

INTERVAL CRITERIA FOR FORCED OSCILLATION OF DIFFERENTIAL EQUATIONS WITH p -LAPLACIAN AND NONLINEARITIES GIVEN BY RIEMANN-STIELTJES INTEGRALS

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ABSTRACT. We consider forced second order differential equation with p -Laplacian and nonlinearities given by a Riemann-Stieltjes integrals in the form of

$$(p(t)\phi_\gamma(x'(t)))' + q_0(t)\phi_\gamma(x(t)) + \int_0^b q(t,s)\phi_{\alpha(s)}(x(t))d\zeta(s) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\gamma, b \in (0, \infty)$, $\alpha \in C[0, b]$ is strictly increasing such that $0 \leq \alpha(0) < \gamma < \alpha(b-)$, $p, q_0, e \in C([t_0, \infty), \mathbb{R})$ with $p(t) > 0$ on $[t_0, \infty)$, $q \in C([0, \infty) \times [0, b))$, and $\zeta : [0, b) \rightarrow \mathbb{R}$ is nondecreasing. Interval oscillation criteria of the El-Sayed type and the Kong type are obtained. These criteria are further extended to equations with deviating arguments. As special cases, our work generalizes, unifies, and improves many existing results in the literature.

1. Introduction

We are concerned with the oscillatory behavior of forced second order differential equations with p -Laplacian and nonlinearities given by a Riemann-Stieltjes integrals in the form of

$$(1.1) \quad (p(t)\phi_\gamma(x'(t)))' + q_0(t)\phi_\gamma(x(t)) + \int_0^b q(t,s)\phi_{\alpha(s)}(x(t))d\zeta(s) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\gamma, b \in (0, \infty)$, $\alpha \in C[0, b]$ is strictly increasing such that $0 \leq \alpha(0) < \gamma < \alpha(b-)$, $p, q_0, e \in C([t_0, \infty), \mathbb{R})$ with $p(t) > 0$ on $[t_0, \infty)$, $q \in C([0, \infty) \times [0, b))$, and $\zeta : [0, b) \rightarrow \mathbb{R}$ is nondecreasing. Our interest is to establish oscillation criteria for Eq.(1.1) without assuming that $q_0(t)$, $q(t, s)$, and $e(t)$ are of definite sign. Here $\int_0^b f(s)d\zeta(s)$ denotes the Riemann-Stieltjes integral of the function f on $[0, b)$ with respect to ζ .

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We note that as special cases, the integral term in the equation becomes a finite sum when $\zeta(s)$ is a step function and a Riemann integral when $\zeta(s) = s$.

As usual, a solution $x(t)$ of Eq.(1.1) is said to be oscillatory if it is defined on some ray $[T, \infty)$ with $T \geq 0$, and has unbounded set of zeros. Eq.(1.1) is said to be oscillatory if every solution extendible throughout $[t_x, \infty)$ for some $t_x \geq 0$ is oscillatory.

In the last 50 years, there has been extensive work on oscillation and nonoscillation of various differential equations, see [1, 3–11, 13, 16–30, 32–39] and the references cited therein.

Sun and Wong [35] investigated the following forced equation with mixed nonlinearities

$$(p(t)x'(t))' + q_0(t)x(t) + \sum_{j=0}^N q_j(t)\phi_{\alpha_j}(x(t)) = e(t),$$

where p , q_0 and e satisfy the same assumptions as for Eq.(1.1), $q_j \in C[0, \infty)$ and

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_N > 0.$$

Without imposing a restriction on the forcing term $e(t)$ given by Kartsatos and others, see [16, 17], that $e(t)$ is the second derivative of an oscillatory function. Hassan, Erbe and Peterson [14] discussed the oscillation of an equation with p -Laplacian, more specifically, they established oscillation criteria of El-Sayed-type for the equation

$$(p(t)(x'(t))^\gamma)' + q_0(t)x^\gamma(t) + \sum_{j=1}^N q_j(t)\phi_{\alpha_j}(x(t)) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, γ is a quotient of odd positive integers and $\alpha_j > 0$, $j = 1, 2, \dots, N$, such that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_N > 0.$$

Hassan and Kong [15] considered the forced second order differential equations with p -Laplacian in the form of

$$(1.2) \quad (p(t)\phi_\gamma(x'(t)))' + \sum_{j=0}^N q_j(t)\phi_{\alpha_j}(x(t)) = e(t),$$

where γ , $\alpha_j > 0$, $j = 0, 1, 2, \dots, N$, such that

$$(1.3) \quad \alpha_j > \gamma, \quad j = 1, 2, \dots, l; \quad \text{and} \quad \alpha_j < \gamma, \quad j = l + 1, l + 2, \dots, N.$$

They established oscillation criteria of El-Sayed-type and Kong-type for Eq. (1.2). Recently, Sun and Kong [31] considered the equation (1.1) when $\gamma = 1$. Motivated by the above, in this paper, we will establish interval oscillation criteria of both the El-Sayed-type and the Kong-type for the more general equation (1.1). Clearly, our work is of significance because Eq.(1.1) not only

contains a p -Laplacian term but also allows an infinite number of nonlinear terms and even a continuum of nonlinearities determined by the function ζ .

Finally, we will extend the results for Eq.(1.1) to forced nonlinear differential equations with deviating arguments in the form
(1.4)

$$(p(t)\phi_\gamma(x'(t)))' + q_0(t)\phi_\gamma(x(t)) + \int_0^b q(t,s)\phi_{\alpha(s)}(x(g(t,s)))d\zeta(s) = e(t),$$

where, in addition to the assumptions for Eq.(1.1), $g \in C([0, \infty) \times [0, b], [0, \infty))$ such that $\lim_{t \rightarrow \infty} g(t, s) = \infty$ for $s \in [0, b]$.

This paper is organized as follows: after this introduction, we state our main results for Eqs.(1.1) and (1.4) in Section 2. All proofs are given in Section 3.

2. Main results

We denote by $L_\zeta(0, b)$ the set of Riemann-Stieltjes integrable functions on $[0, b]$ with respect to ζ . Let $a \in (0, b)$ such that $\alpha(a) = \gamma$. We further assume that

$$\alpha^{-1} \in L_\zeta(0, b) \quad \text{such that} \quad \int_0^a d\zeta(s) > 0 \quad \text{and} \quad \int_a^b d\zeta(s) > 0.$$

We see that the condition $\alpha^{-1} \in L_\zeta(0, b)$ is satisfied if either $\alpha(0) > 0$ or $\alpha(s) \rightarrow 0$ “slowly” as $s \rightarrow 0^+$, or $\zeta(s)$ is constant in a right neighborhood of 0. To state our main results, we begin with the following lemma which generalizes [31, Lemma 1].

Lemma 2.1. *Let*

$$m := \gamma \left(\int_a^b d\zeta(s) \right)^{-1} \int_a^b \alpha^{-1}(s) d\zeta(s)$$

and

$$n := \gamma \left(\int_0^a d\zeta(s) \right)^{-1} \int_0^a \alpha^{-1}(s) d\zeta(s).$$

Then for any $\delta \in (m, n)$, there exists $\eta \in L_\zeta(0, b)$ such that $\eta(s) > 0$ on $[0, b]$,

$$(2.1) \quad \int_0^b \alpha(s)\eta(s)d\zeta(s) = \gamma \quad \text{and} \quad \int_0^b \eta(s)d\zeta(s) = \delta.$$

We note from the definition of m and n that $0 < m < 1 < n$. In the following, we will use the values of δ in the interval $(m, 1]$ to establish interval criteria for oscillation of Eq.(1.1). Our first result provides an oscillation criterion of the El-Sayed-type.

Theorem 2.1. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ and $(a_1, b_1) \cap (a_2, b_2) = \emptyset$ such that, for $i = 1, 2$*

$$(2.2) \quad q(t, s) \geq 0 \quad \text{for} \quad (t, s) \in [a_i, b_i] \times [0, b]$$

and

$$(2.3) \quad (-1)^i e(t) \geq 0 \text{ for } t \in [a_i, b_i].$$

Assume further that for $i = 1, 2$, there exists $u_i \in C^1 [a_i, b_i]$ satisfying $u_i(a_i) = u_i(b_i) = 0$ and $u_i(t) \neq 0$ on $[a_i, b_i]$ such that

$$(2.4) \quad \sup_{\delta \in (m, 1]} \int_{a_i}^{b_i} [Q(t) |u_i(t)|^{\gamma+1} - p(t) |u_i'(t)|^{\gamma+1}] dt > 0,$$

where

$$(2.5) \quad Q(t) := q_0(t) + \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \exp \left(\int_0^b \eta(s) \ln \frac{q(t,s)}{\eta(s)} d\zeta(s) \right)$$

with $\eta(s)$ implicitly defined as in Lemma 2.1 based on δ . Here we use the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$, and $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ for $\delta = 1$. Then Eq.(1.1) is oscillatory.

Remark 2.1. (i) We observe that in Theorem 2.1, if the supremum in (2.4) is assumed at $\delta = 1$, the effect of $e(t)$ is neglected in some extent. This implies that the magnitude of $e(t)$ in $[a_i, b_i]$ cannot be large. For otherwise, the supremum would have been taken at some $\delta \in (m, 1)$.

(ii) Contrast to the results in the literature, by choosing different values of γ , Eq.(1.1) allows the nonlinearities of the unknown function in the integral term to be all sublinear, all superlinear, or mixed.

Following Philos [24], Kong [19], and Kong [20], we say that for any $a, b \in \mathbb{R}$ such that $a < b$, a function $H(t, s)$ belongs to a function class $\mathcal{H}(a, b)$, denoted by $H \in \mathcal{H}(a, b)$, if $H \in C(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t, s) : b \geq t \geq s \geq a\}$, which satisfies

$$(2.6) \quad H(t, t) = 0, \quad H(b, s) > 0 \text{ and } H(s, a) > 0 \text{ for } b > s > a,$$

and $H(t, s)$ has continuous partial derivatives $\partial H(t, s) / \partial t$ and $\partial H(t, s) / \partial s$ on $[a, b] \times [a, b]$ such that

$$(2.7) \quad \frac{\partial H(t, s)}{\partial t} = (\gamma + 1) h_1(t, s) H^{\frac{\gamma}{\gamma+1}}(t, s)$$

and

$$(2.8) \quad \frac{\partial H(t, s)}{\partial s} = (\gamma + 1) h_2(t, s) H^{\frac{\gamma}{\gamma+1}}(t, s),$$

where $h_1, h_2 \in L_{loc}(\mathbb{D}, \mathbb{R})$. Next, we use the function class $\mathcal{H}(a, b)$ to establish an oscillation criterion for Eq.(1.1) of the Kong-type.

Theorem 2.2. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that (2.2) and (2.3) hold. Assume further that for $i = 1, 2$, there exists $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$ such that*

$$\sup_{\delta \in (m, 1]} \left\{ \frac{1}{H_i(c_i, a_i)} \int_{a_i}^{c_i} [Q(s) H_i(s, a_i) - p(s) |h_{i1}(s, a_i)|^{\gamma+1}] ds \right.$$

$$(2.9) \quad \left. + \frac{1}{H_i(b_i, c_i)} \int_{c_i}^{b_i} [Q(s) H_i(b_i, s) - p(s) |h_{i2}(b_i, s)|^{\gamma+1}] ds \right\} > 0,$$

where $Q(t)$ is defined by (2.5). Then Eq.(1.1) is oscillatory.

Remark 2.2. (i) When $\gamma = 1$, Theorems 2.1 and 2.2 unify and improve Theorems 2.1 and 2.2 in [31].

(ii) For $N \in \mathbb{N}$ and $s \in [0, N + 1]$ we let

$$\zeta(s) = \sum_{j=1}^N \chi(s - j) \quad \text{with} \quad \chi(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0; \end{cases}$$

$\alpha \in C[0, N + 1]$ such that $\alpha(j) = \alpha_j, j = 0, 1, \dots, N$, satisfying (1.3); and $q(t, j) = q_j(t) \in C[0, \infty)$ for $j = 0, 1, \dots, N$. Then by a simple computation, we see that Lemma 2.1 and Theorems 2.1 and 2.2 reduce to Lemma 2.1, and Theorems 2.1 and 2.2 in [15], respectively.

In the following, we state the interval oscillation criteria for Eq.(1.4).

Theorem 2.3. *Assume*

$$g_*(t) := \inf_{s \in [0, b)} \{t, g(t, s)\} \quad \text{and} \quad g^*(t) := \sup_{s \in [0, b)} \{t, g(t, s)\}$$

exist. Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants $a_i, b_i \in [T, \infty)$ with $T \leq a_i < b_i$ such that

$$(2.10) \quad q_0(t) \geq 0 \quad \text{for } t \in [g_*(a_i), g^*(b_i)],$$

$$(2.11) \quad q(t, s) \geq 0 \quad \text{for } (t, s) \in [g_*(a_i), g^*(b_i)] \times [0, b),$$

and

$$(2.12) \quad (-1)^i e(t) \geq 0, \quad \text{for } t \in [g_*(a_i), g^*(b_i)].$$

Assume further that there exists $u \in C^1[a_i, b_i]$ satisfying $u(a_i) = u(b_i) = 0$, $i = 1, 2$, $u(t) \not\equiv 0$ on $[a_i, b_i]$ such that for $i = 1, 2$,

$$\sup_{\delta \in (m, 1]} \int_{a_i}^{b_i} [Q_i(t) |u_i(t)|^{\gamma+1} - p(t) |u'_i(t)|^{\gamma+1}] dt > 0,$$

where

$$(2.13) \quad Q_i(t) := q_0(t) + \left[\frac{|e(t)|}{1 - \delta} \right]^{1-\delta} \exp \left(\int_0^b \eta(s) \ln \frac{q(t, s) \psi_i^{\alpha(s)}(t, s)}{\eta(s)} d\zeta(s) \right),$$

and

$$(2.14) \quad \psi_i(t, s) := \begin{cases} \delta_i(t, s), & g(t, s) < t \\ 1, & g(t, s) = t \\ \zeta_i(t, s), & g(t, s) > t \end{cases}$$

with

$$\delta_i(t, s) := \int_{g(a_i, s)}^{g(t, s)} \frac{ds}{p^{\frac{1}{\gamma}}(s)} \left(\int_{g(a_i, s)}^t \frac{ds}{p^{\frac{1}{\gamma}}(s)} \right)^{-1}$$

and

$$\zeta_i(t, s) := \int_{g(t, s)}^{g(b_i, s)} \frac{ds}{p^{\frac{1}{\gamma}}(s)} \left(\int_t^{g(b_i, s)} \frac{ds}{p^{\frac{1}{\gamma}}(s)} \right)^{-1}$$

and with η_j as in Lemma 2.1 based on δ . Here we use the convention that $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ when $\delta = 1$. Then Eq.(1.4) is oscillatory.

Theorem 2.4. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that (2.10)-(2.12) hold. Assume further that there exist $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$ such that*

$$\sup_{\delta \in (m, 1]} \left\{ \frac{1}{H_i(c_i, a_i)} \int_{a_i}^{c_i} [Q_i(s) H_i(s, a_i) - p(s) |h_{i1}(s, a_i)|^{\gamma+1}] ds + \frac{1}{H_i(b_i, c_i)} \int_{c_i}^{b_i} [Q_i(s) H_i(b_i, s) - p(s) |h_{i2}(b_i, s)|^{\gamma+1}] ds \right\} > 0,$$

where $Q_i(t)$ is defined by (2.13). Then Eq.(1.4) is oscillatory.

3. Proofs

Proof of Lemma 2.1. Let

$$\eta_1(s) := \begin{cases} 0, & s \in (0, a) \\ \gamma \alpha^{-1}(s) \left(\int_a^b d\zeta(s) \right)^{-1}, & s \in [a, b) \end{cases}$$

and

$$\eta_2(s) := \begin{cases} \gamma \alpha^{-1}(s) \left(\int_0^a d\zeta(s) \right)^{-1}, & s \in (0, a) \\ 0, & s \in [a, b). \end{cases}$$

Clearly for $i = 1, 2$, $\eta_i \in L_\zeta(0, b)$ and

$$\int_0^b \alpha(s) \eta_i(s) d\zeta(s) = \gamma.$$

Moreover,

$$\int_0^b \eta_1(s) d\zeta(s) = m \quad \text{and} \quad \int_0^b \eta_2(s) d\zeta(s) = n.$$

For $k \in [0, 1]$ let

$$\eta(s, k) := (1 - k) \eta_1(s) + k \eta_2(s), \quad s \in [0, b).$$

Then it is easy to see that

$$\int_0^b \alpha(s) \eta(s, k) d\zeta(s) = \gamma.$$

Furthermore, since $\eta(s, 0) = \eta_1(s)$ and $\eta(s, 1) = \eta_2(s)$, we have

$$\int_0^b \eta(s, 0) d\zeta(s) = m \quad \text{and} \quad \int_0^b \eta(s, 1) d\zeta(s) = n.$$

By the continuous dependence of $\eta(s, k)$ on k there exists $k^* \in (0, 1)$ such that $\eta(s) := \eta(s, k^*)$ satisfies that

$$\int_0^b \eta(s) d\zeta(s) = \delta.$$

Note that $\eta(s) > 0$ for $s \in [0, b)$ and $\int_0^b \alpha(s) \eta(s) d\zeta(s) = \gamma$. □

The next lemma is a generalized Arithmetic-Geometric mean inequality established in [31].

Lemma 3.1. *Let $u \in C[0, b)$ and $\eta \in L_\zeta(0, b)$ satisfying $u \geq 0$, $\eta > 0$ on $[0, b)$ and $\int_0^b \eta(s) d\zeta(s) = 1$. Then*

$$\int_0^b \eta(s) u(s) d\zeta(s) \geq \exp\left(\int_0^b \eta(s) \ln[u(s)] d\zeta(s)\right),$$

where we use the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

Proof of Theorem 2.1. Assume Eq.(1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, assume $x(t) > 0$ for all $t \geq T \geq 0$, where T depends on the solution $x(t)$. When $x(t)$ is an eventually negative, the proof follows the same way except that the interval $[a_2, b_2]$, instead of $[a_1, b_1]$, is used. Define

$$(3.1) \quad z(t) := \frac{p(t)\phi_\gamma(x'(t))}{\phi_\gamma(x(t))}, \quad t \geq T.$$

It follows from (1.1) that for $t \geq T$, $z(t)$ satisfies the first order nonlinear Riccati equation

$$(3.2) \quad z'(t) = -q_0(t) - \int_0^b q(t, s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) + e(t)x^{-\gamma}(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)}.$$

From the assumption, there exists a nontrivial interval $[a_1, b_1] \subset [T, \infty)$ such that (2.2) and (2.3) hold with $i = 1$.

(I) We first consider the case where the supremum in (2.4) is assumed at $\delta = 1$. From (2.3) and (3.2), we have that for $t \in [a_1, b_1]$

$$(3.3) \quad z'(t) \leq -q_0(t) - \int_0^b q(t, s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)}.$$

Let $\eta \in L_\zeta(0, b)$ be defined as in Lemma 2.1 with $\delta = 1$. Then η satisfies (2.1) with $\delta = 1$. This follows that

$$\int_0^b \eta(s) [\alpha(s) - \gamma] d\zeta = 0.$$

Then, from Lemma 3.1, we get, for $t \in [a_1, b_1]$

$$\begin{aligned} & \int_0^b q(t, s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) \\ &= \int_0^b \eta(s) \frac{q(t, s)}{\eta(s)} [x(t)]^{\alpha(s)-\gamma} d\zeta(s) \\ &\geq \exp\left(\int_0^b \eta(s) \ln\left(\frac{q(t, s)}{\eta(s)} [x(t)]^{\alpha(s)-\gamma}\right) d\zeta(s)\right) \\ &= \exp\left(\int_0^b \eta(s) \ln\left[\frac{q(t, s)}{\eta(s)}\right] d\zeta(s) + \ln(x(t)) \int_0^b \eta(s) [\alpha(s) - \gamma] d\zeta(s)\right) \\ &= \exp\left(\int_0^b \eta(s) \ln\left[\frac{q(t, s)}{\eta(s)}\right] d\zeta(s)\right). \end{aligned}$$

This together with (3.3) shows that

$$(3.4) \quad z'(t) \leq -Q(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} \quad \text{for } t \in [a_1, b_1],$$

where $Q(t)$ is defined by (2.5) with $\delta = 1$. Multiplying both sides of (3.4) by $|u_1(t)|^{\gamma+1}$, integrating from a_1 to b_1 , and using integration by parts, we find that

$$\begin{aligned} (3.5) \quad & \int_{a_1}^{b_1} Q(t) |u_1(t)|^{\gamma+1} dt \\ & \leq \int_{a_1}^{b_1} \left[(\gamma + 1) \phi_\gamma(u_1(t)) u_1'(t) z(t) - \frac{\gamma |u_1(t)|^{\gamma+1}}{p^{\frac{1}{\gamma}}(t)} |z(t)|^{\frac{\gamma+1}{\gamma}} \right] dt \\ & \leq \int_{a_1}^{b_1} \left[(\gamma + 1) |u_1(t)|^\gamma |u_1'(t)| |z(t)| - \frac{\gamma |u_1(t)|^{\gamma+1}}{p^{\frac{1}{\gamma}}(t)} |z(t)|^{\frac{\gamma+1}{\gamma}} \right] dt. \end{aligned}$$

Let $\lambda := \frac{\gamma+1}{\gamma}$. Define A and B by

$$A^\alpha := \frac{\gamma |u_1(t)|^{\gamma+1}}{p^{\frac{1}{\gamma}}(t)} |z(t)|^\lambda \quad \text{and} \quad B^{\alpha-1} := |u_1'(t)| (\gamma p(t))^{\frac{1}{\gamma+1}}.$$

Using the inequality in [12] we have

$$(3.6) \quad \alpha AB^{\alpha-1} - A^\alpha \leq (\alpha - 1) B^\alpha,$$

i.e.,

$$(\gamma + 1) |u_1'(t)| |u_1(t)|^\gamma |z(t)| - \frac{\gamma |u_1(t)|^{\gamma+1}}{p^{\frac{1}{\gamma}}(t)} |z(t)|^\lambda \leq p(t) |u_1'(t)|^{\gamma+1},$$

which together with (3.5) implies that

$$\int_{a_1}^{b_1} Q(t) |u_1(t)|^{\gamma+1} dt \leq \int_{a_1}^{b_1} p(t) |u_1'(t)|^{\gamma+1} dt.$$

This leads to a contradiction to (2.4).

(II) Now, we consider the case where the supremum in (2.4) is assumed at $\delta \in (m, 1)$. Then from (2.3) we see that for $t \in [a_1, b_1]$

$$(3.7) \quad z'(t) = -q_0(t) - \int_0^b q(t, s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) - |e(t)| x^{-\gamma}(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)}.$$

Let $\tilde{\eta}(s) = \delta^{-1} \eta(s)$. Then from (2.1) we have

$$(3.8) \quad \int_0^b \tilde{\eta}(s) d\zeta(s) = 1 \quad \text{and} \quad \int_0^b \tilde{\eta}(s) [\delta\alpha(s) - \gamma] d\zeta(s) = 0.$$

Hence for $t \in [a_1, b_1]$

$$(3.9) \quad \begin{aligned} & \int_0^b q(t, s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) + |e(t)| x^{-\gamma}(t) \\ &= \int_0^b \tilde{\eta}(s) \left(\delta \eta^{-1}(s) q(t, s) [x(t)]^{\alpha(s)-\gamma} + |e(t)| x^{-\gamma}(t) \right) d\zeta(s). \end{aligned}$$

Using the Arithmetic-Geometric mean inequality, see [2, Page 17],

$$ch + dk \geq c^h d^k, \quad \text{where } c, d \geq 0, h, k > 0 \text{ and } h + k = 1,$$

with

$$c = \eta^{-1}(s) q(t, s) [x(t)]^{\alpha(s)-\gamma}, \quad d = \frac{1}{1-\delta} |e(t)| x^{-\gamma}(t), \quad h = \delta \text{ and } k = 1 - \delta,$$

we have that for $t \in [a_1, b_1]$ and $s \in [0, b]$

$$\begin{aligned} & \delta \eta^{-1}(s) q(t, s) [x(t)]^{\alpha(s)-\gamma} + (1-\delta) \frac{|e(t)|}{1-\delta} x^{-\gamma}(t) \\ & \geq \left[\frac{q(t, s)}{\eta(s)} \right]^\delta \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} [x(t)]^{\delta\alpha(s)-\gamma}. \end{aligned}$$

Substituting this into (3.9) and using Lemma 3.1 and (3.8), we see that for $t \in [a_1, b_1]$

$$(3.10) \quad \int_0^b q(t, s) [x(t)]^{\alpha(s)-\gamma} d\zeta(s) + |e(t)| x^{-\gamma}(t)$$

$$\begin{aligned}
 &\geq \exp \left(\int_0^b \tilde{\eta}(s) \ln \left(\left[\frac{q(t, s)}{\eta(s)} \right]^\delta \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} [x(t)]^{\delta\alpha(s)-\gamma} \right) d\zeta(s) \right) \\
 &= \exp \left(\int_0^b \tilde{\eta}(s) \left(\ln \left[\frac{q(t, s)}{\eta(s)} \right]^\delta + \ln \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} + [\delta\alpha(s) - \gamma] \ln x(t) \right) d\zeta(s) \right) \\
 &= \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \exp \left(\int_0^b \eta(s) \ln \frac{q(t, s)}{\eta(s)} d\zeta(s) \right).
 \end{aligned}$$

It follows from (3.7) and (3.10) that for $t \in [a_1, b_1]$

$$\begin{aligned}
 z'(t) &\leq -q_0(t) - \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \exp \left(\int_0^b \eta(s) \ln \frac{q(t, s)}{\eta(s)} d\zeta(s) \right) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} \\
 (3.11) \quad &= -Q(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)},
 \end{aligned}$$

where $Q(t)$ is defined by (2.5) with $\delta \in (m, 1)$. The rest of the proof is similar to Part (I) and hence is omitted. \square

Proof of Theorem 2.2. Assume Eq.(1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, assume $x(t) > 0$ for all $t \geq T \geq 0$, where T depends on the solution $x(t)$. Define $z(t)$ by (3.1). From (3.4) and (3.11), we get that

$$(3.12) \quad z'(t) \leq -Q(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)}.$$

Multiplying both sides of (3.12), with t replaced by s , by $H_1(b_1, s)$ and integrating with respect to s from c_1 to b_1 , we find that

$$\begin{aligned}
 &\int_{c_1}^{b_1} Q(s) H_1(b_1, s) ds \\
 &\leq - \int_{c_1}^{b_1} z'(s) H_1(b_1, s) ds - \int_{c_1}^{b_1} \frac{\gamma |z(s)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(s)} H_1(b_1, s) ds.
 \end{aligned}$$

Using integration by parts and from (2.6) and (2.8), we obtain that

$$\begin{aligned}
 &\int_{c_1}^{b_1} Q(s) H_1(b_1, s) ds \\
 &\leq z(c_1) H_1(b_1, c_1) + \int_{c_1}^{b_1} \left[(\gamma + 1) h_{12}(b_1, s) H_1^{\frac{\gamma}{\gamma+1}}(b_1, s) z(s) \right. \\
 &\quad \left. - \frac{\gamma |z(s)|^{\frac{\gamma+1}{\gamma}} H_1(b_1, s)}{p^{\frac{1}{\gamma}}(s)} \right] ds
 \end{aligned}$$

$$(3.13) \quad \leq z(c_1) H_1(b_1, c_1) + \int_{c_1}^{b_1} \left[(\gamma + 1) |h_{12}(b_1, s)| H_1^{\frac{\gamma}{\gamma+1}}(b_1, s) |z(s)| - \frac{\gamma |z(s)|^{\frac{\gamma+1}{\gamma}} H_1(b_1, s)}{p^{\frac{1}{\gamma}}(s)} \right] ds.$$

Let $\lambda = \frac{\gamma+1}{\gamma}$. Define A and B by

$$A^\alpha := \frac{\gamma |z(s)|^\lambda H_1(b_1, s)}{p^{\frac{1}{\gamma}}(s)} \quad \text{and} \quad B^{\alpha-1} := (\gamma p(s))^{\frac{1}{\gamma+1}} |h_{12}(b_1, s)|.$$

Then, using the inequality (3.6), we get that

$$\begin{aligned} & (\gamma + 1) |h_{12}(b_1, s)| H_1^{\frac{\gamma}{\gamma+1}}(b_1, s) |z(s)| - \frac{\gamma |z(s)|^{\frac{\gamma+1}{\gamma}} H_1(b_1, s)}{p^{\frac{1}{\gamma}}(s)} \\ & \leq p(s) |h_{12}(b_1, s)|^{\gamma+1}. \end{aligned}$$

This together with (3.13) shows that

$$(3.14) \quad \frac{1}{H_1(b_1, c_1)} \int_{c_1}^{b_1} \left[Q(s) H_1(b_1, s) - p(s) |h_{12}(b_1, s)|^{\gamma+1} \right] ds \leq z(c_1).$$

Similarly, multiplying both sides of (3.12), with t replaced by s , by $H_1(s, a_1)$ and integrating by parts from a_1 to c_1 , we see that

$$(3.15) \quad \frac{1}{H_1(c_1, a_1)} \int_{a_1}^{c_1} \left[Q(s) H_1(s, a_1) - p(s) |h_{11}(s, a_1)|^{\gamma+1} \right] ds \leq -z(c_1).$$

Combining (3.14) and (3.15) we get that

$$\begin{aligned} & \frac{1}{H_1(c_1, a_1)} \int_{a_1}^{c_1} \left[Q(s) H_1(s, a_1) - p(s) |h_{11}(s, a_1)|^{\gamma+1} \right] ds \\ & + \frac{1}{H_1(b_1, c_1)} \int_{c_1}^{b_1} \left[Q(s) H_1(b_1, s) - p(s) |h_{12}(b_1, s)|^{\gamma+1} \right] ds \leq 0. \end{aligned}$$

This contradicts (2.9) with $i = 1$. \square

The following lemma, which was established in [15], plays a key role in the proof of the oscillation criteria for Eq.(1.4).

Lemma 3.2. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that (2.10)-(2.12) hold. Assume Eq.(1.4) has a nonoscillatory solution $x(t)$ on $[0, \infty)$. Then for $t \in [a_i, b_i]$ with $i = 1, 2$,*

$$\frac{x(g(t, s))}{x(t)} \geq \psi_i(t, s) \quad \text{for } i = 1, 2,$$

where $\psi_i(t, s)$ is defined by (2.14).

Proof of Theorems 2.3 and 2.4. Without loss of generality, we may assume $x(t), x(g(t, s)) > 0$ for all $s \in [0, b)$ and $t \geq T \geq 0$. Define $z(t)$ by (3.1). Then

$$z'(t) = -q_0(t) - \int_0^b q(t, s) \frac{x^{\alpha(s)}(g(t, s))}{x^\gamma(t)} d\zeta(s) + e(t)x^{-\gamma}(t) - \frac{\gamma |z(t)|^{\frac{\gamma+1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)}.$$

From the assumption, there exist constants a_1 and b_1 with $a_1 < b_1$ and $[g_*(a_1), g^*(b_1)] \subset [t_0, \infty)$ such that (2.10)-(2.12) hold with $i = 1$. Then from Lemma 3.2 we have that for $t \in [a_1, b_1]$ and $j = 0, 1, 2, \dots, N$

$$\frac{x^{\alpha(s)}(g(t, s))}{x^\gamma(t)} \geq \psi_1^{\alpha(s)}(t, s) [x(t)]^{\alpha(s)-\gamma}.$$

The rest of the proof is similar to those of Theorem 2.1 and 2.2, and is hence omitted. \square

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