INTERTWINING OPERATORS ASSOCIATED TO THE GROUP S_3

CHARLES F. DUNKL

ABSTRACT. For any finite reflection group G on an Euclidean space there is a parametrized commutative algebra of differential-difference operators with as many parameters as there are conjugacy classes of reflections in G. There exists a linear isomorphism on polynomials which intertwines this algebra with the algebra of partial differential operators with constant coefficients, for all but a singular set of parameter values (containing only certain negative rational numbers). This paper constructs an integral transform implementing the intertwining operator for the group S_3, the symmetric group on three objects, for parameter value $\geq \frac{1}{2}$. The transform is realized as an absolutely continuous measure on a compact subset of $M_2(\mathbf{R})$, which contains the group as a subset of its boundary. The construction of the integral formula involves integration over the unitary group $U(3)$.

Associated to any finite reflection group G on an Euclidean space there is a parametrized commutative algebra of differential-difference operators with as many parameters as there are conjugacy classes of reflections in G. It has been shown that there exists a linear isomorphism on polynomials which intertwines this algebra with the algebra of partial differential operators with constant coefficients, for all but a "singular set" of parameter values. This singular set contains only negative values and is closely linked to the zero-set of the Poincaré series of G. This paper constructs an integral transform implementing the intertwining map for the group S_3, the symmetric group on three objects, for positive parameter values. Previously this had been done only for the group Z_2 (acting by sign-change on $\mathbf{R}$) where the transform is a classical fractional integral. The transform in this paper has its origin in the adjoint action of the unitary group $U(3)$ on the linear space of real diagonal 3×3 matrices (the complexification of the maximal torus). This will lead to a transform realized as an absolutely continuous measure on a certain compact subset of $M_2(\mathbf{R})$.

Here is a concise statement of the main result (rephrased from formulas (5.1), (5.6)): the operator V intertwines the differential-difference operators

with parameter α for the symmetry group G of the triangle

$$\left\{ (1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \right\} \text{ in } \mathbf{R}^2.$$

For $w = (w_{ij})^2_{i,j=1} \in M_2(\mathbf{R})$, let

$$\tilde{J}(w) := \left(\frac{1}{27} \right) \left[1 - 3 \sum_{i,j} w_{ij}^2 - 3(\det(w))^2 \right.$$

$$\left. + 2w_{11} \left(w_{11}^2 - 3w_{12}^2 - 3w_{21}^2 + 3w_{22}^2 \right) + 12w_{21}w_{12}w_{22} \right].$$

The region of integration is

$$\Omega_w, \text{ the closure of } \left\{ w \in M_2(\mathbf{R}) : -\frac{1}{2} < w_{11} < 1, \right.$$

$$\left. |w_{21}| < (1 - w_{11})/\sqrt{3}, \text{ and } \tilde{J}(w) \geq 0 \right\}$$

(a compact subset of $M_2(\mathbf{R})$ that has an elliptical cross-section for fixed w_{11}, w_{21} and contains G in its boundary).

The transform is

$$Vf(x) = (h_\alpha/9) \int_{\Omega_w} f(xw)\phi(w)\tilde{J}(w)^{\alpha - \frac{3}{2}} dw_{11}dw_{21}dw_{12}dw_{22}$$

(for smooth functions f and $x \in \mathbf{R}^2$), where $\phi(w) = 1 + 2\operatorname{Tr}(w) + \det(w)$ and $\alpha > \frac{1}{2}$. The normalization constant h_α (from Proposition 3.2) equals

$$3^{3\alpha - \frac{1}{2}} \left(\alpha - \frac{1}{2} \right) \Gamma \left(\alpha + \frac{1}{3} \right) \Gamma \left(\alpha + \frac{2}{3} \right) /(\pi\Gamma(\alpha))^2.$$

If $\phi(w)$ is replaced by 1, the formula implements the operator V for G-invariant functions. Note that this is a positive operator. It is possible (indeed, conjectured) that V is always positive (see the discussion in Section 5 where V is exhibited as an integral over a subset of $\mathbf{R}^2$), but $-\frac{2}{9} \leq \phi(w) \leq 6$ with $\phi(w_0) = -\frac{2}{9}$ for $w_0 = \left(-\frac{1}{3} \right) I$ and w_0 is in the boundary of Ω_w.

The major part in proving the validity of the formula is contained in Theorem 4.4.

In related work, Beerends [Be] found a transmutation operator also associated to S_3 which maps the ordinary Laplacian to a second-order invariant differential operator on a Riemannian symmetric space. This operator is of "global" type, in the terminology of Heckman [Hec2] while ours are "infinitesimal" type. We choose to call our transform "intertwining" rather than "transmutation" (the terminology of Carroll [C]) because the latter emphasizes the linking of two specific differential operators to each other, while we are concerned with algebras of operators.

This paper begins with some general facts about how an integral transform can be proven to coincide with the algebraically defined intertwining operator, then proceeds to motivate the study of $U(3)$ by interpreting a formula of Harish-Chandra (see Helgason [Hel]). The research for this work involved some experimentation ("educated guesses") and testing of conjectures on low-degree polynomials with computer symbolic algebra (specifically, "Maple V" on a 486-type machine). Along the way, a number of different approaches, coordinate systems, etc. were tried and discarded. Also, Maple made it possible to quickly (and accurately) evaluate formulas (in several variables and a parameter) to test cases. In this presentation the reader may find it easier to carry out the change-of-variables calculations with computer algebra, but otherwise the approach is conceptual; certainly a motivation for the various coordinate changes was to shorten the intermediate expressions.

1. GENERAL RESULTS

The following material is valid for any finite reflection group. The results will be subsequently used for the S_3-case. We recall the definitions from [Du1-3].

Let $\mathscr{P}_n$ be the space of polynomials homogeneous of degree n on $\mathbf{R}^N$, $n = 0, 1, 2, \ldots$. Let

$$A(\mathbf{R}^N) := \left\{ f = \sum_{n=0}^{\infty} f_n : f_n \in \mathscr{P}_n \text{ each } n, \ \|f\|_A := \sum_{n=0}^{\infty} \sup_{|x| \leq 1} |f_n(x)| < \infty \right\},$$

the algebra of "absolutely convergent homogeneous series."

Let $\{v_i : i = 1, \ldots, m\} \subset \mathbf{R}^N$ be the set of positive roots of a finite Coxeter group G. Thus the reflections in G are σ_i, $1 \leq i \leq m$ with $x\sigma_i := x - 2(\langle x, v_i \rangle / |v_i|^2)v_i$, $x \in \mathbf{R}^N$, and they generate G, which is a finite subgroup of the orthogonal group $O(N)$ (the inner product $\langle x, y \rangle := \sum_{i=1}^{N} x_i y_i$ and $|x|^2 = \langle x, x \rangle$). The multiplicity function (parameter) α is an m-tuple of real numbers α_i, $1 \leq i \leq m$, such that $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in G. The differential-difference operators associated to G and α are defined by

$$(1.1) \qquad T_i f(x) := \frac{\partial f}{\partial x_i}(x) + \sum_{j=1}^{m} \alpha_j \frac{f(x) - f(x\sigma_j)}{\langle x, v_j \rangle} (v_j)_i,$$

$1 \leq i \leq N$. The set $\{T_i\}$ generates a commutative algebra (Theorem 1.9 in [Du1]). It was shown in [Du2] that when $\alpha_i \geq 0$ there exists a unique linear operator V such that $V1 = 1$, $V\mathscr{P}_n = \mathscr{P}_n$ (a linear isomorphism) each $n = 0, 1, \ldots$, and

$$T_i V f(x) = V\left(\frac{\partial f}{\partial x_i}\right)(x), \text{ each } i,$$

for all polynomials (for another proof of the existence, and a thorough discussion of the values of α for which V fails to exist, see [DJO]). Further, if $f \in A(\mathbf{R}^N)$, then $Vf \in A(\mathbf{R}^N)$ and $\|Vf\|_A \leq \|f\|_A$ (Theorem 2.7 in [Du3]).

This proves the existence of the analogue of the exponential function (the symmetrization of this function is a type of Bessel function, see formula (1.6)

in the sequel), $K(x, y)$, $(x, y \in \mathbf{R}^N)$, which is real-entire and satisfies

(i) $\qquad\qquad T_i^x K(x, y) = y_i K(x, y);$

(ii) $\qquad\qquad K(xw, yw) = K(x, y)$ for all $w \in G;$

(iii) $\qquad\qquad K(x, y) = \sum_{n=0}^{\infty} V_x(\langle x, y \rangle^n / n!);$

(iv) $\qquad\qquad K(x, y) = K(y, x)$

(the "x" appearing in the superscript of T_i and subscript of V indicates the action of the respective operators).

Let $(f_1, \ldots, f_N)$ be an "α-exact" 1-form, that is, $T_i f_j = T_j f_i$ for $1 \leq i, j \leq N$ and each $f_i \in A(\mathbf{R}^N)$. It was shown in [Du2] that there exists a unique $F \in A(\mathbf{R}^N)$ such that $T_i F = f_i$ and $F(0) = 0$. This allows a simplified condition equivalent to the defining properties of V.

1.1 Lemma. *Let V_1 be a linear operator such that $V_1 1 = 1$, $V_1 \mathscr{P}_n \subset \mathscr{P}_n$ for each $n = 0, 1, 2, \ldots$, and which satisfies*

(1.2)
$$\sum_{j=1}^{m} \alpha_j (V_1 f(x) - V_1 f(x\sigma_j)) = \sum_{i=1}^{N} x_i \left\{ V_1 \left(\frac{\partial}{\partial x_i} f \right)(x) - \frac{\partial}{\partial x_i}(V_1 f)(x) \right\},$$

for each polynomial f, then $V_1 f = V f$ for all polynomials f.

Proof. We use induction on the degree. Assume $V_1 g = V g$ for all $g \in \mathscr{P}_n$, $0 \leq n \leq k$ and let $f \in \mathscr{P}_{k+1}$. By Proposition 2.5 in [Du1],

(1.3) $\qquad \sum_{i=1}^{N} x_i T_i V_1 f(x) = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} V_1 f(x) + \sum_{j=1}^{m} \alpha_j (V_1 f(x) - V_1 f(x\sigma_j)).$

By the hypothesis on V_1,

$$\sum_{i=1}^{N} x_i T_i V_1 f(x) = \sum_{i=1}^{N} x_i V_1 \left(\frac{\partial}{\partial x_i} f \right)(x),$$

and by the inductive hypothesis

$$V_1 \left(\frac{\partial}{\partial x_i} f \right)(x) = V \left(\frac{\partial}{\partial x_i} f \right)(x) = T_i V f(x),$$

and so

$$\sum_{i=1}^{N} x_i T_i V_1 f(x) = \sum_{i=1}^{N} x_i T_i V f(x).$$

Corollary 3.9 of [Du2] shows $T_i V_1 f(x) = T_i V f(x)$, $1 \leq i \leq N$. Both $(T_i V_1 f)_{i=1}^{N}$ and $(T_i V f)_{i=1}^{N}$ are α-exact 1-forms. By Theorem 3.10 of [Du2], $V_1 f - V f$ is a constant, which must be 0 since $V_1 f - V f \in \mathscr{P}_{k+1}$. This shows $V_1 f = V f$ for each polynomial f. $\quad\square$

1.2 Corollary. *It suffices to prove the validity of* (1.4) *for each polynomial* $x \mapsto \langle x, y \rangle^n / n!$ $(y \in \mathbf{R}^N, \ n = 1, 2, 3, \ldots)$. *Further, if* V_1 *is a bounded operator on* $A(\mathbf{R}^N)$, *then* (1.3) *follows for all* $f \in A(\mathbf{R}^N)$ *if it holds for* $f_y(x) = \exp(\langle x, y \rangle)$, *each* $y \in \mathbf{R}^N$.

Proof. The span of $\{x \mapsto \langle x, y \rangle^n : y \in \mathbf{R}^N\}$ is $\mathscr{P}_n$. The homogeneous components of the formula for f_y are the required identities (and, of course, $f_y \in A(\mathbf{R}^N)$ with $\|f_y\|_A = e^{|y|}$). $\square$

An obvious way of constructing endomorphisms on each $\mathscr{P}_n$ is to use a measure on $M_N(\mathbf{R})$ (the space of real $N \times N$ matrices). We postulate the formula

$$(1.4) \qquad V f(x) = \int_\Omega f(x\tau) \phi(\tau) d\mu(\tau),$$

where Ω is a closed subset of the unit ball $M_N(\mathbf{R})$ (for the natural norm), μ is a positive Baire measure on Ω, and ϕ is a polynomial in the entries of τ. Some G-invariance properties will be imposed on Ω, μ, and ϕ; they are derived from the commutation relationship $V(R(w)f) = R(w)Vf$ (all $w \in G$, where the right translation is $R(w)f(x) := f(xw)$, $x \in \mathbf{R}^N$).

1.3 Proposition. *The operator* V *is given by the formula* (1.5) *if each of the following hold:*

(i) $\int_\Omega d\mu(\tau) = 1$;

(ii) $w\Omega = \Omega w = \Omega$, *and* $d\mu(w\tau) = d\mu(\tau w) = d\mu(\tau)$ *for* $w \in G$, $\tau \in \Omega$;

(iii) $\phi(w\tau w^{-1}) = \phi(\tau)$, *for* $w \in G$, $\tau \in \Omega$ *and* $\sum_{w \in G} \phi(w\tau) = |G|$ *(the cardinality of* G*)*;

(iv) $\displaystyle \int_\Omega (\langle x, y \rangle - \langle x\tau, y \rangle) \exp(\langle x\tau, y \rangle) \phi(\tau) d\mu(\tau)$

$$= \int_\Omega \exp(\langle x\tau, y \rangle) \left(\sum_{j=1}^m \alpha_j (\phi(\tau) - \phi(\sigma_j \tau)) \right) d\mu(\tau)$$

for all $x, y \in \mathbf{R}^N$.

Proof. Let $V_1 f(x) := \int_\Omega f(x\tau) \phi(\tau) d\mu(\tau)$. Clearly, $V_1 \mathscr{P}_n \subset \mathscr{P}_n$ each $n = 1, 2, \ldots$. Let $f \in A(\mathbf{R}^N)$ and $w \in G$, then

$$R(w)(V_1 f)(x) = V_1 f(xw)$$

$$= \int_\Omega f(xw\tau) \phi(\tau) d\mu(\tau)$$

$$= \int_{w\Omega w^{-1}} f(x\tau_1 w) \phi(w^{-1}\tau_1 w) d\mu(w^{-1}\tau_1 w)$$

$$= \int_\Omega f(x\tau_1 w) \phi(\tau_1) d\mu(\tau_1)$$

(by (ii) and (iii), where $\tau = w^{-1}\tau_1 w$), which equals $V_1(R(w)f)(x)$. Further, if f is G-invariant, then $V_1 f(x) = \int_\Omega f(x\tau)d\mu(\tau)$ because

$$V_1 f(x) = (1/|G|) \sum_{w \in G} V_1(R(w)f)(x)$$

$$= (1/|G|) \sum_{w \in G} \int_\Omega f(x\tau)\phi(\tau w^{-1})d\mu(\tau w^{-1})$$

$$= \int_\Omega f(x\tau)d\mu(\tau)$$

(by (ii) and (iii)). In particular, $V_1 1 = 1$.

By Corollary 1.2, it remains to establish (1.4) for $f_y(x) = \exp(\langle x, y \rangle)$, each $y \in \mathbf{R}^N$. Using (iv) we have

$$\sum_{j=1}^{m} \alpha_j(V_1 f_y(x) - V_1 f_y(x\sigma_j))$$

$$= \sum_{j=1}^{m} \alpha_j \left(\int_\Omega f_y(x\tau)\phi(\tau)d\mu(\tau) - \int_\Omega f_y(x\sigma_j\tau)\phi(\tau)d\mu(\tau) \right)$$

$$= \int_\Omega \exp(\langle x\tau, y \rangle) \sum_{j=1}^{m} \alpha_j(\phi(\tau) - \phi(\sigma_j\tau))d\mu(\tau),$$

while

$$\sum_{i=1}^{N} x_i \left(V_1 \left(\frac{\partial}{\partial x_i} f_y \right)(x) - \frac{\partial}{\partial x_i}(V_1 f_y)(x) \right)$$

$$= \int_\Omega \left[\left(\sum_{i=1}^{N} x_i y_i \right) \exp(\langle x\tau, y \rangle) - \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} \exp(\langle x\tau, y \rangle) \right] \phi(\tau)d\mu(\tau)$$

$$= \int_\Omega (\langle x, y \rangle - \langle x\tau, y \rangle) \exp(\langle x\tau, y \rangle)\phi(\tau)d\mu(\tau),$$

and thus V_1 satisfies (1.4). Note that

$$\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} \exp(\langle x\tau, y \rangle) = \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} \exp(\langle x, y\tau^* \rangle)$$

$$= \langle x, y\tau^* \rangle \exp(\langle x, y\tau^* \rangle)$$

$$= \langle x\tau, y \rangle \exp(\langle x\tau, y \rangle),$$

where τ^* is the transpose of τ. $\quad\square$

The last general ingredient is a formula linking $K(x, y)$ to the same function for a contiguous multiplicity function. Opdam [O] has characterized (in his

notation J_G, the generalized Bessel function)

$$(1.5) \qquad\qquad K_G(x, y) := \frac{1}{|G|} \sum_{w \in G} K(xw, y)$$

as the real-entire solution to the system of differential equations

$$(1.6) \qquad\qquad q(T^x)f(x, y) = q(y)f(x, y),$$

and $f(x, y) = f(xw, y) = f(x, yw)$ all $w \in G$; for each G-invariant polynomial q and $x, y \in \mathbf{R}^N$ (where $q(T^x)$ denotes the operator polynomial $q(T_1, T_2, \ldots, T_N)$ acting on x).

Heckman [Hec1] showed that $q(T^x)$ is a purely differential operator when restricted to G-invariant functions. We state the formula in the general case where G may have two or more conjugacy classes of reflections. Suppose that $\{\sigma_j : j \in E\}$ is such a class (thus $E \subset \{1, 2, 3, \ldots, m\}$). Let $p_E(x) = \prod_{j \in E} \langle x, v_j \rangle$ (a relative G-invariant) and for a given multiplicity function α, let $\alpha' := \alpha + 1_E$ (that is, $\alpha'_j = \alpha_j + 1$ if $j \in E$ and $\alpha'_j = \alpha_j$ otherwise). Let $K'_G(x, y)$ and T'_j denote the corresponding objects for α'.

1.4 Proposition. $K'_G(x, y) = \gamma_\alpha (\sum_{w \in G} \chi_E(w)K(xw, y))/(p_E(x)p_E(y))$ *for some constant* γ_α, *where* χ_E *is the linear character of* G *associated to* p_E.

Proof. Let $f(x, y) = \sum_{w \in G} \chi_E(w)K(xw, y)/(p_E(x)p_E(y))$. First we show f is real-entire in x, y and satisfies $f(xw_0, y) = f(x, y) = f(x, yw_0) = f(y, x)$ all $w_0 \in G$. The symmetry $f(x, y) = f(y, x)$ is obvious. For $w_0 \in G$,

$$f(xw_0, y) = \sum_{w \in G} \chi_E(w)K(xw_0 w, y)/(p_E(xw_0)p_E(y))$$

$$= \sum_{w \in G} \chi_E(w_0^{-1}w)K(xw, y)/(\chi_E(w_0)p_E(x)p_E(y))$$

$$= f(x, y).$$

For fixed y, each homogeneous component of $\sum_{w \in G} \chi_E(w)K(xw, y)$ is divisible by $p_E(x)$, thus $f(x, y)$ has no singularities on $\mathbf{R}^N \times \mathbf{R}^N$.

Let q be a G-invariant polynomial. It remains to show that $q(T'^x)f(x, y) = q(y)f(x, y)$, for all $y \in \mathbf{R}^N$. By a result of Heckman (Corollary 3.5 in [Hec1]),

$$q(T'^x)f(x, y) = \frac{1}{p_E(x)}q(T^x)(p_E(x)f(x, y))$$

$$= \frac{1}{p_E(x)p_E(y)} \sum_{w \in G} \chi_E(w)q(T^x)K(x, yw^{-1})$$

$$= \frac{1}{p_E(x)p_E(y)} \sum_{w \in G} \chi_E(w)q(yw^{-1})K(x, yw^{-1})$$

$$= q(y)f(x, y)$$

(because q is G-invariant). $\quad\square$

This formula and proof are due to Eric Opdam (private communication) and the author gratefully acknowledges this key insight. It is this formula with $\alpha = 0$ which appears in a theorem of Harish-Chandra in the context of Weyl groups of compact Lie groups (Theorem 5.35 in Helgason's treatise [Hel], p. 328), thus giving us a starting point for the investigation of V for Weyl groups.

2. THE INTEGRAL ASSOCIATED TO THE UNITARY GROUP

Henceforth $G = S_3$ and the multiplicity function is a constant α. The group S_3 is the Weyl group of $U(3)$ and is realized as the group of permutations of coordinates on $\mathbf{R}^3$, usually restricted to $\{x : x_1 + x_2 + x_3 = 0\}$, the alternating polynomial is $p(x) = \prod_{1 \le i < j \le 3}(x_i - x_j)$, and the positive roots are $(1, -1, 0)$, $(0, 1, -1)$, $(1, 0, -1)$. By means of Harish-Chandra's formula we will construct the set $\Omega \subset M_2(\mathbf{R})$ and the measure μ (of Proposition 1.3) for $\alpha = 1$.

We identify $\mathbf{R}^3$ with (a subspace of) the complexification of the Lie algebra of a maximal torus in $U(3)$ by $x \mapsto \delta(x) = \operatorname{diag}(x_1, x_2, x_3)$ (diagonal matrix). Then the adjoint action of $U(3)$ is $Ad(u)\delta(x) = u\delta(x)u^{-1}$. The formula (5.35) (Helgason [Hel]) specializes to

$$\int_{U(3)} \exp(Tr(u\delta(x)u^{-1}\delta(y)))dm_3(u)$$
$$= 2 \sum_{w \in S_3} \operatorname{sgn}(w) \exp(\langle xw, y \rangle)/(p(x)p(y))$$

(where m_3 is the normalized Haar measure on $U(3)$). Gross and Richards [GR] used this formula for $U(N)$ to prove a total positivity result. By Proposition 1.4 the left side is an integral formula for $K_G(x, y)$ for $\alpha = 1$. We now interpret this integral as one over a subset of $M_3(\mathbf{R})$. Indeed,

$$Tr(u\delta(x)u^{-1}\delta(y)) = \sum_{i=1}^{3} \sum_{j=1}^{3} |u_{ij}|^2 x_i y_j.$$

We see that the set of $(|u_{ij}|^2)_{i,j=1}^3$, $u \in U(3)$, is a four-dimensional subset of $M_3(\mathbf{R})$. Introduce a parametrization $b = (b_1, b_2, b_3, b_4) \in \mathbf{R}^4$ and let

$$\tau(b) := \begin{bmatrix} b_1 & b_3 & 1 - b_1 - b_3 \\ b_2 & b_4 & 1 - b_2 - b_4 \\ 1 - b_1 - b_2 & 1 - b_3 - b_4 & b_1 + b_2 + b_3 + b_4 - 1 \end{bmatrix}.$$

The important set Ω (supporting the integral transform for V) is defined to be $\{b \in \mathbf{R}^4 : \tau(b)_{ij} = |u_{ij}|^2$ for some $u \in U(3)$, $(i, j = 1, 2, 3)\}$. There are obvious bounds such as $b_i \ge 0$, $b_1 + b_2 \le 1$, etc., $b_1 + b_2 + b_3 + b_4 \ge 1$, but these alone do not define Ω. We now establish the formula for the integral of functions of $(|u_{ij}|^2)_{i,j=1}^3$ over $U(3)$. Let

$$(2.1) \qquad J(b) := 4b_1 b_2 b_3 b_4 - (b_1 + b_2 + b_3 + b_4 - 1 - b_1 b_4 - b_2 b_3)^2,$$

and define Ω to be the closure of the connected component of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in $\{b \in \mathbf{R}^4 : J(b) > 0\}$ (also $\Omega = \{b \in \mathbf{R}^4 : b_1 \ge 0, b_2 \ge 0, b_1 + b_2 \le 1, J(b) \ge 0\}$).

2.1 Theorem. *Let f be a continuous function on $[0, 1]^4 \subset \mathbf{R}^4$, then*

$$(2.2) \qquad \int_{U(3)} f(|u_{11}|^2, |u_{21}|^2, |u_{12}|^2, |u_{22}|^2) dm_3(u)$$

$$= \frac{2}{\pi} \int_\Omega f(b_1, b_2, b_3, b_4) J(b)^{-\frac{1}{2}} db_1 db_2 db_3 db_4.$$

Proof. The integral over $U(3)$ will be the result of integrating over a subgroup $U(2)$ and then over the homogeneous space $U(3)/U(2)$ ($\cong$ unit sphere in $\mathbf{C}^2$). We use the subgroup $\begin{bmatrix} 1 & 0 \\ 0 & U(2) \end{bmatrix}$ and write $A_2 g(u_0) = \int_{U(2)} g(u_0 u) dm_2(u)$ for the first step (m_2 is the Haar measure on $U(2)$, and g is continuous on $U(3)$). In this calculation assume $0 < b_1 < 1$ (the exceptions have measure zero). Let f be a function of $|u_{ij}|^2$, $1 \le i, j \le 2$, then

$$(2.3) \quad A_2 f = \frac{1}{2\pi} \int_0^\pi \int_0^\pi f(b_1, b_2, (1 - b_1)(1 + \cos\theta)/2,$$

$$(1 - b_2)/2 + h_1 \cos\theta + h_2 \sin\theta \cos\phi) \sin\theta \, d\phi d\theta,$$

where

$$h_1 := (b_1 + b_2 + b_1 b_2 - 1)/(2(1 - b_1)),$$

and

$$h_2 := (b_1 b_2 (1 - b_1 - b_2))^{\frac{1}{2}}/(1 - b_1).$$

The Haar measure on $U(2)$ is

$$dm_2(u') = \frac{1}{2\pi^3} d\psi \, d\xi_1 d\xi_2 \sin\theta \cos\theta \, d\theta$$

in terms of

$$u' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i(\psi+\xi_1)} \cos\theta & -e^{i(\psi-\xi_2)} \sin\theta \\ 0 & e^{i(\psi+\xi_2)} \sin\theta & e^{i(\psi-\xi_1)} \cos\theta \end{bmatrix},$$

the typical element of $U(2)$, with $-\pi < \xi_1, \xi_2 \le \pi$, $0 \le \psi < \pi$, $0 \le \theta \le \pi/2$. For arbitrary $u \in U(3)$, $j = 1$ or 2

$$|(uu')_{j2}|^2 = |u_{j2}|^2 \cos^2\theta + |u_{j3}|^2 \sin^2\theta$$

$$+ \sin\theta \cos\theta \left(u_{j2} \bar{u}_{j3} e^{i(\xi_1 - \xi_2)} + \bar{u}_{j2} u_{j3} e^{i(\xi_2 - \xi_1)} \right).$$

The integral over ψ is trivial and $\xi_1 - \xi_2$ can be replaced by one variable ω with $-\pi < \omega \le \pi$. Also, let $\theta = \phi/2$ and write u_{kj} in polar form $u_{kj} = r_{kj} e^{i\theta_{kj}}$, $k = 1, 2$; $j = 1, 2, 3$ (so that $r_{12}^2 = b_3$, $r_{22}^2 = b_4$). Then

$$A_2 f = \frac{1}{4\pi} \int_{-\pi}^\pi \int_0^\pi f(r_{11}^2, r_{21}^2, (r_{k2}^2 (1 - \cos\phi)/2 + r_{k3}^2 (1 + \cos\phi)/2$$

$$+ r_{k2} r_{k3} \cos(\theta_{k2} - \theta_{k3} + \omega))_{k=1}^2) \sin\phi \, d\phi \, d\omega.$$

But this is the surface measure m_S for the unit sphere S in $\mathbf{R}^3$ in terms of spherical polar coordinates $(v_1, v_2, v_3) := (\cos\phi, \sin\phi \cos\omega, \sin\phi \sin\omega)$

(with $0 \le \phi \le \pi$, $-\pi < \omega \le \pi$). We use the rotation-invariance to simplify the integral and to remove the apparent dependence on $\theta_{k2} - \theta_{k3}$, $k = 1, 2$.

Define two points ζ_1, $\zeta_2 \in \mathbf{R}^3$ by

$$\zeta_j := ((1 - b_j)/2 - b_{j+2}, \, r_{j2}r_{j3}\cos(\theta_{j2} - \theta_{j3}), \, -r_{j2}r_{j3}\sin(\theta_{j2} - \theta_{j3})).$$

Then

$$A_2 f = \int_S f\left(b_1, \, b_2, \, \frac{1}{2}(1 - b_1) + \langle v, \zeta_1 \rangle, \, \frac{1}{2}(1 - b_2) + \langle v, \zeta_2 \rangle\right) dm_S(v),$$

(note $r_{12}^2 + r_{13}^2 = 1 - b_1$ and $r_{22}^2 + r_{23}^2 = 1 - b_2$). The integral depends on $|\zeta_1|^2$, $|\zeta_2|^2$ and $\langle \zeta_1, \zeta_2 \rangle$, and

$$\begin{aligned}
|\zeta_1|^2 &= ((1 - b_1)/2 - b_3)^2 + r_{12}^2 r_{13}^2 \\
&= ((1 - b_1)/2 - b_3)^2 + b_3(1 - b_1 - b_3) = ((1 - b_1)/2)^2;
\end{aligned}$$

$$|\zeta_2|^2 = ((1 - b_2)/2)^2,$$

and

$$\begin{aligned}
\langle \zeta_1, \zeta_2 \rangle &= ((1 - b_1)/2 - b_3)((1 - b_2)/2 - b_4) \\
&\quad + r_{12}r_{22}r_{13}r_{23}\cos(\theta_{12} - \theta_{13} - \theta_{22} + \theta_{23}).
\end{aligned}$$

By the adjoint formula for the matrix inverse and the fact that $u^* = u^{-1}$ we have $u_{12}u_{23} - u_{22}u_{13} = \bar{u}_{31}$ (det u). Multiply this identity by its complex conjugate to get

$$r_{12}^2 r_{23}^2 + r_{22}^2 r_{13}^2 - 2r_{12}r_{23}r_{22}r_{13}\cos(\theta_{12} + \theta_{23} - \theta_{13} - \theta_{22}) = |u_{31}|^2 = 1 - b_1 - b_2.$$

This leads to $\langle \zeta_1, \zeta_2 \rangle = (b_1 b_2 + b_1 + b_2 - 1)/4$. Rotate the coordinates so that $\zeta_2 = ((1 - b_1)/2, 0, 0)$ and $\zeta_2 = (h_1, h_2, 0)$ with $h_1^2 + h_2^2 = |\zeta_2|^2 = ((1 - b_2)/2)^2$, $h_1(1 - b_1)/2 = \langle \zeta_1, \zeta_2 \rangle = (b_1 b_2 + b_1 + b_2 - 1)/4$. This proves formula (2.3), since it suffices to integrate over $0 \le \phi \le \pi$ because the integrand ($\cos\phi$) is even in ϕ.

The integration over $U(3)/U(2)$ is done with the formula

$$\int_{U(3)} g(|u_{11}|^2, |u_{21}|^2, |u_{31}|^2) dm_3(u) = 2 \int\int_R g(b_1, b_2, 1 - b_1 - b_2) db_1 \, db_2,$$

where $R := \{(b_1, b_2) \in \mathbf{R}^2 : b_1 \ge 0, \, b_2 \ge 0, \, b_1 + b_2 \le 1\}$ (for example, see [DR, Chapter 10]).

The last step is to change the variables of integration to (b_1, b_2, b_3, b_4). The Jacobian

$$\left| \frac{\partial(b_1, b_2, b_3, b_4)}{\partial(b_1, b_2, \theta, \phi)} \right| = \frac{h_2(1 - b_1)}{2} \sin^2\theta \, \sin\phi$$

and $\sin^2\theta = b_3(1 - b_1 - b_3)/(1 - b_1)^2$. Further

$$\sin^2\phi = \frac{(1 - b_1)^2 J(b)}{4b_1 b_2 b_3(1 - b_1 - b_2)(1 - b_1 - b_3)},$$

where $J(b)$ was defined in (2.1). The result is formula (2.2) in the theorem. The region Ω of integration is unambiguously defined by the constraints $b_1 \geq 0$, $b_2 \geq 0$, $b_1 + b_2 \leq 1$, $J(b) \geq 0$. □

We have motivated the definition of the weight function J and its domain Ω by means of integration over $U(3)$.

3. Tools for integration with respect to powers of J

In this section we determine the integrals of monomials in $b = (b_1, b_2, b_3, b_4)$ with respect to powers of J over Ω, preparatory to establishing a formula for V, for general $\alpha > \frac{1}{2}$. To help in the visualization of Ω, we point out that the cross-section for fixed (b_1, b_2) with $b_1 > 0$, $b_2 > 0$, $b_1 + b_2 < 1$ is an ellipse which is tangent to the six lines $b_3 = 0$, $b_4 = 0$, $b_3 + b_4 = 1$, $b_3 + b_4 = 1 - b_1 - b_2$, $b_3 = 1 - b_1$, $b_4 = 1 - b_2$ with contact points

$$(0, (1 - b_1 - b_2)/(1 - b_1)), \quad ((1 - b_1 - b_2)/(1 - b_2), 0),$$

$$(b_2/(b_1 + b_2), b_1(b_1 + b_2)),$$

$$(b_1(1 - b_1 - b_2)/(b_1 + b_2), b_2(1 - b_1 - b_2)/(b_1 + b_2)),$$

$$(1 - b_1, b_1b_2/(1 - b_1)), \quad (b_1b_2/(1 - b_2), 1 - b_2),$$

respectively. The points b corresponding to elements of S_3 are on the boundary of J; in fact, the boundary is the image of the real group $O(3)$ embedded in $U(3)$.

We make a change of variables motivated by the formulas in §6. Essentially we keep the variable $\cos\phi$ (now "r") and directly exhibit the constraints $0 \leq b_2 \leq 1 - b_1$ and $0 \leq b_3 \leq 1 - b_1$ for fixed b_1.

$$(3.1) \qquad b_2 = s(1 - b_1) \qquad (0 \leq s \leq 1),$$
$$b_3 = t(1 - b_1) \qquad (0 \leq t \leq 1),$$
$$b_4 = b_4(b_1, s, t, r) := (1 - s)(1 - t) + b_1 st$$
$$+ 2(b_1 st(1 - s)(1 - t))^{\frac{1}{2}} r, \qquad -1 \leq r \leq 1.$$

Now $J = 4b_1(1 - b_1)^2 s(1 - s)t(1 - t)(1 - r^2)$ and the Jacobian

$$\left| \frac{\partial(b_1, b_2, b_3, b_4)}{\partial(b_1, s, t, r)} \right| = 2(1 - b_1)^2 (b_1 st(1 - s)(1 - t))^{\frac{1}{2}}.$$

(This shows that the maximum value of J is $\frac{1}{27}$ at $b = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ corresponding to $s = t = \frac{1}{2}$, $r = 0$.) We will write db for $db_1 db_2 db_3 db_4$.

3.1 Proposition. *For $\alpha > \frac{1}{2}$, and any continuous function for Ω,*

$$\int_\Omega f(b) J(b)^{\alpha - \frac{1}{2}} db = \int_0^1 \int_0^1 \int_0^1 \int_{-1}^1 f(b_1, s(1 - b_1), t(1 - b_1), b_4(b_1, s, t, r))$$

$$\cdot b_1^{\alpha - 1}(1 - b_1)^{2\alpha - 1}(4st(1 - s)(1 - t))^{\alpha - 1}(1 - r^2)^{\alpha - \frac{3}{2}} dr\, ds\, dt\, db_1.$$

3.2 Proposition. *The normalization constant $h_\alpha := \left(\int_\Omega J(b)^{\alpha - \frac{1}{2}} db \right)^{-1}$ has the value $3^{3\alpha - \frac{1}{2}}(\alpha - \frac{1}{2})\Gamma(\alpha + \frac{1}{3})\Gamma(\alpha + \frac{2}{3})/(\pi\Gamma(\alpha))^2$ (for $\alpha > \frac{1}{2}$).*

Proof. The integral equals

$$\frac{4^{\alpha-1}\Gamma(\alpha)^4\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma(3\alpha)\Gamma(2\alpha)} = \frac{\pi^2\Gamma(\alpha)^2}{3^{3\alpha-\frac{1}{2}}\left(\alpha-\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{3}\right)\Gamma\left(\alpha+\frac{2}{3}\right)}$$

by use of the duplication and triplication formulas for Γ. □

The linear functional on polynomials in b given by integration with $h_\alpha J^{\alpha-\frac{3}{2}}$ over Ω involves a balanced terminating $_4F_3$-series.

3.3 Proposition. *Let* $n_1, n_2, n_3, n_4 = 0, 1, 2, \ldots ,$ *then*

$$h_\alpha \int_\Omega b_1^{n_1} b_2^{n_2} b_3^{n_3} b_4^{n_4} J(b)^{\alpha-\frac{3}{2}} db$$

$$= \frac{(\alpha)_{n_1}(\alpha)_{n_2}(\alpha)_{n_3}(\alpha)_{n_4}(2\alpha)_{n_2+n_3}(2\alpha)_{n_1+n_2+n_4}}{(3\alpha)_{n_1+n_2+n_3+n_4}(2\alpha)_{n_2+n_4}(2\alpha)_{n_1+n_2}(2\alpha)_{n_3}}$$

$$\cdot\, _4F_3\left(\begin{matrix} -n_1, -n_2, -n_4, \alpha+n_3 \\ \alpha, 2\alpha+n_3, 1-2\alpha-n_1-n_2-n_4 \end{matrix}; 1\right)$$

(and the integral is invariant under the transpositions $(n_1, n_4) \mapsto (n_4, n_1)$ *and* $(n_2, n_3) \mapsto (n_3, n_2))$.

Proof. Change variables as in Proposition 3.1 and expand $b_4(b_1, s, t, r)^{n_4}$ as a trinomial. Then the integral of each term in the sum is a product of beta functions. The result is

$$\sum_{i_1+i_2+2j=n_4} \binom{n_4}{i_1, i_2, 2j} \frac{\left(\frac{1}{2}\right)_j 2^{2j}(\alpha)_{n_1+i_2+j}(\alpha)_{n_2+i_2+j}(\alpha)_{n_3+i_2+j}}{(\alpha)_j(3\alpha)_{n_1+n_2+n_3+i_2+j}(2\alpha)_{n_2+n_4}} \frac{(2\alpha)_{n_2+n_3}(\alpha)^2_{j+i_1}}{(2\alpha)_{n_3+n_4}}.$$

Note that the term "$2j$" comes from $\int_{-1}^1 r^{2j}(1-r^2)^{\alpha-\frac{3}{2}}dr$. In the sum change variables, letting $i_1 = n_4 - k - j$, $i_2 = k - j$, so that $0 \le k \le n_4$ and $0 \le j \le \min(k, n_4 - k)$ and obtain

$$\frac{(2\alpha)_{n_2+n_3}(\alpha)_{n_1}(\alpha)_{n_2}(\alpha)_{n_3}}{(3\alpha)_{n_1+n_2+n_3}(2\alpha)_{n_2+n_4}(2\alpha)_{n_3+n_4}}$$

$$\cdot \sum_{k=0}^{n_4} \sum_{j=0}^{\min(k, n_4-k)} \frac{n_4!(\alpha+n_1)_k(\alpha+n_2)_k(\alpha)^2_{n_4-k}(\alpha+n_3)_k}{(n_4-k-j)!(k-j)!(\alpha)_j j!(3\alpha+n_1+n_2+n_3)_k}.$$

The sum of the terms involving j equals

$$\frac{1}{(n_4-k)!k!}\, _2F_1\left(\begin{matrix} k-n_4, -k \\ \alpha \end{matrix}; 1\right) = \frac{(\alpha)_{n_4}}{(\alpha)_k(\alpha)_{n_4-k}k!(n_4-k)!},$$

and so the desired integral equals

$$\frac{(\alpha)_{n_1}(\alpha)_{n_2}(\alpha)_{n_3}(\alpha)^2_{n_4}(2\alpha)_{n_2+n_3}}{(2\alpha)_{n_2+n_4}(2\alpha)_{n_3+n_4}(3\alpha)_{n_1+n_2+n_3}}\, _4F_3\left(\begin{matrix} -n_4, \alpha+n_1, \alpha+n_2, \alpha+n_3 \\ \alpha, 1-\alpha-n_4, 3\alpha+n_1+n_2+n_3 \end{matrix}; 1\right)$$

(note that $(\alpha)_{n_4-k} = (\alpha)_{n_4}(-1)^k/(1-\alpha-n_4)_k$).

The series is balanced ((sum of the numerator parameters)+1 = (sum of the denominator parameters)). The transformation 7.2(i) in Bailey's treatise ([B],

p. 56) leads to the formula of the proposition. Another application of this transformation exhibits the symmetry under $(n_2, n_3) \mapsto (n_3, n_2)$.   □

The linear functional is defined for any $\alpha > 0$. The limit as $\alpha \to 0$ is the uniform discrete distribution on $S_3 \subset U(3)$. In the b-coordinates, the identity corresponds to $(1,0,0,1)$, the transpositions to $(0,1,1,0)$, $(0,0,0,1)$, $(1,0,0,0)$ and the rotations to $(0,1,0,0)$, $(0,0,1,0)$. The sum of $b_1^{n_1} b_2^{n_2} b_3^{n_3} b_4^{n_4}$ over this set, divided by 6 is (write $n = (n_1, n_2, n_3, n_4)$)

    1 for $n = (0, 0, 0, 0)$,

  1/3 for $(m, 0, 0, 0)$, $(0, m, 0, 0)$, $(0, 0, m, 0)$, $(0, 0, 0, m)$ $(m \geq 1)$,

  1/6 for $(k, 0, 0, m)$, $(0, k, m, 0)$ for $(k \geq 1$ and $m \geq 1)$,

    0 else.

For the case $\alpha = \frac{1}{2}$, note that the measure $\frac{\Gamma(\alpha)}{\Gamma(\frac{1}{2})\Gamma(\alpha - \frac{1}{2})}(1 - r^2)^{\alpha - \frac{3}{2}} dr$ tends (weak-*) to $\frac{1}{2}(\delta_1 + \delta_{(-1)})$ (the point masses at $r = 1$ and $r = -1$), and the integral of continuous functions over Ω becomes

$$\frac{1}{2\pi^2} \sum_{r = \pm 1} \int_0^1 \int_0^1 \int_0^1 f(b_1, s(1 - b_1), t(1 - b_1), b_4(b_1, s, t, r))$$

$$\cdot b_1^{-\frac{1}{2}}(4st(1 - s)(1 - t))^{-\frac{1}{2}} ds \, dt \, db_1.$$

(It is not hard to show that this is in fact $\int_{O(3)} f(u_{11}^2, u_{21}^2, u_{12}^2, u_{22}^2) dm(u)$ for the Haar measure m on the real orthogonal group $O(3)$.)

The linear functional fails to be positive when $0 < \alpha < \frac{1}{2}$ because the value at the polynomial J^2 is

$$\frac{h_\alpha}{h_{\alpha+2}} = \frac{\alpha^2(\alpha + 1)^2(2\alpha - 1)}{9(2\alpha + 3)(3\alpha + 1)(3\alpha + 2)(3\alpha + 4)(3\alpha + 5)}.$$

4. THE INTERTWINING OPERATOR

The adjoint action of $U(3)$ described in §2 is realized on $\mathbf{R}^3$ leaving the subspace $\mathbf{R}(1, 1, 1)$ invariant. The linear transformation $\tau(b)$ corresponding to $b = (b_1, b_2, b_3, b_4)$ is given by

$$x\tau(b) = (x_1, x_2, x_3) \begin{bmatrix} b_1 & b_3 & 1 - b_1 - b_3 \\ b_2 & b_4 & 1 - b_2 - b_4 \\ 1 - b_1 - b_2 & 1 - b_3 - b_4 & b_1 + b_2 + b_3 + b_4 - 1 \end{bmatrix}.$$

We use the non-orthogonal coordinate system enabling $GL_2(\mathbf{Z})$ representations of S_3; indeed, let $u_1 = x_1 - x_3$, $u_2 = x_3 - x_2$, $u_3 = x_1 + x_2 + x_3$. In these coordinates,

$$u\tau(b) = u \begin{bmatrix} 2b_1 + b_3 - 1 & 1 - 2b_3 - b_1 & 0 \\ 1 - 2b_2 - b_4 & b_2 + 2b_4 - 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further the inner product $\langle x\tau(b), y\rangle$ transforms to

$$u \begin{bmatrix} b_1 - \frac{1}{3} & \frac{1}{3} - b_3 & 0 \\ \frac{1}{3} - b_2 & b_4 - \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} v^T,$$

where $v = (y_1 - y_3, y_3 - y_2, y_1 + y_2 + y_3)$.

Write $\sigma_{12}, \sigma_{23}, \sigma_{13}$ for the transpositions (12), (23), (13) (acting on the coordinates of x), then in the (u_1, u_2) basis

$$\sigma_{12} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \sigma_{23} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \sigma_{13} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The left and right actions of S_3 on $\tau(b)$ are homogenized by a change of origin, namely, let $c_i := b_i - \frac{1}{3}$, $1 \le i \le 4$. Write

$$\tau_3(c) = \begin{bmatrix} c_1 & c_3 & -c_1 - c_3 \\ c_2 & c_4 & -c_2 - c_4 \\ -c_1 - c_2 & -c_3 - c_4 & c_1 + c_2 + c_3 + c_4 \end{bmatrix},$$

and

$$\tau(c) = \begin{bmatrix} 2c_1 + c_3 & -c_1 - 2c_3 \\ -2c_2 - c_4 & c_2 + 2c_4 \end{bmatrix},$$

(the restriction of $\tau_3(c)$ to the subspace $\{x : x_1 + x_2 + x_3 = 0\}$. Our concern is with functions of c which are invariant under $w \mapsto w\tau_3(c)w^{-1}$ or $w \mapsto w\tau_3(c)$ for $w \in S_3$. For each transposition σ we write the result as $\sigma c\sigma$ or σc so that $\sigma\tau_3(c)\sigma = \tau_3(\sigma c\sigma)$ or $\sigma\tau_3(c) = \tau_3(\sigma c)$ respectively:

$$\sigma_{12}c\sigma_{12} = (c_4, c_3, c_2, c_1),$$
$$\sigma_{23}c\sigma_{23} = (c_1, -c_1 - c_2, -c_1 - c_3, c_1 + c_2 + c_3 + c_4),$$
$$\sigma_{13}c\sigma_{13} = (c_1 + c_2 + c_3 + c_4, -c_2 - c_4, -c_3 - c_4, c_4),$$
$$\sigma_{12}c = (c_2, c_1, c_4, c_3),$$
$$\sigma_{23}c = (c_1, -c_1 - c_2, c_3, -c_3 - c_4),$$
$$\sigma_{13}c = (-c_1 - c_2, c_2, -c_3 - c_4, c_4).$$

It turns out that the two needed $w\tau_3(c)w^{-1}$ invariants are

$$\phi_1(c) := \text{trace}\,(\tau(c)) = 2c_1 + c_2 + c_3 + 2c_4$$

and

$$\phi_2(c) := \frac{1}{3}\det(\tau(c)) = c_1 c_4 - c_2 c_3.$$

The discovery of the transform was somewhat experimental, in a sense similar to a proof by induction, where one first has to make a good guess. In the present situation we first calculated $K_n(x, y)$ and $K_{G,n}(x, y) = \frac{1}{6}\sum_{w \in S_3} K_n(xw, y)$ for $n \le 6$. By a special case of the formula in (Proposition 3.2(v) [Du3] or

(1.4) in [Du4], p. 126),

$$K_{n+1}(x, y) = \langle x, y \rangle K_n(x, y)/(3\alpha + n + 1)$$

$$+ \sum_{\substack{w \in S_3 \\ \det(w)=1}} \frac{3\alpha^2}{(n+1)(3\alpha + n + 1)(6\alpha + n + 1)} \langle xw, y \rangle K_n(xw, y)$$

$$+ \sum_{\substack{w \in S_3 \\ \det(w)=-1}} \frac{\alpha}{(n+1)(6\alpha + n + 1)} \langle xw, y \rangle K_n(xw, y),$$

$n = 0, 1, 2, 3, \ldots$, and $K_0 = 1$, this can be done (with computer algebra assistance). The first step was to try to produce the Bessel function components $(K_{G,n}$, see §1) by integration of $(\langle x\tau(b), y \rangle^n/n!)$ with powers of J, and $J^{\alpha - \frac{3}{2}}$ gave the right result for $n \le 6$.

To illustrate the dependence on α and the invariant structure of S_3 we list $K_{G,n}$ for $n \le 6$. The invariants $\|u\|^2 = \frac{2}{3}(u_1^2 + u_1 u_2 + u_2^2)$ and $p_3(u) = (u_1 - u_2)(2u_1 + u_2)(u_1 + 2u_2)$ will appear. Of course, $K_{G,0} = 1$ and $K_{G,1} = 0$ (restricted to the two-dimensional subspace on which S_3 acts irreducibly). Further

$$K_{G,2}(u, v) = \frac{1}{4(3\alpha + 1)} \|u\|^2 \|v\|^2,$$

$$K_{G,3}(u, v) = \frac{1}{162(3\alpha + 1)(3\alpha + 2)} p_3(u) p_3(v),$$

$$K_{G,4}(u, v) = \frac{1}{32(3\alpha + 1)(3\alpha + 2)} \|u\|^4 \|v\|^4,$$

$$K_{G,5}(u, v) = \frac{1}{648(3\alpha + 1)(3\alpha + 2)(3\alpha + 4)} p_3(u) \|u\|^2 p_3(v) \|v\|^2,$$

$$K_{G,6}(u, v) = \frac{1}{(3\alpha + 1)(3\alpha + 2)(3\alpha + 4)(3\alpha + 5)}$$

$$\cdot \left\{ \frac{1}{72(2\alpha + 1)} (u_1 u_2 (u_1 + u_2) v_1 v_2 (v_1 + v_2))^2 \right.$$

$$\left. + \frac{1}{52488} p_3(u)^2 p_3(v)^2 + \frac{\alpha + 2}{128} \|u\|^6 \|v\|^6 \right\}.$$

The next step was to evaluate the integrals

$$h_\alpha \int_\Omega (\langle u\tau(b), v \rangle^n/n!) \phi(b) J(b)^{\alpha - \frac{3}{2}} db,$$

where ϕ is a linear combination of spanning $w\tau_3(c)w^{-1}$-type invariants of degree ≤ 4 and matching them to the known $K_n(u, v)$, $n \le 6$. (This used the integrals of monomials in b, from Proposition 3.3.) This led to the formula $\phi(c) = 1 + 2(2c_1 + c_2 + c_3 + 2c_4) + 3(c_1 c_4 - c_2 c_3)$. This function satisfies $-\frac{2}{9} \le$

$\phi(c) \leq 6$, with the minimum at $c = \left(-\frac{2}{9}, \frac{1}{9}, \frac{1}{9}, -\frac{2}{9}\right)$ and the maximum at $c = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right)$, both on the boundary of Ω.

It remains to show that this satisfies the necessary conditions for realizing the operator V stated in Proposition 1.2. Thus we need to show

$$h_\alpha \int_{\Omega_0} \phi(c) J(c)^{\alpha - \frac{3}{2}} dc = 1$$

and

$$\begin{aligned}
& \int_{\Omega_0} (\langle u, v \rangle - \langle u\tau(c), v \rangle) \exp(\langle u\tau(c), v \rangle) \phi(c) J(c)^{\alpha - \frac{3}{2}} d \\
(4.1) \quad & \quad = h_\alpha \alpha \int_{\Omega_0} \exp(\langle u\tau(c), v \rangle) \sum_{\substack{\sigma \in S_3 \\ \det \sigma = -1}} (\phi(c) - \phi(\sigma c)) J(c)^{\alpha - \frac{3}{2}} dc,
\end{aligned}$$

for arbitrary $u, v \in \mathbf{R}^2$ (where we use $J(c)$ to denote $J(b)$ with $b_i = c_i + \frac{1}{3}$, $1 \leq i \leq 4$, and Ω_0 is the translate $\Omega - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; further $\langle u, v \rangle = \left(\frac{1}{3}\right)(2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2)$, because of the present non-orthogonal coordinate system). The transformation properties $\sum_\sigma \phi_1(\sigma c) = 0$ and $\phi_2(\sigma c) = -\phi_2(c)$ (σ denotes the transpositions) show that $\int_{\Omega_0} \phi_1(c) J(c)^{\alpha - \frac{3}{2}} dc = 0 = \int_{\Omega_0} \phi_2(c) J(c)^{\alpha - \frac{3}{2}} dc$, and thus $h_\alpha \int_{\Omega_0} (1 + 2\phi_1 + 3\phi_2) J^{\alpha - \frac{3}{2}} dc = 1$. Further

$$\psi(c) := \sum_\sigma (\phi(c) - \phi(\sigma c)) = 6\phi_1(c) + 18\phi_2(c).$$

The identity (4.1) as a function of (u, v) is invariant under $(u, v) \mapsto (uw, vw)$ for each $w \in S_3$ (because $\phi(wcw^{-1}) = \phi(c)$), thus it suffices to prove (4.1) for the case $u_1 > 0$, $u_2 > 0$ (the fundamental chamber) or $u_1 \neq 0$, $u_2 = 0$, and arbitrary $v \in \mathbf{R}^2$; since for any $u \neq 0$ there exists $w \in S_3$ so that uw satisfies one of these conditions (depending on whether $u_1 u_2 (u_1 + u_2) = 0$).

For future convenience we note that (4.1) is equivalent to

$$\begin{aligned}
& \alpha \int_{\Omega_0} \exp(\langle u\tau(c), v \rangle) \psi(c) J(c)^{\alpha - \frac{3}{2}} dc \\
(4.2) \quad & \\
& \quad - \int_{\Omega_0} (\langle u, v \rangle - \langle u\tau(c), v \rangle) \exp(\langle u\tau(c), v \rangle) \phi(c) J(c)^{\alpha - \frac{3}{2}} dc = 0.
\end{aligned}$$

4.1 Lemma. *The identity* (4.2) *is valid for* $u_2 = 0$, *arbitrary* $v \in \mathbf{R}^2$, *and* $\alpha > \frac{1}{2}$.

Proof. Recall the coordinate system (3.1) and let $\xi = (b_1 st(1 - s)(1 - t))^{\frac{1}{2}}$. Then

$$\left| \frac{\partial(b_1, b_2, b_3, b_4)}{\partial(b_1, s, t, r)} \right| = 2\xi(1 - b_1)^2,$$

$$\langle u\tau(b), v \rangle = u_1 \left(b_1(v_1 + tv_2) - \frac{1}{3}v_1 - \left(t - \frac{1}{3}\right)v_2 \right)$$

while

$$\langle u, v\rangle - \langle u\tau(b), v\rangle = (1 - b_1)u_1(v_1 + tv_2) = (1 - b_1)\frac{\partial}{\partial b_1}\langle u\tau(b), v\rangle.$$

Further

$$\phi_1 = \frac{1}{2}(3b_1 - 1) + \frac{1}{2}(2s - 1)(2t - 1)(1 + b_1) + 4\xi r,$$

and

$$\phi_2 = \frac{1}{3}(2b_1 - 1)(2s - 1)(2t - 1) + \frac{2}{3}\xi r(3b_1 - 1).$$

In the left side of the identity the part involving $\langle u\tau(c), v\rangle$ is independent of s and r. Integrate over these variables and observe that products of odd powers of $(2s - 1)$ and r vanish. The left side of (4.2) is a constant (in α) times

$$\int_0^1 \int_0^1 \{\exp(\langle u\tau(b), v\rangle)3\alpha(3b_1 - 1) - (\langle u, v\rangle - \langle u\tau(b), v\rangle)\exp(\langle u\tau(b), v\rangle))3b_1\}$$
$$\cdot b_1^{\alpha-1}(1 - b_1)^{2\alpha-1}(t(1 - t))^{\alpha-1}db_1 dt.$$

The inner integral equals

$$\int_0^1 \Big\{ \exp(\langle u\tau(b), v\rangle)3\alpha(3b_1 - 1)b_1^{\alpha-1}(1 - b_1)^{2\alpha-1}$$

$$-3(1 - b_1)\frac{\partial}{\partial b_1}\exp(\langle u\tau(b), v\rangle)b_1^{\alpha}(1 - b_1)^{2\alpha-1}\Big\} db_1$$

$$= -3\int_0^1 \frac{\partial}{\partial b_1}(\exp(\langle u\tau(b), v\rangle)b_1^{\alpha}(1 - b_1)^{2\alpha})db_1 = 0$$

(integration by parts). $\square$

This was the easy case: the S_3 orbit of $(u_1, 0)$ consists of three points (namely $(u_1, 0)$, $(0, -u_1)$, $(-u_1, u_1)$). It does foreshadow the technique for the other case: find a change of variables which allows two steps of integration not involving $\langle u\tau(c), v\rangle$, and then do some integration by parts.

We introduce a linear change of variables designed to change J to a quadratic in one of the variables (which is independent of $\langle u\tau(c), v\rangle$). Henceforth, fix $u_1 > 0$ and $u_2 > 0$.

The new variables z_j, $1 \le j \le 4$, are

(4.3)
$$z_1 = u_1 c_1 - u_2 c_2 - u_1 c_3 + u_2 c_4,$$

$$z_2 = u_1 c_1 - u_2 c_2 + u_1 c_3 - u_2 c_4 + \frac{2}{3}(u_1 - u_2),$$

$$z_3 = u_1 c_1 + u_2 c_2 + u_1 c_3 + u_2 c_4,$$

$$z_4 = u_1 c_1 + u_2 c_2 - u_1 c_3 - u_2 c_4.$$

(The shift-of-origin in z_2 simplifies some later expressions.) The Jacobian is

$$\left|\frac{\partial(c_1, c_2, c_3, c_4)}{\partial(z_1, z_2, z_3, z_4)}\right| = \frac{1}{16u_1^2 u_2^2}.$$

Further

$$\langle u\tau(c), v\rangle = \left(\frac{1}{2}(z_1 + z_2) - \frac{1}{3}(u_1 - u_2)\right)v_1 + \left(\frac{1}{2}(z_1 - z_2) + \frac{1}{3}(u_1 - u_2)\right)v_2.$$

The task is to integrate over z_3 and z_4 in the proposed identity (4.2), and to describe the region of integration for (z_1, z_2) (for given u). Performing the change of variable in J leads to $J(c) = J_z(z_1, z_2, z_3, z_4)/u_1^2 u_2^2$ where J_z is quadratic in z_4, with leading coefficient

$$(4.4) \quad -a_2 = -\frac{1}{16}\left(2z_3(u_1 + u_2) + \frac{4}{3}(u_1 + u_2)^2 - (2u_1 - z_2)(2u_2 + z_2)\right).$$

The inner quantity must be positive (the value at $c = 0$ is $\frac{4}{9}(u_1^2 + u_1 u_2 + u_2^2)$) and to simplify J_z the change

$$(4.5) \qquad z_3 = (3r^2 + 3(2u_1 - z_2)(2u_2 + z_2) - 4(u_1 + u_2)^2)/(6(u_1 + u_2))$$

is made. With this change the coefficient of z_4 in J_z is

$$a_1 := \frac{z_1 z_2}{16(u_2 + u_3)}(r^2 + (2u_1 - z_2)(2u_2 + z_2)).$$

Completing the square we let $z_4' = z_4 - \frac{a_1}{2a_2}$ (where $a_2 = \frac{1}{16}r^2$), then

$$(4.6) \qquad J_z = -\frac{1}{16}r^2 z_4'^2 + \frac{1}{64}\frac{\delta_0}{r^2(u_1 + u_2)^2}, \quad \text{where}$$

$$\delta_0 := (r^2 - z_1^2)(r^2 - z_2^2)((2u_1 - z_2)^2 - r^2)((2u_2 + z_2)^2 - r^2).$$

In the region of integration it is required that $\delta_0 \geq 0$ which by the factorization is equivalent to $\max(|z_1|, |z_2|) \leq r \leq \min(2u_1 - z_2, 2u_2 + z_2)$ (this includes the restriction regarding the connected component of $c = 0$, that is, the point $z_1 = z_3 = z_4 = 0$, $z_2 = \frac{2}{3}(u_1 - u_2)$, and $r^2 = \frac{4}{9}(u_1^2 + u_1 u_2 + u_2)$, in the set $\{(z_1, z_2, r) : \delta_0 > 0\}$). Denote by Ω_u the transform of Ω_0 into the z-coordinates.

4.2 Proposition. *For fixed u (with $u_1 > 0$, $u_2 > 0$) the region Ω_u', the projection of Ω_u onto $\{(z_1, z_2) : z_1, z_2 \in \mathbf{R}\}$ consists of the convex hull of $\{(u_1, u_1), (-u_1, u_1), (u_2, -u_2), (-u_2, -u_2), (u_1 + u_2, u_1 - u_2), (-u_1 - u_2, u_1 - u_2)\}$ and is defined by the inequalities*

$$-u_2 \leq z_2 \leq u_1, \quad -2u_2 \leq z_1 + z_2 \leq 2u_1, \quad -2u_2 \leq z_2 - z_1 \leq 2u_1.$$

Proof. The collection of inequalities is equivalent to $\max(|z_1|, |z_2|) \leq \min(2u_1 - z_2, 2u_2 + z_2)$. For any $(z_1, z_2) \in \Omega_u'$ the variable r has the previously described range, and $\left|z_4 - \frac{a_1}{2a_2}\right| \leq \frac{\delta_0^{\frac{1}{2}}}{2r^2(u_1 + u_2)}$. In the special case $z_1 = 0 = z_2$ (and so $a_1 = 0$)

$$J_z = r^2\left(-\frac{1}{16}z_4^2 + \frac{(4u_1^2 - r^2)(4u_2^2 - r^2)}{64(u_1 + u_2)^2}\right)$$

and

$$0 < r < \min(2u_1, 2u_2); \quad |z_4| \leq \frac{((4u_1^2 - r^2)(4u_2^2 - r^2))^{\frac{1}{2}}}{2(u_1 + u_2)}. \quad \square$$

Note that $u\tau(c) = \left(\frac{1}{2}(z_1 + 3z_2) + u_2 - u_1, \frac{1}{2}(z_1 - 3z_2) + u_1 - u_2\right)$, and the vertices of Ω_u' correspond to the S_3-orbit of u. We perform integration of 1, z_4', $z_4'^2$ in a lemma. The formulas can be made more concise by writing $J_z = -a_2 z_4'^2 + \delta/4a_2$, so that δ denotes the discriminant of J_z as a quadratic in z_4 and $\delta = \delta_0/(256(u_1 + u_2)^2)$ (see (4.6)).

4.3 Lemma. *For* $(z_1, z_2) \in \Omega_u'$,

$$\max(|z_1|, |z_2|) \leq r \leq \min(2u_1 - z_2, 2u_2 + z_2), \qquad \beta > -\frac{1}{2},$$

$$\int_{-\gamma}^{\gamma} (A_0 + A_1 z_4' + A_2 z_4'^2) J_z^{\beta - \frac{1}{2}} dz_4'$$

$$= B\left(\beta + \frac{1}{2}, \frac{1}{2}\right) \frac{1}{\sqrt{a_2}} \left(\frac{\delta}{4a_2}\right)^{\beta} (A_0 + A_2 \delta/(8(\beta + 1)a_2^2)),$$

for arbitrary $A_0, A_1, A_2 \in \mathbf{R}$ *and* $\gamma = \delta^{\frac{1}{2}}/(2a_2)$ *(B denotes the beta function).*
Proof. Let $z_4' = \gamma s$, then $-1 \leq s \leq 1$, and the terms of the integral are of the form $\int_{-1}^{1} s^n (1 - s^2)^{\beta - \frac{1}{2}} ds$. $\square$

4.4 Theorem. *The identity* (4.2)

$$\alpha \int_{\Omega_0} \exp(\langle u\tau(c), v \rangle) \psi(c) J(c)^{\alpha - \frac{3}{2}} dc$$

$$- \int_{\Omega_0} (\langle u, v \rangle - \langle u\tau(c), v \rangle) \exp(\langle u\tau(c), v \rangle) \phi(c) J(c)^{\alpha - \frac{3}{2}} dc = 0$$

is valid for $u_1 > 0$, $u_2 > 0$, *arbitrary* $v \in \mathbf{R}^2$, *and* $\alpha > \frac{1}{2}$.
Proof. Write

$$p_0(u, v, z) = \langle u\tau(c), v \rangle$$

$$= \left(\frac{1}{2}(z_1 + z_2) - \frac{1}{3}(u_1 - u_2)\right) v_1 + \left(\frac{1}{2}(z_1 - z_2) + \frac{1}{3}(u_1 - u_2)\right) v_2,$$

and

$$p_1(u, v, z) = \langle u, v \rangle - \langle u\tau(c), v \rangle$$

$$= \left(u_1 - \frac{1}{2}(z_1 + z_2)\right) v_1 + \left(u_2 - \frac{1}{2}(z_1 - z_2)\right) v_2.$$

Let $16(u_1^2 u_2^2)^{-\alpha+\frac{1}{2}} F(u, v)$ denote the left-hand side of the proposed identity. Change variables from c to z, and factor out $(16u_1^2 u_2^2)^{-1}(u_1^2 u_2^2)^{-\alpha+\frac{3}{2}}$ to obtain

$$F(u, v) = \alpha \int_{\Omega_0} \exp(p_0(u, v, z)) g_1(z) J_z(z)^{\alpha - \frac{3}{2}} dz$$

$$- \int_{\Omega_0} p_1(u, v, z) \exp(p_0(u, v, z)) g_0(z) J_z(z)^{\alpha - \frac{3}{2}} dz,$$

where $g_0(z) = \phi(c) = 1 + 2\phi_1(c) + 3\phi_2(c)$ and $g_1(z) = \psi(c) = 6\phi_1(c) + 18\phi_2(c)$ (we postpone the explicit statement of the result of the substitutions).

We get rid of $p_1(u, v, z)$ by means of integration by parts. Indeed,

$$p_1(u, v, z) = \left((u_1 + u_2 - z_1)\frac{\partial}{\partial z_1} + (u_1 - u_2 - z_2)\frac{\partial}{\partial z_2} \right) p_0(u, v, z).$$

Observe that

$$\int_{\Omega_u} \sum_{i=1}^{4} (A_i + B_i z_i)\frac{\partial}{\partial z_i}(F_1(z)) F_2(z) dz$$

$$= - \int_{\Omega_u} F_1(z) \sum_{i=1}^{4} \frac{\partial}{\partial z_i}((A_i + B_i z_i) F_2(z)) dz$$

$$= - \int_{\Omega_u} \left(\left(\sum_{i=1}^{4} B_i \right) F_1(z) F_2(z) + F_1(z) \sum_{i=1}^{4}(A_i + B_i z_i)\frac{\partial}{\partial z_i} F_2(z) \right) dz$$

for constants A_i, B_i, $1 \leq i \leq 4$, and smooth functions F_1, F_2 such that F_2 vanishes on the boundary of Ω_u; by integrating with respect to z_i first in the corresponding summand.

We apply this to $F_1(z) = \exp(p_0(u, v, z))$ with a trick of adding a term in $\frac{\partial}{\partial z_3}$. Define the differential operator

$$D := (u_1 + u_2 - z_1)\frac{\partial}{\partial z_1} + (u_1 - u_2 - z_2)\frac{\partial}{\partial z_2} + (-(u_1 + u_2)/3 - 2z_3)\frac{\partial}{\partial z_3} ;$$

thus

$$\int_{\Omega_u} (DF_1(z)) F_2(z) dz = - \int_{\Omega_u} (-4F_1(z) F_2(z) + F_1(z) DF_2(z)) dz.$$

The purpose of the $\frac{\partial}{\partial z_3}$ term is to simplify subsequent calculations because $Dr^2 = -2r^2$. Indeed,

$$Dr^2 = D\left(2z_3(u_1 + u_2) + \left(\frac{4}{3}\right)(u_1 + u_2)^2 - (2u_1 - z_2)(2u_2 + z_2) \right)$$

$$= -2r^2.$$

The result of integration by parts is

$$(4.7) \qquad F(u, v) = \int_{\Omega_u} \exp(p_0(u, v, z))$$

$$\cdot \left(\alpha g_1(z) J_z^{\alpha - \frac{3}{2}} - 4 g_0(z) J_z^{\alpha - \frac{3}{2}} + D(g_0(z) J_z^{\alpha - \frac{3}{2}}) \right) dz.$$

It remains to show that the integral over z_3 and z_4 vanishes, for each fixed $(z_1, z_2) \in \Omega'_u$. For now we assume $\alpha > \frac{3}{2}$.

We will integrate over z_4 directly and then for the remaining step change the variable z_3 to r and finish by another integration by parts. The substitutions $g_0(z) = \phi(c)$ and $g_1(z) = \psi(c)$ lead to linear dependence on z_4, and we let

$$g_0 = g_{01} z'_4 + g_{00}, \qquad g_1 = g_{11} z'_4 + g_{10}$$

(so that g_{ij} is independent of $z'_4 = z_4 - a_1/2a_2$). Recall (from (4.6)) $J_z = -a_2 z_4'^2 + \frac{\delta}{4a_2}$ where $a_2 = \frac{1}{16} r^2$ and $\delta = \delta_0/(256(u_1 + u_2)^2)$ the discriminant of J_z as a quadratic in z_4. Discard the part of the integrand in (4.7) which is odd in z'_4 (and omit the (z_1, z_2) part $\exp(p_0(u, v, z))$) and obtain

$$\left\{ (\alpha g_{10} - 4 g_{00} + D g_{00} + g_{01}(D z'_4)) \left(-a_2 z_4'^2 + \frac{\delta}{4a_2} \right) \right.$$

$$+ \left(\alpha - \frac{3}{2} \right) \left(g_{00} \left(2 a_2 z_4'^2 + D(\delta/4a_2) \right) - 2 a_2 g_{01} z_4'^2 D(z'_4) \right) \left. \right\} \left(-a_2 z_4'^2 + \frac{\delta}{4a_2} \right)^{\alpha - \frac{5}{2}}$$

because $D J_z = -D(a_2) z_4'^2 - 2 a_2 z'_4 D(z'_4) + D(\delta/4a_2)$ and $D(z'_4) = D(-a_1/2a_2)$. Further $D(a_2) = -2a_2$. Performing the integral over z'_4 by Lemma 4.3 (with $\beta = \alpha - 2$) gives the value

$$B\left(\alpha - \frac{3}{2}, \frac{1}{2} \right) \frac{1}{\sqrt{a_2}} \left(\frac{\delta}{4a_2} \right)^{\alpha - 2}$$

$$\cdot \left\{ \left(\frac{\alpha - \frac{3}{2}}{\alpha - 1} \right) \left(\frac{\delta}{4a_2} \right) (\alpha g_{10} - 4 g_{00} + D g_{00} + g_{01} D z'_4) \right\}$$

$$+ \left(\alpha - \frac{3}{2} \right) \left(g_{00} D \left(\frac{\delta}{4a_2} \right) + (2 a_2 g_{00} - 2 a_2 g_{01} D(z'_4)) \frac{\delta}{8 a_2^2 (\alpha - 1)} \right\}$$

$$= \frac{(\alpha - \frac{3}{2})}{(\alpha - 1)} B\left(\alpha - \frac{3}{2}, \frac{1}{2} \right) \frac{1}{\sqrt{a_2}} \left(\frac{\delta}{4a_2} \right)^{\alpha - 2}$$

$$\cdot \left\{ \left(\frac{\delta}{4a_2} \right) (g_{10} - 4 g_{00} + D g_{00} + g_{01} D z'_4 + g_{00} - g_{01} D z'_4) \right.$$

$$\left. + (\alpha - 1) \left(g_{10} \frac{\delta}{4a_2} + g_{00} D \left(\frac{\delta}{4a_2} \right) \right) \right\}$$

$$= B\left(\alpha - \frac{1}{2}, \frac{1}{2}\right) \frac{1}{\sqrt{a_2}} \left\{ \left(\frac{\delta}{4a_2}\right)^{\alpha-1} (g_{10} - 3g_{00} + Dg_{00}) \right.$$

$$\left. + (\alpha - 1)\left(\frac{\delta}{4a_2}\right)^{\alpha-2} \left(g_{10}\frac{\delta}{4a_2} + g_{00}D\left(\frac{\delta}{4a_2}\right)\right) \right\}.$$

(We wrote $\alpha g_{10} = (\alpha - 1)g_{10} + g_{10}$ in the calculation.) It remains to integrate this function over z_3 and show the result vanishes. This will be done by changing the variable of integration to r and showing the integrand is a constant times $\frac{\partial}{\partial r}\left(\left(\frac{\delta}{4a_2}\right)^{\alpha-1}h\right)$ for a function h satisfying

(4.8)
$$\frac{\partial}{\partial r}h = g_{10} - 3g_{00} + Dg_{00},$$

and

(4.9)
$$h\frac{\partial}{\partial r}\left(\frac{\delta}{4a_2}\right) = g_{10}\frac{\delta}{4a_2} + g_{00}D\left(\frac{\delta}{4a_2}\right).$$

By (4.5), $dz_3 = r\,dr/(u_1 + u_2)$ and this cancels the factor $\sqrt{a_2} = r/4$, to a constant. The equation (4.9) is equivalent to

$$\left(h\frac{\partial\delta}{\partial r} - 2\frac{h}{r}\delta\right)\left(\frac{1}{4a_2}\right) = (g_{10}\delta + g_{00}D(\delta) + 2g_{00}\delta)\left(\frac{1}{4a_2}\right)$$

(because $\frac{\partial a_2}{\partial r} = \frac{2}{r}a_2$ and $Da_2 = -2a_2$), which in turn is equivalent to

(4.10)
$$g_{10} + 2g_{00} + \frac{2h}{r} = \frac{1}{\delta}\left(h\frac{\partial\delta}{\partial r} - g_{00}D\delta\right).$$

The right side is a logarithmic derivative and the factorization of δ allows a simplification of the calculation.

Here are some useful new substitutions:

$$\xi_1 := 2u_1 - z_2, \quad \xi_2 := 2u_2 + z_2,$$

which yield $\delta = \frac{1}{256(u_1+u_2)^2}(r^2 - z_1^2)(r^2 - z_2^2)(\xi_1^2 - r^2)(\xi_2^2 - r^2)$ and $a_1 = z_1z_2(r^2 + \xi_1\xi_2)/(8(\xi_1 + \xi_2))$. In the variables $z_1, z_2, \xi_1, \xi_2, r$ the operator

$$D = \frac{1}{2}(\xi_1 + \xi_2 - 2z_1)\frac{\partial}{\partial z_1} + \frac{1}{2}(\xi_1 - \xi_2)\left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial \xi_2} - \frac{\partial}{\partial \xi_1}\right) - r\frac{\partial}{\partial r}.$$

Further, let $q_2 := r^2 + z_1(\xi_1 + \xi_2) + \xi_1\xi_2$ and $q_3 := (\xi_1 + \xi_2 + z_1)r^2 + \xi_1\xi_2 z_1$.

We can now state the result of the substitutions on g_0, g_{10} (recall $g_0(z) = \phi(c)$ and $\psi(c) = g_{11}(z_1, z_2, z_3)z_4' + g_{10}(z_1, z_2, z_3)$):

$$g_0 = \frac{1}{u_1u_2}\left\{-\frac{3}{4}z_2z_4' + \frac{3}{4}\frac{(r^2 - z_2^2)}{r^2(\xi_1 + \xi_2)}q_3\right\},$$

and

$$g_{10} = \frac{1}{u, u_2} \cdot \frac{3}{4r^2(\xi_1 + \xi_2)}$$

$$\cdot \{(\xi_1 - \xi_2)z_2q_3 - (\xi_1 + \xi_2)q_2r^2 + 2r^2q_3 + 2(r^2 - z_2^2)(q_3 + z_1(r^2 + \xi_1\xi_2))\}.$$

We will show that

$$h := \frac{1}{u_1 u_2} \cdot \frac{3}{8} \cdot \frac{(r^2 - z_2^2)}{r} q_2$$

satisfies the equations (4.8) and (4.10). First,

$$\frac{\partial h}{\partial r} = -\frac{1}{u_1 u_2} \frac{3}{8r^2} \{2r^2 q_2 + (r^2 - z_2^2)(2r^2 - q_2)\}.$$

$$Dg_{00} = \frac{3}{4u_1 u_2 r^2 (\xi_1 + \xi_2)} \left\{ - q_3(2r^2 + z_2(\xi_1 - \xi_2)) \right.$$

$$\left. + (r^2 - z_2^2) \left(-q_3 + \frac{1}{2}(\xi_1 + \xi_2)(2r^2 + q_2) \right) \right\}$$

By use of the relation $-q_3 + z_1(r^2 + \xi_1\xi_2) = -r^2(\xi_1 + \xi_2)$ this shows that $g_{10} - 3g_{00} + Dg_{00} = \frac{\partial h}{\partial r}$.

Write $\delta = f_1 f_2 f_3 f_4 / (256(u_1 + u_2)^2)$ with

$$f_1 := r^2 - z_1^2, \quad f_2 := r^2 - z_2^2,$$
$$f_3 := \xi_1^2 - r^2, \quad f_4 := \xi_2^2 - r^2.$$

To prove (4.10) we need to show $g_{10} + 2g_{00} + 2h/r = \sum_{i=1}^{4} k_i / f_i$, where $k_i := \left(h \frac{\partial}{\partial r} f_i - g_{00} D f_i \right)$. (In this formulation the divisions can actually be carried out by hand, without computer algebra assistance.) Now

$$Df_1 = -2r^2 + 2z_1^2 - z_1(\xi_1 + \xi_2), \quad k_1 = \frac{3}{4} \frac{(r^2 - z_1^2)(r^2 - z_2^2)}{u_1 u_2(\xi_1 + \xi_2)r^2},$$

$$Df_2 = -2r^2 - z_2(\xi_1 - \xi_2)(2q_3 - (\xi_1 + \xi_2)q_2 + (\xi_1 + \xi_2)^2 z_1),$$

$$k_2 = \frac{3}{4} \frac{(r^2 - z_1^2)(r^2 - z_2^2)}{u_1 u_2(\xi_1 + \xi_2)r^2} \left(z_2(\xi_1 - \xi_2)q_3 + 2q_3 r^2 - (\xi_1 + \xi_2)q_2 r^2 \right)$$

$$Df_3 = 2r^2 - \xi_1(\xi_1 - \xi_2), \quad k_3 = \frac{3}{4} \frac{(r^2 - z_2^2)(\xi_1^2 - r^2)}{u_1 u_2(\xi_1 + \xi_2)r^2}(q_3 + z_1(r^2 - \xi_2^2)),$$

$$Df_4 = 2r^2 + \xi_2(\xi_1 - \xi_2), \quad k_4 = \frac{3}{4} \frac{(r^2 - z_2^2)(\xi_2^2 - r^2)}{u_1 u_2(\xi_1 + \xi_2)r^2}(q_3 + z_1(r^2 - \xi_1^2)).$$

The verification of (4.10) can now be carried out directly. During the latter part of the proof we assumed $\alpha > \frac{3}{2}$. When the identity (4.2) is written as a sum of homogeneous terms in v (expanding $\exp(u\tau(c), v\rangle)$), the dependence on α in each term is rational with no poles for $\alpha > \frac{1}{2}$. Thus the identity holds for all $\alpha > \frac{1}{2}$. $\square$

5. EXPRESSIONS FOR THE INTERTWINING OPERATOR

Having established the validity of the transform we use the various coordinate systems to produce several formulae. Here is a notation for the result of the integration over z_3 and z_4: for real variables z_1, z_2, ξ_1, ξ_2 subject to

$\max(|z_1|, |z_2|) \le \min(\xi_1, \xi_2)$ and a parameter $\alpha > \frac{1}{2}$, let

$$F_\alpha(z_1, z_2, \xi_1, \xi_2) = \frac{3}{8} \int_{\max(|z_1|, |z_2|)}^{\min(\xi_1, \xi_2)} (r^2(\xi_1 + \xi_2 + z_1) + \xi_1\xi_2 z_1)$$
$$\cdot \left(\frac{r^2 - z_2^2}{r^2}\right)^\alpha ((r^2 - z_1^2)(\xi_1^2 - r^2)(\xi_2^2 - r^2))^{\alpha-1} dr.$$

This function is positively homogeneous of degree $6\alpha - 2$. In the formula

(5.1)
$$Vf(u) = h_\alpha \int_{\Omega_0} f(u\tau(c))\phi(c)J(c)^{\alpha-\frac{3}{2}} dc,$$

when $u_1 > 0$, $u_2 > 0$, we use the change of variables (4.3), integrate over z_4 with Lemma 4.3, and again write $\phi(c) = g_0(z) = g_{01}(z_1, z_2, z_3)z_4' + g_{00}(z_1, z_2, z_3)$. The part odd in z_4' vanishes, and g_{00} has been incorporated into F_α. Thus

(5.2)
$$Vf(u) = \frac{1}{(u_1 u_2(u_1 + u_2))^{2\alpha}} m_\alpha$$
$$\cdot \int_{\Omega_{u'}} f\left(\frac{1}{2}z_1 + \frac{3}{2}z_2 + u_2 - u_1, \frac{1}{2}z_1 - \frac{3}{2}z_2 + u_1 - u_2\right)$$
$$\cdot F_\alpha(z_1, z_2, 2u_1 - z_2, 2u_2 + z_2) dz_1 dz_2,$$

where the normalizing constant is

$$m_\alpha := \left(\frac{27}{64}\right)^\alpha \left(\frac{16}{\pi\sqrt{3\pi}}\right) \frac{\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{3}\right)\Gamma\left(\alpha + \frac{2}{3}\right)}{\Gamma(\alpha)^3}.$$

Let $H(u)$ denote the closed convex hull of the S_3-orbit of u. It is described by the inequalities (when $u_1 > 0$, $u_2 > 0$): $(\zeta_1, \zeta_2) \in H(u)$ if and only if $-2u_1 - u_2 \le \zeta_1 - \zeta_2$, $\zeta_1 + 2\zeta_2$, $-2\zeta_1 - \zeta_2 \le u_1 + 2u_2$.

With the linear change, $\zeta = u\tau(c)$, that is,

$$z_1 = \zeta_1 + \zeta_2, \quad z_2 = \frac{1}{3}(\zeta_1 - \zeta_2 + 2u_1 - 2u_2)$$

the formula (5.2) is transformed to

(5.3)
$$Vf(u) = \frac{1}{(u_1 u_2(u_1 + u_2))^{2\alpha}} \frac{2}{3} m_\alpha$$
$$\cdot \int_{H(u)} f(\zeta_1, \zeta_2) \cdot F_\alpha\left(\zeta_1 + \zeta_2, \frac{1}{3}(\zeta_1 - \zeta_2 + 2u_1 - 2u_2),\right.$$
$$\frac{1}{3}(2u_1 + 4u_2 + \zeta_1 - \zeta_2),$$
$$\left.\frac{1}{3}(4u_1 + 2u_2 - \zeta_1 + \zeta_2)\right) d\zeta_1 d\zeta_2.$$

The values of $Vf(u)$ for the other cases when $u_1 u_2(u_1 + u_2) \neq 0$ can be obtained from $Vf(u) = R(w)Vf(uw^{-1}) = V(R(w)f)(uw^{-1})$, choosing $w \in S_3$ so that $(uw^{-1})_1 > 0$ and $(uw^{-1})_2 > 0$.

The formula (5.3) bears some resemblance to the transform in Beerends' paper ((2.3) in [Be]), which is of noncompact type. No attempt has been made to apply limiting arguments to Beerends' transform to get the G-invariant case of the operator V.

For the case $u_2 = 0$ we use the coordinate system from (3.1) and after a chain of calculations similar to that of Lemma 4.1, we obtain

$$Vf(u_1, 0) = \left(\frac{27}{4}\right)^\alpha \frac{2}{\pi\sqrt{3}} \frac{\Gamma\left(\alpha + \frac{2}{3}\right)\Gamma\left(\alpha + \frac{1}{3}\right)}{\Gamma(\alpha)^2}$$

$$(5.4) \qquad \cdot \int_0^1 \int_0^1 f(u_1(t - 1 + (2 - t)b_1), u_1(1 - 2t)(1 - b_1))$$

$$\cdot 3b_1(b_1^{\alpha-1}(1 - b_1)^{2\alpha-1})(4t(1 - t))^{\alpha-1}dt\, db_1.$$

Again the coordinates could be changed so that this becomes an integral over $H(u)$, which in this case has the vertices $(u_1, 0)$, $(0, -u_1)$, $(-u_1, u_1)$.

We return to an orthogonal coordinate system so that S_3 is embedded in $O(2)$:

$$(5.5) \qquad y_1 := \frac{1}{\sqrt{6}}(2u_1 + u_2), \quad y_2 := \frac{1}{\sqrt{2}}u_2.$$

In this system (using $\tau'(c)$ for the matrix corresponding to $\tau(c)$)

$$y\tau'(c) = y \begin{bmatrix} \frac{3}{2}c_1 & -\frac{\sqrt{3}}{2}(c_1 + 2c_3) \\ -\frac{\sqrt{3}}{2}(c_1 + 2c_2) & \frac{1}{2}c_1 + c_2 + c_3 + 2c_4 \end{bmatrix}.$$

The reflections are

$$\sigma_{12} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \sigma_{13} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad \sigma_{23} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The positive roots are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $(0, 1)$. The fundamental chamber $(u_1 > 0,\ u_2 > 0)$ becomes $\{(y_1, y_2) \in \mathbf{R}^2 : 0 < y_2 < \sqrt{3}y_1\}$. In formula (5.3) perform the same change of variables on (ζ_1, ζ_2), namely $\omega_1 := \frac{1}{\sqrt{6}}(2\zeta_1 + \zeta_2)$, $w_2 := \frac{1}{\sqrt{2}}\zeta_2$, so that

$$z_1 = \frac{1}{\sqrt{2}}(\sqrt{3}\omega_1 + \omega_2), \quad z_2 = \frac{1}{\sqrt{6}}(\omega_1 - \sqrt{3}\omega_2) + \sqrt{\frac{2}{3}}(y_1 - \sqrt{3}y_2).$$

The result is (for $0 < y_2 < \sqrt{3}y_1$):

(5.6)
$$Vf(y_1, y_2) = \frac{1}{(y_2(3y_1^2 - y_2^2))^{2\alpha}} m'_\alpha \int_{H(y)} \int f(\omega_1, \omega_2)$$

$$\cdot F_\alpha\left(\frac{\sqrt{3}}{2}(\sqrt{3}\omega_1 + \omega_2), \frac{1}{2}(\omega_1 - \sqrt{3}\omega_2) + (y_1 - \sqrt{3}y_2),\right.$$

$$\left. 2y_1 - \frac{1}{2}(\omega_1 - \sqrt{3}\omega_2), (y_1 + \sqrt{3}y_2) + \frac{1}{2}(\omega_1 - \sqrt{3}\omega_2)\right) d\omega_1\, d\omega_2,$$

where $H(y)$ has the vertices

$$(y_1, \pm y_2), \quad \left(-\frac{1}{2}y_1 - \frac{\sqrt{3}}{2}y_2, \pm\left(-\frac{\sqrt{3}}{2}y_1 + \frac{1}{2}y_2\right)\right),$$

$$\left(-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2, \pm\left(\frac{\sqrt{3}}{2}y_1 + \frac{1}{2}y_2\right)\right),$$

and
$$m'_\alpha = \frac{16}{2^{2\alpha}\sqrt{\pi}} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{3}\right)\Gamma\left(\alpha + \frac{2}{3}\right)}{\Gamma(\alpha)^3}.$$

(We see that the orthogonal coordinates do not make the formulas more attractive.)

Finally, we display a formula for $Vf(y_1, 0)$ (using $(y_1, 0)$ as a typical point on a wall). In (5.4), let $\omega_1 = \frac{1}{2}(3b_1 - 1)$, $\omega_2 = (1 - 2t)(1 - b_1)(\sqrt{3}/2)$. Then

$$Vf(y_1, 0) = \frac{3}{\pi} \frac{\Gamma\left(\alpha + \frac{1}{3}\right)\Gamma\left(\alpha + \frac{2}{3}\right)}{\Gamma(\alpha)^2} \int_{H((1,0))} f(y_1(\omega_1, \omega_2))$$

$$\cdot(1 + 2\omega_1)^\alpha(1 - \omega_1 - \sqrt{3}\omega_2)^{\alpha-1}(1 - \omega_1 + \sqrt{3}\omega_2)^{\alpha-1} d\omega_1\, d\omega_2.$$

The region $H((1, 0))$ has the vertices $(1, 0)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. In this degenerate case the formula is reminiscent of the one-variable integral for V, namely,

$$B\left(\alpha, \frac{1}{2}\right)^{-1} \int_{-1}^{1} f(x\omega)(1 - \omega)^{\alpha-1}(1 + \omega)^\alpha d\omega$$

(see Theorem 5.1 in [Du3]).

Observe that the normalizing constants m_α and m'_α indicate the poles $\alpha = -\frac{1}{2} - n$, $-\frac{1}{3} - n$, $-\frac{2}{3} - n$, $n = 0, 1, 2, \ldots$, which were known to occur in the algebraic form of V, from the general theory of singular polynomials (Dunkl, De Jeu, Opdam [DJO]), and the Bessel function (Opdam [O]).

The limiting case $\alpha \to \frac{1}{2}+$ for Vf (expressed by (5.1)) is also an integral transform. The measure is supported by the boundary of Ω (see the discussion at the end of §3). For $\alpha = \frac{1}{2}$ the transform V is related to the Cartan motion group associated with the pair $(SL(3, \mathbf{R}), SO(3))$, see Remark 6.12 in Opdam [O] and Remark 4.27 in de Jeu [J]. The integral in Lemma 4.3 reduces to a sum of values at the endpoints ($\beta \to -\frac{1}{2}+$). The function F_α, for $\alpha = \frac{1}{2}$, has infinite values if $|z_1| = |z_2|$ or $z_2 = u_1 - u_2$ (that is, $\xi_1 = \xi_2$).

We already know the effect of V on the harmonic polynomials $(y_1 + iy_2)^m$ by the formulas in (Section 3, [Du1]). Indeed,

(5.7)
$$V\left((y_1 + iy_2)^{(3n+\ell)}\right) = \frac{(\alpha+1)_n(3n+\ell)!}{(2\alpha+1)_n(3\alpha+1)_{3n+\ell}} (y_1 + iy_2)^\ell C_n^{(\alpha,\alpha+1)}((y_1 + iy_2)^3),$$

$n = 0, 1, 2, 3, \ldots$, $\ell = 0, 1, 2$; where $C_n^{(\alpha,\alpha+1)}$ is the Heisenberg polynomial

$$C_n^{(\alpha,\alpha+1)}(\zeta) := \sum_{j=0}^n \frac{(\alpha)_j(\alpha+1)_{n-j}}{j!(n-j)!} \bar{\zeta}^j \zeta^{n-j}, \quad (\zeta \in \mathbf{C}).$$

These polynomials are annihilated by $\frac{1}{2}(T_1 + iT_2)$, the analogue of $\frac{\partial}{\partial \bar{y}}$.

One problem remains: is $F_\alpha(z_1, z_2, \xi_1, \xi_2) \geq 0$? If this is always true, then V is a positive transform, and the function $K(x, y)$ (see (1.3)) satisfies: $x \mapsto K(x, iy)$ is of positive type for each $y \in \mathbf{R}^2$, in particular, $|K(x, iy)| \leq 1$, for $x, y \in \mathbf{R}^2$. This kernel is used in a generalized Fourier transform ([Du4], and de Jeu [J]). Our results do show that the Bessel function K_G (see (1.6)) has this positivity property, for $\alpha \geq \frac{1}{2}$. By direct computation $F_\alpha \geq 0$ for $\alpha = 1$ and $\alpha = 2$. Also, $F_\alpha \geq 0$ when $z_1 \geq 0$. The problem is that the integrand has the factor $r^2(\xi_1 + \xi_2 + z_1) + \xi_1\xi_2 z_1$ which equals $z_1(\xi_1 + z_1)(\xi_2 + z_1)$ at $r = -z_1 > 0$.

We can, however, bound the L^1-norm of ϕ by the L^2-norm with respect to the measure $h_\alpha J(c)^{\alpha-\frac{3}{2}} dc$ on Ω. Indeed,

$$\|A_0 + A_1\phi_1 + A_2\phi_2\|_2^2 = A_0^2 + \frac{A_1^2}{3\alpha+2} + \frac{A_2^2(\alpha+2)}{9(2\alpha+1)(3\alpha+1)(3\alpha+2)}$$

(for arbitrary $A_0, A_1, A_2, \in \mathbf{R}$ by Proposition 3.3). In particular, let $A_0 = 1$, $A_1 = 2$, $A_2 = 3$, then $\|\phi\|_2^2 = 1 + \frac{4}{3\alpha+2} + \frac{3}{(2\alpha+1)(3\alpha+1)}$, which is decreasing in α (for $\alpha > 0$). Thus for $\alpha \geq 1$, $\|\phi\|_1 \leq \|\phi\|_2 \leq \sqrt{\frac{41}{20}}$. This shows $|K(x, iy)| \leq 1.43$ (De Jeu [J] has already proven the upper bound $\sqrt{6}$).

REFERENCES

[Ba]   N. Bailey, *Generalized hypergeometric series*, Cambridge Univ. Press, Cambridge, 1935.

[Be]   R. J. Beerends, *A transmutation property of the generalized Abel transform associated with root system A_2*, Indag. Math. (N.S.) **1** (1990), 155–168.

[C]   R. Carroll, *Transmutation theory and applications*, North-Holland, Amsterdam, 1985.

[Du1]   C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.

[Du2]   ———, *Operators commuting with Coxeter group actions on polynomials*, Invariant Theory and Tableaux, (D. Stanton, ed.), Springer-Verlag, New York–Berlin–Heidelberg, 1990, pp. 107–117.

[Du3]   ———, *Integral transforms with reflection group invariance*, Canad. J. Math. **43** (1991), 1213–1227.

[Du4]   ———, *Hankel transforms associated to finite reflection groups*, Contemporary Math., vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 123–138.

[DJO]   C. F. Dunkl, M. de Jeu, and E. Opdam, *Singular polynomials for finite reflection groups*, Trans. Amer. Math. Soc. **346** (1994), 237–256.

[DR]   C. F. Dunkl and D. E. Ramirez, *Topics in harmonic analysis*, Appleton–Century–Crofts, New York, 1971.

[GR]   K. Gross and D. Richards, *Total positivity, finite reflection groups, and a formula of Harish-Chandra*, J. Approx. Theory (to appear)

[Hec1]   G. Heckman, *A remark on the Dunkl differential-difference operators*, Harmonic Analysis on Reductive Groups, (W. Barker and P. Sally, eds.), Birkhäuser, Basel, 1991, pp. 181–191.

[Hec2]   ——— , *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. **103** (1991), 341–350.

[Hel]   S. Helgason, *Groups and geometric analysis*, Academic Press, New York, 1984.

[J]   M. de Jeu, *The Dunkl transform*, Invent. Math. **113** (1993), 147–162.

[O]   E. Opdam, *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group*, Compositio Math. **85** (1993), 333–373.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903
E-mail address: `cfd5z@virginia.edu`