Interval Completion is Fixed Parameter Tractable*

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Abstract

We present an algorithm with runtime $O(k^{2k}n^3m)$ for the following NP-complete problem [9, problem GT35]: Given an arbitrary graph G on n vertices and m edges, can we obtain an interval graph by adding at most k new edges to G? This resolves the long-standing open question [17, 7, 25, 14], first posed by Kaplan, Shamir and Tarjan, of whether this problem was fixed parameter tractable. The problem has applications in Profile Minimization for Sparse Matrix Computations [10, 26], and our results show tractability for the case of a small number k of zero elements in the envelope. Our algorithm performs bounded search among possible ways of adding edges to a graph to obtain an interval graph, and combines this with a greedy algorithm when graphs of a certain structure are reached by the search.

Keywords: Interval graphs, profile minimization, edge completion, FPT algorithm, branching

1 Introduction and motivation

Interval graphs are the intersection graphs of intervals of the real line and have a wide range of applications [13]. Connected with interval graphs is the following problem: Given an arbitrary graph G, what is the minimum number of edges that must be added to G in order to obtain an interval graph? This problem is NP-hard [18, 9]. The problem arises in Sparse Matrix Computations, where one of the standard methods for reordering a matrix to get as few non-zero elements as possible during Gaussian elimination, is to permute the rows and columns of the matrix so that non-zero elements are gathered close to the main diagonal [10]. The *profile* of a matrix is the smallest number of entries that can be enveloped within off-diagonal non-zero elements of the matrix. Translated to graphs, the profile of a graph G is exactly the minimum number of edges in an interval supergraph of G [26]. Originally, Physical Mapping of DNA was another motivation for this problem [12].

In this paper, we present an algorithm with runtime $O(k^{2k}n^3m)$ for the k-*Interval Completion* problem of deciding whether a graph on n vertices and m edges can be made into an interval graph by adding at most k edges. A parameterized problem with parameter k and input size x that can be solved by an algorithm having runtime $f(k) \cdot x^{O(1)}$ is called *fixed parameter tractable (FPT)* (see [7] for an introduction to fixed parameter tractability and bounded search

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tree algorithms). The k-Interval Completion problem is thus FPT, which settles a long-standing open problem [17, 7, 25, 14]. An early paper (first appearance FOCS '94 [16]) in this line of research by Kaplan, Shamir and Tarjan [17] gives FPT algorithms for k-Chordal Completion, k-Strongly Chordal Completion, and k-Proper Interval Completion. In all these cases a bounded search tree algorithm suffices, that identifies a subgraph which is a witness of non-membership in the desired class of graphs, and branches recursively on all possible ways of adding an inclusion-minimal set of edges that gets rid of the witness. For example, since a graph is chordal iff it has no induced cycle on more than 3 vertices, the k-Chordal completion algorithm [17, 2] takes as witness a subset C of vertices inducing a d-cycle, $4 \leq d \leq k+3$, and branches on all ways of adding the $d-3$ edges needed to make the subgraph induced by C chordal. The existence of an FPT algorithm for solving k-Interval Completion was left as an open problem by [17], with the following explanation for why a bounded search tree algorithm seemed unlikely: "An arbitrarily large obstruction X could exist in a graph that is not interval but could be made interval with the addition of any one out of $O(|X|)$ edges". Our FPT algorithm for this problem is nevertheless based heavily on applying the bounded search tree technique, supplemented with a greedy algorithm to circumvent the obstructions mentioned in the quote.

Let us mention some related work. Ravi, Agrawal and Klein gave an $O(\log^2 n)$-approximation algorithm for Minimum Interval Completion, subsequently improved to an $O(\log n \log\log n)$-approximation by Even, Naor, Rao and Schieber [8] and finally to an $O(\log n)$-approximation by Rao and Richa [23]. Heggernes, Suchan, Todinca and Villanger showed that an inclusion-minimal interval completion can be found in polynomial time [15]. Kuo and Wang [20] gave an $O(n^{1.722})$ algorithm for Minimum Interval Completion of a tree, subsequently improved to an $O(n)$ algorithm by Díaz, Gibbons, Paterson and Torán [5]. Cai [2] proved that k-completion into any hereditary graph class having a finite set of forbidden subgraphs is FPT. Some researchers have been misled to think that this settled the complexity of k-Interval Completion, however, interval graphs do not have a finite set of forbidden subgraphs [21]. Gutin, Szeider and Yeo [14] gave an FPT algorithm for deciding if a graph G has profile at most $k + |V(G)|$, but the more natural parameterization of the profile problem is to ask if G has profile at most $k + |E(G)|$, which is equivalent to the k-Interval Completion problem on G. Similar questions, asking if we can add/remove at most k vertices/edges to a graph such that a certain property is satisfied, have been investigated in the litterature for various graph properties, see *e.g.* [3].

Our algorithm for k-Interval Completion circumvents the problem of large obstructions (witnesses) by first getting rid of all small witnesses, in particular witnesses for the existence of an asteroidal triple (AT) of vertices. Three non-adjacent vertices a, b, c form an AT if there exists a path from any two of them avoiding the neighborhood of the third. Since a graph is an interval graph if and only if it is both chordal and AT- free [21], to complete into an interval graph we must destroy witnesses for non-chordality and witnesses for existence of an AT. Witnesses for non-chordality (chordless cycles of length > 3) must have size $O(k)$ and do not present a problem. Likewise, if an AT is witnessed by an induced subgraph S of size $O(k)$ it does not present a problem, as shown in Section 3 of the paper. In Section 4, we show that in every induced subgraph S witnessing the existence of an AT, one of the vertices of the AT is *shallow*, meaning that there is a short path from it to each of the other two vertices of the AT. We give a branching rule for the case when G has no AT witnessed by a small subgraph, but it has at least $k+1$ shallow vertices. The most difficult case is when we have a chordal non-interval graph G

with no AT having a small witness, and with at most k shallow vertices. For this case we introduce *thick* AT-witnesses in Section 5, consisting of an AT and all vertices on any chordless path between any two vertices of the AT avoiding the neighborhood of the third vertex of the AT. We define minimality for thick AT-witnesses, and show that also in every minimal thick AT-witness one of the vertices of the AT is shallow. In Section 6 we show how to carefully compute the set C of shallow vertices so that removing C from the graph gives an interval graph. Based on the cardinality of C, we show how to further continue branching in a bounded way. When no bounded branching is possible we show that the instance has enough structure that the best solution is a completion computed in a greedy manner. The presented algorithm consists of 4 branching rules in addition to the greedy completion.

2 Preliminaries

We work with simple and undirected graphs $G = (V, E)$, with vertex set $V(G) = V$ and edge set $E(G) = E$, and $n = |V|$, $m = |E|$. For $X \subset V$, $G[X]$ denotes the subgraph of G induced by the vertices in X. We will use $G - x$ to denote $G[V \setminus \{x\}]$ for $x \in V$, and $G - S$ to denote $G[V \setminus S]$ for $S \subseteq V$.

For neighborhoods, we use $N_G(x) = \{y \mid xy \in E\}$, and $N_G[x] = N_G(x) \cup \{x\}$. For $X \subseteq V$, $N_G[X] = \bigcup_{x \in X} N_G[x]$ and $N_G(X) = N_G[X] \setminus X$. We will omit the subscript when the graph is clear from the context. A vertex set X is a *clique* if $G[X]$ is complete, and a *maximal clique* if no superset of X is a clique. A vertex x is *simplicial* if $N(x)$ is a clique.

We will say that a path $P = v_1, v_2, ..., v_p$ is *between* v_1 and v_p, and we call it a *v_1, v_p-path.* The *length* of P is p. We will use $P - v_p$ and $P + v_{p+1}$ to denote the paths $v_1, v_2, ..., v_{p-1}$ and $v_1, v_2, ..., v_p, v_{p+1}$, respectively. We say that a path P *avoids* a vertex set S if P contains no vertex of S. A *chord* of a cycle (path) is an edge connecting two non-consecutive vertices of the cycle (path). A *chordless* cycle (path) is an induced subgraph that is isomorphic to a cycle (path). A graph is *chordal* if it contains no chordless cycle of length at least 4.

A graph is an *interval graph* if intervals can be associated to its vertices such that two vertices are adjacent if and only if their corresponding intervals overlap. Three non-adjacent vertices form an *asteroidal triple (AT)* if there is a path between every two of them that does not contain a neighbor of the third. A graph is *AT-free* if it contains no AT. A graph is an interval graph if and only if it is chordal and AT-free [21]. A vertex set $S \subseteq V$ is called *dominating* if every vertex not contained in S is adjacent to some vertex in S. A pair of vertices $\{u, v\}$ is called a *dominating pair* if every u, v-path is dominating. Every interval graph has a dominating pair [4], and thus also a dominating chordless path.

A *clique tree* of a graph G is a tree T whose nodes (also called *bags*) are maximal cliques of G such that for every vertex v in G, the subtree T_v of T that is induced by the bags that contain v is connected. A graph is chordal if and only if it has a clique tree [1]. A *clique path* Q of a graph G is a clique tree that is a path. A graph G is an interval graph if and only if has a clique-path [11]. An interval graph has at most n maximal cliques.

Given two vertices u and v in G, a vertex set S is a *u, v-separator* if u and v belong to different connected components of $G - S$. A u, v-separator S is *minimal* if no proper subset of S is a u, v-separator. S is a *minimal separator* of G if there exist two vertices u and v in G with

S a minimal u, v-separator. For a chordal graph G, a set of vertices S is a minimal separator of G if and only if S is the intersection of two neighboring bags in any clique tree of G [1].

An interval supergraph $H = (V, E \cup F)$ of a given graph $G = (V, E)$, with $E \cap F = \emptyset$, is called an *interval completion* of G. H is called a *k-interval completion* of G if $|F| \leq k$. The set F is called the set of *fill edges* of H. On input G and k, the *k-Interval Completion problem* asks whether there is an interval completion of G with at most k fill edges.

3 Non-chordality and small simple AT-witnesses: Rules 1, 2

Branching Rule 1:
If G not chordal, find a chordless cycle C of length at least 4. If $|C| > k+3$ answer no, otherwise:

- Branch on the at most $4^{|C|}$ different ways to add an inclusion minimal set of edges (of cardinality $|C| - 3$) between the vertices of C to make C chordal.

It is shown in [17, 2] that there are at most $4^{|C|}$ minimal set of edges for making C chordal. If Rule 1 applies we branch by creating at most $4^{|C|}$ recursive calls, each with new parameter value $k - (|C| - 3)$. The correctness of Rule 1 is well understood [17, 2]. Let us remark that each invocation of the recursive search tree subroutine will apply only one of four branching rules. Thus, if Rule 1 applies we apply it and branch, else if Rule 2 applies we apply it and branch, else if Rule 3 applies we apply it and branch, else apply Rule 4. Rules 2, 3 and 4 will branch on single fill edges, dropping the parameter by one. Also Rule 1 could have branched on single fill edges, simply by taking the set of non-edges of the induced cycle and branching on each non-edge separately. We continue with Rule 2.

Observation 3.1 *Given a graph G, let $\{a, b, c\}$ be an AT in G. Let P'_{ab} be the set of vertices on a path between a and b in $G - N[c]$, let P'_{ac} be the set of vertices on a path between a and c in $G - N[b]$, and let P'_{bc} be the set of vertices on a path between b and c in $G - N[a]$. Then any interval completion of G contains at least one fill edge from the set $\{cx \mid x \in P'_{ab}\} \cup \{ax \mid x \in P'_{bc}\} \cup \{bx \mid x \in P'_{ac}\}$.*

Proof. Otherwise $\{a, b, c\}$ would still be an independent set of vertices with a path between any two avoiding the neighborhood of the third, in other words it would be an AT. ∎

We introduce simple AT-witnesses and give a branching rule for small such witnesses.

Definition 3.2 *Let $\{a, b, c\}$ be an AT in a graph G. There are three paths P_{ab}, P_{bc}, P_{bc}, where P_{ab} is the set of vertices on a* shortest *path between a and b in $G - N[c]$, P_{ac} is the set of vertices on a* shortest *path between a and c in $G - N[b]$, and P_{bc} is the set of vertices on a* shortest *path between b and c in $G - N[a]$. The induced subgraph $G_{abc} = G[P_{ab} \cup P_{bc} \cup P_{ac}]$ of G is called a* simple AT-witness *of this AT. We call $\{P_{ab}, P_{bc}, P_{bc}\}$ the* core *of G_{abc}.*

Observe that a simple AT-witness for $\{a, b, c\}$ and the mentioned shortest paths of its core exist if and only if $\{a, b, c\}$ is an AT. Furthermore, from the definition of an AT-witness G_{abc} for $\{a, b, c\}$, a, b, and c are vertices of G_{abc}.

Branching Rule 2:
If G is chordal: For each triple $\{a, b, c\}$ check if $\{a, b, c\}$ is an AT. For each AT $\{a, b, c\}$, find a simple AT-witness G_{abc} for it with core $\{P_{ab}, P_{bc}, P_{bc}\}$. If there exists an AT $\{a, b, c\}$, such that $|\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}| \leq k + 15$ for the simple AT-witness G_{abc}, then:

- Branch on each of the fill edges in the set $\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}$.

By Observation 3.1, any interval completion contains at least one edge from the set branched on by Rule 2.

Lemma 3.3 *Let G be a graph to which Rule 1 cannot be applied (i.e. G is chordal). There exists a polynomial time algorithm that finds a simple AT-witness G_{abc} with core $\{P_{ab}, P_{bc}, P_{bc}\}$, where $|\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}| \leq k + 15$, if such an AT-witness exists.*

Proof. A simple AT-witness can be found in polynomial time: for a triple of vertices, check if there exists a shortest path between any two of them that avoids the neighborhood of the third vertex. Since shortest paths are used to define simple AT-witnesses and all shortest paths between a pair of vertices have the same length, then $|\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}|$ will be the same for all cores $\{P_{ab}, P_{bc}, P_{bc}\}$ defining simple AT-witnesses for an AT $\{a, b, c\}$. ■

4 Minimal simple AT-witnesses and shallow vertices: Rule 3

In this section we consider chordal graphs to which Rule 2 cannot be applied, which means chordal graphs containing no AT of small enough size. We introduce minimal simple AT-witnesses and show that they each have a shallow vertex.

Definition 4.1 *A simple AT-witness G_{abc} for an AT $\{a, b, c\}$ is* minimal *if $G_{abc} - x$ is AT-free for any $x \in \{a, b, c\}$.*

Since the vertices of an AT belong to any AT-witness for that AT, it follows that not every AT has a minimal simple AT-witness. However, clearly, for each AT $\{a', b', c'\}$, we can find an AT $\{a, b, c\}$ that has a minimal simple AT-witness. Hence, as long as there is an AT in a chordal graph, there is also an AT that has a minimal simple AT-witness. Interestingly, by the following result of Lekkerkerker and Boland [21] and since minimal simple AT-witnesses are induced subgraphs, a minimal simple AT-witness of a chordal graph is either of constant size or it is a member of one of the two families of graphs shown in Figure 1.

Theorem 4.2 ([21]) *Let G be a chordal graph with more than 7 vertices. Then G contains an AT, and no induced subgraph of G contains an AT, if and only if G belongs to the family of graphs shown in Figure 1.*

Since the chordal graphs that we study in this section contain no AT of size less than 15, the minimal simple AT-witnesses that we encounter from now on will always be one of the two types given in Figure 1. (The reader might be interested to know that, by the results of Lekkerkerker and Boland [21], any other possible minimal AT-witness in a chordal graph is of size exactly 7 and is one of two different graphs.)

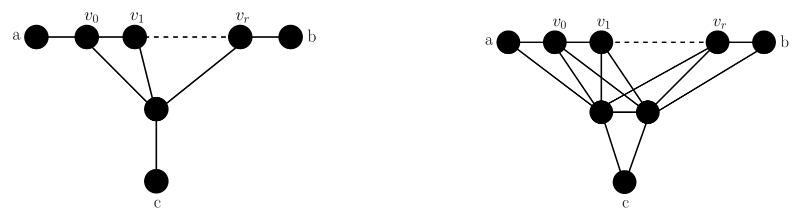

Figure 1: These two families of graphs ($r \geq 1$) are the only minimal simple AT-witnesses of non-constant size that survive Rules 1 and 2.

Definition 4.3 *Let $\{a, b, c\}$ be an AT in a chordal graph G. Vertex c is called* shallow *if shortest a, c-paths and shortest b, c-paths are of length at most 4.*

Observation 4.4 *Let G be a graph to which neither Rule 1 (i.e., G is chordal) nor Rule 2 can be applied. Let G_{abc} be a minimal simple AT-witness for an AT $\{a, b, c\}$ in G. Then the following statements are true.*

- *Each of a, b, c is a simplicial vertex in G_{abc}.*
- *For any $x \in \{a, b, c\}$, $G_{abc} - x$ is an interval graph, and in this interval graph, $\{a, b, c\} \setminus \{x\}$ is a dominating pair.*
- *Let $\{P_{ab}, P_{bc}, P_{ac}\}$ be the core of G_{abc}, where $|P_{ab}| \geq |P_{bc}| \geq |P_{ac}|$. If $|P_{ab}| \geq 6$ then c is shallow.*

Proof. Since neither Rule 1 nor Rule 2 apply to G and G_{abc} is minimal then by Theorem 4.2 we know that G_{abc} belongs to one of the families in Figure 1. Each of a, b, c is either the end vertex of the long path or the vertex at the bottom for each of the graphs in Figure 1. Hence the observation follows. The vertex at the bottom is the shallow vertex. ∎

We are ready to give Branching Rule 3.

Branching Rule 3:
Let $C(G)$ be the set of vertices of G each of which is shallow in some AT of G. This rule applies if Rules 1 and 2 do not apply and $|C(G)| > k$, in which case we let B be a subset of $C(G)$ with $|B| = k + 1$. For each $c \in B$ find a simple AT-witness G_{abc} with core $\{P_{ab}, P_{bc}, P_{ac}\}$, where c is shallow.

- For each $c \in B$, branch on the at most 8 fill edges $\{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}$.
- Branch on the at most $|B|(|B| - 1)/2$ possible fill edges $\{uv \mid u, v \in B \text{ and } uv \notin E\}$.

Observe that Rule 3 only needs a subset of C of size $k + 1$, and thus an algorithm can stop the computation of C when this size is reached.

Lemma 4.5 *If Rule 3 applies to G then any k-interval completion of G contains a fill edge which is branched on by Rule 3.*

Proof. In a k-interval completion we cannot add more than k fill edges. Thus, since $|B| = k+1$ any k-interval completion H of G either contains a fill edge between two vertices in B (and all these are branched on by Rule 3), or there exists a vertex $c \in B$ with no fill edge incident to it (since the opposite would require $k + 1$ fill edges). If $c \in B$ does not have a fill edge incident to it, then by Observation 3.1 one of the edges in $\{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}$ must be a fill edge (and all these are branched on for each $c \in B$ by Rule 3). ∎

Lemma 4.6 *Let G be a graph to which Rule 1 and Rule 2 cannot be applied. There exists a polynomial time algorithm that finds the unique maximal set $C(G)$ of shallow vertices in G, and applies Rule 3 if $|C(G)| \geq k + 1$.*

Proof. A minimal simple AT-witness can be found in polynomial time: for a triple of vertices, check if there exists a shortest path between any two of them that avoids the neighborhood of the third vertex. If these three paths exist and induces a minimal simple AT-witness for the triple add the shallow vertex to the set $C(G)$. Notice that every vertex that is shallow in some minimal simple AT-witness will be added to the set $C(G)$. Rule 3 is used on the set $C(G)$, when vertex number $k + 1$ is added. ∎

5 Thick AT-witnesses

In this section we introduce thick AT-witnesses and show that minimal thick AT-witnesses have a shallow vertex. These shallow vertices will be important for the fourth and final rule given in the next section.

We now consider graphs G to which none of the Rules 1, 2, or 3 can be applied. This means that G is chordal (Rule 1), the set $C(G)$ of shallow vertices in G has cardinality at most k (Rule 3), implying that (the connected components of) $G[C(G)]$ is an interval graph (Rule 2).

Definition 5.1 *Let $\{a, b, c\}$ be an AT in a chordal graph G, and let $W = \{w \mid w$ is a vertex of a chordless a, b-path, a, c-path, or b, c-path in $G\}$. The graph $G_{Tabc} = G[W]$ is the (unique)* thick AT-witness *for the AT $\{a, b, c\}$.*

We denote the neighborhoods of a, b, and c in G_{Tabc} by respectively S_a, S_b, and S_c, since these are minimal separators in G_{Tabc} and also in G by the following two observations.

Observation 5.2 *Let G_{Tabc} be a thick AT-witness in a chordal graph G. For any $x \in \{a, b, c\}$, x is a simplicial vertex and $S_x = N_{G_{Tabc}}(x)$ is a minimal separator in G_{Tabc}.*

Proof. We prove the observation for $x = a$; the other possibilities are symmetric. Because of the existence of a shortest b, c-path avoiding S_a, it follows that b and c are contained in the same connected component of $G_{Tabc} \setminus S_a$. Every neighbor of a in G_{Tabc} appears in a chordless path from a to either b or c or both. By the fact that b and c appears in the same connected component of $G_{Tabc} \setminus S_a$ and that neighbor of a in G_{Tabc} appears in a chordless path from a to either b or c, we can conclude that S_a is both a minimal a, b-separator and a minimal a, c-separator. In a chordal graph, every minimal separator is a clique [6]. Hence a is simplicial in G_{Tabc}. ∎

Observation 5.3 *Let G_{Tabc} be a thick AT-witness in a chordal graph G. Then the set of minimal separators of G_{Tabc} are exactly the set of minimal a,b-separators, a,c-separators, and b,c-separators of G.*

Proof. Let S be a minimal a,b-separator in G. Note that $S \subseteq G_{Tabc}$. There exist two connected components C_a and C_b of $G-S$, containing respectively a and b, such that $N_G(C_a) = N_G(C_b) = S$. For any vertex $z \in S$ we can now find a chordless shortest path in G from z to each of a and b, where every intermediate vertex is contained in respectively C_a and C_b. By joining these two paths, we get a chordless path from a to b that contains z. Since this holds for any vertex in S, it follows by the way we defined G_{Tabc} that any minimal a,b-separator of G is a minimal a,b-separator of G_{Tabc}. The argument can be repeated with a,c and b,c to show that every minimal a,c-separator or b,c-separator of G is also a minimal separator of G_{Tabc}.

Every minimal separator of G_{Tabc} separates two simplicial vertices appearing in two different leaf bags of any clique tree of G_{Tabc}. Since a,b,c are the only simplicial vertices in G_{Tabc} (every other vertex being an internal vertex of a chordless path), every minimal separator of G_{Tabc} is a minimal a,b-separator, b,c-separator, or a,c-separator. Let S be a minimal a,b-separator in G_{Tabc}. Vertex set S is a subset of a minimal a,b-separator of G, since the same chordless paths exist in G. But S cannot be a proper subset of a minimal a,b-separator of G, since every minimal a,b-separator of G is a minimal a,b-separator in G_{Tabc}, and thus S would not be a minimal separator in G_{Tabc} otherwise. The argument can be repeated with a,c and b,c. ■

Definition 5.4 *A thick AT-witness G_{Tabc} is* minimal *if $G_{Tabc} - x$ is AT-free for every $x \in \{a,b,c\}$.*

Observation 5.5 *Let G_{Tabc} be a minimal thick AT-witness in a chordal graph G. Then $G_{Tabc}-c$ is an interval graph, and in this interval graph $\{a,b\}$ is a dominating pair.*

Proof. The graph $G' = G_{Tabc} - c$ is by definition an interval graph, since G_{Tabc} is a minimal thick AT-witness. For a contradiction assume that $\{a,b\}$ is not a dominating pair, and thus there exists a path P'_{ab} from a to b in $G' - N[y]$ for some vertex $y \in V(G') \setminus \{a,b\}$. Let Q be a clique path of G'. Vertex y does not appear in any bag of Q that contains a or b, and it does not appear in any bags between the subpaths Q_a and Q_b of Q. Let us without loss of generality assume that Q_a appears between Q_y and Q_b in Q. We show that y is then not in any chordless path between any pair of a,b,c, giving the contradiction. Due to the above assumptions, y is not contained in the component C_b of $G' - N[a]$ that contains b. Furthermore, a is a simplicial vertex by Observation 5.2, and P'_{ab} contains vertices from $N_{G'}(a)$, thus $y \notin N_{G'}(a)$ since P'_{ab} would not avoid the neighborhood of y otherwise. The path $P_{bc} - c$ is contained in C_b since it contains no vertex of $N[a]$, and thus y is not adjacent to any vertex in $P_{bc} - c$. We know that $cy \notin E(G_{Tabc})$, since by Observation 5.2, $N_{G_{Tabc}}(c)$ is a clique, and thus y would be adjacent to the neighbor of c in P_{bc} if cy were an edge in $E(G_{Tabc})$. Now we have a contradiction since y is not in any chordless path between any pair of a,b,c. ■

Lemma 5.6 *Let G be a graph to which neither Rule 1 nor Rule 2 can be applied, and let G_{Tabc} be a minimal thick AT-witness in G where c is shallow. Then for every vertex $u \in S_c$ we have $S_a \cup S_b \subseteq N[u]$.*

Proof. Let $E' = E(G_{Tabc})$, and let us on the contrary and without loss of generality assume that $c'a' \notin E'$ for $c' \in S_c$ and $a' \in S_a$. Let $P_{ab} = (a = v_1, v_2, ..., v_r = b), P_{bc}$, and P_{ac} be the shortest paths used to define a simple AT-witness for $\{a, b, c\}$. We will show that either $\{a', b, c\}$ or $\{a, v_{r-1}, c\}$ is an AT in a subgraph of G_{Tabc}, contradicting its minimality.

Vertex set $\{a', b, c\}$ is an independent set since $cb \notin E'$, $a'b \notin E'$ due to $|P_{ab}| > 15 - 8$ (Rule 2), and $a'c \notin E'$ because c is simplicial in G_{Tabc}, and thus $c'a' \in E'$ if $a'c \in E'$. Either $v_2 = a'$, or $a'v_2 \in E'$ since a is simplicial in G_{Tabc}. $P_{ab} - a + a'$ is a path from a' to b that avoids the neighborhood of c. In the same way $P_{ac} - a + a'$ is a path from a' to c, and since we have already seen that $a'b \notin E'$ this path avoids the neighborhood of b. By Observation 5.5, c' is adjacent to some vertex on the path $P_{ab} = (a = v_1, v_2, ..., v_r = b)$. If c' is adjacent to some vertex v_i where $i > 3$, then there is a path $c, c', v_i, ..., v_r = b$ that avoids the neighborhood of a', and we have a contradiction since a', b, c would be an AT in $G_{Tabc} - a$. We can therefore assume that there is a $j \in \{2, 3\}$ such that $v_jc' \in E'$, and that there exists no $v_ic' \in E'$ for any $i > 3$. The set $\{a, v_{r-1}, c\}$ is an independent set, since $cv_{r-1}, av_{r-1} \notin E'$. The path $a, v_2, ..., v_{r-1}$ avoids the neighborhood of c, the path $c, c', v_j, ..., a$ avoids the neighborhood of v_{r-1}, and $P_{bc} - b + v_{r-1}$ is a path from c to v_{r-1} that avoids the neighborhood of a, since b is simplicial in G_{Tabc}. This is a contradiction since $G_{Tabc} - b$ contains the AT $\{a, v_{r-1}, c\}$. ■

Lemma 5.7 *Let $G = (V, E)$ be a graph to which neither Rule 1 nor Rule 2 can be applied. Let G_{Tabc} be a minimal thick AT-witness in G where c is shallow. Let C_c be the connected component of $G - S_c$ that contains c. Then every vertex of C_c has in G the same set of neighbors S_c outside C_c, in other words $\forall u \in C_c : N_G(u) \setminus C_c = S_c$.*

Proof. By definition $N_G(u) \setminus C_c \subseteq S_c$. Let us assume for a contradiction that $ux \notin E$ for some $x \in S_c$ and $u \in C_c$. Since C_c is a connected component there exists a path from u to c inside C_c. Let u', c' be two consecutive vertices on this path, such that $S_c \subseteq N_G(c')$ and $u'x' \notin E$ for some $x' \in S_c$. By Lemma 5.6 x' creates a short path from a to b that avoids the neighborhood of u', and by using $P_{ac} - c$ and $P_{bc} - c$ and the vertices c' and u' we can create short paths from a to u' and from b to u' that avoid the neighborhoods of b and a, respectively. This is now a contradiction, since $\{a, b, u'\}$ is an AT with a simple AT-witness where the number of branching fill edges are 5 for the path a, a', x', b', b, 5 for $P_{ac} - c$ and c', u', and 5 for $P_{bc} - c$ and c', u', giving a total of 15 branching edges. ■

The following simple observations are needed for the proof of Lemma 5.10.

Observation 5.8 *A vertex v is simplicial only if v is an end vertex of every chordless path that contains v.*

Proof. Any vertex that appears as a non end vertex in a chordless path, has two neighbors that are not adjacent. ■

Observation 5.9 *Let G_{Tabc} be a minimal thick AT-witness in a graph G to which neither Rule 1 nor Rule 2 can be applied. Then at least one of the vertices in the AT $\{a, b, c\}$ is shallow, and there exists a minimal simple AT-witness G_{abc} for $\{a, b, c\}$, where $V(G_{abc}) \subseteq V(G_{Tabc})$.*

Proof. Let P_{ab}, P_{ac}, P_{bc} be shortest chordless paths contained in G_{Tabc}, and let G_{abc} be defined by P_{ab}, P_{ac}, P_{bc}. It is clear that G_{Tabc} is minimal only if G_{abc} is minimal. By Observation 4.4, Rule 2, and the fact that G_{abc} is a minimal AT-witness for $\{a, b, c\}$, we know that at least one of the vertices in $\{a, b, c\}$ is shallow. ∎

Lemma 5.10 *Let G be a graph to which neither Rule 1 nor Rule 2 can be applied, and let G_{Tabc} be a thick AT-witness in G. Then there exists a minimal thick AT-witness G_{Txyz} in G, where $V(G_{Txyz}) \subseteq V(G_{Tabc})$ and z is shallow, such that $z \in \{a, b, c\}$.*

Proof. G_{Txyz} will be obtained from G_{Tabc} by deleting one of the simplicial vertices in the AT that defines G_{Tabc}, and repeat this until a minimal thick AT-witness G_{Txyz} is obtained. Note that only neighbors of the deleted vertex can become simplicial after each deletion, by Observation 5.8. As a result, the deleted vertices induce at most three connected components. Actually the number of components will be exactly three, since otherwise the connected components contains a chordless path between two of the vertices in $\{x, y, z\}$ which is a contradiction to the definition of a minimal thick AT-witness. Thus, each connected component is adjacent to one of the vertices x, y, z. By Observation 5.9 one of the vertices x, y, z is shallow. Let us without loss of generality assume that z is the shallow vertex in G_{Txyz}. By Lemma 5.3, minimal separators of G_{Txyz} are also minimal separators of G_{Tabc}, so let us assume without loss of generality that z and c are contained in the same connected component of $G_{Tabc} - N_{G_{Txyz}}(z)$. Notice that z and c might be the same vertex. By Lemma 5.7, c is shallow in the minimal thick AT-witness G_{Txyc}. ∎

Like Rules 2 and 3, Rule 4 will branch on single fill edges, but it will also consider minimal separators, based on the following two basic observations.

Observation 5.11 *If G has a minimal thick AT-witness G_{Tabc} in which P_{ac} and P_{bc} are shortest a, c and b, c-paths avoiding $N(b)$ and $N(a)$ respectively, then any interval completion of G either contains a fill edge from the set $\{bx \mid x \in P_{ac}\} \cup \{ax \mid x \in P_{bc}\}$ or contains one of the edge sets $\{\{cx \mid x \in S\} \mid S$ is a minimal a, b-separator in $G_{Tabc}\}$.*

Proof. By Observation 3.1, we know that at least one of the edges in $\{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\} \cup \{cx \mid x \in P_{ab}\}$ for the paths P_{ab}, P_{ac}, P_{bc} defined in the proof of Observation 5.9, is a fill edge of any interval completion of G. If an interval completion H does not contain any fill edge from the set $\{bx \mid x \in P_{ac}\} \cup \{ax \mid x \in P_{bc}\}$, then H contains at least one fill edge from the set $\{cx \mid x \in P'_{ab}\}$, where P'_{ab} is any chordless a, b-path in G that avoids the neighborhood of c. Thus, $N_H(c)$ contains a minimal a, b-separator in G (which by Observation 5.3 is also a minimal a, b-separator in G_{Tabc}) since every chordless and thus every a, b-path in $G - N[c]$ contains a vertex of $N_H(c)$. ∎

Observation 5.12 *Let G be a graph to which neither Rule 1 nor 2 can be applied, and let G_{Tabc} be a minimal thick AT-witness in G where c is shallow. Then $S_c \subset S$ for every minimal a, b-separator S different from S_a and S_b.*

Proof. Let S be a minimal a, b-separator different from S_a and S_b. No minimal a, b-separator contains another minimal a, b-separator as a subset, thus there exists a vertex $a' \in S_a \setminus S$ and a vertex $b' \in S_b \setminus S$. S is then also a minimal a', b'-separator because of the edges aa' and bb'. It then follows from Lemma 5.6 that $S_c \subset N(a') \cap N(b')$, and thus $S_c \subset S$. ∎

6 Partitioning the shallow vertices: Rule 4

In this section we present the fourth and final rule and prove correctness of the resulting search tree algorithm. We start by detailing the computation of the set $C(G)$ of shallow vertices which will give us a partition of $C(G)$ that we will use in our branching rule 4.

Definition 6.1 *Given a graph G to which Rules 1 and 2 do not apply, we compute a set $C(G) = C_1 \cup C_2 \cup ... \cup C_r$ of vertices that are shallow in some minimal thick AT-witness, with $G \setminus C(G) = R_r$ an interval graph, as follows:*

$R_0 := G;\ i := 0;\ C(G) := \emptyset;$
while *R_i is not an interval graph* **do**
  $i := i + 1;$
  Find $G_{Ta_ib_ic_i}$ a minimal thick AT-witness in R_{i-1} with c_i shallow;
  Let C_i be the connected component of $R_{i-1} - N_{G_{Ta_ib_ic_i}}(c_i)$ that contains c_i;
  for *each $c \in C_i$* **do** $G_{Ta_ib_ic} := G_{Ta_ib_ic_i} - c_i + c;$
  $R_i := R_{i-1} - C_i;$
  $C(G) := C(G) \cup C_i;$
end-while
$r := i$

The minimal thick AT-witness $G_{Ta_ib_ic_i}$ is found by first finding an AT $\{a, b, c\}$, then removing simplicial vertices different from a, b, c according to Observation 5.8 to get a thick AT-witness, and then applying the procedure in the proof of Lemma 5.10.

A priori we have no guarantee that there are no edges between a vertex in C_i and a vertex in C_j, for some $i \neq j$, but when $|C(G)| \leq k$ (which is ensured by Rule 3) this indeed holds, as shown in the following lemma.

Lemma 6.2 *Let $G = (V, E)$ be a graph to which none of Rules 1, 2, 3 can be applied, and let $C(G) = C_1 \cup C_2 \cup ... \cup C_r$ from Definition 6.1. Then C_i induces an interval graph that is a connected component of $G[C(G)]$, for each $1 \leq i \leq r$.*

Proof. Firstly, $|C_i| \leq k$ since Rule 3 does not apply and since and Rules 1, 2 do not apply, it must induce an interval graph. To argue that it is a connected component, note first that by definition $G[C_i]$ is connected and $C_i \cap C_j = \emptyset$ for any $i \neq j$. For a contradiction we assume that $cz \in E$ for some $c \in C_i$ and $z \in C_j$ with $i < j$. Notice that $cz \in E$ implies that $z \in S_c$. Let G_{Tabc} be the minimal thick AT-witness in R_{i-1} with c the shallow vertex and $S_c = N_{G_{Tabc}}(c)$, and let likewise G_{Txyz} be the minimal thick AT-witness in R_{j-1} with z shallow and $S_z = N_{G_{Txyz}}(z)$. Let P_{ab} be a path from a to b in $G_{Tabc} \setminus N(c)$. There are now two cases:

Case I: There is a vertex $w \in P_{ab} \cap S_z$. Note that S_c and S_z are minimal separators in the chordal graphs R_{i-1} and R_{j-1} respectively, and thus by Observation 5.3 S_c and S_z are cliques [6]. Thus, since $cw \notin E$ we must have $c \notin S_z$. But then we have c and z in the same component D_z of $G \setminus S_z$. By Lemma 5.7 c and z must therefore have the same neighbors outside D_z. But this contradicts the fact that $zw \in E$ while $cw \notin E$.

Case II: $P_{ab} \cap S_z = \emptyset$. Let D_z be the connected component of $G \setminus S_z$ that contains z. By Observation 5.5 $G_{Tabc} \setminus \{c\}$ is an interval graph where a, b is a dominating pair, thus $zw \in E$ for some $w \in P_{ab}$ since $z \in S_c$ and therefore $V(P_{ab}) \subseteq D_z$.

Since c is shallow we know that $|P_{ac}| + |P_{bc}| \leq 8$ and since Rule 2 cannot be applied we know that $|P_{ab}| + |P_{ac}| + |P_{bc}| \geq k + 16$. Thus we have at least $k + 16 - 8$ vertices in P_{ab} and thus $|D_z| \geq |P_{ab}| > k$. Assuming we can show the subset-property $D_z \subseteq C_1 \cup C_2 \cup ... \cup C_j$ we are done with the proof since this will lead to the contradiction $k < |D_z| \leq |C_1 \cup C_2 \cup ... \cup C_j| \leq |C(G)| \leq k$. Let us prove the subset-property. G has a perfect elimination ordering starting with the vertices of C_1, as these vertices are a component resulting from removing a minimal separator from G. By induction, we have that G has a perfect elimination ordering α starting with the vertices in $C_1 \cup C_2 \cup ... \cup C_{j-1}$. For a contradiction assume there exists a vertex $t \in D_z \setminus (C_1 \cup C_2 \cup ... \cup C_j)$. As $t \in D_z$ there is a shortest chordless t, z-path P_{tz} in D_z. The edge $tz \notin E$ since this would make t a member of C_j (as $t \notin C_j$). Only vertices in $C_1 \cup C_2 \cup ... \cup C_{j-1}$ are removed from the graph, and these separate z from t in $G[D_z]$ (since $t \notin C_j$). Let s be the lowest numbered vertex in the ordering α that belongs to the path P_{tz}. This is now a contradiction, since a non-end vertex of a chordless path cannot be simplicial if two adjacent vertices eliminated later in the perfect elimination ordering are non adjacent. ∎

Rule 4 will branch on a bounded number of single fill edges and it will also compute a greedy completion by choosing for each shallow vertex a minimal separator minimizing fill and making the shallow vertex adjacent to all vertices of that separator. We will prove that if none of the single fill edges branched on in Rule 4 are present in any k-interval completion, then the greedy completion gives an interval completion with the minimum number of edges. The greedy choices of separators are made as follows:

Definition 6.3 *Let G be a graph to which none of Rules 1, 2, 3 can be applied. Let Definition 6.1 give $C(G) = C_1 \cup C_2 \cup ... \cup C_r$, representative vertices $c_1, c_2, ..., c_r$ and minimal thick AT-witnesses $G_{Ta_ib_ic_i}$ and graphs $G = R_0 \supset R_1 \supset ... \supset R_r$, with R_r interval. Let M_i, for $i = 1, 2, ..., r$ be a minimal a_i, b_i-separator S in $G_{Ta_ib_ic_i}$ different from S_{a_i} and S_{b_i} and $N(C_j)$ for all $1 \leq j \leq r$, satisfying $S \cap C(G) = \emptyset$, and minimizing $|S \setminus N(C_i)|$. If no such S exists, define $M_i = null$.*

Lemma 6.4 *If $M_i \neq null$ then M_i is a minimal separator in R_r.*

Proof. The vertex set M_i is a minimal separator in $G_{Ta_ib_ic_i}$ by construction and since $G_{Ta_ib_ic_i}$ is a subgraph of the chordal graph R_i it is by Observation 5.3 also a minimal separator of R_i. We prove that M_i is also a minimal separator in R_j for any $i + 1 \leq j \leq r$ by induction on j. Recall that R_j is obtained by removing C_j from R_{j-1}, where C_j is a component of $R_{j-1} \setminus S_{c_i}$ for a minimal separator S_{c_i} of R_{j-1}, and $S_{c_i} = N(C_j)$ by Lemma 5.7. Consider a clique tree of R_{j-1} and observe that any minimal separator of R_{j-1} that is not a minimal separator of R_j is either equal to $N(C_j)$ or it contains a vertex of C_j. Finally, note that the minimal separator M_i has been chosen so that it is not of this type. ∎

Branching Rule 4:
Rule 4 applies if none of Rules 1, 2, 3 apply, in which case we compute, as in Definitions 6.1 and 6.3, $C_1, C_2, ..., C_r$ (which are connected components of $G[C(G)]$ by Lemma 6.2), the minimal thick AT-witnesses $G_{Ta_ib_ic}$ with c shallow for each $c \in C_i$, and $M_1, ..., M_r$ (which are minimal separators of R_r by Lemma 6.4). For each $1 \leq i \leq r$ and each $c \in C_i$ choose $a'_i \in S_{a_i} \setminus S_c$ and $b'_i \in S_{b_i} \setminus S_c$ and find P_{a_ic} and P_{b_ic} (shortest paths in $G_{Ta_ib_ic}$ avoiding $N(b_i)$ and $N(a_i)$, respectively, of length at most 4 by Observation 5.9). For each pair $1 \leq i \neq j \leq r$, choose a vertex $v_{i,j} \in N(C_j) \setminus N(C_i)$ (if it exists).

- For $1 \leq i \leq r$ and $c \in C_i$, branch on the at most 8 fill edges $\{a_ix \mid x \in P_{b_ic}\} \cup \{b_ix \mid x \in P_{a_ic}\}$ and also on the 2 fill edges $\{ca'_i, cb'_i\}$.
- Branch on the at most $|C(G)|(|C(G)| - 1)/2$ fill edges $\{uv \mid u, v \in C(G) \text{ and } uv \notin E\}$.
- Branch on the at most $|C(G)|r$ fill edges $\bigcup_{1 \leq i \neq j \leq r}\{cv_{i,j} \mid c \in C_i\}$.
- Finally, compute $H = (V,\ E \cup \bigcup_{1 \leq i \leq r}\{cx \mid c \in C_i \text{ and } x \in M_i\})$ and check if it is a k-interval completion of G (note that we do not branch on H.)

Lemma 6.5 *If G has a k-interval completion, and Rules 1, 2, and 3 do not apply to G, and no k-interval completion of G contains any single fill edge branched on by Rule 4, then the graph H, which Rule 4 obtains by adding fill edges from every vertex in C_i to every vertex in M_i for every $1 \leq i \leq r$, is a k-interval completion of G.*

Proof. By Observation 5.11, for each $c \in C_i$ either one of the edges in $\{a_ix \mid x \in P_{b_ic}\} \cup \{b_ix \mid x \in P_{a_ic}\}$ is a fill edge (and all these are branched on by Rule 4) or else the k-interval completion contains the edge set $\{cx \mid x \in S\}$ for some minimal a_i, b_i-separator S in $G_{Ta_ib_ic}$. Such an edge set in a k-interval completion is one of four types (listed below) depending on the separator S used to define it. For each type and any $c \in C_i$ we argue that Rule 4 considers it. Observe that $N(C_i) \setminus C_i = N(c) \setminus C_i$ by Lemma 5.7, and thus the fill edges from c will go to vertices in $S \setminus N(C_i)$, which is nonempty since there is an a_i, b_i-path avoiding $N(c)$. We now give the four types of minimal separators S, and show that the first three are branched on by a single fill edge:

1. $S \cap C(G) \neq \emptyset$. Since $N(C_i) \cap C(G) = \emptyset$ by Lemma 6.2, we have in this case a fill edge between two vertices in $C(G)$ (between $c \in C_i$ and a vertex in $C(G) \cap S \setminus N(C_i)$) and all these are branched on by Rule 4.
2. $S = S_{a_i}$ or $S = S_{b_i}$, where S_{a_i}, S_{b_i}, S_c defined by $G_{Ta_ib_ic}$. We found in Rule 4 a pair of vertices $a'_i \in S_{a_i} \setminus S_c$ and $b'_i \in S_{b_i} \setminus S_c$ and branched on the fill edges ca'_i and cb'_i.
3. $S = N(C_j)$ for some $1 \leq j \leq r$. If $S = N(C_j)$ then $N(C_j) \setminus N(C_i) \neq \emptyset$ and we found in Rule 4 a vertex $v_{i,j} \in N(C_j) \setminus N(C_i)$ and branched on the fill edge $cv_{i,j}$.
4. S is neither of the three types above. Note that M_i was chosen in Definition 6.3 by looping over all minimal a_i, b_i-separators S in $G_{Ta_ib_ic_i}$ (which by Lemma 5.7 are exactly the minimal a_i, b_i-separators of $G_{Ta_ib_ic}$) satisfying $S \cap C(G) = \emptyset$, $S \neq S_a$, $S \neq S_b$, and

$S \neq N(C_j)$ for any j. Thus, of all separators of this fourth type, M_i is the one minimizing the fill.

The assumption is that G has a k-interval completion but no single edge branched on by Rule 4 is present in any k-interval completion. This means that only separators of the fourth type are used in any k-interval completion. Since H added the minimum possible number of fill edges while using only separators of the fourth type any interval completion of G must add at least $|E(H) \setminus E(G)|$ edges. It remains to show that H is an interval graph. H is constructed from an interval graph R_r and the components $G[C_1], ..., G[C_r]$ of $G[C(G)]$, which are interval graphs by Lemma 6.2, and $M_1, ..., M_r$ which are minimal separators of R_r by Lemma 6.4. Since $M_i \neq S_{a_i}$ and $M_i \neq S_{b_i}$ we have by Observation 5.12 that $S_c = N(C_i) \subset M_i$ so that adding all edges between C_i and M_i for $1 \leq i \leq r$ gives the graph H. We show that H is an interval graph by induction on $0 \leq i \leq r$. Let $H_0 = R_r$ and let H_i for $i \geq 1$ be the graph we get from H_{i-1} and C_i by making all vertices of C_i adjacent to all vertices of the minimal separator M_i of R_r. H_0 is an interval graph by induction, and its minimal separators include all minimal separators of R_r. If $(K_1, K_2, ...K_q)$ is a clique path of H_{i-1} with $M_i = K_j \cap K_{j+1}$, and $(K'_1, K'_2, ..., K'_p)$ is a clique path of $G[C_i]$ then $(K_1, K_2, ..., K_j, K'_1 \cup M_i, K'_2 \cup M_i, ..., K'_p \cup M_i, K_{j+1}, ..., K_q)$ is a clique path of H_i, and hence H_i is an interval graph. Finally, observe that the minimal separators of H_{i-1} and hence of R_r are also minimal separators of H_i. ■

Theorem 6.6 *The search tree algorithm applying Rules 1, 2, 3, 4 in that order decides in $O(k^{2k} n^3 m)$ time whether an input graph G on n vertices and m edges can be completed into an interval graph by adding at most k edges.*

Proof. At least one of the rules will apply to any graph which is not interval. The correctness of Rule 1 is well understood [17, 2], that of Rules 2 and 3 follow by Observations 3.1 and 5.11 and of Rule 4 by Lemma 6.5. Each branching of Rules 2, 3 and 4 add a single fill edge and drops k by one. As already mentioned, also Rule 1 could have added a single fill edge in each of its then at most k^2 branchings. The height of the tree is thus no more than k, before k reaches 0 and we can answer "no". If an interval graph is found we answer "yes".

Let us argue for the runtime. The graph we are working on never has more than $m+k$ edges. In Rule 1 we decide in linear time if the graph has a large induced cycle. In Rule 2 we may have to try all triples when searching for an AT with a small simple AT-witness, taking $O(n^3(m+k))$ time. In Rule 3 and 4 we need to find a minimal thick AT-witness at most $k+1$ times. As observed earlier, the minimal thick AT-witness is found by first finding an AT $\{a, b, c\}$, which can be done in time $O(m+k)$ since G is a chordal graph [19], then remove simplicial vertices different from a, b, c to find the thick AT-witness, and then make it minimal. Using a clique tree we find in this way a single minimal thick AT-witness in time $O(n^3)$ and at most k of them in time $O(n^3 k)$. Hence each rule takes time at most $O(n^3(m+k))$ and has branching factor at most k^2 (e.g. in Rule 1 and also in Rule 3 when branching on all fill edges between pairs of shallow vertices). The height of the search tree is at most k and the number of nodes therefore at most k^{2k}. We can assume $k \leq n \leq m$ since otherwise a brute-force algorithm easily solves minimum interval completion in n^{2n} steps. Thus each rule takes time $O(n^3 m)$ for total runtime $O(k^{2k} n^3 m)$. ■

7 Concluding remarks

We have shown that k-Interval Completion is FPT. The runtime of our algorithm can probably be improved somewhat, at the expense of much more complicated data structures. In an earlier version of this paper, in STOC 2007 Proceedings, we asked if there was a hereditary graph class recognizable in polynomial time for which the k-completion problem into this graph class was not FPT. This question has been answered [22], since for the complements of wheel-free graphs the k-completion problem is $W[2]$-complete. It is still an open problem whether the complexity of k-completions into perfect graphs is FPT.

An alternative equivalent definition of the complexity class FPT relates to kernelization. In this formulation a parameterized problem is FPT if there exists a polynomial-time algorithm that for any instance outputs an equivalent 'kernelized' instance whose size is a function of the parameter only. The quest for the smallest possible kernel size is orthogonal to the quest for the fastest possible FPT algorithm. The FPT algorithm given here for k-interval completion implies that this problem has an exponential sized kernel. We leave it as an open problem if k-interval completion has a polynomial-sized kernel.

References

[1] P. Buneman. A characterization of rigid circuit graphs. *Discrete Math.*, 9:205–212, 1974.

[2] L. Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Information Processing Letters*, 58(4):171–176, 1996.

[3] L. Cai. Parameterized complexity of vertex colouring. *Discrete Applied Mathematics* 127, 3, 415 - 429, 2003

[4] D. G. Corneil, S. Olariu, and L. Stewart. Asteroidal Triple-Free Graphs. *SIAM J. Discrete Math.*, 10(3):399–430, 1997.

[5] J. Díaz, A.M. Gibbons, M.S. Paterson, and J. Torán. The minsumcut problem. In *Proceedings WADS 1991*, 65-79

[6] G. A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, 25:71–76, 1961.

[7] R. G. Downey and M. R. Fellows. *Parameterized Complexity.* Springer Verlag, New York, 1999.

[8] G. Even, J. Naor, S. Rao, and B. Scheiber. Divide-and-conquer approximation algorithms via spreading metrics. In *Proceedings FOCS 1995*, 62-71.

[9] M. Garey and D. Johnson. *Computers and intractability: a guide to the theory of NP-completeness* W.H.Freeman and co, San Fransisco, 1979.

[10] J. A. George and J. W. H. Liu. *Computer Solution of Large Sparse Positive Definite Systems.* Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1981.

[11] P. C. Gilmore and A. J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canadian Journal of Mathematics*, 16:539–548, 1964.

[12] P. W. Goldberg, M. C. Golumbic, H. Kaplan, and R. Shamir. Four strikes against physical mapping of DNA. *J. Comput. Bio.*, 2(1):139–152, 1995.

[13] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Second edition. Annals of Discrete Mathematics 57. Elsevier, 2004.

[14] G. Gutin, S. Szeider, and A. Yeo. Fixed-parameter complexity of minimum profile problems. In *Proceedings IWPEC 2006*, pages 60–71, 2006. Springer LNCS 4169.

[15] P. Heggernes, K. Suchan, I. Todinca, and Y. Villanger. Minimal interval completions. In *Proceedings ESA 2005*, pages 403 – 414, 2005. Springer LNCS 3669.

[16] H. Kaplan, R. Shamir, and R. E. Tarjan. Tractability of parameterized completion problems on chordal and interval graphs: minimum fill-in and physical mapping. In *Proceedings FOCS 1994*, 780-791.

[17] H. Kaplan, R. Shamir, and R. E. Tarjan. Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs. *SIAM J. Comput.*, 28(5):1906–1922, 1999.

[18] T. Kashiwabara and T. Fujisawa. An NP-complete problem on interval graphs. *IEEE Symp. of Circuits and Systems*, pages 82–83, 1979.

[19] D. Kratsch, R. McConnell, K. Mehlhorn, and J. Spinrad. Certifying algorithms for recognizing interval graphs and permutation graphs *SIAM J. Comput.*, 36(2):326–353, 2006.

[20] D. Kuo, and G.J. Chang. The profile minimization problem in trees. *SIAM J. Comput.* 23, 71-81, 1994

[21] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962. 003-1010 (2006)

[22] D. Lokshtanov. Wheel-free deletion is W[2]-Hard. In *Proceedings IWPEC 2008*, pages 141–147, 2008. Springer LNCS 5018.

[23] S. Rao and A. Richa. New approximation techniques for some linear ordering problems. *SIAM J. Comput.* 34(2):388-404 (2004)

[24] R. Ravi, A. Agrawal, and P. Klein. Ordering problems approximated: single processor scheduling and interval graph completion. In *Proceedings ICALP 1991* 751-762, 1991

[25] M. Serna and D. Thilikos. Parameterized complexity for graph layout problems. *Bulletin of the EATCS*, 86:41–65, 2005.

[26] R. E. Tarjan. Graph theory and Gaussian elimination. In *Sparse Matrix Computations* (eds. J. R. Bunch and D. J. Rose). Academic Press, 1976.