 \leq j \leq 2g$. Then the pairing $Q(a_2, \dots, a_n, b_2^1, \dots, b_n^{2g}) \exp(f_2)[\mathcal{M}(n, d)]$ is given by

$$\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\bar{\omega}}) = \frac{(-1)^{n_+(g-1)}}{n!} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \times \left(\frac{\sum_{w \in W_{n-1}} e^{\langle [w\bar{c}], X \rangle} \int_{T_n^{2g}} \eta e^\omega}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (\exp(Y_j) - 1)} \right),$$

where $n_+ = \frac{1}{2}n(n-1)$ is the number of positive roots of $K = \text{SU}(n)$ and $X \in T_n$ has coordinates $Y_1 = X_1 - X_2, \dots, Y_{n-1} = X_{n-1} - X_n$ defined by the simple roots, while $W_{n-1} \cong S_{n-1}$ is the Weyl group of $\text{SU}(n-1)$ embedded in $\text{SU}(n)$ in the standard way using the first $n-1$ coordinates. The element \bar{c} was defined in Remark 2.3: it is the unique element of \mathfrak{t}_n which satisfies $e^{2\pi i \bar{c}} = c$ and belongs to the fundamental domain defined by the simple roots for the translation action on \mathfrak{t}_n of the integer lattice Λ^I . Also, the notation $[[\gamma]]$ (introduced in Definition 2.1) means the unique element which is in the fundamental domain defined by the simple roots for the translation action on \mathfrak{t}_n of the integer lattice and for which $[[\gamma]]$ is equal to γ plus some element of the integer lattice.

Remark 8.2. Here the integral

$$\int_{T_n^{2g}} \eta e^\omega$$

is to be interpreted as the integral of the restriction of ηe^ω over a connected component

$$T_n^{2g} \times \{\lambda\}$$

(for some $\lambda \in \mathfrak{t}_n$ satisfying $c \exp \lambda = 1$) of the fixed point set of the action of T_n on $M_n(c)$. It does not matter which component we choose here, because η and ω are invariant under the translation maps s_{Λ_0} defined in Lemma 4.5 for Λ_0 in the integer lattice of \mathfrak{t}_n .

Remark 8.3(a). We can substitute $-X$ for X in Theorem 8.1 to get

$$\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\bar{\omega}}) = \frac{(-1)^{n_+(g-1)}}{n!} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{\sum_{w \in W_{n-1}} e^{-\langle [w\bar{c}], X \rangle} \int_{T_n^{2g}} \eta(-X) e^\omega}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (1 - \exp(-Y_j))} \right).$$

(b) When η is a polynomial in a_2, \dots, a_n then

$$\int_{T_n^{2g}} \eta e^\omega = \eta \int_{T_n^{2g}} e^\omega = n^g \eta$$

(see Lemma 10.7 below). Since \tilde{a}_r is represented by the polynomial $\tau_r(-X)$ for $2 \leq r \leq n$ (see Proposition 4.4 or Section 9 below), this means that, by (a)

above, Theorem 8.1 combined with Proposition 2.2 gives us Witten’s formula (2.4).

(c) We can also replace the symplectic form ω by any nonzero scalar multiple $\varepsilon\omega$. Then the moment map μ is multiplied by the same scalar ε , and the proof of Theorem 8.1 yields

$$\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\varepsilon\bar{\omega}}) = \frac{(-1)^{n+(g-1)}}{n!} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{\sum_{w \in W_{n-1}} e^{\langle \varepsilon[[w\bar{c}], X] \rangle} \int_{T_n^{2g}} \eta e^{\varepsilon\omega}}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (\exp(\varepsilon Y_j) - 1)} \right).$$

If the degree of η is equal to the dimension of $\mathcal{M}(n, d)$ then the left-hand side of this equation is equal to

$$\int_{\mathcal{M}(n,d)} \Phi(\eta)$$

and hence is independent of ε . Thus in this case we can take any nonzero value of ε on the right-hand side, or let ε tend to zero, to give alternative formulas for $\int_{\mathcal{M}(n,d)} \Phi(\eta)$.

Recall from Lemma 6.1 that

$$(8.2) \quad \int_{\mathcal{M}(n,d)} \Phi(\eta e^{\bar{\omega}}) = \frac{1}{n!} \int_{N_n(c)} \Phi_n(\mathcal{D}_n \eta e^{\bar{\omega}}).$$

Proposition 7.14 tells us that $\int_{N_n(c)} \Phi_n(\mathcal{D}_n \eta e^{\bar{\omega}})$ can be expressed in terms of iterated integrals of the same form for smaller values of n , but with c no longer central in $K = \text{SU}(n)$. We shall therefore obtain Theorem 8.1 from the following result involving values of c which are not central (cf. Remark 6.4), which will be proved by induction on n .

PROPOSITION 8.4. *Let $c = \text{diag}(c_1, \dots, c_n) \in T_n$ be such that the product of no proper subset of c_1, \dots, c_n is 1. If $\eta(X)$ is a polynomial in the $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$, so that $s_{\tilde{e}_i}^* \eta = \eta$, then*

$$\int_{N_n(c)} \Phi_n(\mathcal{D}_n \eta e^{\bar{\omega}}) = (-1)^{n+(g-1)} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{\sum_{w \in W_{n-1}} e^{\langle [[w\bar{c}], X] \rangle} \int_{T_n^{2g}} \eta e^{\omega}}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (\exp(Y_j) - 1)} \right),$$

where $W_{n-1} \cong S_{n-1}$ is the Weyl group of $\text{SU}(n-1)$, embedded in $\text{SU}(n)$ in the standard way using the first $n-1$ coordinates, and $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n) \in \mathfrak{t}_n$ satisfies $e^{2\pi i \tilde{c}} = c$ and belongs to the fundamental domain defined by the simple roots for the translation action on \mathfrak{t}_n of the integer lattice Λ^I .

Proof of Theorem 8.1 from Proposition 8.4. Note that when $c = \text{diag}(e^{2\pi i d/n}, \dots, e^{2\pi i d/n})$ we had introduced an element $\tilde{c} \in \mathfrak{t}_n$ (see Remark 2.3)

which satisfies $e^{2\pi i \bar{c}} = c$ and belongs to the fundamental domain defined by the simple roots for the translation action on T_n of the integer lattice Λ^I . Thus Theorem 8.1 follows immediately from (8.2) and Proposition 8.4. \square

Proof of Proposition 8.4. The proof is by induction on n . When $n = 1$ then both $SU(n)$ and the torus T_n are trivial, $\mathcal{D}_n = 1$ and both $M_n(c)$ and $N_n(c)$ are single points. Thus in this case Proposition 8.4 reduces to the tautology $\eta = \eta$ for any $\eta \in H_{SU(1)}^*(M_1(c))$.

Now let us assume that $n > 1$ and that the result is true for all smaller values of n . By Proposition 7.14,

$$\int_{N_n(c)} \Phi_n(\mathcal{D}_n \eta e^{\bar{\omega}}) = \sum_{1 \leq r \leq n-1} \sum_{I \subseteq \{3, \dots, n\}, |I|=r-1} (-1)^{r(n-r)(g-1)} \text{Res}_{Y_1=0} \int_{F_I} \Phi_{n-1} \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta e^{\bar{\omega}}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right)$$

where

$$\text{Res}_{Y_1=0} \int_{F_I} \Phi_{n-1} \left(\frac{\mathcal{D}_r^{[I_1]} \mathcal{D}_{n-r}^{[I_2]} \eta e^{\bar{\omega}}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right)$$

is equal to $(r(n-r))^{-2g}$ times the iterated integral

$$\int_{N_r(c(I_1, \delta_I))} \Phi_r \left(\mathcal{D}_r e^{\bar{\omega}_r} \text{Res}_{Y_1^I=0} \int_{N_{n-r}(c(I_2, -\delta_I))} \times \Phi_{n-r} \left(\mathcal{D}_{n-r} e^{\bar{\omega}_{n-r}} \int_{(S^1)^{2g}} e^{\Omega} \tilde{\Psi}_I^* \left(\frac{\eta e^{-\delta_I Y_1}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \right) \right)$$

and also to $(r(n-r))^{-2g}$ times the iterated integral

$$\int_{N_{n-r}(c(I_2, -\delta_I))} \Phi_{n-r} \left(\mathcal{D}_{n-r} e^{\bar{\omega}_{n-r}} \text{Res}_{Y_1^I=0} \int_{N_r(c(I_1, \delta_I))} \times \Phi_r \left(\mathcal{D}_r e^{\bar{\omega}_r} \int_{(S^1)^{2g}} e^{\Omega} \tilde{\Psi}_I^* \left(\frac{\eta e^{-\delta_I Y_1}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \right) \right),$$

for $c(I_1, \delta_I)$ and $c(I_2, -\delta_I)$ defined as in Proposition 7.3 with δ_I as in Remark 7.4 and τ_I as in Remark 7.6. Here $\Omega \in H^2((S^1)^{2g})$ satisfies

$$\tilde{\Psi}_I^*(\bar{\omega}) = \bar{\omega}_r + \bar{\omega}_{n-r} + \Omega - \delta_I \hat{e}_1$$

as in Remark 7.13.

We need to consider separately those I containing n and those for which n is not an element of I ; first let us suppose that n is not an element of I . Note that

$$(-1)^{(r(n-r) + \frac{1}{2}r(r-1) + \frac{1}{2}(n-r)(n-r-1))(g-1)} = (-1)^{\frac{1}{2}n(n-1)(g-1)},$$

and

$$\frac{e^{-\delta_I Y_1}}{1 - e^{-Y_1}} = \frac{e^{(1-\delta_I)Y_1}}{e^{Y_1} - 1}.$$

The finite cover $\rho_I: S^1 \times T_r \times T_{n-r} \rightarrow T_n$ is $r(n-r)$ -to-one, so that it induces an $(r(n-r))^{2g}$ -to-one surjection from $(S^1)^{2g} \times T_r^{2g} \times T_{n-r}^{2g}$ to T_n^{2g} and we have

$$\int_{T_n^{2g}} \eta e^\omega = \int_{(S^1)^{2g} \times T_r^{2g} \times T_{n-r}^{2g}} \eta e^{\omega_r + \omega_{n-r} + \Omega}.$$

Moreover this finite cover $\rho_I: S^1 \times T_r \times T_{n-r} \rightarrow T_n$ takes the coordinate $Y_1 = X_1 - X_2$ on \mathfrak{t} to the coordinate Y_1^I on the Lie algebra of S^1 . Since $\tilde{\Psi}_I^*$ was defined using ρ_I (see Remark 7.5), we deduce using Remark 7.13 and Remark 7.6 and induction on n that $(-1)^{r(n-r)(g-1)}$ times the iterated integral

$$\int_{N_r(c(I_1, \delta_I))} \Phi_r \left(\mathcal{D}_r e^{\bar{\omega}_r} \text{Res}_{Y_1^I=0} \int_{N_{n-r}(c(I_2, -\delta_I))} \right. \\ \left. \times \Phi_{n-r} \left(\mathcal{D}_{n-r} e^{\bar{\omega}_{n-r}} \int_{(S^1)^{2g}} e^{\Omega} \tilde{\Psi}_I^* \left(\frac{\eta e^{-\delta_I Y_1}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \right) \right)$$

equals $(-1)^{n+(g-1)}(r(n-r))^{2g}$ times the iterated residue

$\text{Res}_{X_{i_1}-X_{i_2}=0} \dots \text{Res}_{X_{i_{r-1}}-X_{i_r}=0} \text{Res}_{X_1-X_2=0} \text{Res}_{X_{i_{r+1}}-X_{i_{r+2}}=0} \dots \text{Res}_{X_{i_1}-X_{i_2}=0}$

$$\sum_{w_1 \in W_{r-1}} \sum_{w_2 \in W_{n-r-1}} \frac{e^{\langle [w_1 \tilde{c}(I_1, \delta_I)], Y_{I_1} \rangle} e^{\langle [w_2 \tilde{c}(I_2, -\delta_I)], Y_{I_2} \rangle} e^{(1-\delta_I)Y_1} \int_{T_n^{2g}} \eta e^\omega}{\mathcal{D}_n^{2g-2} (e^{Y_1} - 1) \prod_{j \neq r} (\exp(X_{i_j} - X_{i_{j+1}}) - 1)}$$

where Y_{I_1} and Y_{I_2} are the projections of X onto the Lie algebras of the maximal tori T_r and T_{n-r} of $SU(r)$ and $SU(n-r)$ embedded in $SU(n)$ via the decomposition of $\{1, \dots, n\}$ as $I_1 \cup I_2$, and W_{r-1} and W_{n-r-1} are the Weyl groups of $SU(r-1)$ and $SU(n-r-1)$ embedded in $SU(r)$ and $SU(n-r)$ using all but the last coordinates.

There is no need to assume that $i_1 < i_2 < \dots < i_r$ and $i_{r+1} < i_{r+2} < \dots < i_n$ here. We simply need that $I_1 = I \cup \{1\} = \{i_1, \dots, i_r\}$ and $I_2 = \{1, \dots, n\} - I_1 = \{i_{r+1}, \dots, i_n\}$. So let us assume that

$$i_r = 1$$

and

$$i_{r+1} = 2.$$

We are also supposing that n is not an element of I (i.e. that $n \in I_2$) so we may assume in addition that $i_n = n$. Then we can apply the Weyl transformation $w_I^1 \in W_{n-1}$ given by the permutation

$$\begin{bmatrix} 1 & \dots & r & \dots & n-1 \\ i_1 & \dots & i_r & \dots & i_{n-1} \end{bmatrix}$$

together with Remark 7.15 to identify the iterated residue above with

$$(-1)^{n+(g-1)}(r(n-r))^{2g} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \\ \times \sum_{w_1 \in W_{r-1}} \sum_{w_2 \in W_{n-r-1}} \frac{e^{\langle [w_1^1 w_1 w_2(\tilde{c})], X \rangle} \int_{T_n^{2g}} \eta e^\omega}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (1 - \exp(-Y_j))}.$$

When $n \in I$ the argument is similar but we apply induction to $(-1)^{r(n-r)(g-1)}$ times

$$\int_{N_{n-r}(c(I_2, -\delta_I))} \Phi_{n-r} \left(\mathcal{D}_{n-r} e^{\tilde{\omega}_{n-r}} \text{Res}_{Y_1^I=0} \int_{N_r(c(I_1, \delta_I))} \right. \\ \left. \times \Phi_r \left(\mathcal{D}_r e^{\tilde{\omega}_r} \int_{(S^1)^{2g}} e^{\Omega} \tilde{\Psi}_I^* \left(\frac{\eta e^{-\delta_I Y_1}}{\tau_I^{2g-2} (1 - e^{-Y_1})} \right) \right) \right)$$

and observe that

$$\text{Res}_{X_1-X_2=0} \frac{e^{-\delta_I(X_1-X_2)}}{1 - e^{-(X_1-X_2)}} = -\text{Res}_{X_2-X_1=0} \frac{e^{\delta_I(X_2-X_1)}}{1 - e^{X_2-X_1}}$$

(see the Remark after Corollary 3.2). As $I_1 = I \cup \{1\} = \{i_1, \dots, i_r\}$ and $I_2 = \{1, \dots, n\} - I_1 = \{i_{r+1}, \dots, i_n\}$ and $n \in I$ we can assume that $i_1 = 1$, $i_r = n$ and $i_n = 2$. Then we use the Weyl transformation $w_I^2 \in W_{n-1}$ given by the permutation

$$\begin{bmatrix} 1 & \dots & n-r & n-r+1 & \dots & n-1 \\ i_{r+1} & \dots & i_n & i_1 & \dots & i_{r-1} \end{bmatrix}$$

together with Remark 7.15 to equate the iterated integral above with

$$\text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \sum_{w_1 \in W_{r-1}} \sum_{w_2 \in W_{n-r-1}} \frac{e^{\langle [w_1^2 w_1 w_2(\tilde{c})], X \rangle} \int_{T_n^{2g}} \eta e^\omega}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (\exp(Y_j) - 1)}.$$

Thus it suffices to prove:

LEMMA 8.5. *For each subset I of $\{3, \dots, n\}$ with $r - 1$ elements, fix i_1, \dots, i_n such that $I \cup \{1\} = \{i_1, \dots, i_r\}$ and $\{2, \dots, n\} - I = \{i_{r+1}, \dots, i_n\}$ and also $i_r = 1$, $i_{r+1} = 2$ and $i_n = n$ (if $n \notin I$) or $i_1 = 1$, $i_r = n$ and $i_n = 2$ (if $n \in I$). Define permutations w_I^1 (for I such that $n \notin I$) and w_I^2 (for I such that $n \in I$) as above. Then as*

- (i) r runs over $\{1, \dots, n - 1\}$,
- (ii) w_1 runs over permutations of $\{1, \dots, r\}$ fixing r ,
- (iii) w_2 runs over permutations of $\{r + 1, \dots, n\}$ fixing n and
- (iv) I runs over subsets of $\{3, \dots, n\}$ with $r - 1$ elements not containing

n ,

the product $w_I^1 w_1 w_2$ runs over the set of permutations w of $\{1, \dots, n\}$ fixing n such that

$$w^{-1}(1) < w^{-1}(2).$$

Moreover if instead of (iv) I runs over subsets of $\{3, \dots, n\}$ with $r - 1$ elements containing n , then the product $w_1^2 w_1 w_2$ runs over the set of permutations w of $\{1, \dots, n\}$ fixing n such that

$$w^{-1}(1) > w^{-1}(2).$$

Proof. If $w \in W_{n-1}$ satisfies $w^{-1}(1) < w^{-1}(2)$ let $r = w^{-1}(1)$ and $I = \{j: w^{-1}(j) < r\}$. On the other hand if $w \in W_{n-1}$ satisfies $w^{-1}(1) > w^{-1}(2)$ let $r = n - w^{-1}(2)$ and $I = \{j > 1: w^{-1}(j) > n - r\} \cup \{n\}$. In each case it is easy to check that there exist unique choices of w_1 and w_2 such that $w_1^2 w_1 w_2 = w$ or $w_1^2 w_1 w_2 = w$.

This completes the proof of the lemma and hence of Proposition 8.4. \square

Remark 8.6. It is shown in Proposition 3.4 of [33] that the multivariable residue (multiplied by the constant C_K) of Theorem 3.1 and formula (6.1) can be replaced by the iterated one-variable residue

$$\text{Res}_{Y_1=0}^+ \dots \text{Res}_{Y_{n-1}=0}^+$$

multiplied by the Jacobian (in this case $1/n$) of the change of coordinates from an orthonormal system to (Y_1, \dots, Y_{n-1}) . Here, if $\text{Res}_{y=0} g(y)$ denotes the coefficient of y^{-1} in the Laurent expansion about 0 of a meromorphic function $g(y)$ of one complex variable y , then Res^+ is defined for meromorphic functions of the special form $\sum_{1 \leq i \leq s} e^{\lambda_i y} q_i(y)$, where $\lambda_1, \dots, \lambda_s$ are real numbers and q_1, \dots, q_s are rational functions of one variable with complex coefficients, by

$$\text{Res}_{y=0}^+ \left(\sum_{1 \leq i \leq s} e^{\lambda_i y} q_i(y) \right) = \sum_{1 \leq i \leq s, \lambda_i > 0} \text{Res}_{y=0} (e^{\lambda_i y} q_i(y)).$$

Since

$$\frac{e^{\gamma y}}{e^y - 1}$$

can be formally expanded as

$$- \sum_{m \in \mathbb{Z}, m + \gamma > 0} e^{(m + \gamma)y}$$

when $0 < \gamma < 1$, the formula (6.1) can be formally rewritten as

$$\prod_{r=2}^n a_r^{m_r} \exp(f_2)[\mathcal{M}(n, d)] = \frac{(-1)^{n+(g-1)}}{n} \text{Res}_{Y_1=0}^+ \dots \text{Res}_{Y_{n-1}=0}^+ \frac{e^{\langle \tilde{c}, X \rangle} \int_{T_n^{2g}} e^{\omega} \prod_{r=2}^n \tau_r^{m_r}}{\mathcal{D}_n^{2g-2} \prod_{1 \leq j \leq n-1} (e^{Y_j} - 1)}.$$

Moreover the multivariable residue Res is invariant under the action of the Weyl group, as are all the other ingredients of the right-hand side of (6.1)

except for \tilde{c} . Thus by averaging (6.1) over the Weyl group we obtain a special case of Theorem 8.1.

9. Residue formulas for general intersection pairings

In order to obtain explicit formulas for all the pairings, Witten observes that they can be obtained from those for the a_r and f_r via his formula [50, (5.20)]. In this section we shall generalize our version of his formula ([50, (4.74)], which is our Theorem 8.1 via the results of Section 2) to give formulas for $\int_{\mathcal{M}(n,d)} \Phi(\eta e^{\tilde{\omega}})$ where η is an equivariant cohomology class that does not simply involve the $\tilde{a}_r(X)$ but also involves the $\tilde{b}_r^j(X)$ and the $\tilde{f}_r(X)$ (see Theorems 9.11 and 9.12 below). The key step in the proof is Lemma 9.9, combined with the argument used in Sections 5–8 to prove Theorem 8.1.

In the next section we shall see that Theorem 9.11 yields Witten's formula [50, (5.20)]. This will follow from certain equations satisfied by the formula given in Theorem 9.10 (Propositions 10.2 and 10.3).

The next lemma (from [30]) will give an explicit formula for an equivariant cohomology class $\tilde{f}_r(X)$ on $M(c)$ such that $\Phi(\tilde{f}_r(X)) = f_r$ (cf. Proposition 4.4). In order to state it, we introduce the following notation.

Definition 9.1 (the moment). If θ is the Maurer-Cartan form on K , the moment $J(\theta) \in (\Omega^1(K) \otimes \mathfrak{k}^*)^K$ is defined for $X \in \mathfrak{k}$ by

$$(9.1) \quad J(\theta)(X)_k = -\iota_{X^\#} \theta = -\text{Ad}(k^{-1})X,$$

where $X^\#$ is the vector field on K given by the left action of X on K .

Remark 9.2. See [7, Ch. 7] for an explanation of the role of the moment in the construction of equivariant characteristic classes, via an equivariant version of Chern-Weil theory. Given a principal bundle over a K -manifold equipped with a compatible action of K on the total space of the bundle, the moment J plays the same role as the symplectic moment map plays for a principal $U(1)$ bundle \mathcal{L} over a Hamiltonian K -manifold with $c_1(\mathcal{L}) = [\omega]$ (and with a lift of the action of K to the total space of \mathcal{L}). In particular, the appropriate notion of "equivariant curvature" is the sum of the usual curvature and the moment J .

In the next few paragraphs we provide a brief outline of the use of the Bott-Shulman construction (see for instance [10] and other references given in [30]) to obtain equivariant differential forms representing the equivariant characteristic classes $\tilde{f}_r(X)$. This material is summarized from [30], which gives a construction of de Rham representatives for equivariant characteristic classes giving rise to the characteristic classes of the universal bundle over $\mathcal{M}(n, d) \times \Sigma$.

This was accomplished by regarding this bundle (and the classifying space for it) as *simplicial manifolds*. For more details see [30].

Let $\Delta^2 = \{(t_0, t_1, t_2) \in [0, 1]^3: t_0 + t_1 + t_2 = 1\}$ be the standard 2-simplex. There is a principal K -bundle

$$\Delta^2 \times K^3 \xrightarrow{\pi_2} \Delta^2 \times K^2$$

for which the bundle projection $\pi_2: K^3 \rightarrow K^2$ is given by

$$\pi_2(g_0, g_1, g_2) = (g_0g_1^{-1}, g_1g_2^{-1}) \quad ([30, (3.9)]) .$$

We define a connection $\Theta^{(2)}$ on the total space of this bundle by

$$\Theta^{(2)} = \sum_{i=0}^2 t_i \theta^{(i)} \in \Omega^1(\Delta^2 \times K^3) \otimes \mathbf{k},$$

where $\theta^{(i)} \in \Omega^1(K^3) \otimes \mathbf{k}$ is the Maurer-Cartan form on the i^{th} copy of K . The curvature

$$F_{\Theta^{(2)}} \in \Omega^2(\Delta^2 \times K^3) \otimes \mathbf{k}$$

of the bundle is

$$(9.2) \quad F_{\Theta^{(2)}} = \sum_i d(t_i \theta^{(i)}) + [\Theta^{(2)}, \Theta^{(2)}].$$

We use this connection and curvature and the Chern-Weil theory of equivariant characteristic classes (see for instance Chapter 7 of [7]) to define an equivariant form on the total space $\Delta^2 \times K^3$ of the bundle, which represents the equivariant characteristic class associated to τ_r in equivariant cohomology. We then integrate this equivariant form over the simplex Δ^2 . Finally, we may pull this form back to the base space K^2 via a section $\sigma_2: K^2 \rightarrow K^3$ given by

$$\sigma_2(k_1, k_2) = (k_1k_2, k_2, 1) \quad ([30, (4.3)]).$$

Explicitly, we make the following definition:

Definition 9.3. Let $\Phi_2^K(\tau_r) = \sigma_2^* \bar{\Phi}_2^K(\tau_r) \in \Omega_K^{2r-2}(K \times K)$ (see [30], above (4.3)) where the section σ_2 was as defined above, and

$$(9.3) \quad \bar{\Phi}_2^K(\tau_r) = \int_{\Delta^2} \tau_r(F_{\theta(t)} + J(\theta(t))).$$

Let $\Delta^1 = \{(t_0, t_1) \in [0, 1]^2: t_0 + t_1 = 1\} \cong [0, 1]$ be the standard 1-simplex. We shall perform a similar construction using a principal K -bundle

$$\Delta^1 \times K^2 \xrightarrow{\pi_1} \Delta^1 \times K.$$

The bundle projection $\pi_1: K^2 \rightarrow K$ is defined by

$$\pi_1(g_0, g_1) = g_0g_1^{-1}.$$

A section $\sigma_1: K \rightarrow K^2$ of the bundle is given by $\sigma_1(k) = (k, 1)$.

On the total space $\Delta^1 \times K^2$ we define a connection

$$\Theta^{(1)} = \sum_{i=0}^1 t_i \theta^{(i)} \in \Omega^1(\Delta^1 \times K^2) \otimes \mathfrak{k},$$

where $\theta^{(i)} \in \Omega^1(K^2) \otimes \mathfrak{k}$ is the Maurer-Cartan form on the i^{th} copy of K . The definition of the curvature

$$F_{\Theta^{(1)}} \in \Omega^2(\Delta^1 \times K^2) \otimes \mathfrak{k}$$

is similar to (9.2). As before, we evaluate the invariant polynomial τ_r on the equivariant curvature and integrate over the simplex Δ^1 to get an equivariant form over $K \times K$, and finally we pull this form back to K using the section σ_1 : explicitly, we make the following:

Definition 9.4. We define

$$\Phi_1^K(\tau_r) = \sigma_1^* \bar{\Phi}_1^K(\tau_r) \in \Omega^{2r-1}(K),$$

where

$$(9.4) \quad \bar{\Phi}_1^K(\tau_r) = \int_{\Delta^1} \tau_r(F_{\theta(t)} + J(\theta(t))) \in \Omega_K^*(K \times K).$$

Definition 9.5 (equivariant chain homotopy). We define a chain homotopy

$$I_K: \Omega_K^{*+1}(\mathfrak{k}) \rightarrow \Omega_K^*(\mathfrak{k})$$

as follows: when $v \in \mathfrak{k}$,

$$(9.5) \quad (I_K \beta)_v = \int_0^1 F_t^*(\iota_{\bar{v}} \beta) dt,$$

where $F_t: \mathfrak{k} \rightarrow \mathfrak{k}$ is multiplication by t and \bar{v} is the vector field on \mathfrak{k} which takes the constant value v .

LEMMA 9.6 ([30, Th. 8.1]). *The equivariant cohomology class of the equivariant differential form*

$$\tilde{f}_r(X) = \text{pr}_1^* \tilde{f}_r(X)_1 + \text{pr}_2^* \tilde{f}_r(X)_2$$

is a lift of $f_r \in H^{2r-2}(\mathcal{M}(n, d))$ to $H_K^{2r-2}(M(c))$. Here, the maps pr_1 and pr_2 are the projection maps from $M(c)$ to K^{2g} and \mathfrak{k} defined at (4.4). Also, from ([30, (7.13)]),

$$(9.6) \quad \begin{aligned} \tilde{f}_r(X)_1 &= \left(\sum_{j=1}^g (-\text{ev}_{\gamma_j^1} \times \text{ev}_{x_j} + \text{ev}_{\gamma_{j+g}^0} \times \text{ev}_{x_{j+g}}) \right)^* \Phi_2^K(\tau_r)(X) \\ &+ \left(\sum_{j=1}^g (-\text{ev}_{\gamma_{j+g}^1} \times \text{ev}_{x_{j+g}} + \text{ev}_{\gamma_j^0} \times \text{ev}_{x_j}) \right)^* \Phi_2^K(\tau_r)(X) \\ &\in \Omega_K^*(K^{2g}) \end{aligned}$$

and

$$(9.7) \quad \tilde{f}_r(X)_2 = -I_K(e_c^* \Phi_1^K(\tau_r)) \in \Omega_K^*(\mathbf{k})$$

where γ_j^α (for $\alpha = 0, 1$ and $j = 1, \dots, 2g$) are certain elements of \mathbb{F}^{2g} (the free group on $2g$ generators x_1, \dots, x_{2g} , as in Section 4), whose definition is as given in (7.12) of [30], and for any $z \in \mathbb{F}^{2g}$, $ev_z: K^{2g} \rightarrow K$ denotes the evaluation map on z . Here, $e_c: \mathbf{k} \rightarrow K$ is defined by $e_c(\Lambda) = c \exp \Lambda$ where the central element $c = e^{2\pi i \text{id}/n} \text{diag}(1, \dots, 1)$ is as defined in (2.6).

By (9.4),

$$(9.8) \quad \bar{\Phi}_1^K(\tau_r)(-X) = \int_{t \in [0,1]} \tau_r \left(dt(\theta^{(0)} - \theta^{(1)}) + t d\theta^{(0)} + (1-t)d\theta^{(1)} \right. \\ \left. + \frac{1}{2}[t\theta^{(0)} + (1-t)\theta^{(1)}, t\theta^{(0)} + (1-t)\theta^{(1)}] \right. \\ \left. + t \text{Ad}(g_0^{-1})X + (1-t)\text{Ad}(g_1^{-1})X \right).$$

Now

$$\bar{\Phi}_1^K(\tau_r)|_{T \times T}(-X) = \int_{t \in [0,1]} \tau_r(dt(\theta^{(0)} - \theta^{(1)}) + X)$$

since

$$d\theta^{(i)} + \frac{1}{2}[\theta^{(i)}, \theta^{(i)}] = 0$$

and the restrictions of $[\theta^{(i)}, \theta^{(i)}]$ to T vanish. Further

$$\sigma_1^* \bar{\Phi}_1^K(\tau_r)|_T(-X) = \int_{t \in [0,1]} \tau_r(dt\theta + X),$$

where θ is the Maurer-Cartan form on T . If $\tau_r(Z_1, \dots, Z_{n-1}) = \sum_I (\tau_r)_I Z^I$ where $I = (i_1, \dots, i_{n-1})$ is a multi-index and $Z^I = Z_1^{i_1} \dots Z_{n-1}^{i_{n-1}}$ (in terms of a coordinate system $\{Z_a = \langle \hat{u}_a, X \rangle, a = 1, \dots, n-1\}$ on \mathfrak{t} , specified by an oriented orthonormal basis \hat{u}_a for \mathfrak{t} for which $\theta_a, a = 1, \dots, n-1$ are the corresponding components of the Maurer-Cartan form $\theta \in \Omega^1(T) \otimes \mathfrak{t}$), then we have

$$(9.9) \quad \Phi_1^K(\tau_r)|_T(-X) = \sum_I \int_{t \in [0,1]} (\tau_r)_I (dt\theta_1 + Z_1)^{i_1} \dots (dt\theta_{n-1} + Z_{n-1})^{i_{n-1}} \\ = \sum_{a=1}^{n-1} \theta_a \partial \tau_r / \partial Z_a.$$

LEMMA 9.7. For $\Lambda \in \mathfrak{t}$ (in terms of the Maurer-Cartan form $\theta \in \Omega^1(T) \otimes \mathfrak{t}$),

$$I_K(e_c^* \theta)_\Lambda = \Lambda.$$

Proof. We have

$$I_K(e_c^* \theta)_\Lambda = \int_0^1 F_t^*(e_c^* \theta(\bar{\Lambda})) dt = \Lambda$$

since $e_c^* \theta(\bar{\Lambda}): \mathfrak{t} \rightarrow \mathbb{R}$ is the function with constant value Λ . □

Let $q \in S(\mathfrak{k}^*)^K$ be an invariant polynomial which is given in terms of the elementary symmetric polynomials τ_j by

$$(9.10) \quad q(X) = \tau_2(X) + \sum_{r=3}^n \delta_r \tau_r(X).$$

The associated element $\tilde{f}_{(q)}$ of $H_K^*(M(c))$ is defined by

$$(9.11) \quad \tilde{f}_{(q)} = \tilde{f}_2 + \sum_{r=3}^n \delta_r \tilde{f}_r.$$

Here, the δ_r are formal nilpotent parameters; we expand $\exp \tilde{f}_{(q)}$ as a formal power series in the δ_r . We can alternatively regard the δ_r as real parameters and $\exp \tilde{f}_{(q)}$ as a formal equivariant cohomology class; the integral

$$\int_{\mathcal{M}(n,d)} \Phi(\exp \tilde{f}_{(q)})$$

and the integral appearing in (10.4) are well defined and are polynomial functions of the δ_j , since $\int_{\mathcal{M}(n,d)} \Phi(\eta) = 0$ unless $2\deg(\eta) = \dim \mathcal{M}(n, d)$.

Note that by Lemma 9.6 we can write $\tilde{f}_{(q)}(X) = \text{pr}_1^* \tilde{f}_{(q)}(X)_1 + \text{pr}_2^* \tilde{f}_{(q)}(X)_2$ where $\tilde{f}_{(q)}(X)_1 \in \Omega_K^*(K^{2g})$ and $\tilde{f}_{(q)}(X)_2 \in \Omega_K^*(\mathfrak{k})$.

Then we have

LEMMA 9.8. *For $X \in \mathfrak{t}$, the restriction of $\tilde{f}_{(q)}(-X)_2$ to $\mu^{-1}(\mathfrak{t})$ is given at $(h_1, \dots, h_{2g}, \Lambda) \in \mu^{-1}(\mathfrak{t}) \subset K^{2g} \times \mathfrak{k}$ by*

$$\tilde{f}_{(q)}(-X)_2|_{\mu^{-1}(\mathfrak{t})}(h_1, \dots, h_{2g}, \Lambda) = -(dq)_X(\Lambda).$$

Proof.

$$\begin{aligned} \tilde{f}_{(q)}(-X)_2|_{\mu^{-1}(\mathfrak{t})} &= -I_K e_c^* \Phi_1^K(q)(-X) \\ &= -I_K \left(\sum_{a=1}^{n-1} \theta^a \partial q / \partial Z^a \right) \text{ by (9.9)} \\ &= -(dq)_X(\Lambda) \text{ by Lemma 9.7.} \end{aligned} \quad \square$$

LEMMA 9.9. *Assume that $X \in \mathfrak{t}$. Let $\Lambda = \sum_{a=1}^{n-1} m_a \hat{e}_a \in \Lambda^I$ for $m_a \in \mathbb{Z}$ (where the simple roots \hat{e}_a are as in (6.2))¹⁵ and let s_Λ denote the homeomorphism of $M_{\mathfrak{t}}(c)$ given by Lemma 4.5. Then on $M_{\mathfrak{t}}(c)$*

$$(9.12) \quad s_\Lambda^* \tilde{f}_{(q)}(-X) = \tilde{f}_{(q)}(-X) - (dq)_X(\Lambda),$$

¹⁵Note that the \hat{e}_a are a basis of \mathfrak{t} , but not an orthonormal basis.

or equivalently

$$s_{\Lambda}^* \tilde{f}_{(q)}(X) = \tilde{f}_{(q)}(X) + (dq_{(o)})_X(\Lambda),$$

where the notation

$$q_{(o)}(X) = q(-X).$$

is introduced.

Remark. This result generalizes (4.9).

Proof of Lemma 9.9. Since $\tilde{f}_{(q)}(X) = \text{pr}_1^* \tilde{f}_{(q)}(X)_1 + \text{pr}_2^* \tilde{f}_{(q)}(X)_2$, we need to prove the formula for $s_{\Lambda}^* \text{pr}_2^* \tilde{f}_{(q)}(X)_2$ where $\tilde{f}_{(q)}(X)_2 = -I_K e_c^* \Phi_1^K(q)$ for $\Phi_1^K(q) \in \Omega_K^*(K)$. Lemma 9.9 then follows from Lemma 9.8. \square

THEOREM 9.10. *Suppose η is a polynomial in the $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$. Let $q \in S(\mathbf{k}^*)^K$. Then for any $X \in \mathfrak{t}$,*

$$\begin{aligned} \int_{N_n(V)} \Phi \left(\eta e^{\tilde{f}_{(q)}} \left(e^{(dq_{(o)})_X(\hat{e}_1)} - 1 \right) \alpha \right) \\ = - \sum_{F \in \mathcal{F}: -\|\hat{e}_1\|^2 < \langle \hat{e}_1, \mu(F) \rangle < 0} \text{Res}_{Y_1=0} \int_F \frac{\Phi_{n-1}(\eta e^{\tilde{f}_{(q)}} \alpha)}{e_F}. \end{aligned}$$

Here, we sum over the components F of the fixed point set of \hat{T}_1 in $P^{-1}(V) \cap \mu^{-1}(\hat{\mathfrak{t}}_1)/T_{n-1}$; the notation is as in the statement of Lemma 6.7. The notation $q_{(o)}$ was introduced in the statement of Lemma 9.9. We have defined the map Φ_{n-1} in Proposition 6.3, and after (7.3).

Proof. This follows from the same proof as for Lemma 6.7, when (4.9) is replaced by its generalization Lemma 9.9. \square

We aim to prove the following result by induction:

THEOREM 9.11 (a). *For the particular q defined in (9.10),*

$$\begin{aligned} \int_{N_n(c)} \Phi(e^{\tilde{f}_{(q)}} \mathcal{D}_n \eta) &= \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \\ &\frac{\int_{T^{2g} \times \{-[w\hat{c}]\}} \left(e^{\tilde{f}_{(q)}(X)} \eta(X) \right)}{\mathcal{D}(X)^{2g-2} \prod_{j=1}^{n-1} (\exp -B(-X)_j - 1)}, \end{aligned}$$

where η is a polynomial in the \tilde{a}_r and \tilde{b}_r^j and $B(X)_j = -(dq)_X(\hat{e}_j)$.

Here we have used the fixed invariant inner product on \mathfrak{k} to identify $dq_X: \mathfrak{t} \rightarrow \mathbb{R}$ with an element of \mathfrak{t} and thus define the map $B: \mathfrak{t} \rightarrow \mathfrak{t}$. The notation $[[\gamma]]$ was introduced in Definition 2.1.

Substitution of $-X$ for X on the right-hand side of the equation in Theorem 9.11 (a) gives the equivalent formulation:

THEOREM 9.11 (b). *In the notation of Theorem 9.11 (a),*

$$\int_{N_n(c)} \Phi(e^{\tilde{f}(a)} \mathcal{D}_n \eta) = \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \frac{\int_{T^{2g} \times \{-[[w\tilde{c}]]\}} (e^{\tilde{f}(a)}(-X) \eta(-X))}{\mathcal{D}(X)^{2g-2} \prod_{j=1}^{n-1} (1 - \exp -B(X)_j)}.$$

Finally we may use Lemma 10.9 and Lemma 10.12 (a) where the restrictions to T^{2g} of the equivariant cohomology classes $\tilde{f}_r(X)$ and $\tilde{b}_r^j(X)$ are expressed in terms of the basis ζ_a^j for $H^1(T^{2g})$ (for $a = 1, \dots, n-1$ and $j = 1, \dots, 2g$). We also use Lemma 10.10, where the symplectic volume of T^{2g} is calculated. These lemmas enable us to compute $\int_{T^{2g}} e^{\tilde{f}(a)}(-X) \eta(-X)$ and rephrase Theorem 9.11 (b) as follows. (Here we have also reformulated the left-hand side of Theorem 9.11 (b) in terms of the pairings on $\mathcal{M}(n, d)$, using Lemma 6.1.)

THEOREM 9.12. *In the notation of Theorem 9.11,*

(a)

$$\begin{aligned} (9.13) \quad & \int_{\mathcal{M}(n,d)} \exp(f_2 + \delta_3 f_3 + \dots + \delta_n f_n) \prod_{r=2}^n a_r^{m_r} \prod_{k_r=1}^{2g} (b_r^{k_r})^{p_{r,k_r}} \\ &= \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\frac{e^{dq_X([[w\tilde{c}]])} (\prod_{r=2}^n \tau_r(X)^{m_r})}{\mathcal{D}(X)^{2g-2} \prod_{j=1}^{n-1} (1 - \exp -B(X)_j)} \right. \\ & \quad \times \int_{T^{2g}} \exp \left\{ - \sum_{a,b} \sum_{j=1}^g \zeta_a^j \zeta_b^{j+g} \partial^2 q_X(\hat{u}_a, \hat{u}_b) \right\} \\ & \quad \left. \times \prod_{r=2}^n \prod_{k_r=1}^{2g} \left(\sum_{a=1}^{n-1} (d\tau_r)_X(\hat{u}_a) \zeta_a^{k_r} \right)^{p_{r,k_r}} \right). \end{aligned}$$

(b) *In particular,*

$$\begin{aligned} (9.14) \quad & \int_{\mathcal{M}(n,d)} \exp(f_2 + \delta_3 f_3 + \dots + \delta_n f_n) \prod_{r=2}^n a_r^{m_r} \\ &= (-1)^{n+(g-1)} \frac{n^g}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \frac{e^{dq_X([[w\tilde{c}]])} \prod_{r=2}^n \tau_r(X)^{m_r} (\det H_{\mathbf{t}}(X))^g}{\mathcal{D}(X)^{2g-2} \prod_{j=1}^{n-1} (1 - \exp -B(X)_j)}. \end{aligned}$$

Remark 9.13. In the preceding theorem, we have used the following notation. The a_r, f_r and b_r^j (for $r = 2, \dots, n$ and $j = 1, \dots, 2g$) are generators of the cohomology ring, introduced in Section 2. The τ_r are the elementary symmetric polynomials, and the δ_r are formal nilpotent parameters which were introduced in (9.10). The polynomial $q = \tau_2 + \sum_{r=3}^n \delta_r \tau_r$ was introduced in (9.10). Its derivative $dq_X: \mathfrak{t} \rightarrow \mathbb{R}$ is identified with an element of \mathfrak{t} via the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{t} , and hence $dq: \mathfrak{t} \rightarrow \mathfrak{t}^*$ is identified with a map $B: \mathfrak{t} \rightarrow \mathfrak{t}$ (see the statement of Theorem 9.11(a)). If $\gamma \in \mathfrak{t}$, the notation $[[\gamma]]$ is as introduced in Definition 2.1; it is the unique element in the fundamental domain defined by the simple roots for the translation action on \mathfrak{t} of the integer lattice which is equivalent to γ under translation by the integer lattice. The ζ_a^j are derived from the components of the Maurer-Cartan form $\theta \in \Omega^1(T) \otimes \mathfrak{t}$ in terms of an orthonormal basis $\{\hat{u}_a, a = 1, \dots, n-1\}$ of \mathfrak{t} : they have been identified with a basis of $H^1(T^{2g})$. (See Definition 10.6 below.) Finally $\det H_{\mathfrak{t}}(X)$ is the determinant of the Hessian of $q: \mathfrak{t} \rightarrow \mathbb{R}$, in terms of the coordinates on \mathfrak{t} given by the orthonormal basis $\{\hat{u}_a\}$; it is independent of the choice of orthonormal basis.

We note that in Theorem 9.12 the orthonormal basis introduced above could be replaced by a general basis, provided one defines the ζ_a^j using that basis, and multiplies the Hessian by a factor due to the change of basis; see Remark 10.1 below.

Remark 9.14. We can replace f_2 by any nonzero constant scalar multiple εf_2 provided we replace the polynomial q by q^ε where

$$q^\varepsilon(X) = \varepsilon \tau_2(X) + \delta_3 \tau_3(X) + \dots + \delta_n \tau_n(X)$$

(cf. Remark 8.3 (b)).

In order to prove Theorem 9.11 and hence Theorem 9.12 we follow the proof of Theorem 8.1 using the following:

LEMMA 9.15. *Suppose η is a polynomial in the $\tilde{a}_r(X)$ and $\tilde{b}_r^j(X)$. Then for any $X \in \mathfrak{t}$ and $q \in S(\mathfrak{k}^*)^K$ chosen as in (9.10),*

$$(9.15) \quad \int_{N_n(V)} \Phi \left(\eta \mathcal{D} e^{\tilde{f}(q)} \alpha \right) = - \sum_{F \in \mathcal{F}: -\|\hat{e}_1\|^2 < \langle \hat{e}_1, \mu(F) \rangle < 0} \text{Res}_{Y_1=0} \int_F \frac{\Phi_{n-1}(\mathcal{D}\eta(X) e^{\tilde{f}(q)} \alpha)}{e_F \left(e^{(dq_{(q)})_X(\hat{e}_1)} - 1 \right)},$$

where the notation is as in the statement of Lemma 6.7.

Proof. This follows from Theorem 9.10 by replacement of η by

$$(9.16) \quad \frac{\eta \mathcal{D}}{\left(e^{(dq_{(q)})_X(\hat{e}_1)} - 1 \right)} = \frac{\eta \bar{D} Y_1}{\left(e^{(dq_{(q)})_X(\hat{e}_1)} - 1 \right)},$$

where we have defined $\bar{\mathcal{D}} = \mathcal{D}/Y_1$. Notice that $(dq_{(o)})_X(\hat{e}_1)$ is divisible by Y_1 : to see this, we observe that if we define the generating functional

$$P(X_1, \dots, X_n) = \prod_{j=1}^n (1 + tX_j) = \sum_{r=0}^n \tau_r(X_1, \dots, X_n) t^r$$

(where the τ_r are the elementary symmetric polynomials) then

$$dP = ((1 + tX_1)t dX_2 + (1 + tX_2)t dX_1) \prod_{j=3}^n (1 + tX_j) + \mathcal{P}$$

where \mathcal{P} is a collection of terms involving dX_3, \dots, dX_n . Evaluating dP on $\hat{e}_1 = (1, -1, 0, \dots, 0)$ we thus obtain

$$t^2(-Y_1) \prod_{j=3}^n (1 + tX_j) = \sum_{r=0}^n t^r (d\tau_r)_X(\hat{e}_1).$$

It follows that the $(d\tau_r)_X(\hat{e}_1)$ (and hence $(dq_{(o)})_X(\hat{e}_1)$) are divisible by Y_1 . Thus $-(dq_{(o)})_X(\hat{e}_1) = -Y_1(1 + \nu)$ where $\nu \in H_T^*$ has degree at least 1; so we have

$$(e^{(dq_{(o)})_X(\hat{e}_1)} - 1) = -Y_1(1 - \tilde{\nu})$$

where $\tilde{\nu} = \sum_{j \geq 1} \tilde{\nu}_j$ is a formal sum of classes $\tilde{\nu}_j$ with degree at least 1 in (a completion of) H_T^* . Then the expression

$$\Phi \left(\frac{\eta \mathcal{D}}{(e^{(dq_{(o)})_X(\hat{e}_1)} - 1)} \right)$$

(which appears on the left-hand side of the equation in Theorem 9.10) is well defined. On the right hand side, we may replace

$$\frac{Y_1}{(e^{+(dq_{(o)})_X(\hat{e}_1)} - 1)}$$

by $-(1 - \tilde{\nu})^{-1} = -\sum_{s \geq 0} \tilde{\nu}^s$. □

We now use Lemma 9.15 to prove Theorem 9.11 (a) by induction on n . The proof follows the outline of the proof of Theorem 8.1 when $q = q_2$, with the following modifications:

1. $e^{\tilde{\omega}}$ is replaced by $e^{\tilde{f}(q)}$ (and e^{ω} replaced by $e^{\tilde{f}(q) - \tilde{f}_2} e^{\omega}$).
2. $(dq_{(o)})_X(\hat{e}_1)$ replaces $-Y_1$, so $e^{(dq_{(o)})_X(\hat{e}_1)} - 1$ replaces $e^{-Y_1} - 1$.

3. In particular, $-(dq_{(o)})_X(\hat{e}_1) = -B(-X)_1$ replaces Y_1 in the identity

$$\frac{e^{-\delta_I Y_1}}{1 - e^{-Y_1}} = \frac{e^{(1-\delta_I)Y_1}}{e^{Y_1} - 1}$$

which is used in the proof of Proposition 8.4.

We also use the elementary fact that $(dq_{(o)})_X = -(dq)_{-X} = B(-X)$.

10. Witten's formulas for general intersection pairings

In this section we state and prove Witten's formulas (Propositions 10.2 and 10.3 below; cf. [50, §5], in particular the calculations (5.11)–(5.20)), which enabled him to calculate general intersection pairings in terms of those of the form

$$\int_{\mathcal{M}(n,d)} \prod_r a_r^{m_r} e^{f_2}.$$

We shall prove these formulas starting from our explicit formulas for the general intersection pairings (see Theorem 9.11).

Some of the notation in the statements of Propositions 10.2 and 10.3 was introduced at the beginning of Section 9. The invariant polynomial q was defined by (9.10). Using the invariant metric on \mathbf{k} , the map $-dq : \mathbf{k} \rightarrow \mathbf{k}^*$ may be regarded as a map $B = B^{(2)} + \sum_{r \geq 3} \delta_r B^{(r)} : \mathbf{k} \rightarrow \mathbf{k}$, where $B^{(r)} = -d\tau_r : \mathbf{k} \rightarrow \mathbf{k}$; we find $B^{(2)} = -d\tau_2 = \text{id} : \mathbf{k} \rightarrow \mathbf{k}$. (Note that we have put $\tau_2(X) = -\frac{1}{2}\langle X, X \rangle$ in terms of the inner product $\langle \cdot, \cdot \rangle$ defined at (2.2).) The other maps $B^{(r)}$ are not linear.

The Hessian of $-q$ is H ; it is a function from \mathbf{k} to symmetric bilinear forms on \mathbf{k} . If k, l run over an orthonormal basis $\{\hat{v}_k\}$ of \mathbf{k} then the Hessian at X is the matrix

$$(10.1) \quad H(X)_{kl} = -(\partial^2 q)_X(\hat{v}_k, \hat{v}_l).$$

Remark 10.1. In most places in Sections 9 and 10, the orthonormal basis $\{\hat{u}_a\}$ for \mathfrak{t} may be replaced by any basis for \mathfrak{t} (including the basis $\{\hat{e}_a, a = 1, \dots, n - 1\}$, which is of course not orthonormal), and similarly for the orthonormal basis $\{\hat{v}_l\}$ for \mathbf{k} . However it is more convenient to define the determinant of the Hessian (given in (10.1)) in terms of an *orthonormal* basis, since one must otherwise include a normalization factor proportional to the square of the determinant of a matrix whose columns are the basis elements. The second place where it is useful to introduce an orthonormal basis is in the definition of the symplectic form in terms of the generators ζ_a^j for the cohomology of T^{2g} ; the symplectic form is defined using the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{t} , and the formula (Lemma 10.8) for the restriction of the symplectic form to T^{2g} is cleaner in terms of an orthonormal basis.

For these reasons we have chosen to use an orthonormal basis for \mathfrak{t} throughout Sections 9 and 10, although in many specific instances this basis may be replaced by a general basis. In particular in the statement of our main theorem Theorem 9.12, it is easy to check that the orthonormal basis may be replaced by a general basis, provided that the ζ_a^j are also defined using this basis, and that the Hessian is multiplied by the appropriate factor.

We assume the δ_r are formal *nilpotent* parameters; then the invertibility of B is guaranteed. We write $B^{-1}: \mathfrak{k} \rightarrow \mathfrak{k}$ as the inverse of B . (If the δ_r are nilpotent, the inverse of B may be written as a formal power series in the δ_r .)

PROPOSITION 10.2. *For any invariant polynomial $\tau \in S(\mathfrak{k}^*)^K$, the integral*

$$(10.2) \quad \int_{\mathcal{M}(n,d)} \Phi(\tau(-X) \exp \tilde{f}_{(q)})$$

is equal to the integral

$$(10.3) \quad \int_{\mathcal{M}(n,d)} \Phi \left(\tau(B^{-1}(-X)) \left(\det H(B^{-1}(-X)) \right)^{g-1} \right) \exp f_2$$

which is of the form that may be calculated by Theorem 8.1.

PROPOSITION 10.3. *Let $\tau \in S(\mathfrak{k}^*)^K$ be an invariant polynomial, so that*

$$\tau = \sum_{m_2, \dots, m_n} c_{m_2, \dots, m_n} \prod_{r=2}^n \tilde{a}_r^{m_r}$$

is a polynomial in the $\tilde{a}_2, \dots, \tilde{a}_n$. Let s_r^j be real parameters (for $r = 2, \dots, n$ and $j = 1, \dots, 2g$). Then,

$$(10.4) \quad \int_{\mathcal{M}(n,d)} \Phi \left(\tau(X) \exp \left(\sum_{r=2}^n \sum_{j=1}^{2g} s_r^j \tilde{b}_r^j(X) \right) \exp \tilde{f}_{(q)} \right) \\ = \int_{\mathcal{M}(n,d)} \Phi \left(\tau(X) \exp \hat{\tau}(X) \exp \tilde{f}_{(q)} \right).$$

Here, the invariant polynomial $\hat{\tau}$ on \mathfrak{k} is defined (for $X \in \mathfrak{t}$) by

$$\hat{\tau}(-X) = - \sum_{a,b=1}^{n-1} \sum_{r,s=2}^n \sum_{j=1}^g s_r^j s_s^{j+g} (d\tau_r)_X(\hat{u}_a) (d\tau_s)_X(\hat{u}_b) (\partial^2 q)_{ab}^{-1},$$

where $\{\hat{u}_a: a = 1, \dots, n-1\}$ denotes an oriented orthonormal basis of \mathfrak{t} ; see (10.19) for the definition.

Remark 10.4. Notice that in our conventions on the equivariant cohomology differential and the moment, the construction of [30] described at the

beginning of Section 9 yields

$$\tilde{a}_r(X) = \tau_r(-X).$$

Thus $\tau(X) = \sum_{m_2, \dots, m_n} c_{m_2, \dots, m_n} \prod_{r=2}^n \tau_r(-X)^{m_r}$.

Proposition 10.2 is proved by comparing Theorem 9.12 (b) (applied to (10.2)) with Theorem 8.1 (applied to (10.3)). Proposition 10.3 is obtained by applying Theorem 9.11 (b) to both sides of (10.4) and examining the restrictions to T^{2g} (which are computed in Lemmas 10.9 and 10.13).

Propositions 10.2 and 10.3 enable us to extract formulas for all pairings, by differentiating the formulas (10.3) and (10.4) with respect to the parameters δ_r and s_r^j and then setting these parameters equal to zero. In fact, for any nonnegative integers n_r (for $r \geq 3$) we have

$$\begin{aligned} & \left(\prod_{r=3}^n \left(\frac{\partial}{\partial \delta_r} \right)^{n_r} \int_{\mathcal{M}(n,d)} \Phi(\tau(X) \exp \tilde{f}_{(q)}) \right)_{\delta_3 = \dots = \delta_n = 0} \\ &= \int_{\mathcal{M}(n,d)} \prod_{r=3}^n f_r^{n_r} \Phi(\tau(X)) \exp f_2, \end{aligned}$$

and likewise for any nonnegative integers n_r (with $n_2 = 0$) and any choices of $p_{r,j} = 0, 1$ we have

$$\begin{aligned} & \left(\prod_{r=2}^n \left(\frac{\partial}{\partial \delta_r} \right)^{n_r} \prod_{j=1}^{2g} \left(\frac{\partial}{\partial s_r^j} \right)^{p_{r,j}} \int_{\mathcal{M}(n,d)} \Phi \left(\tau(X) \exp \left(\sum_{r=2}^n \sum_{j=1}^{2g} s_r^j \tilde{b}_r^j(X) \right) \exp \tilde{f}_{(q)} \right) \right)_{\delta_r=0, s_{r,j}=0} \\ &= \int_{\mathcal{M}(n,d)} \Phi(\tau(X)) \exp f_2 \prod_{r=2}^n f_r^{n_r} \prod_{j=1}^{2g} (b_r^j)^{p_{r,j}} \end{aligned}$$

(where the parameters δ_r and s_r^j on the left-hand side run over $r = 2, \dots, n$ and $j = 1, \dots, 2g$).

We can use Proposition 10.3 to give an explicit formula for pairings of the form

$$(10.5) \quad \int_{\mathcal{M}(n,d)} \Phi \left(\prod_{r=2}^n \prod_{k_r=1}^{2g} (\tilde{b}_r^{k_r}(X))^{p_{r,k_r}} \tau(X) \right) e^{f_2}$$

where $p_{r,k_r} = 0$ or 1. We note that by Proposition 10.3 this equals

$$(10.6) \quad \prod_{r=2}^n \prod_{k_r=1}^{2g} \left(\frac{\partial}{\partial s_r^{k_r}} \right)^{p_{r,k_r}} \int_{\mathcal{M}(n,d)} \Phi(\tau(X) \exp \hat{\tau}(X)) e^{f_2} \Big|_{s_r^j=0 \forall r,j}$$

where

$$(10.7) \quad \hat{\tau}(-X) = - \sum_{a,b=1}^{n-1} \sum_{j=1}^g \sum_{r,s=2}^n s_r^j s_s^{j+g} (d\tau_r)_X(\hat{u}_a) (d\tau_s)_X(\hat{u}_b) (\partial^2 q)_{ab}^{-1}.$$

Here the \hat{u}_a are an oriented orthonormal basis of \mathfrak{t} . We introduce $T_{rs}: \mathfrak{k} \rightarrow \mathbb{R}$ given by¹⁶

$$T_{rs}(-X) = - \sum_{a,b=1}^{n-1} (d\tau_r)_X(\hat{u}_a) (d\tau_s)_X(\hat{u}_b) (\partial^2 q)_{ab}^{-1}.$$

Thus we may rewrite (10.7) as

$$(10.8) \quad \hat{\tau}(X) = \sum_{r,s=2}^n T_{rs}(X) \left(\sum_{j=1}^g s_r^j s_s^{j+g} \right).$$

We observe that in order for the pairing (10.5) to be nonzero, one requires $p_{r,j} = 0$ or 1 for all r and j (since the b_r^j are of odd degree). Further, in order for the expression (10.6) to yield a nonzero answer, we require for each $j = 1, \dots, g$ that

$$p_{2,j} + \dots + p_{n,j} = p_{2,j+g} + \dots + p_{n,j+g} = l_j$$

for some l_j . We may then rewrite (10.6) as

$$(10.9) \quad \left(\prod_{j=1}^g \left(\frac{\partial}{\partial s_{r_1}^j} \dots \frac{\partial}{\partial s_{r_{l_j}}^j} \right) \left(\frac{\partial}{\partial s_{s_1}^{j+g}} \dots \frac{\partial}{\partial s_{s_{l_j}}^{j+g}} \right) \int_{\mathcal{M}(n,d)} \Phi(\tau(X) \exp \hat{\tau}(X)) e^{f_2} \right)_{s_r^j=0 \forall r,j}$$

Because $\hat{\tau}$ is quadratic in the s_r^j and we are setting all the s_r^j to zero in the end, for each j we may represent the symbols $\frac{\partial}{\partial s_r^j}$ and $\frac{\partial}{\partial s_r^{j+g}}$ as 1-valent vertices (labelled by r) in a bipartite graph: there must be exactly one edge coming out of each of these vertices, and these edges must connect the symbol $\frac{\partial}{\partial s_r^j}$ with a symbol $\frac{\partial}{\partial s_s^{j+g}}$ for some s . Such bipartite graphs of course correspond to permutations σ_j of $\{1, \dots, l_j\}$.

It follows from (10.8) that

$$\frac{\partial}{\partial s_r^j} \frac{\partial}{\partial s_s^{j+g}} \hat{\tau}(X) = T_{rs}(X) \text{ for any } j$$

¹⁶Notice that T_{rs} is an invariant polynomial on \mathfrak{k} .

so that

$$(10.10) \quad \prod_{j=1}^g \left(\frac{\partial}{\partial s_{r_1}^j} \cdots \frac{\partial}{\partial s_{r_{l_j}}^j} \right) \left(\frac{\partial}{\partial s_{s_1}^{j+g}} \cdots \frac{\partial}{\partial s_{s_{l_j}}^{j+g}} \right) \int_{\mathcal{M}(n,d)} \Phi(\tau(X) \exp \hat{\tau}(X)) e^{f_2}$$

$$= \int_{\mathcal{M}(n,d)} \Phi \left(\prod_{j=1}^g \sum_{\sigma_j} T_{r_1 s_{\sigma_j(1)}}(X) \cdots T_{r_{l_j} s_{\sigma_j(l_j)}}(X) \tau(X) \right) e^{f_2}$$

where we sum over all permutations σ_j of $\{1, \dots, l_j\}$. Hence we obtain by Remark 8.2 (a) and Lemma 6.1:

THEOREM 10.5.

$$(10.11) \quad \int_{\mathcal{M}(n,d)} \prod_{j=1}^g b_{r_1}^j \cdots b_{r_{l_j}}^j b_{s_1}^{j+g} \cdots b_{s_{l_j}}^{j+g} \Phi(\tau(X)) e^{f_2}$$

$$= \int_{\mathcal{M}(n,d)} \Phi \left(\prod_{j=1}^g \sum_{\sigma_j} T_{r_1 s_{\sigma_j(1)}}(X) \cdots T_{r_{l_j} s_{\sigma_j(l_j)}}(X) \tau(X) \right) e^{f_2}$$

which equals $(-1)^{n+(g-1)\frac{n^g}{n}}$ times the iterated residue

$$\sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \cdots \text{Res}_{Y_{n-1}=0} \frac{\prod_{j=1}^g \sum_{\sigma_j} T_{r_1 s_{\sigma_j(1)}}(-X) \cdots T_{r_{l_j} s_{\sigma_j(l_j)}}(-X) \tau(-X) e^{-\langle [w\hat{c}], X \rangle}}{\mathcal{D}^{2g-2}(X) (1 - \exp(-Y_1)) \cdots (1 - \exp(-Y_{n-1}))}$$

Let \hat{u}_a ($a = 1, \dots, n-1$) denote an oriented orthonormal basis on \mathfrak{t} . For $X \in \mathfrak{t}$ define coordinates Z_a by $Z_a = (X, \hat{u}_a)$ so that $X = \sum_a Z_a \hat{u}_a$. Write the Maurer-Cartan form θ on T as $\theta = \sum_a \theta_a \hat{u}_a$; then the θ_a form a set of generators of $H^1(T)$.

Definition 10.6. A set of generators $\{\zeta_a^j\}$ ($j = 1, \dots, 2g; a = 1, \dots, n-1$) for $H^1(T^{2g})$ is defined by specifying that $\zeta_a^j = \pi_j^* \theta_a$ where $\pi_j: T^{2g} \rightarrow T$ is the projection onto the j^{th} copy of T .

LEMMA 10.7. $\int_T \theta_1 \wedge \cdots \wedge \theta_{n-1} = \text{vol}(T)$. Here, $\text{vol}(T)$ is the Riemannian volume of $T = \mathfrak{t}/\Lambda^I$ in the metric $\langle \cdot, \cdot \rangle$; in other words it is given by $(\det E)^{\frac{1}{2}} = \sqrt{n}$ where E is the $(n-1) \times (n-1)$ matrix (known as the Cartan matrix) given by $E_{ab} = \langle \hat{e}_a, \hat{e}_b \rangle$ in terms of the basis for the integer lattice $\Lambda^I \subset \mathfrak{t}$ over \mathbb{Z} given by the simple roots $\{\hat{e}_a\}$, $a = 1, \dots, n-1$.

LEMMA 10.8. *The restriction of $\tilde{f}_{(q)}(X)_1$ to T^{2g} is given in terms of the generators ζ_a^j of $H^1(T^{2g})$ by*

$$\tilde{f}_{(q)}(X)_1|_{T^{2g}} = \frac{1}{2} \sum_{a,b} \partial^2 q(\hat{u}_a, \hat{u}_b) \sum_{j=1}^g (-\zeta_a^j \zeta_b^{j+g} + \zeta_a^{j+g} \zeta_b^j),$$

where the $\{\hat{u}_a\}$ are an oriented orthonormal basis of \mathfrak{t} .

Proof. We need to understand the restriction of $\tilde{f}_{(q)}$ to T^{2g} . As in (9.6), we have

$$\begin{aligned} \tilde{f}_{(q)}(X)_1 &= \left(\sum_{j=1}^g (-\text{ev}_{\gamma_j^1} \times \text{ev}_{x_j} + \text{ev}_{\gamma_{j+g}^0} \times \text{ev}_{x_{j+g}}) \right)^* \Phi_2^K(q)(X) \\ &\quad + \left(\sum_{j=1}^g (-\text{ev}_{\gamma_{j+g}^1} \times \text{ev}_{x_{j+g}} + \text{ev}_{\gamma_j^0} \times \text{ev}_{x_j}) \right)^* \Phi_2^K(q)(X) \in \Omega_K^*(K^{2g}) \end{aligned}$$

where (after restricting to $T \times T \times T$)

$$\bar{\Phi}_2^K(q)|_{T \times T \times T}(-X) = \int_{(t_0, t_1, t_2) \in \Delta^2} q \left(\sum_{k=0}^2 dt_k \theta^{(k)} + X \right) \in H_T^*(T \times T \times T)$$

([30, above (5.6)]) and $\bar{\Phi}_2^K(q) = \sigma_2^* \bar{\Phi}_2^K(q)$ where $\sigma_2: (g_1, g_2) \mapsto (g_1 g_2, g_2, 1)$. By (10.12),

$$(10.13) \quad \bar{\Phi}_2^K(q)|_{T \times T}(-X) = -\frac{1}{2} \sum_{a,b} \partial^2 q(\hat{u}_a, \hat{u}_b) \zeta_a^1 \zeta_b^2 \in H_T^*(T \times T).$$

For purposes of evaluation on T^{2g} the generators γ_j^r in (9.6) reduce to

$$\gamma_j^0 = \gamma_{j+g}^1 = 1, \quad \gamma_j^1 = x_{j+g}, \quad \gamma_{j+g}^0 = x_j,$$

where x_1, \dots, x_{2g} are the chosen generators of \mathbb{F}^{2g} . So we get from (9.6)

$$\tilde{f}_{(q)}(X)_1|_{T^{2g}} = \sum_{j=1}^g (-\text{ev}_{x_{j+g}} \times \text{ev}_{x_j} + \text{ev}_{x_j} \times \text{ev}_{x_{j+g}})^* \Phi_2^K(q)(X).$$

We find that

$$\begin{aligned} \tilde{f}_{(q)}(-X)_1|_{T^{2g}} &= -\frac{1}{2} \sum_{a,b} \partial^2 q(\hat{u}_a, \hat{u}_b) \sum_{j=1}^g (\zeta_a^j \zeta_b^{j+g} - \zeta_a^{j+g} \zeta_b^j) \\ &= -\sum_{a,b} \partial^2 q(\hat{u}_a, \hat{u}_b) \sum_{j=1}^g \zeta_a^j \zeta_b^{j+g}. \quad \square \end{aligned}$$

Similarly for $\Lambda \in \mathfrak{t} \subset \mathfrak{k}$,

$$\tilde{f}_{(q)}(-X)_2(\Lambda) = -(dq)_X(\Lambda)$$

(see Lemma 9.8).

As a result we see immediately that:

LEMMA 10.9. *Suppose $c \exp \Lambda = 1$. Then*

$$\begin{aligned} \int_{T^{2g} \times \{\Lambda\}} \exp \tilde{f}_{(q)}(-X) &= \int_{T^{2g} \times \{\Lambda\}} \exp \tilde{f}_{q_2}(-X) (\det H_{\mathbf{t}}(X))^g \\ &= e^{-(dq)_X(\Lambda)} \int_{T^{2g}} (\det H_{\mathbf{t}}(X))^g \exp \omega \end{aligned}$$

where ω is the standard symplectic form on T^{2g} and the quadratic form $H_{\mathbf{t}}(X)$ is the Hessian of the restriction of $-q$ to \mathbf{t} (evaluated on an oriented orthonormal basis of \mathbf{t}). In other words,

$$H_{\mathbf{t}}(X)_{ab} = -(\partial^2 q)_X(\hat{u}_a, \hat{u}_b)$$

where $\{\hat{u}_a: a = 1, \dots, n-1\}$ is an oriented orthonormal basis for \mathbf{t} .

Proof. This follows by integrating

$$\exp - \sum_{a,b} \partial^2 q(\hat{u}_a, \hat{u}_b) \sum_{j=1}^g (\zeta_a^j \zeta_b^{j+g})$$

over T^{2g} . (Notice that $\partial^2 q(\hat{u}_a, \hat{u}_b)$ is symmetric in a and b). □

LEMMA 10.10.

$$\int_{T^{2g}} \exp \omega = n^g.$$

Proof. This follows from Lemmas 10.7 and 10.8. □

In order to prove Proposition 10.2, note that by Theorem 9.11(b),

$$\int_{N_n(c)} \Phi(e^{\tilde{f}_{(q)}} \mathcal{D}_n \eta)$$

equals $\frac{(-1)^{n+(g-1)}}{n!}$ times the iterated residue

$$\sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \frac{\int_{T^{2g} \times \{-[[w\bar{c}]]\}} (e^{\tilde{f}_{(q)}(-X)} \eta(-X))}{\mathcal{D}(X)^{2g-2} (1 - e^{-B(X)_{n-1}}) \dots (1 - e^{-B(X)_1})}$$

This applies in particular when $\eta(X) = \tau(-X)$ is a linear combination of monomials $\prod_r \tilde{a}_r^{m_r}$ in the \tilde{a}_r which does not involve the \tilde{b}_r^j ; since $\tilde{a}_r(X) = \tau_r(-X)$, it is natural to write $\eta(-X) = \tau(X)$. For η of this form, the expression above equals

$$\begin{aligned} &\frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \\ &\frac{\int_{T^{2g} \times \{-[[w\bar{c}]]\}} e^{\tilde{f}_{q_2}(-X)} (\det H_{\mathbf{t}}(X))^g \tau(X)}{\mathcal{D}(X)^{2g-2} (1 - e^{-B(X)_{n-1}}) \dots (1 - e^{-B(X)_1})} \end{aligned}$$

by Lemma 10.9.

We now replace X by $B^{-1}(X)$ (where the transformation $B^{-1} : \mathfrak{k} \rightarrow \mathfrak{k}$ was defined above Proposition 10.2). This change of variables produces a Jacobian $(\det H_{\mathfrak{t}}(B^{-1}(X)))^{-1}$. Thus we obtain

$$(10.14) \quad \int_{N_n(c)} \Phi(\mathcal{D}e^{\tilde{f}(q)}\eta) = \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(\left(\det H_{\mathfrak{t}}(B^{-1}(X)) \right)^{g-1} \times \frac{\int_{T^{2g} \times \{-[[w\tilde{c}]]\}} e^{\tilde{f}_{q_2}(-B^{-1}(X))} \tau(B^{-1}(X))}{\mathcal{D}^{2g-2}(B^{-1}(X))(1 - e^{-Y_{n-1}}) \dots (1 - e^{-Y_1})} \right).$$

Now we have the statement:

LEMMA 10.11.

$$\mathcal{D}^2(B^{-1}(X)) = \mathcal{D}^2(X)(\det(-\partial^2 q)_{\mathfrak{t}^\perp})^{-1},$$

where $(\partial^2 q)_{\mathfrak{t}^\perp}$ denotes the restriction of the symmetric bilinear form $(\partial^2 q)_X$ on \mathfrak{k} to a symmetric bilinear form on \mathfrak{t}^\perp , which is then identified with a linear map from \mathfrak{t}^\perp to itself using the fixed invariant inner product.

This lemma will be used in establishing Proposition 10.2 since the Hessian H appearing in that proposition is the Hessian of the (K -invariant) function $-q: \mathfrak{k} \rightarrow \mathbb{R}$, which is block diagonal with one block being the Hessian $H_{\mathfrak{t}}$ of the restriction of this function to \mathfrak{t} and the other block being $-(\partial^2 q)_{\mathfrak{t}^\perp}$.

Proof. We introduce the (orthonormal) basis X_γ, Y_γ for \mathfrak{t}^\perp corresponding to the positive roots γ , and a corresponding system of coordinates x_γ, y_γ on \mathfrak{t}^\perp ; we have

$$[X_\gamma, X] = \gamma(X)Y_\gamma, \quad [Y_\gamma, X] = -\gamma(X)X_\gamma.$$

We observe that the map B and its inverse B^{-1} on \mathfrak{k} are K -equivariant, and map \mathfrak{t} to \mathfrak{t} and \mathfrak{t}^\perp to \mathfrak{t}^\perp . Hence

$$(10.15) \quad (dB^{-1})_X([X_\gamma, X]) = [X_\gamma, B^{-1}(X)]$$

and

$$B^{-1}(\text{Ad exp}(X_\gamma)(X)) = \text{Ad exp}(X_\gamma)(B^{-1}(X)),$$

and similarly for Y_γ . We find

$$(10.16) \quad \frac{\gamma(B^{-1}(X))}{\gamma(X)} = (d(B^{-1})_{y_\gamma})_X(Y_\gamma) = (d(B^{-1})_{x_\gamma})_X(X_\gamma)$$

where $(B^{-1})_{x_\gamma}, (B^{-1})_{y_\gamma} : \mathfrak{t}^\perp \rightarrow \mathbb{R}$ are the coordinate functions in the directions x_γ and y_γ . Thus,

$$(10.17) \quad \begin{aligned} \mathcal{D}^2(B^{-1}(X)) &= \mathcal{D}^2(X)(\det d(B^{-1})_\perp) \\ &= \mathcal{D}^2(X)(\det -\partial^2 q)_{\mathfrak{t}^\perp}^{-1} \end{aligned}$$

where $d(B^{-1})_\perp$ is the square matrix of partial derivatives of the \mathfrak{t}^\perp components of B^{-1} in the directions along \mathfrak{t}^\perp . This completes the proof of Lemma 10.11. \square

Proposition 10.2 now follows immediately by using (10.14) and Lemma 10.11 to express (10.2) as an iterated residue, and observing that Theorem 8.1 (in the version given by Remark 8.3 (a)) yields the same iterated residue for (10.3).

Let us now consider the proof of Proposition 10.3. For the rest of this section let $a = 1, \dots, n - 1$ index an oriented orthonormal basis $\{\hat{u}_a\}$ of \mathfrak{t} . We have $\tilde{b}_r^j(X) = \text{pr}_1^* \tilde{b}_r^{j,1}$ where $\tilde{b}_r^{j,1} = \text{ev}_{x_j}^* \Phi_1^K(\tau_r)$ and $\Phi_1^K(\tau_r) = \sigma_1^* \bar{\Phi}_1^K(\tau_r)$ where $\bar{\Phi}_1^K(\tau_r)$ is as defined by (9.8). Also, x_j (for $j = 1, \dots, 2g$) are our chosen set of generators of $H_1(\Sigma)$.

Theorem 8.1 applies when $\eta(X) = \tau(X) \exp \sum_{j=1}^{2g} \sum_{r \geq 2} s_r^j \tilde{b}_r^j(X)$ for $s_r^j \in \mathbb{C}$ and $\tau \in S(\mathfrak{k}^*)^K$. Define $S^j \in S(\mathfrak{k}^*)^K$ by $S^j(X) = \sum_{r \geq 2} s_r^j \tau_r(X)$; we then define $\tilde{b}_{S^j}^j$ by $\tilde{b}_{S^j}^j(X) = \sum_{r \geq 2} s_r^j \tilde{b}_r^j(X)$.

LEMMA 10.12 (a). *The restriction to T^{2g} of $\tilde{b}_r^j(-X)$ is $\sum_{a=1}^{n-1} (d\tau_r)_X(\hat{u}_a) \zeta_a^j$ where ζ_a^j (for $a = 1, \dots, n - 1$ and $j = 1, \dots, 2g$) are the elements of the basis of $H^1(T^{2g})$ corresponding to an oriented orthonormal basis $\{\hat{u}_a\}$ for \mathfrak{t} .*

(b) *The restriction to T^{2g} of $\tilde{b}_{S^j}^j(-X)$ is $\sum_{a=1}^{n-1} (dS^j)_X(\hat{u}_a) \zeta_a^j$.*

Proof. We have by (8.21) of [30] that

$$\tilde{b}_r^j(X)_1 = \text{ev}_{x_j}^* \Phi_1^K(\tau_r).$$

By (9.9),

$$\Phi_1^K(\tau_r)|_{T^{2g}}(-X) = \sigma_1^* \bar{\Phi}_1^K(\tau_r)|_{T^{2g}}(-X) = \sum_{a=1}^{n-1} (d\tau_r)_X(\hat{u}_a) \theta_a$$

so that

$$\tilde{b}_r^j(-X)_1 |_{T^{2g}} = \text{ev}_{x_j}^* \sigma_1^* \bar{\Phi}_1^K(\tau_r)|_{T^{2g}}(-X) = \sum_{a=1}^{n-1} (d\tau_r)_X(\hat{u}_a) \zeta_a^j$$

since the generators ζ_a^j of $H^1(T^{2g})$ become identified with the components θ_a of the Maurer-Cartan form on the j^{th} copy of T in T^{2g} . \square

LEMMA 10.13. *In the notation introduced just before Lemma 10.12,*

$$(10.18) \quad \int_{T^{2g}} \exp \tilde{f}_{(q)}(-X) \exp \sum_{j,r} s_r^j \tilde{b}_r^j(-X) = \int_{T^{2g}} \exp \tilde{f}_{(q)}(-X) \exp \hat{\tau}(X)$$

where

$$\begin{aligned}
 (10.19) \quad \hat{\tau}(X) &= - \sum_{a,b=1}^{n-1} \sum_{j=1}^g (dS^j)_X(\hat{u}_a)(dS^{j+g})_X(\hat{u}_b)(\partial^2 q)_{ab}^{-1} \\
 &= - \sum_{a,b=1}^{n-1} \sum_{r,s=2}^n \sum_{j=1}^g s_r^j s_s^{j+g} (d\tau_r)_X(\hat{u}_a)(d\tau_s)_X(\hat{u}_b)(\partial^2 q)_{ab}^{-1}.
 \end{aligned}$$

Here, $\{\hat{u}_a: a = 1, \dots, n - 1\}$ denotes an oriented orthonormal basis of \mathfrak{t} .

Proof. We need to consider the left-hand side of (10.18), which is

$$(10.20) \quad \int_{T^{2g}} \exp \tilde{f}_{(q)}(-X) \exp \sum_{j,r} s_r^j \tilde{b}_r^j(-X).$$

By Lemma 10.8 the restriction of $\tilde{f}_{(q)}(-X)$ to T^{2g} is $\exp -\frac{1}{2} \sum_{a,b} \sum_{j=1}^g (\partial^2 q)_X(\hat{u}_a, \hat{u}_b)(\zeta_a^j \zeta_b^{j+g} - \zeta_a^{j+g} \zeta_b^j)$ while $\exp \sum_{j,r} s_r^j \tilde{b}_r^j(-X)$ restricts on T^{2g} (by Lemma 10.12) to

$$\exp \sum_{r,j} s_r^j \sum_{a=1}^{n-1} (d\tau_r)_X(\hat{u}_a) \zeta_a^j.$$

Thus for any given $j = 1, \dots, g$ we must compute the integral

$$(10.21) \quad \int_{T^2} \exp \left(\sum_{\sigma,\tau} y_\sigma A^{\sigma\tau} y_\tau / 2 + \sum_\sigma y_\sigma B_j^\sigma \right)$$

where σ runs over pairs (a, i) for $a = 1, \dots, n - 1$ and $i = 0, 1$ (where $i = 0$ corresponds to j and $i = 1$ to $j + g$) and $y_{a,i} = \zeta_a^{j+gi}$. Here, the matrix A is given by

$$(10.22) \quad A^{a0,b1} = -(\partial^2 q)_X(\hat{u}_a, \hat{u}_b) = -A^{a1,b0}; \quad A^{a0,b0} = A^{a1,b1} = 0;$$

thus the Pfaffian of A (whose square is $\det A$) is given by

$$\text{Pf}(A) = \det \left(-\partial^2 q|_{\mathfrak{t}} \right).$$

For $j = 1, \dots, g$ the vector B_j is

$$(10.23) \quad B_j^{a0} = -(\partial S^j)_X(\hat{u}_a); \quad B_j^{a1} = -(\partial S^{j+g})_X(\hat{u}_a).$$

The result is that

$$\begin{aligned}
 (10.24) \quad &\int_{T^2} \exp \left(\sum_{\sigma,\tau} y_\sigma A^{\sigma\tau} y_\tau / 2 + \sum_\sigma y_\sigma B_j^\sigma \right) \\
 &= \int_{T^2} \exp \left(\sum_{\sigma,\tau} y_\sigma A^{\sigma\tau} y_\tau / 2 \right) \exp(-B_j^t A^{-1} B_j) / 2 \\
 &= \text{Pf}(A) \exp(-B_j^t A^{-1} B_j) / 2,
 \end{aligned}$$

where B_j^t denotes the transpose of the vector B_j . Thus we find that (10.20) becomes

$$(10.25) \quad \det(-\partial^2 q|_{\mathfrak{t}})^g \exp \left(\sum_{a,b=1}^{n-1} \sum_{j=1}^g (dS^j)_X(\hat{u}_a)(dS^{j+g})_X(\hat{u}_b)(\partial^2 q)_{a,b}^{-1} \right),$$

which equals the right-hand side of (10.18). This completes the proof of Lemma 10.13. \square

Proposition 10.3 follows from Theorem 9.11 once we have shown that

$$\int_{T^{2g}} \exp \tilde{f}_{(q)}(-X) \exp \sum_{r \geq 2} s_r^j \tilde{b}_r^j(-X) = \int_{T^{2g}} \exp \tilde{f}_{(q)}(-X) \exp \hat{\tau}(X)$$

where $\hat{\tau}$ is given by (10.7). This is now clear from Lemma 10.13. \square

11. The Verlinde formula

The Verlinde formula is a formula for the dimension $D_{n,d}(g, k)$ of the space of holomorphic sections of powers of \mathcal{L} , where \mathcal{L} is a particular line bundle over $\mathcal{M}(n, d)$; it has been proved by Beauville and Laszlo [6], Faltings [20], Kumar, Narasimhan and Ramanathan [38] and Tsuchiya, Ueno and Yamada [48]. In this section we show how the Verlinde formula follows from our formula (Theorem 8.1) for intersection pairings in $\mathcal{M}(n, d)$.

A line bundle \mathcal{L} over $\mathcal{M}(n, d)$ may be defined for which $c_1(\mathcal{L}) = n f_2$, since $n f_2 \in H^2(\mathcal{M}(n, d), \mathbb{Z})$ (see [16]). As described in Section 1, this bundle is the *determinant line bundle*. Whenever k is a positive integer divisible by n , we then define

$$(11.1) \quad D_{n,d}(g, k) = \dim H^0(\mathcal{M}(n, d), \mathcal{L}^{k/n}).$$

Let us introduce

$$r = k + n;$$

let us also introduce the highest root γ_{\max} , which is given by $\gamma_{\max}(X) = X_n - X_1$ or $\gamma_{\max} = e_1 + e_2 + \dots + e_{n-1}$. We then make the following definition:

Definition 11.1. The Verlinde function $V_{n,d}(g, k)$ is given by

$$V_{n,d}(g, k) = \sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+ : \langle \lambda, \gamma_{\max} \rangle < r} \frac{e^{-2\pi i \langle \lambda - \rho, \vec{c} \rangle}}{(S_{0\lambda}(k))^{2g-2}}$$

where ρ is half the sum of the positive roots and

$$S_{0\lambda}(k) = \frac{1}{\sqrt{n} r^{(n-1)/2}} \prod_{\gamma > 0} 2 \sin \pi \langle \gamma, \lambda \rangle / r.$$

(See [21, (A.44)] and [49, (3.16)].) Verlinde’s conjecture says that the Verlinde function specifies the dimension of the space of holomorphic sections of $\mathcal{L}^{k/n}$:

THEOREM 11.2 (Verlinde’s conjecture).

$$D_{n,d}(g, k) = V_{n,d}(g, k).$$

We shall show how to extract Verlinde’s conjecture from our previous results: an outline of the method we use was given by Szenes ([45, (§4.2)]).

In fact $H^i(\mathcal{M}(n, d), \mathcal{L}^m) = 0$ for all $i > 0$ and $m > 0$ by an argument using the Kodaira vanishing theorem and the facts that \mathcal{L} is a positive line bundle and the canonical bundle of $\mathcal{M}(n, d)$ is equal to \mathcal{L}^{-2} (see [5, §5] and Théorème F of [16]); now $D_{n,d}(g, k)$ is given for $k > 0$ by the Riemann-Roch formula:

$$(11.2) \quad D_{n,d}(g, k) = \int_{\mathcal{M}(n,d)} \text{ch}\mathcal{L}^{k/n} \text{td}\mathcal{M}(n, d).$$

We use the following results to convert (11.2) into a form to which we may apply our previous results.

LEMMA 11.3. *For any complex manifold M the Todd class of M is given by*

$$\text{td}(M) = e^{c_1(M)/2} \hat{A}(M)$$

where $c_1(M)$ is the first Chern class of the holomorphic tangent bundle of M , and $\hat{A}(M)$ is the A -roof genus of M .

Proof. See for example [22, pages 97–99]. □

PROPOSITION 11.4.

$$\hat{A}(\mathcal{M}(n, d)) = \Phi \left(\prod_{\gamma>0} \frac{\gamma(X)/2}{\sinh \gamma(X)/2} \right)^{2g-2}.$$

Proof. This is proved by Newstead¹⁷ in [42].

LEMMA 11.5. *We have*

$$c_1(\mathcal{M}(n, d)) = 2n f_2.$$

¹⁷Newstead writes the details of the proof only for $n = 2$ but the same proof yields the result for general n .

Proof. This is proved in [16, Th. F]. □

Of course the Chern character of $\mathcal{L}^{k/n}$ is given by $\text{ch}\mathcal{L}^{k/n} = e^{kf_2}$. Thus we obtain:

COROLLARY 11.6. *The quantity $D_{n,d}(g, k)$ is given by*

$$D_{n,d}(g, k) = \int_{\mathcal{M}(n,d)} e^{(k+n)f_2} \Phi \left(\prod_{\gamma>0} \frac{\gamma(X)}{e^{\gamma(X)/2} - e^{-\gamma(X)/2}} \right)^{2g-2}.$$

Proof. This follows immediately from (11.2), Lemmas 11.3 and 11.5 and Proposition 11.4. □

THEOREM 11.7. *We have*

$$(11.3) \quad D_{n,d}(g, k) = \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Y_1=0} \dots \text{Res}_{Y_{n-1}=0} \left(e^{r\langle [[w\tilde{c}]], X \rangle} \int_{T^{2g}} e^{r\omega} \prod_{\gamma>0} \left(\frac{\gamma(X)}{e^{\gamma(X)/2} - e^{-\gamma(X)/2}} \right)^{2g-2} \times \frac{1}{\prod_{j=1}^l (e^{rY_j} - 1) \mathcal{D}(X)^{2g-2}} \right).$$

Proof. This comes straight from Corollary 11.6 and Theorem 8.1. Note that because the factor e^{f_2} in the statement of Theorem 8.1 has been replaced by e^{rf_2} , it is necessary to replace $e^{\langle [[w\tilde{c}]], X \rangle}$ by $e^{r\langle [[w\tilde{c}]], X \rangle}$, and $e^{Y_j} - 1$ by $e^{rY_j} - 1$ (cf. Remark 8.3 (c)). □

We introduce $Z_j = \exp Y_j$. Since for any $w \in W_{n-1}$,

$$[[w\tilde{c}]] = [[w\tilde{c}]]_1 \hat{e}_1 + [[w\tilde{c}]]_2 \hat{e}_2 + \dots + [[w\tilde{c}]]_{n-1} \hat{e}_{n-1}$$

(as in the statement of Proposition 2.2) with $n[[w\tilde{c}]]_j \in \mathbb{Z}$ for all j , and $0 \leq [[w\tilde{c}]]_j < 1$ for all j , we obtain

$$e^{r\langle [[w\tilde{c}]], X \rangle} = Z_1^{[[w\tilde{c}]]_1 r} Z_2^{[[w\tilde{c}]]_2 r} \dots Z_{n-1}^{[[w\tilde{c}]]_{n-1} r}.$$

(Recall that k and r are divisible by n so $e^{r\langle \tilde{c}, X \rangle}$ is a well-defined single-valued function of Z_1, \dots, Z_{n-1} .) Thus we can equate $D_{n,d}(g, k)$ with

$$(11.4) \quad \frac{(-1)^{n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Z_1=1} \dots \text{Res}_{Z_{n-1}=1} \left(\left(\prod_{j=1}^{n-1} \frac{1}{Z_j} \right) \int_{T^{2g}} e^{r\omega} \times \frac{Z_1^{[[w\tilde{c}]]_1 r} Z_2^{[[w\tilde{c}]]_2 r} \dots Z_{n-1}^{[[w\tilde{c}]]_{n-1} r}}{\prod_{\gamma>0} (\tilde{\gamma}^{1/2} - \tilde{\gamma}^{-1/2})^{2g-2} (Z_1^r - 1) \dots (Z_{n-1}^r - 1)} \right)$$

$$= \frac{(-1)^{n-1+n+(g-1)}}{n!} \sum_{w \in W_{n-1}} \text{Res}_{Z_1=1} \dots \text{Res}_{Z_{n-1}=1} \left(\left(\prod_{j=1}^{n-1} \frac{1}{Z_j} \right) \times \int_{T^{2g}} e^{r\omega} \frac{Z_1^{-[[w\tilde{c}]]_1 r} Z_2^{-[[w\tilde{c}]]_2 r} \dots Z_{n-1}^{-[[w\tilde{c}]]_{n-1} r}}{\prod_{\gamma>0} (\tilde{\gamma}^{1/2} - \tilde{\gamma}^{-1/2})^{2g-2} (Z_1^{-r} - 1) \dots (Z_{n-1}^{-r} - 1)} \right).$$

Here, we have introduced $\tilde{\gamma}$ defined (for the root $\gamma = e_r + e_{r+1} + \dots + e_{s-1}$) by

$$\tilde{\gamma}(Z_1, \dots, Z_{n-1}) = Z_r \dots Z_{s-1}.$$

We also have:

LEMMA 11.8.

$$\int_{T^2} e^\omega = n$$

and hence

$$\int_{T^{2g}} e^{r\omega} = r^{(n-1)g} n^g.$$

Proof. This follows from Lemma 10.10. □

The following may be proved by the same method as in Section 2 (see [45]):

PROPOSITION 11.9. *Suppose f is the meromorphic function on the complexification $T^{\mathbb{C}}$ of T defined by*

$$(11.5) \quad f(Z) = (-1)^{n-1} (-1)^{n+(g-1)} r^{(n-1)(g-1)} n^{g-1} \frac{Z_1^{-\tilde{c}_1 r} \dots Z_{n-1}^{-\tilde{c}_{n-1} r}}{\prod_{\gamma>0} (\tilde{\gamma}^{1/2} - \tilde{\gamma}^{-1/2})^{2g-2}}.$$

Then,

$$(11.6) \quad \frac{1}{(n-1)!} \text{Res}_{Z_1=1} \dots \text{Res}_{Z_{n-1}=1} \sum_{w \in W_{n-1}} \prod_{j=1}^{n-1} \left(\frac{r}{Z_j} \right) \frac{[[wf]](Z)}{\prod_{j=1}^{n-1} (Z_j^{-r} - 1)} = \sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+ : \langle \lambda, \gamma_{\text{max}} \rangle < r} f(\exp 2\pi i \lambda / r).$$

Here, W_{n-1} is the permutation group on $\{1, \dots, n-1\}$ which is (isomorphic to) the Weyl group of $SU(n-1)$, and $[[wf]]$ is the function

$$(11.7) \quad [[wf]](Z) = (-1)^{n-1} (-1)^{n+(g-1)} r^{(n-1)(g-1)} n^{g-1} \frac{Z_1^{-[[w\tilde{c}]]_1 r} \dots Z_{n-1}^{-[[w\tilde{c}]]_{n-1} r}}{\prod_{\gamma>0} (\tilde{\gamma}^{1/2} - \tilde{\gamma}^{-1/2})^{2g-2}}.$$

Remark. Notice that

$$\sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+ : \langle \lambda, \gamma_{\text{max}} \rangle < r} f(\exp 2\pi i \lambda / r) = \frac{1}{n-1} \sum_{m_j=1}^{r-1} f \left(e^{2\pi i (\sum_j m_j w_j) / r} \right).$$

(Here, the w_j are the fundamental weights, which are dual to the simple roots.) The set $\{X \in \mathfrak{t}: X = \sum_j \lambda_j \hat{e}_j, 0 \leq \lambda_j < 1, j = 1, \dots, n-1\}$ is a fundamental domain for the action of the integer lattice Λ^I on \mathfrak{t} , while the set $\{X \in \mathfrak{t}_+ \subset \mathfrak{t}: \gamma_{\max}(X) < 1\}$ is a fundamental domain for the affine Weyl group W_{aff} (the semidirect product of the Weyl group and the integer lattice), and Λ^I has index $(n-1)!$ (rather than $n!$) in W_{aff} (in other words a fundamental domain for Λ^I contains $(n-1)!$ fundamental domains for W_{aff}). This difference accounts for the factor $1/(n-1)!$ in Proposition 11.9 which replaces the factor $1/n!$ in its analogue Proposition 2.2.

Applying Proposition 11.9 we find (recalling from Section 2 that $(-1)^{n-1} = c^\rho$ when n and d are coprime) that

$$(11.8) \quad D_{n,d}(g, k) = (-1)^{n+(g-1)r^{(n-1)(g-1)}} n^{g-1} c^\rho \sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+: \langle \lambda, \gamma_{\max} \rangle < r} e^{-2\pi i \langle \tilde{c}, \lambda \rangle} \times \frac{1}{\prod_{\gamma > 0} (e^{2\pi i \langle \frac{\gamma}{2r}, \lambda \rangle} - e^{-2\pi i \langle \frac{\gamma}{2r}, \lambda \rangle})^{2g-2}}.$$

This gives

$$(11.9) \quad D_{n,d}(g, k) = (-1)^{n+(g-1)r^{(n-1)(g-1)}} n^{g-1} \sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+: \langle \lambda, \gamma_{\max} \rangle < r} e^{-2\pi i \langle \tilde{c}, \lambda - \rho \rangle} \times \frac{1}{\prod_{\gamma > 0} (2i \sin \pi \langle \gamma, \lambda \rangle / r)^{2g-2}}$$

$$(11.10) \quad = r^{(n-1)(g-1)} n^{g-1} \sum_{\lambda \in \Lambda_{\text{reg}}^w \cap \mathfrak{t}_+: \langle \lambda, \gamma_{\max} \rangle < r} e^{-2\pi i \langle \tilde{c}, \lambda - \rho \rangle} \times \frac{1}{\prod_{\gamma > 0} \left(2 \sin \frac{\pi \langle \gamma, \lambda \rangle}{r} \right)^{2g-2}}.$$

Comparing this with Definition 11.1, we see that $D_{n,d}(g, k) = V_{n,d}(g, k)$. This completes the proof of Theorem 11.2. □

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