INTERVAL ESTIMATION FOR THE MEANS OF BINOMIAL, NEGATIVE BINOMIAL, AND TAKACS DISTRIBUTIONS

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Abstract *In this paper, we extend some methods for obtaining the Wald score and likelihood ratio confidence intervals for the mean of the Takacs (generalized negative binomial) distribution. Also, we present these confidence intervals for the mean of the binomial and negative binomial distributions as the special cases of the Takacs distribution. The Takacs distribution is a member of the natural exponential family with cubic variance function (NEF-CVF). The coverage probabilities for these confidence intervals are estimated by means of a simulation study. The results show that the score and likelihood ratio intervals are better than Wald interval, and the Wald interval has the poorest performance.*

Keywords: *Coverage probability, Cubic variance function (CVF), Interval estimation, Natural exponential family (NEF), Takacs distribution.*

1. INTRODUCTION

Interval estimations for natural exponential family (NEF) are the important topics that discussed in many references such as Santner (1998), Agresti and Coull (1998), Brown et al. (2001, 2002), Cai (2005), Sun et al. (2008), Arefi et al. (2008), and Cai and Wang (2009). In these references we can study some confidence intervals such as: Wald, score, Agresti-Coull, likelihood ratio, and Jefferys. Also Brown et al. (2003) investigated these intervals and their coverage probabilities for natural exponential family with quadratic variance function (NEF-QVF).

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The NEF-QVF contains six distributions as: normal, gamma, generalized hyperbolic secant (GHS), binomial, negative binomial and Poisson. The confidence intervals for the parameters of the binomial distribution have garnered a substantial amount of attention in recent years; among the others, see Reiczigel (2003), Zhou et (2004), Brown and Li (2005), Roths and Tebbs (2006), and Wang (2007). Also, Arefi et al. (2008) studied some confidence intervals for the mean of an inverse Gaussian distribution. This distribution is a member of the natural exponential family.

In present work, some methods are investigated to obtain the confidence intervals for the mean of the binomial, negative binomial, and Takacs distributions. Note that the Takacs distribution is an important member of the NEF, whose variance is a cubic function of the mean μ. We estimate the coverage probabilities for these confidence intervals by means of a simulation study.

This paper is organized as follows: In Section 2, we introduce the NEF. In Section 3, the confidence intervals for the means of the binomial, negative binomial and Takacs distributions are studied. In Section 4, the coverage probabilities of the proposed intervals are estimated by means of a simulation study. In Section 5, we apply the proposed intervals for the mean of a Takacs distribution on a real data set. A brief conclusion is provided in Section 6.

2. NATURAL EXPONENTIAL FAMILY

The natural exponential family (NEF) represents a very important class of distributions in probability and statistical theory. Excellent accounts of exponential family theory are contained in Barndorff-Nielsen (1978) and Letac (1992). Here, we present and study a special case of NEF where the variance of the distribution is a function of the mean.

Let X be a random variable with the probability (density) function $f(x;\xi)$. X is said to have a distribution belonging to NEF if

$$f(x;\xi) = \exp(\xi x - \psi(\xi))h(x) \qquad (1)$$

where, ξ is the natural parameter and $h(x)$ is a real valued (normalized) function of x (see Bickel and Doksum, 1977). It is possible to show that, if $f(x;\xi)$ is as (1), then $\mu = E(X) = \psi'(\xi)$ and $\sigma^2 = Var(X) = \psi''(\xi)$. In a special case, if the variance can be based on the mean μ as

$$\sigma^2 = Var(X) = a_0 + a_1\mu + a_2\mu^2 + a_3\mu^3 \qquad (2)$$

where a_0, a_1, a_2, and a_3 are suitable constants, then it is said to be a “NEF with cubic variance function". Morris (1982, 1983) and Brown (1986) studied some properties for the NEF with quadratic variance function ($a_3 = 0$ in (2)). Normal, gamma, generalized hyperbolic secant (GHS), binomial, negative binomial, and Poisson distributions are the members of the natural exponential family with quadratic variance function (NEF-QVF).

Also, Letac and Mora (1990) studied some properties for the natural exponential family with cubic variance function (NEF-CVF). Ressel, inverse Gaussian, Abel, Takacs, strict arcsine, and large arcsine distributions are the members of NEF-CVF. In this paper, we study some confidence intervals for the mean of the binomial, negative binomial, and Takacs distributions. Hence, we list some necessary facts for these distributions as follows.

- **Binomial**: A binomial (Bernoulli) distribution $Bin(1,p)$ has the probability mass function

  $$f(x) = p^x(1-p)^{1-x} \qquad x = 0,1 \qquad 0 < p < 1.$$

  Here, $\mu = E(X) = p$ and $Var(X) = p(1-p) = \mu - \mu^2$. Hence, $a_0 = a_3 = 0$, $a_1 = 1$, and $a_2 = -1$.

- **Negative Binomial**: A negative binomial distribution $NBin(1,p)$ has the probability mass function

  $$f(x) = p^x(1-p) \qquad x = 0,1,... \qquad 0 < p < 1.$$

  In this case, $\mu = \frac{p}{1-p}$ and $Var(X) = \frac{p}{(1-p)^2} = \mu + \mu^2$. Hence, $a_0 = a_3 = 0$ and $a_1 = a_2 = 1$.

- **Takacs** (generalized negative binomial): A random variable X has a Takacs distribution $GNB(1,m,p)$, if its probability mass function is

  $$f(x) = \frac{1}{1+mx}\binom{1+mx}{x} p^x(1-p)^{1+(m-1)x} \qquad x = 0,1,2,... \tag{3}$$

  where, $0 < p < 1$ and $0 \leq m < p^{-1}$. In this case, $\mu = \frac{p}{1-mp}$ and $Var(X) = \mu(1+m\mu)(1+(m-1)\mu)$. Hence, $a_0 = 0$, $a_1 = 1$, $a_2 = 2m-1$, and $a_3 = m(m-1)$.

Note that, the binomial and negative binomial distributions are the special cases of the Takacs distribution with $m = 0$ and $m = 1$ in (3), respectively.

3. CONFIDENCE INTERVALS FOR THE MEANS OF BINOMIAL, NEGATIVE BINOMIAL, AND TAKACS DISTRIBUTIONS

Let $X_1, X_2, ..., X_n$ be a random sample of size n from $f(x;\xi)$ given by (1). The maximum likelihood estimation (MLE) for the mean μ is the sample mean $\hat{\mu} = \bar{X}$. Now, we want to obtain some confidence intervals for μ, at the confidence level $1-\alpha$ (for more detail about these confidence intervals, see Brown et al., 2003 and Arefi et al., 2008).

Definition 1 *The* **Wald interval** (CI_W) *is based on Slutsky's statistic* $W_n = \frac{n^{1/2}(\hat{\mu}-\mu)}{\hat{\sigma}}$ *as follows:*

$$CI_W : \widehat{\mu} \pm k\widehat{\sigma}n^{-1/2} = \widehat{\mu} \pm k(a_0 + a_1\widehat{\mu} + a_2\widehat{\mu}^2 + a_3\widehat{\mu}^3)^{1/2}n^{-1/2}$$

where $k = z_{1-\alpha/2}$*. Note that, Slutsky's statistic converges to* $N(0,1)$ *in distribution (see Casella and Berger, 2002).*

Definition 2 *The* **score interval** (CI_S) *is based on the central limit theorem with the statistic* $Z = \frac{n^{1/2}(\widehat{\mu}-\mu)}{\sigma}$ *as follows:*

$$-k \leq \frac{n^{1/2}(\widehat{\mu}-\mu)}{(a_0+a_1\mu+a_2\mu^2+a_3\mu^3)^{1/2}} \leq k \qquad (4)$$

where, $k = z_{1-\alpha/2}$*. Note that, this interval is obtained by inverting Rao's equal-tailed score test of* $H_0 : \mu = \mu_0$*.*

Remark 1 *The score interval in* (4) *for the means of the binomial and negative binomial distributions is summarized as follows:*

$$CI_S : \frac{n\widehat{\mu} + k^2/2}{n - k^2a_2} \pm \frac{kn^{1/2}}{n - k^2a_2}\left(\widehat{\mu} + a_2\widehat{\mu}^2 + \frac{k^2}{4n}\right)^{1/2}.$$

Remark 2 *For calculating the score interval for the mean of the Takacs distribution, we should solve some cubic equations in terms of* μ*. The theoretical approach for calculating the score interval is difficult. Hence, we use LINGO software for necessary calculations.*

Definition 3 *Let* $\Lambda_n = \frac{L(\mu_0)}{\sup_\mu L(\mu)}$ *be the likelihood ratio. The* **likelihood ratio interval** (CI_{LR}) *is obtained by solving the equation* $-2\log\Lambda_n \leq \chi^2_{\alpha,1} = k^2$ *in terms of* μ*. It is obtained by inverting the likelihood ratio test under* $H_0 : \mu = \mu_0$ *(for more detail, see Rao, 1973 and Serfling, 1980).*

In the following, we study these confidence intervals for the binomial, negative binomial, and Takacs distributions.

- **Binomial:** Let $X_1, X_2, ..., X_n$ be a random sample of size n from a $Bin(1,p)$. The above intervals for the mean p are calculated as follows ($\widehat{p} = \widehat{\mu} = \overline{X}$ and $\widehat{q} = 1 - \widehat{p}$):

  i) Wald interval: This confidence interval, at the confidence level $1-\alpha$, is obtained as

  $$CI_W : \widehat{\mu} \pm k(\widehat{\mu}(1-\widehat{\mu}))^{1/2} n^{-1/2}.$$

  ii) Score interval: We calculate the score interval, at the confidence level $1-\alpha$, as follows

  $$CI_S : \frac{n\widehat{\mu} + k^2/2}{n+k^2} \pm \frac{kn^{1/2}}{n+k^2}(\widehat{\mu}(1-\widehat{\mu}) + \frac{k^2}{4n})^{1/2}.$$

  iii) Likelihood ratio interval: The likelihood ratio is given by

  $$\Lambda_n = \left(\frac{\mu}{\widehat{\mu}}\right)^{n\widehat{\mu}} \left(\frac{1-\mu}{1-\widehat{\mu}}\right)^{n(1-\widehat{\mu})}.$$

  The equation $-2\log\Lambda_n = k^2$ in terms of μ has two roots. These roots are the limits of the likelihood ratio interval. By substituting $t = \frac{\mu}{\widehat{\mu}} - 1$ in the equation $-2\log\Lambda_n = k^2$ and using Maclaurin series, we obtain the following relation

  $$\frac{1}{2(1-\widehat{\mu})}t^2 + \frac{2\widehat{\mu}-1}{3(1-\widehat{\mu})^2}t^3 - \frac{1-3\widehat{\mu}(1-\widehat{\mu})}{4(1-\widehat{\mu})^3}t^4 - \frac{k^2}{2n\widehat{\mu}} = O(n^{-2}).$$

  If we solve this equation in terms of b_0, b_1, b_2, and b_3 with $t = b_0 + b_1 n^{-1/2} \pm kb_2 n^{-1} + b_3 n^{-3/2}$, then we have:

  $$b_0 = 0, \qquad b_1 = \pm k\left(\frac{(1-\widehat{\mu})}{\widehat{\mu}}\right)^{1/2},$$

  $$b_2 = \pm\frac{k(1-2\widehat{\mu})}{3\widehat{\mu}}, \qquad b_3 = \pm\frac{k^3(1-13\widehat{\mu}(1-\widehat{\mu}))}{\widehat{\mu}^{3/2}(1-\widehat{\mu})^{1/2}}.$$

  So the roots in term of t are approximated as follows:

  $$\begin{cases} \underline{t} = b_0 + b_1 n^{-1/2} - kb_2 n^{-1} + b_3 n^{-3/2} + O(n^{-2}), \\ \overline{t} = b_0 + b_1 n^{-1/2} + kb_2 n^{-1} + b_3 n^{-3/2} + O(n^{-2}). \end{cases}$$

Thus, the likelihood ratio interval for μ, at the confidence level $1-\alpha$, is

$$\widehat{\mu}(\underline{t}+1) \leq p \leq \widehat{\mu}(\overline{t}+1).$$

- **Negative binomial:** Consider a random sample of size n from a negative binomial distribution $NBin(1,p)$. The different intervals for the mean $\mu = \frac{p}{q}$, at the confidence level $1-\alpha$, are calculated as follows

  i*) Wald interval: This confidence interval is obtained as follows

$$CI_W : \widehat{\mu} \pm k(\widehat{\mu}+\widehat{\mu}^2)^{1/2} n^{-1/2}.$$

ii*) Score interval: The score interval for μ is calculated as follows

$$CI_S : \frac{n\widehat{\mu}+k^2/2}{n-k^2} \pm \frac{kn^{1/2}}{n-k^2}(\widehat{\mu}(1+\widehat{\mu})+\frac{k^2}{4n})^{1/2}.$$

iii*) Likelihood ratio interval: The likelihood ratio is given by

$$\Lambda_n = \left(\frac{\mu}{\widehat{\mu}}\right)^{n\widehat{\mu}} \left(\frac{1+\widehat{\mu}}{1+\mu}\right)^{n(1+\widehat{\mu})}.$$

The roots of equation $-2\log\Lambda_n = k^2$ are the limits of the likelihood ratio interval. By substituting $t = \frac{\mu}{\widehat{\mu}} - 1$ in the equation $-2\log\Lambda_n = k^2$ and using Maclaurin series, we obtain the following relation

$$\frac{1}{2\widehat{q}}t^2 + \frac{2\widehat{p}-1}{3\widehat{q}^2}t^3 - \frac{1-3\widehat{p}\widehat{q}}{4\widehat{q}^3}t^4 - \frac{k^2}{2n\widehat{p}} = O(n^{-2}),$$

where $\widehat{p} = \frac{\widehat{\mu}}{1+\widehat{\mu}}$ and $\widehat{q} = 1-\widehat{p}$. If we solve this equation in terms of b_0, b_1, b_2, and b_3 with $t = b_0 + b_1 n^{-1/2} \pm kb_2 n^{-1} + b_3 n^{-3/2}$, then we have

$$b_0 = 0, \qquad b_1 = \pm k\left(\frac{1+\widehat{\mu}}{\widehat{\mu}}\right)^{1/2},$$

$$b_2 = \pm\frac{k(1+2\widehat{\mu})}{3\widehat{\mu}}, \qquad b_3 = \pm\frac{k^3(1+13\widehat{\mu}(1+\widehat{\mu}))}{36\widehat{\mu}^{3/2}(1+\widehat{\mu})^{1/2}}.$$

So the roots in term of t are approximated as follows

$$\begin{cases} \underline{t} = b_0 + b_1 n^{-1/2} - kb_2 n^{-1} + b_3 n^{-3/2} + O(n^{-2}), \\ \overline{t} = b_0 + b_1 n^{-1/2} + kb_2 n^{-1} + b_3 n^{-3/2} + O(n^{-2}). \end{cases}$$

Thus, the likelihood ratio interval for μ, at the confidence level $1-\alpha$, is

$$\widehat{\mu}(\underline{t}+1) \leq \mu \leq \widehat{\mu}(\overline{t}+1).$$

- **Takacs** Let $X_1, X_2, ..., X_n$ be a random sample of size n from a $GNB(1,m,p)$, where m is a known parameter. In the following, Wald interval and the likelihood ratio interval for the mean $\mu = \frac{p}{1-mp}$ are calculated. Note that, the score interval for the mean is based on a numerical method (see Remark 2).

i**) Wald interval: This confidence interval for μ, at the confidence level $1-\alpha$, is obtained as

$$CI_W : \widehat{\mu} \pm kn^{-1/2}\widehat{\mu}^{1/2}(1+m\widehat{\mu})^{1/2}(1+(m-1)\widehat{\mu})^{1/2}.$$

ii**) Likelihood ratio interval: The likelihood ratio is given by:

$$\begin{aligned}\Lambda_n &= \left(\frac{\mu(1+m\widehat{\mu})}{\widehat{\mu}(1+m\mu)}\right)^{n\overline{x}} \left(\frac{1+(m-1)\mu}{1+(m-1)\widehat{\mu}} \cdot \frac{1+m\widehat{\mu}}{1+m\mu}\right)^{n(1+(m-1)\overline{x})} \\ &= \left(\frac{p}{\widehat{p}}\right)^{n\overline{x}} (\frac{1-p}{1-\widehat{p}})^{n(1+(m-1)\overline{x})},\end{aligned}$$

where, $\widehat{p} = \frac{\widehat{\mu}}{1+m\widehat{\mu}}$ and $\widehat{q} = 1-\widehat{p}$. By substituting $t = \frac{\mu(1+m\widehat{\mu})}{\widehat{\mu}(1+m\mu)} - 1$, the following function is a convex function of t

$$-\ln\Lambda_n = -n\overline{x}\left[\ln(1+t) + \frac{\hat{q}}{\hat{p}}\ln\left(1-\frac{\hat{p}}{\hat{q}}t\right)\right].$$

Hence, the equation $-2\ln\Lambda_n = k^2$ has two roots. These roots are the limits of the likelihood ratio interval for t. Now, we want to approximate this limits. Based on Maclaurin series of $\ln(1+t) = -\sum_{i=1}^{\infty}\frac{(-t)^i}{i}$, the equation $-2\ln\Lambda_n = k^2$ can be changed as follows:

$$\frac{1}{2\widehat{q}}t^2 + \frac{2\widehat{p}-1}{3\widehat{q}^2}t^3 - \frac{1-3\widehat{p}(1-\widehat{p})}{4\widehat{q}^3}t^4 - \frac{k^2}{2n\widehat{p}} = O(n^{-2}).$$

If we solve this equation based on b_0, b_1, b_2, and b_3 with $t = b_0 + b_1 n^{-1/2} \pm kb_2 n^{-1} + b_3 n^{-3/2}$, then we have

$$b_0 = 0, \qquad b_1 = \pm k\left(\frac{\widehat{q}}{\widehat{p}}\right)^{1/2},$$

$$b_2 = \pm\frac{k(1-2\widehat{p})}{3\widehat{p}}, \qquad b_3 = \pm\frac{k^3(1-13\widehat{p}\widehat{q})}{\widehat{p}^{3/2}\widehat{q}^{1/2}}.$$

So the roots in term of t are approximated as

$$\begin{cases} \underline{t} = -k(\widehat{q}/\widehat{p})^{1/2}n^{-1/2} + \frac{1}{3}k^2(1-2\widehat{p})(n\widehat{p})^{-1} - \frac{1}{36}k^3(1-13\widehat{p}\widehat{q})\widehat{q}^{-1/2}(n\widehat{p})^{-3/2} + O(n^{-2}), \\ \overline{t} = +k(\widehat{q}/\widehat{p})^{1/2}n^{-1/2} + \frac{1}{3}k^2(1-2\widehat{p})(n\widehat{p})^{-1} - \frac{1}{36}k^3(1-13\widehat{p}\widehat{q})\widehat{q}^{-1/2}(n\widehat{p})^{-3/2} + O(n^{-2}). \end{cases}$$

Thus, the likelihood ratio interval for μ, at the confidence level $1-\alpha$, is

$$\frac{\widehat{p}\ (\underline{t}+1)}{1-m\widehat{p}(\underline{t}+1)} \leq \mu \leq \frac{\widehat{p}\ (\overline{t}+1)}{1-m\widehat{p}(\overline{t}+1)}.$$

Remark 3 *If m is an unknown parameter of the Takacs distribution, then a point estimator should be substituted instead of m in the above relations (for example, the moment estimator for m is obtained as* $\widehat{m} = \frac{1}{2}\left(\sqrt{1+\frac{4S^2}{\overline{X}^3}}+1\right) - \frac{1}{\overline{X}}$, *where* $s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$*).*

4. ESTIMATED COVERAGE PROBABILITY VIA A SIMULATION STUDY

In this section, the coverage probability for the confidence intervals is estimated by means of a simulation study.

Definition 4 *let* $[L(\underline{x}), U(\underline{x})]$ *be a confidence interval for the mean* μ, *at the confidence level* $1-\alpha$, *and based on a random sample of size n. The process (calculating the confidence interval) is repeated with w repetitions. The estimated coverage probability (ECP) is defined as the proportion of confidence intervals containing* μ.

We simulate the ECP based on $w = 10000$ repetitions, and at the confidence level $1-\alpha = 0.95$. Some results of ECP for the proposed intervals are shown in Figures 1-3 based on $n = 5, 6, \ldots, 200$ and the fixed parameters p ($p = 0.2$ in binomial distribution, $p = 0.75$ in negative binomial distribution, and $p = 0.25$ in Takacs distribution). Also, the results of ECP are shown in Figures 4-6 for the different values of p, and the fixed values of the size of random sample. See also Tables 1-4 for the other results of ECP based on some parameters of the binomial, negative binomial, and Takacs distributions.

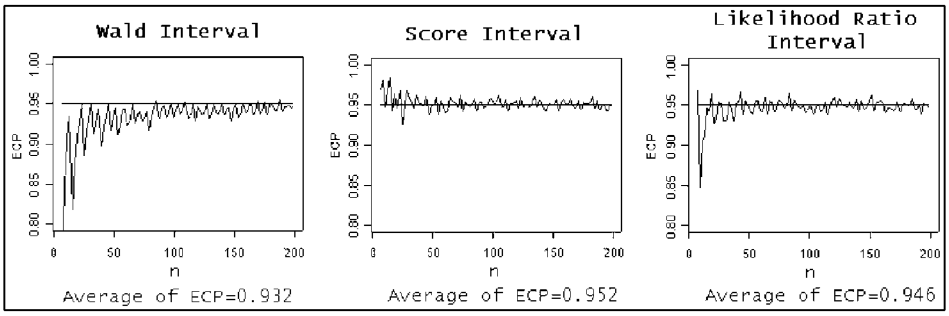

Figure 1: ECP in binomial distribution with *p* = 0.2, and *n* from 5 to 200.

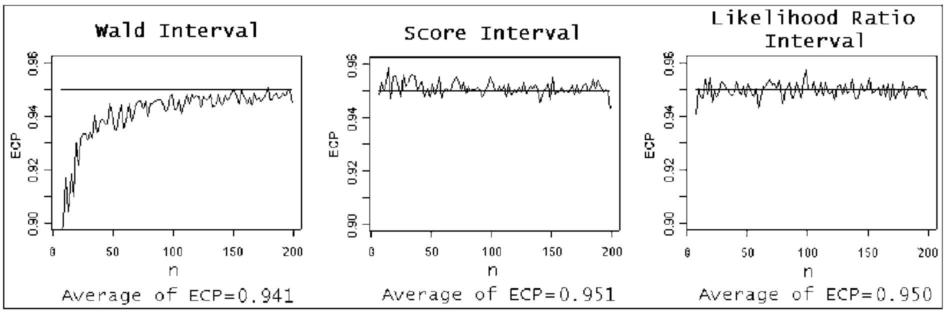

Figure 2: ECP in negative binomial distribution with *p* = 0.75, and *n* from 5 to 200.

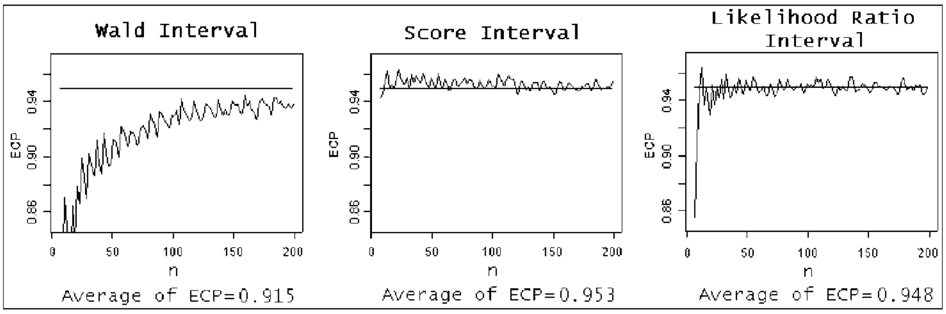

Figure 3: ECP in Takacs distribution with *(m, p)* = (2, 0.25), and *n* from 5 to 200.

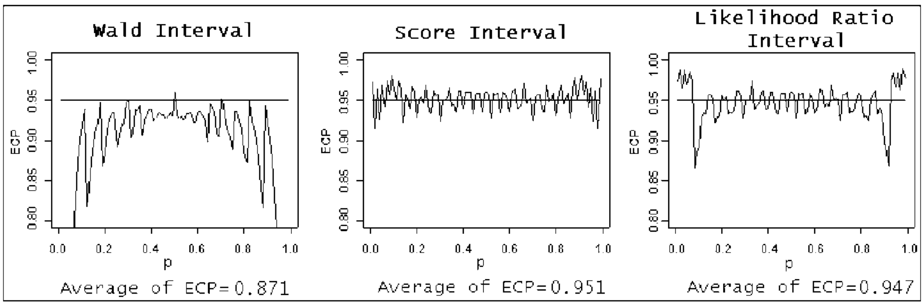

Figure 4: ECP in binomial distribution with *n* = 25, and *p* = 0.01, 0.02, …, 0.99.

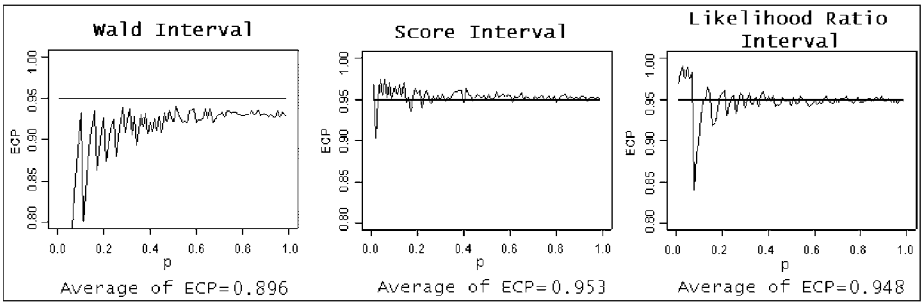

Figure 5: ECP in negative binomial distribution with *n* = 25, and *p* = 0.01, 0.02, …, 0.99.

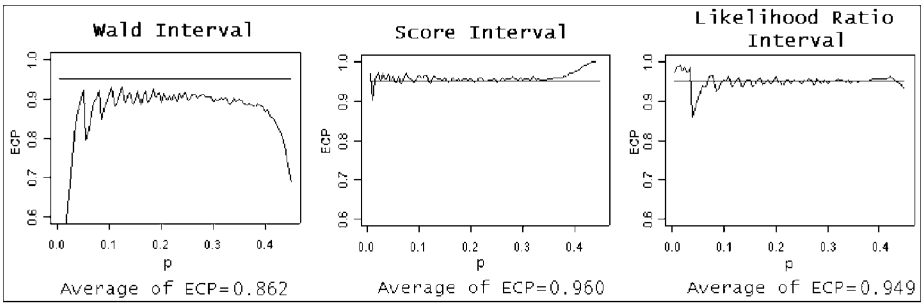

Figure 6: ECP in Takacs distribution with the values *n* = 150, *m* = 0.95, and *p* = 0.005, 0.010, …, 0.490.

Table 1: Some ECPs for the mean of binomial distribution

		$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
$n=25$	ECP_W	0.9159	0.8813	0.9502	0.9411	0.9538	0.9320	0.9463	0.8833	0.9218
	ECP_S	0.9670	0.9242	0.9249	0.9411	0.9538	0.9320	0.9218	0.9258	0.9650
	ECP_{LR}	0.8916	0.9542	0.9502	0.9411	0.9538	0.9320	0.9463	0.9564	0.8971
$n=50$	ECP_W	0.8762	0.9395	0.9370	0.9405	0.9358	0.9429	0.9340	0.9386	0.8798
	ECP_S	0.9650	0.9516	0.9574	0.9405	0.9358	0.9429	0.9572	0.9543	0.9705
	ECP_{LR}	0.9347	0.9516	0.9574	0.9405	0.9358	0.9429	0.9572	0.9543	0.9417
$n=100$	ECP_W	0.9361	0.9328	0.9494	0.9481	0.9418	0.9488	0.9497	0.9353	0.9319
	ECP_S	0.9372	0.9405	0.9388	0.9481	0.9418	0.9488	0.9382	0.9419	0.9372
	ECP_{LR}	0.9579	0.9534	0.9494	0.9481	0.9418	0.9488	0.9497	0.9560	0.9557
$n=200$	ECP_W	0.9288	0.9431	0.9468	0.9493	0.9399	0.9548	0.9444	0.9396	0.9260
	ECP_S	0.9577	0.9617	0.9511	0.9493	0.9399	0.9548	0.9432	0.9577	0.9582
	ECP_{LR}	0.9425	0.9504	0.9511	0.9493	0.9399	0.9548	0.9432	0.9475	0.9427

Table 2: Some ECPs for the mean of negative binomial distribution

		$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=0.9$
$n=25$	ECP_W	0.9281	0.9298	0.9169	0.9280	0.9213	0.9185	0.9329	0.9282	0.9297
	ECP_S	0.9718	0.9537	0.9527	0.9468	0.9504	0.9484	0.9493	0.9460	0.9516
	ECP_{LR}	0.9002	0.9573	0.9399	0.9563	0.9414	0.9435	0.9495	0.9437	0.9522
$n=50$	ECP_W	0.9040	0.9445	0.9304	0.9323	0.9398	0.9360	0.9398	0.9410	0.9413
	ECP_S	0.9599	0.9587	0.9530	0.9500	0.9530	0.9498	0.9510	0.9493	0.9521
	ECP_{LR}	0.9540	0.9549	0.9515	0.9484	0.9478	0.9492	0.9497	0.9483	0.9489
$n=100$	ECP_W	0.9099	0.9378	0.9418	0.9498	0.9378	0.9416	0.9453	0.9470	0.9469
	ECP_S	0.9395	0.9420	0.9524	0.9514	0.9504	0.9521	0.9490	0.9523	0.9540
	ECP_{LR}	0.9329	0.9374	0.9505	0.9506	0.9470	0.9522	0.9482	0.9494	0.9526
$n=200$	ECP_W	0.9341	0.9351	0.9444	0.9428	0.9501	0.9456	0.9486	0.9448	0.9476
	ECP_S	0.9448	0.9496	0.9487	0.9492	0.9547	0.9504	0.9473	0.9496	0.9525
	ECP_{LR}	0.9418	0.9492	0.9480	0.9524	0.9547	0.9513	0.9489	0.9497	0.9520

Table 3: Some ECPs for the mean of Takacs binomial distribution with *m*=2 (0<*p*< $\frac{1}{m}$).

		$p=0.05$	$p=0.10$	$p=0.15$	$p=0.2$	$p=0.25$	$p=0.30$	$p=0.35$	$p=0.40$	$p=0.45$
$n=25$	ECP_W	0.7187	0.7639	0.8315	0.8902	0.8977	0.8656	0.8704	0.8213	0.6918
	ECP_S	0.9310	0.9671	0.9625	0.9558	0.9581	0.9559	0.9556	0.9664	0.9998
	ECP_{LR}	0.9761	0.9086	0.9699	0.9345	0.9501	0.9563	0.9490	0.9518	0.9444
n=50	ECP_W	0.9229	0.9191	0.9064	0.8877	0.9016	0.8981	0.8949	0.8635	0.6854
	ECP_S	0.9657	0.9621	0.9573	0.9549	0.9504	0.9540	0.9581	0.9737	0.9843
	ECP_{LR}	0.9089	0.9526	0.9529	0.9582	0.9436	0.9510	0.9487	0.9539	0.9277
n=100	ECP_W	0.8902	0.9369	0.9427	0.9270	0.9216	0.9269	0.9123	0.8951	0.6039
	ECP_S	0.9572	0.9487	0.9526	0.9559	0.9541	0.9565	0.9566	0.9749	0.9948
	ECP_{LR}	0.9441	0.9432	0.9454	0.9533	0.9417	0.9543	0.9465	0.9571	0.8833
n=200	ECP_W	0.9031	0.9251	0.9412	0.9369	0.9456	0.9363	0.9332	0.9089	0.4248
	ECP_S	0.9501	0.9467	0.9512	0.9553	0.9505	0.9511	0.9523	0.9731	0.9313
	ECP_{LR}	0.9631	0.9451	0.9488	0.9475	0.9497	0.9465	0.9492	0.9576	0.7311

Table 4: Some ECPs for the mean of Takacs binomial distribution with *m*=5 (0<*p*< $\frac{1}{m}$)

		$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$
$n=25$	ECP_W	0.9365	0.9185	0.9365	0.9145	0.9360	0.9295	0.9215	0.8600
	ECP_S	0.9425	0.9715	0.9445	0.9610	0.9415	0.9365	0.9390	0.8945
	ECP_{LR}	0.9200	0.9550	0.9580	0.9405	0.9415	0.9365	0.9390	0.8945
n=50	ECP_W	0.8946	0.9219	0.9266	0.9233	0.9389	0.9324	0.9155	0.8622
	ECP_S	0.9552	0.9556	0.9466	0.9500	0.9578	0.9433	0.9241	0.8903
	ECP_{LR}	0.9509	0.9516	0.9444	0.9336	0.9469	0.9433	0.9241	0.8903
n=100	ECP_W	0.9456	0.9448	0.9419	0.9378	0.9455	0.9419	0.9209	0.8422
	ECP_S	0.9408	0.9455	0.9499	0.9515	0.9472	0.9488	0.9258	0.8634
	ECP_{LR}	0.9567	0.9455	0.9480	0.9438	0.9455	0.9488	0.9258	0.8422
n=200	ECP_W	0.9525	0.9417	0.9438	0.9486	0.9446	0.9405	0.9184	0.7664
	ECP_S	0.9556	0.9521	0.9483	0.9424	0.9492	0.9371	0.9184	0.7888
	ECP_{LR}	0.9556	0.9503	0.9554	0.9485	0.9499	0.9371	0.9184	0.7888

Based on the above results (Fig. 1-6 and Tables 1-4), we can infer the following information for the estimated coverage probability (ECP) in binomial, negative binomial, and Takacs distributions.

- The ECP of the proposed intervals are organized as $ECP_W \leq ECP_{LR} \leq ECP_S$.
- In binomial distribution, when the parameter p goes to 0.5, the ECP of Wald interval increases. Also, when the parameter p goes to 0 or 1, the ECP of the score and likelihood ratio intervals increase (see Fig. 4 and Table 1).
- In negative binomial distribution, when the parameter p goes to 1, the ECP of Wald interval increases. Also, when the parameter p goes to 0, the ECP of score and likelihood ratio intervals increase. (See Fig. 5 and Table 2).
- In Takacs distribution with the known parameter m, the ECP of Wald interval increases for a p about median of parameter space ($p \approx \min\{\frac{1}{2m}, 0.5\}$). Also, when the parameter p goes to 0, the ECP of score and likelihood ratio intervals increase. (See Fig. 6 and Tables 3-4).
- When the size of random sample increases, the ECP of the proposed intervals increases.

5. NUMERICAL ILLUSTRATION

The following data is provided by counts of the number of European red mites on apple leaves (see Bliss and Fisher, 1953). On a day (July 18, 1951), a random sample of 25 leaves were selected from each of 6 McIntosh trees in a single orchard, and the number of adult females counted on each leaf. The frequency distribution of mites on the 150 leaves is given in the first two columns of Table 5.

The MLE of μ is $\widehat{\mu} = \bar{x} = \frac{172}{150}$. Also, based on Remark 3, the estimation of m is $\widehat{m} = 0.95$. Based on the goodness of fit test with the statistic $Y = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i} = 2.7957$ and $Y < \chi^2_{0.95,3} = 7.81$, we can fit a Takacs distribution $GNB(1, \widehat{m} = 0.95, \widehat{p} = 0.5488)$ to this data set.

Now, the confidence intervals for the mean μ, at the confidence level $1 - \alpha = 0.95$, are obtained as follows:

$$\begin{cases} CI_W = [0.9062, 1.3872], \\ CI_S = [0.9385, 1.4298], \\ CI_{LR} = [0.9185, 1.3939]. \end{cases}$$

The lengths of these intervals are obtained as follows:

$$L_{LR} = 0.4754 < L_W = 0.4810 < L_S = 0.4913.$$

Suppose that, $\mu = \frac{172}{150}$ ($p = 0.5488$) and $\beta = 0.95$. Based on the random samples of size $n = 150$, the estimated coverage probabilities for Wald, score, and LR intervals are obtained as follows

$$ECP_W = 0.9472, \quad ECP_{LR} = 0.9480, \quad ECP_S = 0.9540.$$

Based on the above results, the score and likelihood ratio intervals are better than Wald interval. The likelihood ratio interval has the smallest lengths, and the score interval has the most ECPs.

Table 5: Fitting a Takacs model to counts of red mites on apple leaves

No. of mites per leaf x_i	Number of leaves observed f_i	Probability p_i	Expected frequency e_i	$\frac{(f_i-e_i)^2}{e_i}$
0	70	0.4512	67.680	0.0795
1	38	0.2577	38.655	0.0111
2	17	0.1398	20.970	0.7516
3	10	0.0739	11.085	0.1062
4	9	0.0383	5.745	1.8442
5	3	(≥ 5) 0.0391	5.865	0.0031
6	2			
7	1			
Total	$n = 150$	1	150	$Y = 2.7957$

6. CONCLUSIONS

In this paper, we present some confidence intervals (Wald, score, and likelihood ratio intervals) for the mean of the binomial, negative binomial, and Takacs distributions. Also, the estimated coverage probability (ECP) of these intervals is provided by means of a simulation study. The simulation study shows that the ECP of the proposed intervals is organized as $ECP_W \leq ECP_{LR} \leq ECP_S$. Hence, the Wald interval has the lowest ECP and the score interval has the upper ECP.

The results of the confidence intervals for the other distributions of NEF with cubic variance function will be studied in further works.

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