Discrete Comput Geom 13:149–159 (1995)

Discrete & Computational
Geometry

Intersections of Curve Systems and the Crossing Number of $C_5 \times C_5$

R. B. Richter[1] and C. Thomassen[2]

[1] Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6
brichter@math.carleton.ca

[2] Mathematical Institute, Building 303, Technical University of Denmark, 2800 Lyngby, Denmark

Abstract. If $\mathscr{C}_1$ and $\mathscr{C}_2$ are two families of pairwise disjoint simple closed curves in the plane such that each curve in $\mathscr{C}_1$ intersects each curve in $\mathscr{C}_2$, then the total number of points of intersection in $\mathscr{C}_1 \cup \mathscr{C}_2$ is at least $2(m-1)n$, where $m = |\mathscr{C}_1| \le |\mathscr{C}_2| = n$, and this bound is best possible. We use this to show that the cartesian product of two 5-cycles has crossing number 15.

1. Introduction

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise crossings of edges among all drawings of G in the plane. There are very few classes of graphs for which the crossing numbers are known exactly.

The cycle of length n is denoted C_n. Harary *et al.* [H] conjectured that the crossing number of the cartesian product $C_m \times C_n$ is $(m-2)n$, for $3 \le m \le n$. (The *cartesian product* of C_m and C_n is a 4-regular graph on vertices $v_{i,j}$, $1 \le i \le m$, $1 \le j \le n$, with $v_{i,j}$ being adjacent to each of $v_{i \pm 1, j}$ and $v_{i, j \pm 1}$, with the first index being read modulo m and the second modulo n. The $m+n$ cycles obtained by fixing one of the coordinates are the *principal cycles*.)

To date, this conjecture has been verified only for $m = 3, 4$ [B], [R]. Beineke and Ringeisen wrote in 1980, "...it appears to be quite difficult to determine even the crossing number of $C_5 \times C_5$."

As an alternative approach to investigating $cr(\mathscr{C}_m \times \mathscr{C}_n)$, we consider intersection properties of curve systems in the Euclidean plane. If $\mathscr{C}$ is a collection of simple closed curves, then we denote by $i(\mathscr{C})$ the number of points of intersection. The general result mentioned in the abstract says that if $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$, where $\mathscr{C}_1$ and $\mathscr{C}_2$

consist of m and n, respectively, pairwise disjoint curves and every curve in $\mathscr{C}_1$ intersects every curve in $\mathscr{C}_2$, then $i(\mathscr{C}) \geq 2(m-1)n$, if $m \leq n$.

This conclusion is not true if we drop the condition that the members of $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, are pairwise disjoint, even in the case $m = 3$. (Thus, the complete conjecture of Harary *et al.* cannot be proved by these methods.) However, in special cases of interest, such as $m = n = 4$ and $m = n = 5$, this condition can be dropped, and thereby obtain the corollaries that $cr(C_4 \times C_4) = 8$ and $cr(C_5 \times C_5) = 15$.

2. Intersections of Two Pairwise Disjoint Curve Systems

In this section we prove the result mentioned in the abstract about the number of intersections of two families of pairwise disjoint simple closed curves. To be more specific, a *disjoint* (m, n)*-mesh* is a pair $(\mathscr{C}_1, \mathscr{C}_2)$ of families, each consisting of pairwise disjoint simple closed curves, with $|\mathscr{C}_1| = m$ and $|\mathscr{C}_2| = n$, such that every curve in $\mathscr{C}_1$ intersects every curve in $\mathscr{C}_2$. Further, we assume that no point in the plane is in more than two of the curves in $\mathscr{C}_1 \cup \mathscr{C}_2$.

To simplify the notation, for a disjoint (m, n)-mesh $(\mathscr{C}_1, \mathscr{C}_2)$, let $i(\mathscr{C}_1, \mathscr{C}_2)$ denote the number of points of intersection in $\mathscr{C}_1 \cup \mathscr{C}_2$ and let $i(m, n)$ denote the minimum of $i(\mathscr{C}_1, \mathscr{C}_2)$, with $(\mathscr{C}_1, \mathscr{C}_2)$ ranging over all disjoint (m, n)-meshes. We have the following result.

Theorem 1. *Let* $2 \leq m \leq n$. *Then*

$$i(n, m) = i(m, n) = 2(m-1)n.$$

Proof. That $i(m, n) \leq 2(m-1)n$ is seen by providing an appropriate figure, which is left to the reader. The interesting part of the proof is showing that $i(m, n) \geq 2(m-1)n$. This is done by induction on $m + n$. We can use as a base the case $m = 2$, which is trivial, as there are obviously at least $2n$ intersections.

Lemma 2. *Let* $(\mathscr{C}_1, \mathscr{C}_2)$ *be a disjoint* (m, n)*-mesh. Suppose there are distinct curves* $C_1, C_2, C_3 \in \mathscr{C}_1$ *with* C_1 *and* C_2 *in different regions of* $\mathbb{R}^2 \setminus C_3$. *Then*

$$i(\mathscr{C}_1, \mathscr{C}_2) \geq 2n + i(m-1, n).$$

Proof. Since every curve in $\mathscr{C}_2$ has a point in each of C_1 and C_2, each curve in $\mathscr{C}_2$ must meet C_3 in at least two points. Delete C_3 from $\mathscr{C}_1$ to get the result. □

Obviously, the symmetric conclusion holds in Lemma 2 with the roles of $\mathscr{C}_1$ and $\mathscr{C}_2$ interchanged. It is easy to see that if a disjoint (m, n)-mesh has a curve that separates two in the same class, then Lemma 2 and the inductive assumption show that this mesh has at least $2(m-1)n$ intersections. Therefore, the rest of the proof is devoted to dealing with the case that the disjoint (m, n)-mesh $(\mathscr{C}_1, \mathscr{C}_2)$ is *separation-free*, i.e., for $i = 1, 2$, $\mathscr{C}_i$ has that property that, for each $C \in \mathscr{C}_i$, no two curves in $\mathscr{C}_i$ lie in different components of $\mathbb{R}^2 \setminus C$. We prove the following, which completes the proof of Theorem 1.

Proposition 3. *Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a separation-free disjoint (m, n)-mesh, with $3 \le m \le n$. Then $i(\mathscr{C}_1, \mathscr{C}_2) \ge 2m(n-1)$.*

We require some preliminary facts, the first of which is a simple consequence of the Jordan Curve Theorem.

Lemma 4. *Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a (separation-free) disjoint (m, n)-mesh. If $C \in \mathscr{C}_1$ and $C' \in \mathscr{C}_2$ are such that $|C \cap C'| > 1$, then either $|C \cap C'|$ is even or there is a (separation-free) disjoint (m, n)-mesh $(\mathscr{C}'_1, \mathscr{C}'_2)$ such that $i(\mathscr{C}'_1, \mathscr{C}'_2) < i(\mathscr{C}_1, \mathscr{C}_2)$.*

Lemma 5. *Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a separation-free disjoint (m, n)-mesh and suppose $C \in \mathscr{C}_1$ and $C' \in \mathscr{C}_2$ exist such that $|C \cap C'| > 1$. Then either $|C \cap C'| \ge 4$ or there is a separation-free disjoint (m, n)-mesh $(\mathscr{C}'_1, \mathscr{C}'_2)$ such that $i(\mathscr{C}'_1, \mathscr{C}'_2) < i(\mathscr{C}_1, \mathscr{C}_2)$.*

Proof. By Lemma 4, the only other possibility is that $|C \cap C'| = 2$. Let A_1 and A_2 be the two components of $C \setminus C'$. Only one of these, say A_1, is in the component of $\mathbb{R}^2 \setminus C'$ that contains all the other curves in $\mathscr{C}_2$. Therefore, A_2 is disjoint from all the curves in both families (except, of course, C and C'). Similarly, there is an arc A'_2 of C' that is disjoint from all the curves in both families.

Now replace C by $(C \setminus A_2) \cup A'_2$ and C' by $(C' \setminus A'_2) \cup A_2$ and remove one of the two tangential intersections. The resulting separation-free disjoint (m, n)-mesh has fewer intersections than $(\mathscr{C}_1, \mathscr{C}_2)$. □

Proof of Proposition 3. Choose $(\mathscr{C}_1, \mathscr{C}_2)$ to be a separation-free disjoint (m, n)-mesh having fewest intersections. (We remark that it may be that $i(\mathscr{C}_1, \mathscr{C}_2) > i(m, n)$.) Let $C_1, C_2, C_3 \in \mathscr{C}_1$ and let $C'_1, C'_2, C'_3 \in \mathscr{C}_2$. If $i(\{C_1, C_2, C_3\}, \{C'_1, C'_2, C'_3\}) = 9$, then, putting a vertex inside each curve and drawing three arcs from the vertex to the three intersections would yield a planar drawing of $K_{3,3}$. Therefore, $i(\{C_1, C_2, C_3\}, \{C'_1, C'_2, C'_3\}) > 9$, so $i, j \in \{1, 2, 3\}$ exist such that $|C_i \cap C'_j| > 1$. By Lemma 5, $|C_i \cap C'_j| \ge 4$, so that $i(\{C_1, C_2, C_3\}, \{C'_j\}) \ge 6$.

For any $C' \in \mathscr{C}_2$, then $i(\{C_1, C_2, C_3\}, \{C'\})$ is either 3 or at least 6 and, by the preceding paragraph, there are at most two elements of $\mathscr{C}_2$ for which this number is 3. Therefore, $i(\{C_1, C_2, C_3\}, \mathscr{C}_2) \ge 6 + 6(n-2) = 6(n-1)$.

As there are $\binom{m}{3}$ ways of choosing C_1, C_2, C_3, and each intersection occurs in $\binom{m-1}{2}$ of them, there are at least

$$\frac{\binom{m}{3}}{\binom{m-1}{2}} 6(n-1)$$

intersections, as required. □

3. Intersections of (3, n)-Meshes

One of the motivations for considering meshes is that, in any planar drawing of $C_m \times C_n$, the principal cycles have the property that each one in one family (the m-cycles) intersects each one in the other (the n-cycles). Thus, we generalize disjoint meshes to allow more general configurations, which will include drawings of $C_m \times C_n$.

An (m, n)*-mesh* is any pair $(\mathscr{C}_1, \mathscr{C}_2)$ of families of planar closed curves (not necessarily simple, but with only finitely many self-intersections), such that $|\mathscr{C}_1| = m$, $|\mathscr{C}_2| = n$, and each curve in $\mathscr{C}_1$ intersects each curve in $\mathscr{C}_2$ in a non-self-intersection point. We let $i^*(\mathscr{C}_1, \mathscr{C}_2)$ denote the total number of intersections and self-intersections in $\mathscr{C}_1 \cup \mathscr{C}_2$. We also let $i^*(m, n)$ denote the least $i^*(\mathscr{C}_1, \mathscr{C}_2)$ over all (m, n)-meshes $(\mathscr{C}_1, \mathscr{C}_2)$.

An (m, n)-mesh $(\mathscr{C}_1, \mathscr{C}_2)$ is *optimal* if $i^*(\mathscr{C}_1, \mathscr{C}_2) = i^*(m, n)$.

We note the following basic facts.

Lemma 6.

(1) *If $(\mathscr{C}_1, \mathscr{C}_2)$ is an optimal (m, n)-mesh, then every curve in $\mathscr{C}_1 \cup \mathscr{C}_2$ is simple.*
(2) *If $(\mathscr{C}_1, \mathscr{C}_2)$ is an optimal (m, n)-mesh and $C \in \mathscr{C}_1$, $C' \in \mathscr{C}_2$ are such that $|C \cap C'| > 1$, then $|C \cap C'|$ is even.*

In the case $m = 3$, we have complete information.

Theorem 7. *For $n \geq 3$,*

$$i^*(3, n) = \begin{cases} 12, & n = 3, \\ 3n + \left\lceil \dfrac{n+3}{4} \right\rceil + \left\lceil \dfrac{n+4}{4} \right\rceil, & n > 3. \end{cases}$$

Proof. To see that the expression is a lower bound for $i^*(3, n)$, we proceed by induction on n. For the base, we note that any (3, 3)-mesh is a drawing of $C_3 \times C_3$, which has nine vertices and crossing number 3 [H]. Therefore, $i^*(3, 3) = 12$, as required.

Now assume $n \geq 4$. Let $(\mathscr{C}_1, \mathscr{C}_2)$ be any (3, n)-mesh and suppose there is some curve $C \in \mathscr{C}_2$ that has at least four intersections. Then $(\mathscr{C}_1, \mathscr{C}_2 \setminus \{C\})$ is a $(3, n-1)$-mesh. It follows that $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 4 + i^*(3, n-1)$ and we are done by induction.

Therefore, we can assume that every curve in $\mathscr{C}_2$ has only the three intersections that it must have to meet each curve in $\mathscr{C}_1$. It is easy to see that $\mathscr{C}_2$ is separation-free and, viewing $(\mathscr{C}_1, \mathscr{C}_2)$ as a 4-regular plane graph, each curve in $\mathscr{C}_2$ is a triangle bounding a face.

Consider $\mathscr{C}_1$ as a 4-regular plane graph H, having k vertices. Then it has $e = 2k$ edges and $f = k + 2$ faces. If f_i denotes the number of faces of length i, then $\sum_i if_i = 2e = 4k$.

Obviously, the n curves of $\mathscr{C}_2$ must go into faces of H and the faces used must be incident with all three of the curves in $\mathscr{C}_1$. It is an easy induction to see that we cannot put more than $i - 2$ curves from $\mathscr{C}_2$ into a face of length i, so that $n \leq \Sigma_i (i - 2)f_i \leq 4k - 2(k + 2) = 2k - 4$. (We remark that it may be that $\mathscr{C}_1$ is not a connected graph. However, it is easy to see that $n \leq k$ in this case, so that we still have $2k - 4 \geq n$.) Note that k is even, so $k \geq 2\lceil n/4 \rceil + 2$. Since $i^*(\mathscr{C}_1, \mathscr{C}_2) = 3n + k$, we are done.

We leave it to the reader to find the appropriate drawings (extracted from the above proof) to show that the expression is also an upper bound for $i^*(m, n)$. □

We point out that the number of intersections that actually arise from a drawing of $C_3 \times C_n$ is at least $4n$ [R]. Thus, we cannot hope to use these general methods to obtain the crossing number of $C_m \times C_n$, for all pairs (m, n).

For $m > 3$, we have a much less detailed picture. For $m \leq 6$, there is the same phenomenon of, for large n, $i^*(m, n)$ being smaller than the conjectured number of intersections from a drawing of $C_m \times C_n$. Using the methods of Theorem 7 we can prove that $i^*(4, n) = 5n + o(n)$, but we have not got an exact formula. For $m = 5$, 6, we do not have even this asymptotic information.

4. Evaluation of $i^*(4, 4)$, $i^*(4, 5)$, and $i^*(5, 5)$

In this section we prove three specific results, namely, $i^*(4, 4) = 24$, $i^*(4, 5) = 30$, and $i^*(5, 5) = 40$. These imply, in turn, that $cr(C_4 \times C_4) = 8$, $cr(C_4 \times C_5) = 10$, and $cr(C_5 \times C_5) = 15$. The first two are known [D], [B], while the last agrees with the conjecture of Harary *et al.* [H]. Moreover, Lemma 6 implies that, in any optimal drawing of $C_4 \times C_4$, $C_4 \times C_5$, or $C_5 \times C_5$, no principal cycle can have a self-intersection. For $C_4 \times C_4$, every 4-cycle can be taken as a principal cycle, so, in this case, no 4-cycle can have a self-intersection.

In order to prove these results, we need one more observation. An (m, n)-mesh $(\mathscr{C}_1, \mathscr{C}_2)$ is *really optimal* if it is optimal and, for any optimal mesh $(\mathscr{C}_1', \mathscr{C}_2')$, $i(\mathscr{C}_1) + i(\mathscr{C}_2) \leq i(\mathscr{C}_1') + i(\mathscr{C}_2')$. Thus, really optimal means the mesh first minimizes the total number of intersections and, subject to this, it minimizes the total number of intersections among pairs of curves belonging to the same one of $\mathscr{C}_1$ or $\mathscr{C}_2$.

Lemma 8. *Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a really optimal (m, n)-mesh and suppose $C_1, C_2 \in \mathscr{C}_1$ exist such that $|C_1 \cap C_2| = 2$ and there is a $C \in \mathscr{C}_1$ with $C \cap (C_1 \cup C_2) = \emptyset$. Label the regions of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ so that C is in the exterior and, for $k = 1, 2$, let A_k be the arc in $C_k \setminus C_{3-k}$ that is not incident with the exterior region. Suppose no curve in $\mathscr{C}_1$ intersects the interior of $A_1 \cup A_2$. Then $C_1', C_2' \in \mathscr{C}_2$ exist such that, for each $j = 1, 2$, $|C_j' \cap (C_1 \cup C_2)| \geq 3$.*

Proof. Replacing the curves C_1 and C_2 by $\overline{C}_1 = (C_1 \setminus A_1) \cup A_2$ and $\overline{C}_2 = (C_2 \setminus A_2) \cup A_1$ yields a new family of curves in the plane with fewer total intersections (the tangential intersections can be removed). Therefore, optimality implies it is not a mesh, so that there is some curve $C_1' \in \mathscr{C}_2$ and some $j \in \{1, 2\}$ such that

$C_1' \cap \overline{C}_j = \emptyset$. Thus, $C_1' \cap C_j \subseteq A_j$. We note that, since $C_1' \cap C \neq \emptyset$, $|C_1' \cap (C_1 \cup C_2)| \geq 3$. If this were the only such curve, then we could simply adjust the curve C_1' to cross A_j twice and meet A_{3-j}, yielding an (m, n)-mesh with the same number of total crossings (so it is optimal) and having fewer intersections among pairs in the same one of $\mathscr{C}_1$ or $\mathscr{C}_2$, contradicting the assumption that $(\mathscr{C}_1, \mathscr{C}_2)$ is really optimal. □

We remark that Lemma 8 is also true if the roles of $\mathscr{C}_1$ and $\mathscr{C}_2$ are interchanged.

Theorem 9. $i^*(4, 4) = 24$.

Proof. Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a really optimal (4, 4)-mesh. Lemma 2 generalizes, so its conclusion still applies. Therefore, we can assume $(\mathscr{C}_1, \mathscr{C}_2)$ is separation-free. More generally, if any curve has eight or more intersections, then, because $i^*(3, 4) = 16$, $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 24$. It follows that we can assume, for $i = 1, 2$, if $C \in \mathscr{C}_i$, then C intersects at most one curve from $\mathscr{C}_i$, and if it does meet such a curve, then it does so in exactly two points.

If $(\mathscr{C}_1, \mathscr{C}_2)$ is a disjoint mesh, then we are done by Theorem 1. Therefore, we can assume $\mathscr{C}_1$ is not disjoint.

For a pair $C_1, C_2 \in \mathscr{C}_i$ such that $C_1 \cap C_2 \neq \emptyset$, we apply Lemma 8 to obtain the curves C_1' and C_2'. If C_1 intersects both in at least two points, then C_1 has at least eight intersections and we are done. Thus we can assume both C_1 and C_2 have at least seven intersections; if one has eight, then we are done, so we may assume exactly seven.

If some pair C_3', C_4' of curves in $\mathscr{C}_2$ also intersect each other, then the same reasoning implies that they both have seven crossings. Deleting C_1 and C_3', for example, removes at least 12 intersections (seven each, with at most two counted twice) and leaves a (3, 3)-mesh, which has at least 12 intersections remaining. Therefore, the (4, 4)-mesh has at least 24 intersections, as required.

Therefore, we can assume $\mathscr{C}_2$ is disjoint. Let C_3 and C_4 be the other two curves in $\mathscr{C}_1$. If C_3 and C_4 are disjoint, then we can delete C_1 (removing seven intersections) and obtain a separation-free disjoint (3, 4)-mesh, having at least 18 intersections, by Proposition 3. This gives at least 25 intersections for the (4, 4)-mesh.

Finally, if C_3 and C_4 are not disjoint, then each has seven intersections, as for C_1 and C_2. We have, then, a total of $(4 \times 7) - (2 \times 2) = 24$ intersections, as required. □

Theorem 10. $i^*(4, 5) = 30$.

Proof. Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a really optimal (4, 5)-mesh. If some element of $\mathscr{C}_2$ has six or more intersections, then we are easily done, as $30 = 24 + 6$. Therefore, we can assume $\mathscr{C}_2$ is a disjoint family of curves and each intersects at most one element of $\mathscr{C}_1$ in two points.

Similarly, if any element of $\mathscr{C}_1$ has at least 10 intersections, then we are done, as $30 = 20 + 10$. Therefore, no element in $\mathscr{C}_1$ has more than four points of intersection with curves in $\mathscr{C}_1$.

Step 1. Suppose $C_1, C_2 \in \mathscr{C}_1$ are such that $|C_1 \cap C_2| = 4$. Every element of $\mathscr{C}_2$ meets both of C_1 and C_2 in a single point or we are done. Also, the other elements of $\mathscr{C}_1$ are disjoint from both C_1 and C_2. Therefore, there is a single region of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ whose closure R contains all the curves (except C_1 and C_2).

Both C_1 and C_2 intersect R in at most two arcs, which make up the boundary of R. Every one of the curves in $\mathscr{C}_2$ intersects one arc from each. It follows that there are three curves $C'_1, C'_2, C'_3 \in \mathscr{C}_2$ such that C'_2 and C'_3 are in different regions of $R \setminus C'_1$. From this we conclude that the remaining elements of $\mathscr{C}_1$ all intersect C'_1 in at least four points, showing that C'_1 has at least 10 points of intersection, and we are done. Therefore, we can assume that, for any two curves $C_1, C_2 \in \mathscr{C}_1$, $|C_1 \cap C_2| \leq 2$.

Step 2. Now suppose $C_1, C_2, C_3 \in \mathscr{C}_1$ are such that C_1 intersects both C_2 and C_3 in two points. Then every curve in $\mathscr{C}_2$ intersects C_1 in exactly one point and the fourth curve in $\mathscr{C}_1$ is disjoint from C_1. Therefore, this curve and the curves in $\mathscr{C}_2$ all lie in the same region of $\mathbb{R}^2 \setminus C_1$.

Suppose, first, that $C_2 \cap C_3$ is also nonempty. Then each of these curves could play the role of C_1 in the preceding discussion. Since any two of C_1, C_2, and C_3 have exactly two intersections, the five curves in $\mathscr{C}_2$ all lie in the region of $\mathbb{R}^2 \setminus (C_1 \cup C_2 \cup C_3)$ that contains the fourth curve C_4 in $\mathscr{C}_1$. This region is bounded by at most four arcs, with at least one from each of C_1, C_2, and C_3. However, this is impossible, since $\mathscr{C}_2$ is a disjoint family.

Step 3. Thus, we may suppose $C_2 \cap C_3$ is empty. If $C_4 \cap C_2 \neq \emptyset$, then C_1 and C_2 both have nine intersections. All curves in $\mathscr{C}_2$ lie in a single region of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$. Taking any three of the curves in $\mathscr{C}_2$, one of them separates the other two in this region. Thus, each of C_3 and C_4 intersects the separator in at least two points. Thus, each intersects at least three of the curves in $\mathscr{C}_2$ in two points, so C_3, for example, has at least 10 intersections and we are done.

Step 4. Therefore, we can assume C_4 is disjoint from $C_1 \cup C_2 \cup C_3$. Delete C_1 to obtain a separation-free disjoint (3, 5)-mesh, which has, by Proposition 3, at least 24 intersections, so that $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 33$. It follows that we can assume that no element of $\mathscr{C}_1$ has four intersections with other elements of $\mathscr{C}_1$.

Step 5. If there are no points of intersection among the curves in $\mathscr{C}_1$, then we are done: the mesh is disjoint and Theorem 1 applies. Therefore, we can assume C_1, $C_2 \in \mathscr{C}_1$ are such that $|C_1 \cap C_2| = 2$ and that the other two elements, C_3, C_4 of $\mathscr{C}_1$, are pairwise disjoint from each of C_1 and C_2.

If C_3 and C_4 do not intersect each other, then delete C_1 (and at least seven intersections) to obtain a separation-free disjoint (3, 5)-mesh, which has, by Proposition 3, at least 24 intersections, for a total of at least 31, as required. Therefore, we can assume C_3 and C_4 intersect in two points.

We claim that this configuration does not exist, i.e., it is impossible to have:

(1) Separation-free.
(2) The four curves in $\mathscr{C}_1$ partition into two pairs, each intersecting the other in the pair, but no other intersections.

(3) The five curves in $\mathscr{C}_2$ are pairwise disjoint.
(4) No curve in $\mathscr{C}_1$ has 10 intersections.
(5) No curve in $\mathscr{C}_2$ has six intersections.

Pick any four of the five curves in $\mathscr{C}_2$ to obtain a (4, 4)-mesh. This has at least 24 intersections, four of which are accounted for in $C_1 \cap C_2$ and $C_3 \cap C_4$. Therefore, there are 20 intersections of the form $C \cap C'$, with $C \in \mathscr{C}_1$ and $C' \in \mathscr{C}_2$. As no element of $\mathscr{C}_2$ has six or more intersections, every element of $\mathscr{C}_2$ must have exactly five intersections, all of which are with elements of $\mathscr{C}_1$. Thus, for each element C' of $\mathscr{C}_2$, there is a unique element of C of $\mathscr{C}_1$ such that $|C' \cap C| = 2$; for all other elements C'' of $\mathscr{C}_1$, $|C' \cap C''| = 1$.

Hence, one of $|C' \cap (C_3 \cup C_4)|$ and $|C' \cap (C_1 \cup C_2)|$ is 2 and the other is 3. Therefore, we can assume there are C_1', C_2', C_3' in $\mathscr{C}_2$ such that, for $i = 1, 2, 3$, $|C_i' \cap (C_3 \cup C_4)|$ is 2. One of C_1', C_2', C_3' (say C_1') separates the other two in a region of $\mathbb{R}^2 \setminus (C_3 \cup C_4)$. It follows that $|C_1' \cap C_1| \geq 4$, a contradiction. □

Now we move on to the case of greatest interest.

Theorem 11. *$i^*(5, 5) = 40$ and, therefore, $cr(C_5 \times C_5) = 15$.*

Proof. Let $(\mathscr{C}_1, \mathscr{C}_2)$ be a really optimal (5, 5)-mesh.

Claim 1. *If any curve has 10 or more intersections, then $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 40$.*

An immediate consequence of Claim 1 and Lemma 2 is the following.

Claim 2. *If $(\mathscr{C}_1, \mathscr{C}_2)$ is not separation-free, then $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 40$.*

Claim 3. *If there is a curve in $\mathscr{C}_1$ with nine intersections and two curves in $\mathscr{C}_2$ that intersect, then $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 40$.*

To see this, let $C \in \mathscr{C}_1$ have nine intersections and let C_1' and C_2' be intersecting elements of $\mathscr{C}_2$. If C_1' has four intersections with curves in $\mathscr{C}_2$, then C_1' has at least nine intersections. Deleting C and C_1' removes 17 intersections and leaves a (4, 4)-mesh with at least 24 intersections, and we are done. Thus, we can assume C_1' and C_2' are pairwise disjoint from the remaining curves in $\mathscr{C}_2$. By Lemma 8, either one of C_1' or C_2' has at least nine intersections or they both have two intersections with distinct curves from $\mathscr{C}_1$. In the first case, C and the one of C_1' and C_2' having nine intersections combine for a total of at least 16 intersections, by Lemma 4. Deleting them yields a (4, 4)-mesh, having at least 24 intersections and we are done. In the second case, one of C_1' and C_2' meets C in a single point. Again, this one and C account for 16 intersections.

Claim 4. *If there is a curve C in $\mathscr{C}_1$ having four or more intersections with curves in $\mathscr{C}_1$, then $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 40$.*

Note that C has at least nine intersections, so we can assume it has exactly nine intersections. By Claim 3, we can assume $\mathscr{C}_2$ is a disjoint family. We are in the same setting that we had in the proof of Theorem 10. Steps 1–3 are handled exactly as in the proof of Theorem 10. Thus, we can assume that no element of $\mathscr{C}_1$ has four points of intersection with some single element of $\mathscr{C}_1$ and that if C intersects both C_1 and C_2 from $\mathscr{C}_1$, then C_1 and C_2 are disjoint and are pairwise disjoint from the remaining curves in $\mathscr{C}_1$.

Step 4. Let C_3, C_4 be the remaining curves in $\mathscr{C}_1$. If they are disjoint, then delete C to obtain a separation-free disjoint (4, 5)-mesh, which has, by Proposition 3, at least 32 intersections, showing $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 41$. If C_3 and C_4 intersect, then Lemma 8 implies that at least one of them, say C_3, has at least eight intersections. Deleting C and C_3 removes at least 17 intersections and leaves a separation-free disjoint (3, 5)-mesh. By Proposition 3, $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 24 + 17 = 41$.

Claim 5. *If there is a curve C in $\mathscr{C}_1$ that intersects another element of $\mathscr{C}_1$ and has nine intersections, then $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 40$.*

By Claim 4, we can assume no curve in $\mathscr{C}_1$ has four intersections with curves in $\mathscr{C}_1$. Suppose C intersects $C_1 \in \mathscr{C}_1$. If the remaining elements of $\mathscr{C}_1$ are pairwise disjoint, then delete C to get a separation-free disjoint (4, 5)-mesh, which, by Proposition 3, has at least 32 intersections. Together with the nine in C, we have a total of 41. Thus, we may assume that C_2, C_3 are other elements of $\mathscr{C}_1$ that intersect. By Lemma 8, at least one of them has at least eight intersections. We can assume C_2 is such a curve. Deleting C and C_2 removes 17 intersections and leaves a separation-free disjoint (3, 5)-mesh, which has, by Proposition 3, at least 24 intersections, so $i^*(\mathscr{C}_1, \mathscr{C}_2) \geq 41$, completing the proof of Claim 5.

Since a disjoint (5, 5)-mesh has at least 40 intersections, we can assume that some pair of curves C_1, C_2 in $\mathscr{C}_1$ are not disjoint. We apply Lemma 8 to C_1 and C_2 to obtain the curves C'_1, C'_2 in $\mathscr{C}_2$, each having three intersections with $C_1 \cup C_2$.

Suppose C'_1 and C'_2 both have two intersections with C_1. This means that C_1 has at least nine intersections. By Claim 5, we are done. Therefore, we can assume C'_1 has two intersections with C_1, C'_2 has two intersections with C_2, and C_1 and C_2 both have exactly eight intersections.

Let C'_3, C'_4, C'_5 be the remaining three curves in $\mathscr{C}_2$. Each intersects each of C_1 and C_2 in a single point and, therefore, consists of two arcs between these two points. If no two of C'_3, C'_4, C'_5 intersect, then one of them separates the other two in a region of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$ and so must have at least 14 intersections.

Thus, we may suppose $C'_3 \cap C'_4$ is nonempty. Then C'_3 and C'_4 are both disjoint from C'_5. One of the two arcs, say A'_3, in $C'_3 \setminus (C_1 \cup C_2)$ is disjoint from C'_4, and one of the arcs, say A'_4, in $C'_4 \setminus (C_1 \cup C_2)$ is disjoint from C'_3. (Otherwise, $C'_3 \cap C'_4$ has at least four intersections.) Let A'_5 be either of the arcs in $C'_5 \setminus (C_1 \cup C_2)$.

One of the three pairwise disjoint arcs A'_3, A'_4, A'_5 separates the other two. If it were A'_5, then C'_3 and C'_4 would be disjoint, a contradiction. Hence, we can assume A'_3 separates A'_4 from A'_5.

It follows that each of the three curves in $\mathscr{C}_1 \setminus \{C_1, C_2\}$ has at least two points of intersection with C_3'. Therefore, C_3' has at least 10 points of intersection, and we are done. □

5. Comments

We know that, when n is larger than m, the number of intersections in an (m, n)-mesh need not be as large as the crossing number of $C_m \times C_n$. However, we do not know what happens in the case $m = n$. It would be interesting to have a picture of an (n, n)-mesh with fewer than $2(n - 1)n$ intersections, if such exists. On the other hand, we expect that our techniques can be used to show that $cr(C_6 \times C_6) = 24$.

It should be possible to use $cr(C_5 \times C_5) = 15$ as a base for an induction to show that $cr(C_5 \times C_n) = 3n$, for $n \geq 5$. One possibility is to use the method of [B]. It is necessary to generalize Lemma 2 of that paper, but there will be many more than the four cases to consider. It would be interesting to generalize it to arbitrary m and n.

A related problem is to evaluate $i(n)$, the minimum number of intersections in a family of n curves in the plane, any two of which intersect. The following are easy observations:

(1) $i(5) = 12$ and $i(6) = 20$.
(2) $\lim_{n\to\infty} i(n)/\binom{n}{2}$ exists.
(3) $i(n) \leq 2\binom{n}{2}$.
(4) For $m \geq 3$, $i(2m) \geq 3m^2 - 3m + 2$ and $i(2m + 1) \geq 3m^2 + 1$.

We remark that (4) is a straightforward induction and that (3) and (4) imply

$$\tfrac{3}{2} \leq \lim_{n\to\infty} i(n)/\binom{n}{2} \leq 2.$$

Acknowledgments

This work was accomplished while the first author was visiting the second at the Technical University of Denmark in the spring of 1993. He wishes to express his gratitude to both Carsten Thomassen and the Mathematical Institute for their hospitality.

References

[B] L. W. Beineke and R. D. Ringeisen, On the crossing numbers of products of cycles and graphs of order four, *J. Graph Theory* **4** (1980), 145–155.

[D] A. M. Dean and R. B. Richter, The crossing number of $C_4 \times C_4$, *J. Graph Theory*, to appear.
[H] F. Harary, P. C. Kainen, and A. J. Schwenk, Toroidal graphs with arbitrarily high crossing numbers, *Nanta Math.* **6** (1973), 58–67.
[R] R. D. Ringeisen and L. W. Beineke, The crossing number of $C_3 \times C_n$, *J. Combin. Theory* **24** (1978), 134–136.

Received August 30, 1993.