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INTERVAL-ANALYSIS TECHNIQUES IN LINEAR SYSTEMS: AN APPLICATION IN POWER SYSTEMS*

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Abstract. In this paper we apply the technique of interval analysis to get bounds on the initial value response of a linearized single machine infinite bus problem when a parameter is varied. It is generally believed that responses for parameter variations in an interval should lie within the responses for the extremums of the parameter variations. This is not generally true and our example demonstrates this. The interval-analysis technique permits getting the overall bound on the response. Further experimentation also revealed that the method has some limitations particularly involving lightly damped long-term dynamics.

The technique is useful in finding the robustness of a particular design such as the power system stabilizer for parameter variations.

1. Introduction

Interval-analysis techniques have been shown to give satisfactory bounds for the set of initial value responses of linear systems that linearly depend on a parameter belonging to an interval. These include interesting examples such as the tolerance problem in electric circuits, optimal control problems with large tolerances on a parameter, and the like, where sometimes sensitivity methods fail [1]–[4].

We specifically consider here the stabilization of a power-system problem where one of the design parameters can be chosen over a wide interval. Thus parametrization can be done in the form

$$\dot{x} = (G_1 + \theta G_2)x, \qquad x(0) = x_0, \qquad \theta \in [-1.0, 1.0], \tag{1}$$

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where G_1, G_2 are real $n \times n$ matrices. Using partitioning

$$\theta_i = [(-M + 2(i-1))/M, (-M + 2i)/M], \; i = 1, \ldots, M,$$

we generate, for the above initial value problem, M subproblems. We develop an algorithm to generate bounds at any desired point in time t for the interval solutions for the above M subproblems. The interval solution for the (entire) initial value problem is then obtained by taking the union over the subproblem interval solutions, producing envelopes for the set of all solutions associated with the interval vector initial condition x_0 and the perturbation parameter θ.

It is generally believed that responses are monotone with respect to the parameter θ so that only the boundary responses corresponding to $\theta = -1$ and $\theta = +1$ are sufficient to produce the envelope. However, this is not always true, as shown in the example, and the interval-analysis technique captures the variations.

2. Theory

In this section we provide a brief summary of some of the *interval-analysis results* developed in [1]–[3] and we show how these results can be applied to the initial-value problem (1) endowed with a parameter belonging to an interval (see Section 1).

We let $\mathscr{I}$ denote the set of intervals $[a, b]$, $a, b \in R$, $a \leq b$. When $a = b$, we call $I = [a, a]$ a *degenerate* interval. On $\mathscr{I}$ we define the *interval arithmetic operations* $+$, $-$, $\cdot$, $/$ by

$$[a, b] + [c, d] = [a + c, b + d],$$

$$[a, b] - [c, d] = [a - d, b - c],$$

$$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)],$$

$$[a, b]/[c, d] = [a, b] \cdot [1/d, 1/c] \qquad \text{provided that} \quad 0 \notin [c, d].$$

For each $I = [a, b]$, $J = [c, d]$ in $\mathscr{I}$, we define ρ: $\mathscr{I} \times \mathscr{I} \to R^+$ by

$$\rho(I, J) = \max(|a - c|, |b - d|).$$

It is easily shown that $(\mathscr{I}, \rho)$ is a complete metric space [1].

Next, let $I = [a, b]$ be fixed and let $\mathscr{I}_I = \{J \in \mathscr{I} : J \subset I\}$. It can readily be shown that the space $(\mathscr{I}_I, \rho)$ is a complete and compact metric subspace of $(\mathscr{I}, \rho)$ [1].

We can define *interval functions* of an interval variable f: $\mathscr{I} \to \mathscr{I}$. (For example, if $f(x) = x^2$, then $f(J) = J \cdot J$.) Properties of such functions (e.g., continuity) are defined on $(\mathscr{I}, \rho)$, or on $(\mathscr{I}_I, \rho)$ in an obvious way [1].

Now let

$$\mathscr{F} = \{f \mid f: \mathscr{I}_I \to \mathscr{I}, f \text{ is continuous on } \mathscr{I}_I\}.$$

For $f, g \in \mathscr{F}$, define

$$\mu(f, g) = \sup_{J \in \mathscr{I}_I} \{\rho(f(J), g(J))\}.$$

It is an easy matter to show that $(\mathscr{F}, \mu)$ is a complete metric space.

The space $(\mathscr{F}, \mu)$ includes the $C([a, b])$ real function space and the *united extension* of its member functions, f, defined by

$$\bar{f}([a, b]) = \bigcup_{x \in [a, b]} f([x, x]).$$

In [1] we show that the rational interval functions belong to $(\mathscr{F}, \mu)$ and exhibit the inclusion property $f([a, b]) \supset \bar{f}([a, b])$. Defining a partition of the interval $I = [a, b]$ by $I_i^n \triangleq [a(n - i + 1) + b(i - 1),\ (n - i)a + ib]/n$, $i = 1, \ldots, n$, we establish in [1] the *convergence result*

$$f(I) \supset \lim_{n \to \infty} \bigcup_{i=1}^{n} f(I_i^n) = \bar{f}(I).$$

This result states that by using a sufficiently fine partition of I and by computing the union of the interval functions over the partition subintervals, it is possible to approximate the *exact range* of the interval function for $x \in I$, $\bar{f}(I)$, as closely as desired. In [1] we also extend the above convergence result to *rational interval matrix functions* of an interval variable.

In [2] we show that the sequence of partial sums obtained from the infinite series representation of the *interval exponential function* is a Cauchy sequence which converges to a member function of the space $(\mathscr{F}, \mu)$. We devise a technique to compute an approximation $\tilde{g}$ of the interval exponential function $g([a, b]) \triangleq \exp[(a, b])$ which, for $\varepsilon > 0$, provides the error inclusion property

$$[1 - \varepsilon, 1 + \varepsilon] \cdot g([a, b]) \supset \tilde{g}([a, b]) \supset g([a, b]) \supset \bar{g}([a, b]),$$

where $\tilde{g}$ is an *augmented* kth-order partial sum for the exponential function g and ε depends on the size of k. Finally, we apply the *convergence result* derived in [1], and described above, to reduce the conservativeness when obtaining estimates for $\bar{g}$ by the above results. Also, we extend the above results in [2] to *interval matrix exponential functions* of an interval variable.

In order to obtain *true* estimates from our algorithmic results, we employ in [2] and [3] *machine bounding arithmetic* in computing partial sums for the interval matrix exponential.

In [3] we consider the *initial-value problem* (1). As indicated in Section 1, we generate for (1) M *subproblems* and we use the results of [1] and [2] to establish in [3] an algorithm which enables us to obtain bounds at any desired point in time, t, for the interval solution for the above M subproblems. The interval solution for the (entire) initial-value problem (1) (given in Section 1) is then obtained by taking the union over the subproblem interval solutions, producing interval bounds or envelopes for the set of all solutions associated with the interval vector initial condition x_0 and the perturbation parameter θ, including the effects of algorithmic computer truncation or rounding errors.

In [4] we extend the results of [1]–[3] to situations involving more than one parameter.

3. Application to power system

We consider the single machine infinite bus problem (Figure 1) whose nonlinear model is of the form [5]

Machine Equations

$$\dot{E}'_q = -\frac{1}{T'_{d0}}(E'_q + (x_d - x'_d)I_d - E_{fd}), \tag{2}$$

$$\dot{\delta} = \omega - \omega_s, \tag{3}$$

$$\dot{\omega} = \frac{\omega_s}{2H}[T_m - (E'_q I_q + (x_q - x'_q)I_d I_q - D(\omega - \omega_s)]. \tag{4}$$

Stator Algebraic Equations

$$x_q I_q - V_d = 0, \tag{5}$$

$$E'_q - V_q - x'_d I_d = 0. \tag{6}$$

Load Flow Equations

$$R_e I_d - x_e I_q = V_d - V_\infty \sin\delta, \tag{7}$$

$$x_e I_d + R_e I_q = V_q - V_\infty \cos\delta. \tag{8}$$

The following change of variable is now made

$$E'_q = x_{mq} I_{fd} - (x_d - x'_d)I_d. \tag{9}$$

Assume $R_e \triangleq 0$ and let $x'_d = x_d - x^2_{md}/x_{fd}$ and $T'_{d0} \triangleq x_{fd}/(\omega_s R_{fd})$. Lengthy

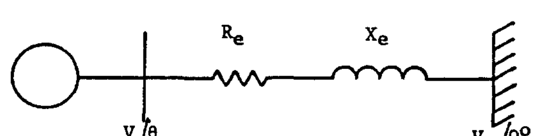

Figure 1. Single machine infinite bus system.

computation yields the following equations:

$$\dot{\delta} = \omega - \omega_s, \tag{10}$$

$$\dot{\omega} = a_1 - a_2(\omega - \omega_s) - a_3 I_{fd} \sin \delta - a_4 \sin 2\delta, \tag{11}$$

$$\dot{I}_{fd} = a_5 E_{fd} + a_6(\omega - \omega_3) \sin \delta - a_7 I_{fd}, \tag{12}$$

$$V_t = [(c_2 \sin \delta)^2 + (c_3 \cos \delta + c_1 I_{fd})^2]^{1/2}, \tag{13}$$

where

$$a_1 = \frac{T_m}{M}, \qquad a_2 = \frac{D}{M}, \qquad a_3 = \frac{x_m V_\infty}{(x_d + x_e)M},$$

$$a_6 = \frac{x_{md} V_\infty}{x_{fd}(x_d + x_e) - x_{md}^2}, \qquad a_7 = \frac{\omega_s R_{fd}(x_d + x_e)}{x_{fd}(x_d + x_e) - x_{md}^2},$$

$$c_1 = \frac{x_e x_{md}}{x_d + x_e}, \qquad c_2 = \frac{V_\infty x_q}{(x_q + x_e)}, \qquad c_3 = \frac{x_d V_\infty}{(x_d + x_e)}.$$

Linearization of (10)–(13) leads to

$$\begin{aligned} \dot{x} &= A_1 x + B_1 \Delta E_{fd} + \bar{B}_1 \Delta T_m, \\ \Delta V_t &= C_1 \Delta\delta + D_1 \Delta I_{fd}, \end{aligned} \tag{14}$$

where

$$x = (\Delta\delta, \Delta\sigma, \Delta I_{fd})^T \quad \text{and} \quad \Delta\sigma = \omega - \omega_s.$$

Augment the system (10)–(12) by the equation

$$\dot{z} = \Delta V_t. \tag{15}$$

In [6] a two-step procedure is used to design the feedback control. The first step is to compute state feedback of the type

$$\Delta E_{fd} = k\Delta\delta + k_2 \Delta\sigma + k_3 \Delta I_{fd} + k_4 z$$

to minimize a quadratic performance index. The parameters in (14) pertain to the prefault system. Since the power system undergoes a structural change due to switching action when there is a fault, this is taken into account by a feedback for the postfault system of the type

$$\Delta E_{fd} = h_1 \Delta\delta + h_2 \Delta\sigma + h_3 \Delta I_{fd} + h_4 \Delta z + h_5 \Delta V_t. \tag{16}$$

If h_5 is considered as a free parameter, the two control laws are equal if

$$\begin{aligned} h_1 + c_1 h_5 &= k_1, \qquad h_2 = k_2, \\ h_3 + D_1 h_5 &= h_5, \qquad h_4 = k_4. \end{aligned} \tag{17}$$

h_5 is chosen so as to have satisfactory closed-loop operation of the postfault system given by

$$\begin{bmatrix} \Delta\dot{\delta} \\ \Delta\dot{\sigma} \\ \Delta\dot{I}_{fd} \\ \Delta\dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_3 I_{fd}^0 \cos\delta^0 & -a_2 & -a_3 \sin\delta^0 & 0 \\ a_5 k_1 & a_5 k_2 + a_6 \sin\delta^0 & a_5 k_3 - a_7 & a_5 k_4 \\ C_2 & 0 & D_2 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\sigma \\ \Delta I_{fd} \\ \Delta z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Delta u, \tag{18}$$

$$\Delta u = h_5 \Delta y, \tag{19}$$

$$\Delta y = [a_5(C_2 - C_1) \quad 0 \quad (D_2 - D_1)a_5 \quad 0] \begin{bmatrix} \Delta\delta \\ \Delta\sigma \\ \Delta I_{fd} \\ \Delta z \end{bmatrix}. \tag{20}$$

Symbolically (18)–(20) are of the form

$$\Delta\dot{x} = A\Delta x + b\Delta u,$$
$$\Delta u = h_5 \Delta y,$$
$$\Delta y = c^T \Delta x, \tag{21}$$

which simplifies to

$$\Delta\dot{x} = (A + h_5 b c^T)\Delta x \tag{22}$$
$$= A_2 \Delta x. \tag{23}$$

Notice that except for h_5 all other parameters are fixed. We now apply interval analysis to this model to see the effect on the response due to wide variations in h_5. The initial conditions are those at $t = t_{cl}$ since (21) pertains to the postfault system.

With the numerical values given in [6], the A_2 matrix works out to be

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5.011 & -0.3044 & -18.96 & 0 \\ 6.095 - 0.0723h_5 & 4.445 & -13.45 + 0.0742h_5 & -22.80 \\ -0.4421 & 0 & 0.3698 & 0 \end{bmatrix}.$$

From the root locus for the eigenvalues of this matrix as h_5 varies from -80

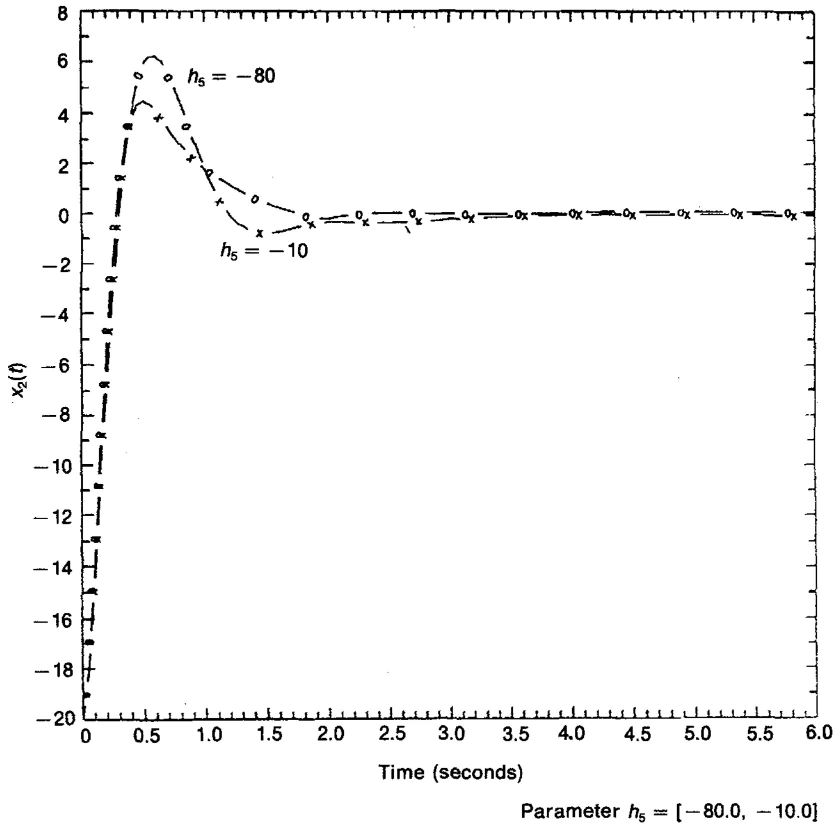

Figure 2. The bounds on the response of $x_2(t)$ by the interval-analysis technique.

to -10, we choose a nominal operating point for $h_5 = -50$ which gives satisfactory damping. $x^T(0) = [0.5, -20.0, 0, -0.1]$ is chosen as the initial condition for the postfault system to illustrate the interval-analysis technique. The response bounds on the variables using the theory of Section 2 is shown in Figures 2 and 3 for subintervals of 100 points for $x_2(t) = \Delta r$ and $x_4(t) = \Delta z$. The response although monotonic between the bounds of the interval for the most part however showed a violation for x_2 around $t = 1.93$ s and for x_4 around $t = 5.25$ s. An expanded version of the actual response near these two instants is shown in Figures 4 and 5. The response for $h_5 = -40$ and $h_5 = -30$ for $x_2(t)$ and $x_4(t)$ respectively do not lie between the responses for $h_5 = [-80, -10]$. Hence the need to find the bounds. The bounds obtained by the interval analysis (Figures 2 and 3) were verified to be good. We observed that with subintervals of 25 the response bounds were somewhat conservative.

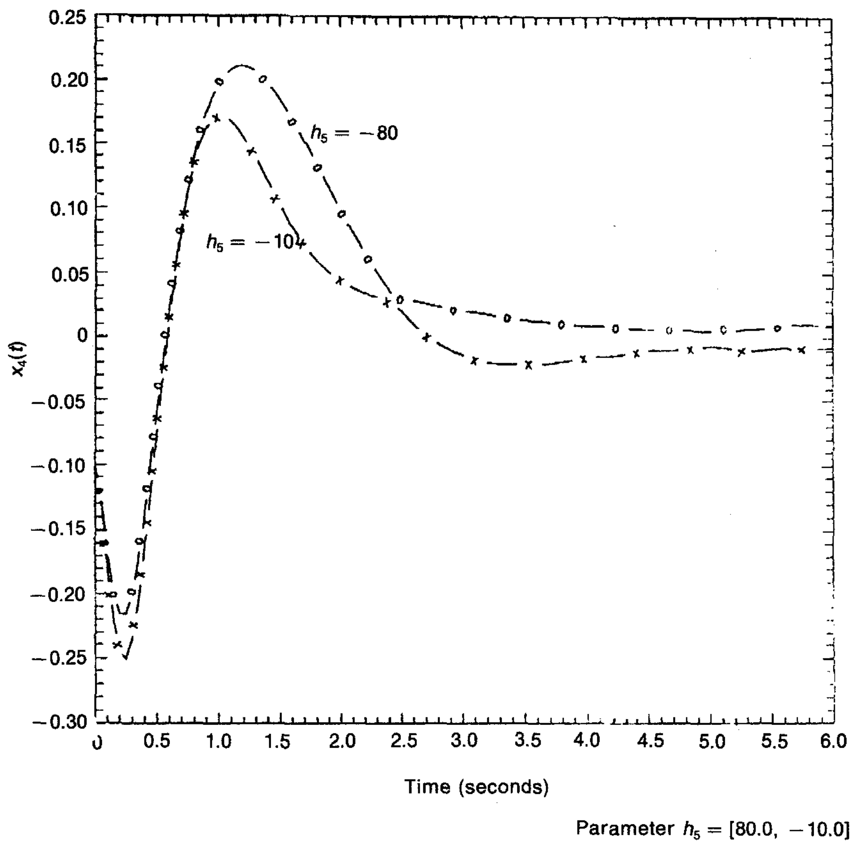

Figure 3. The response bounds of $x_2(t)$ by interval analysis.

We applied the technique on another power system representing interarea response with very low damping. The technique works up to a certain time instant to give the bound but later gives conservative estimates. More research is needed to look into such problems.

4. Conclusion

In robust control of systems when a parameter varies over a wide range it is nice to get bounds on responses for the parameter variations. The applicability of interval-analysis techniques is illustrated through a power-system example. For systems with low damping, some kind of time-windowing technique may be helpful.

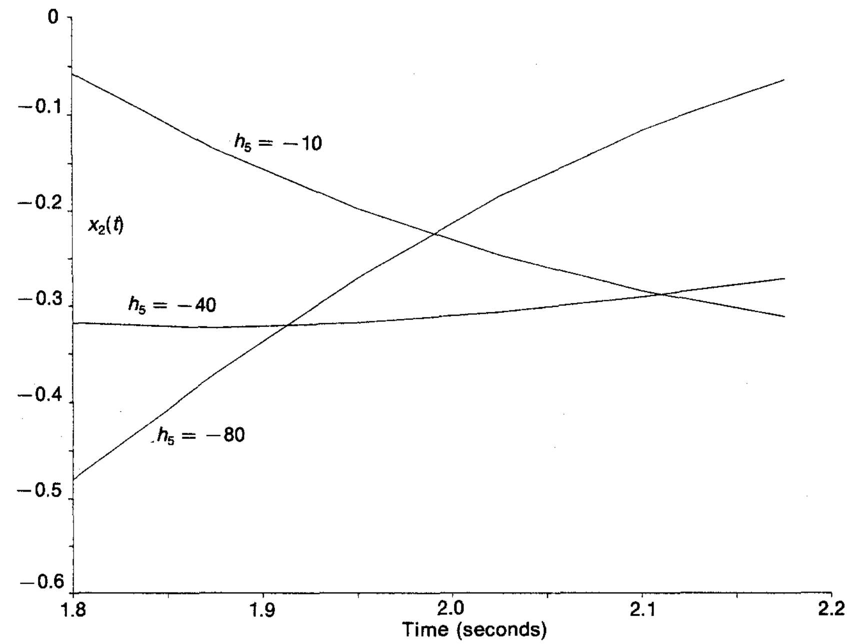

Figure 4. Actual response of $x_2(t)$ near $t = 1.92$ s.

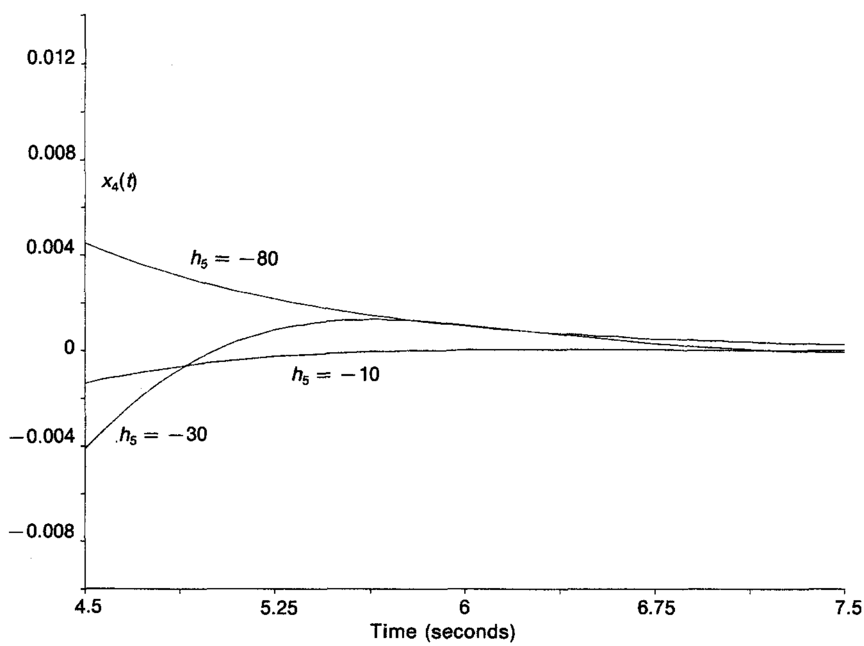

Figure 5. Actual response of $x_4(t)$ near $t = 5.25$ s.

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