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INTERVAL EXCHANGE TRANSFORMATIONS AND SOME SPECIAL FLOWS ARE NOT MIXING

BY

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ABSTRACT

An interval exchange transformation (I.E.T.) is a map of an interval into itself which is one-to-one and continuous except for a finite set of points and preserves Lebesgue measure. We prove that any I.E.T. is not mixing with respect to any Borel invariant measure. The same is true for any special flow constructed by any I.E.T. and any "roof" function of bounded variation. As an application of the last result we deduce that in any polygon with the angles commensurable with π the billiard flow is not mixing on two-dimensional invariant manifolds.

§1. An *interval exchange transformation* (I.E.T.) is a map of an interval $I = [a, b]$ into itself which is one-to-one and continuous except for a finite set of points and preserves Lebesgue measure. It is easy to see that every I.E.T. f can be represented in the following form: there exist a positive integer m and numbers a_i, σ_i, ε_i, $i = 1, \cdots, m$, $a = a_0 < a_1 < a_2 < \cdots < a_{m-1} < a_m = b$, $\varepsilon_i = \pm 1$ such that

$$f(x) = \varepsilon_i x + \sigma_i \quad \text{if } a_{i-1} < x < a_i \qquad (i = 1, \cdots, m). \tag{1}$$

We will not be concerned about any particular definition of f at the points a_i because from the measure theoretical point of view this is unimportant. Thus, every subinterval $I_i = (a_{i-1}, a_i)$ moves as a rigid body without stretches, squeezes and breaks. This visual representation is responsible for the term "Interval exchange transformation." If m is the minimal positive integer such that an I.E.T. f has representation (1) we shall say that f is an *exchange transformation of m intervals*.

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Although an I.E.T. may have invariant measures different from Lebesgue measure, the following simple lemma shows that from the point of view of ergodic theory the general case can be reduced to that particular case.

LEMMA 1. *Let $f: I \to I$ be an exchange transformation of m intervals and μ a Borel non-atomic probability measure invariant with respect to f. Then there exists an exchange transformation of $r \leqq m$ intervals $g:[0,1] \to [0,1]$ such that f considered as an automorphism of the measure space (I, μ) is metrically isomorphic to g with respect to Lebesgue measure λ. Moreover, this metric isomorphism can be effected by a monotone function $R: I \to [0,1]$.*

PROOF. Let us define a map $R: I \to [0,1]$ in the following way:

(2) $$R(x) = \mu([a,x]), \qquad x \in I.$$

Since μ is a non-atomic measure the map R is continuous and surjective. Obviously R is monotone and $R_*\mu = \lambda$. In general, R is not bijective, but, nevertheless it is an isomorphism between the measure spaces (I, μ) and $([0,1], \lambda)$. Let us define a map $g:[0,1] \to [0,1]$ by

$$g(x) = R(f(y))$$

where $y \in I$ is any point such that $Ry = x$. The map g is well defined except possibly for a finite set of points. For, the set $R^{-1}x$ is either a point or an interval. In the first case the choice of y is unqiue. In the second case $fR^{-1}x$ is a union of a finite number of intervals. Actually, for all but a finite number of points x (in fact, for not more than m points) this set may contain only one interval. In this case $RfR^{-1}x$ may be a point or an interval. But $\lambda(RfR^{-1}x) = \mu(fR^{-1}x) = \mu(R^{-1}x) = \lambda(\{x\}) = 0$. Therefore, $RfR^{-1}c$ is a point.

The map g preserves Lebesgue measure. If $R^{-1}x$ does not contain any points of discontinuity of f, then g is continuous at x. Thus, g is continuous and one-to-one except for $r \leqq m$ points and preserves Lebesgue measure. Consequently, g is an exchange map of at most r intervals. □

The following lemma plays a key role in the subsequent proofs as well as in many other considerations concerning interval exchange transformations.

LEMMA 2. *Let $f: I \to I$ be an exchange transformation of m intervals and let $\Delta \subset I$ be a subinterval. Then the induced map F_Δ is an exchange transformation of $r \leqq m+1$ intervals. Moreover, there exists a decomposition*

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_s, \qquad r \leqq s \leqq m+1$$

into disjoint subintervals and positive integers $t_1, \cdots, t_s$ such that for $x \in \Delta_i$

$$f_\Delta(x) = f^{t_i}(x)$$

with f continuous on every interval

$$f^k \Delta_i, \qquad k = 0, \cdots, t_i - 1.$$

PROOF. Let us denote by Π the set of points of discontinuity of f plus the endpoints of the interval Δ. The set Π contains at most $m+1$ points. Let

$$\Omega = \{x \in \Delta : f_\Delta x = f^t x \text{ and } f^k_x \not\in \Pi,\ k = 0, \cdots, t-1,\ t = 1, 2, \cdots\}.$$

Obviously, Ω is an open set. Since Ω contains all but a finite number of points for which the induced map f_Δ is defined, $\lambda(\Delta \setminus \Omega) = 0$ and, consequently, the set Ω is dense in Δ.

Let Σ be one of the (maximal) intervals forming Ω. Then on the interval Σ, $f_\Delta = f^{t_\Sigma}$,

$$\left(\bigcup_{k=0}^{t_\Sigma - 1} f^k \Sigma \right) \cap \Pi = \varnothing$$

but the endpoints of the intervals

$$f^k \Sigma, \qquad k = 0, \cdots, t_\Sigma - 1$$

hit the set Π at least twice. If not, the interval Σ could be extended within the set Ω. On the other hand, all intervals $f^k\Sigma$, $\Sigma \in \Omega$, $k = 0, \cdots, t_\Sigma - 1$ are disjoint so that every point $p \in \Pi$ may serve as an endpoint for at most two intervals of this form. Thus, the set Ω consists of at most $m+1$ intervals. These intervals are disjoint and the union of their closures coincides with Δ. □

THEOREM 1. *Let $f: I \to I$ be an I.E.T., μ — any Borel probability measure invariant with respect to f. Then f, considered as an automorphism of the measure space (I, μ), is not mixing.*

PROOF. It is enough to consider ergodic measures. Such a measure μ is either periodic (concentrated on a finite set) or non-atomic. In the first case f is obviously non-mixing, in the second case we can apply Lemma 1 and reduce the problem to the case when μ is Lebesgue measure.

Let us fix an interval $\Delta \subset I$. Using Lemma 2 we can represent I (ignoring a finite set of points) as a union of disjoint intervals:

$$I = \bigcup_{i=1}^{s} \bigcup_{n=0}^{t_i - 1} f^n(\Delta_i). \tag{3}$$

Let us denote for brevity

$$f^n(\Delta_i) = \Delta_i^n$$

and the partition of I mod 0

$$\{\Delta_i^n, \quad i = 1, \cdots, s, n = 0, \cdots, t_i - 1\}$$

by ξ_Δ. Since the length of every interval Δ_i^n does not exceed the length of Δ, the partition ξ_Δ may be made arbitrarily fine by choosing Δ sufficiently small. Let us consider now the induced map f_{Δ_i} and use Lemma 2 once more. We have the following representations with some positive integers t_{ij}, $i = 1, \cdots, s_i$, $s_i \leqq m + 1$:

$$\Delta_i = \bigcup_{j=1}^{s_i} \Delta_{i,j} = \bigcup_{j=1}^{s_i} f^{t_{ij}} \Delta_{i,j}, \tag{4}$$

$$I = \bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} \bigcup_{n=0}^{t_i - 1} f^n(\Delta_{ij}),$$

where $f^n \Delta_{ij}$ are disjoint intervals and $s_i \leqq m + 1$ ($i = 1, \cdots, s$). Obviously $t_{ij} \geqq t_i$.

Let us denote

$$f^n(\Delta_{ij}) \text{ by } \Delta_{ij}^n \qquad \text{for } i = 1, \cdots, s, \quad j = 1, \cdots, s_i, \quad n = 0, \cdots, t_i - 1.$$

Let us show that

$$f^{t_{ij}}(\Delta_{ij}^n) \subset \Delta_i^n. \tag{5}$$

Note that $f^{t_{ij}}(\Delta_{ij}^n) = f^n f^{t_{ij}}(\Delta_{ij})$. But $f^{t_{ij}} \Delta_{ij} \subset \Delta_i$ (cf. (4)) and consequently $f^{t_{ij}}(\Delta_{ij}^n) \subset f^n(\Delta_i) = \Delta_i^n$.

We have from (4) and (5),

$$\Delta_i^n = \bigcup_{j=1}^{s_i} \Delta_{i,j}^n,$$

$$\Delta_{i,j}^n \subset f^{-t_{ij}} \Delta_i^n,$$

and

$$\Delta_i^n \subset \bigcup_{j=1}^{s_i} f^{-t_{ij}} \Delta_i^n. \tag{6}$$

Now let A be any set measurable with respect to the partition ξ_Δ. Then by (6)

$$A \subset \bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} f^{-t_{ij}} A$$

and since f is measure-preserving and $s \leqq m + 1$, $s_i \leqq m + 1$ there exists t_{ij} such that

$$\mu(A \cap f^{t_{ij}}A) = \mu(f^{-t_{ij}}A \cap A) > \frac{1}{(m+1)^2}\mu(A). \tag{7}$$

Now let us fix a set A such that

$$\mu(A) < \frac{1}{10(m+1)^2} \tag{8}$$

and a positive integer N.

We can choose an interval $\Delta \subset I$ so small that

(i) there exists a set A_Δ measurable with respect to the partition ξ_Δ and such that

$$\mu(A \Delta A_\Delta) < \frac{1}{10}(\mu(A))^2;$$

(ii) all numbers t_i in decomposition (3) corresponding to the interval Δ are bigger than N.

To fulfill (ii) we take a point $x \in I$ such that f is continuous at the points $f^n x$, $n = 0, 1, \cdots, N-1$. All these points are different. For, suppose that for some positive integers $k, l, k > l$ we have $f^k x = f^l x$. Then the map f^{k-l} is continuous at the fixed point $f^l x$. This together with (1) imply that near that point either $f^{k-l}y = y$ or $f^{k-l}y = -y$. In both cases f has a set of positive measure consisting of periodic points.

Thus, we can find an interval Δ_0 containing the point x such that the intervals $f^n\Delta_0$, $n = 0, \cdots, N-1$ are disjoint. Every subinterval of that interval satisfies (ii). Applying (7) to the set A_Δ we conclude that for some $t_{ij} \geqq t_i > N$

$$\begin{aligned}\mu(A \cap f^{t_{ij}}A) &> \mu(A_\Delta \cap f^{t_{ij}}A_\Delta) - 2\mu(A \Delta A_\Delta)\\ &\geqq \frac{1}{(m+1)^2}\mu(A_\Delta) - \frac{1}{5}(\mu(A))^2.\end{aligned}$$

Since $\mu(A_\Delta) > \frac{9}{10}\mu(A)$ we have from (8)

$$\begin{aligned}\mu(A \cap f^{t_{ij}}A) &> \left(\frac{9}{10}\right)^2 \frac{1}{(m+1)^2}\mu(A) - \frac{1}{5}(\mu(A))^2\\ &> (\mu(A))^2\left(\frac{10\cdot 9^2}{10^2} - \frac{1}{5}\right) > 2(\mu(A))^2.\end{aligned}$$

Thus, f is not mixing. Moreover, since A is an arbitrary set satisfying (8) f cannot even have any mixing factors. □

§2. Now let us consider the special flow $\{f_t^h\}$ constructed by an I.E.T. $f: I \to I$ and "roof" function $h: I \to \mathbf{R}^+$. This flow acts on the space

$$I_{h(\cdot)} = \{(x,t) \in I \times \mathbf{R},\ 0 \leqq t \leqq h(x)\}$$

by uniform "vertical" motion with jumps from the point $(x, h(x))$ to $(f(x), 0)$. If the roof function is bounded then every finite f-invariant measure μ generates a finite invariant measure $\mu_{h(\cdot)}$ for the special flow. Namely, $\mu_{h(\cdot)}$ is the restriction of the direct product $\mu \times \lambda$ (λ is the Lebesgue measure on $\mathbf{R}$) to the space $I_{h(\cdot)}$.

If the roof function is bounded away from zero then every finite invariant measure for the special flow has the described form.

THEOREM 2. *Let $f : I \to I$ be an interval exchange transformation, h a positive function on I of bounded variation, ν a Borel probability measure invariant with respect to the special flow $\{f_t^h\}$. Then the flow $\{f_t^h\}$ is not mixing with respect to ν.*

PROOF. We are going to combine the method from §1 with an idea used by A. Kočergin in [1] in the proof of a similar result for special flows over irrational rotations of the circle.

As above we can assume that $\{f_t^h\}$ is ergodic with respect to ν. Then taking (if necessary) a smaller interval $I' \subset I$ we can represent the flow $\{f_t^h\}$ as a special flow over the induced I.E.T. $f_{I'}$ with the roof function $h' > 1$. For $x \in I'$ let

$$f_{I'}x = f^{n(x)}x.$$

Then

$$h'(x) = \sum_{i=0}^{n(x)-1} h(f^i x)$$

so that h' is also a function of bounded variation. So we can assume from the beginning that $h > 1$. Consequently the measure ν has the form $\nu = \mu_{h(\cdot)}$ for some finite Borel ergodic f-invariant measure μ. If the measure μ is discrete the flow $\{f_t^h\}$ is periodic, so we can assume that the measure μ is non-atomic. Then we can use Lemma 1 and conjugate f with another I.E.T. g such that the measure μ goes to Lebesgue measure. This conjugation can be lifted in an obvious manner to a conjugation between the flow $\{f_t^h\}$ and a special flow over g. Since the map R given by (2) is monotone the new roof function also has bounded variation.

Summarizing, we have reduced the general case to a situation when $h > 1$, the measure ν has a form $\lambda_{h(\cdot)}$ where λ is Lebesgue measure on I and the I.E.T. f is ergodic with respect to λ.

Let us fix an interval $\Delta \subset I$ and make all the constructions described in §1. Let $x, y \in \Delta_{i,j}^n$. We want to compare the time of first return of the points $(x, 0)$ and $(y, 0)$ to the set $\Delta_i^n \times \{0\}$. These two times T_x and T_y are given by

$$T_x = \sum_{k=0}^{t_{ij}-1} h(f^k x)$$

and

$$T_y = \sum_{k=0}^{t_{ij}-1} h(f^k y).$$

For $k = 0, \cdots, t_{ij} - n - 1$ the points $f^k x$ and $f^k y$ belong to the interval $f^{n+k}\Delta_{i,j}$. These intervals are pairwise disjoint. For $k = t_{ij} - n, \cdots, t_{ij} - 1$ the points $f^n(x)$ and $f^n(y)$ belong to the interval $\Delta_i^{k+n-t_{ij}}$ which are also disjoint. Therefore

$$\begin{aligned} |T_x - T_y| &= \left| \sum_{k=0}^{t_{ij}-1} h(f^k x) - h(f^k y) \right| \\ &\leqq \sum_{k=0}^{t_{ij}-n-1} |h(f^k x) - h(f^k y)| + \sum_{k=t_{ij}-n-1}^{t_{ij}-1} |h(f^k x) - h(f^k y)| \qquad (9) \\ &\leqq 2 \operatorname{Var} h. \end{aligned}$$

Now, let $x \in \Delta_{i,j}$, $0 \leqq n \leqq t_i - 1$. We want to compare T_x and $T_{f^n x}$. We have

$$T_x - T_{f^n x} = \sum_{k=0}^{n-1} h(f^k x) - \sum_{k=0}^{n-1} h(f^{k+t_{ij}} x).$$

The points $f^k x$ and $f^{k+t_{ij}} x$ belong to the interval Δ_i^k and these intervals with $k = 0, \cdots, n-1$ are pairwise disjoint. Therefore

$$|T_x - T_{f^n x}| \leqq \sum_{k=0}^{n-1} |h(f^k x) - h(f^{k+t_{ij}} x)| \leqq \operatorname{Var} h. \qquad (10)$$

Combining (9) and (10) we see that for every two points $x \in \Delta_{i,j}^n$, $y \in \Delta_{ij}^m$, $0 \leqq n, m \leqq t_i$

$$|T_x - T_y| < 4 \operatorname{Var} h.$$

In other words, there are numbers T_{ij}, $i = 1, \cdots, s$, $j = 1, \cdots, s_i$ such that for every $x \in \Delta_{i,j}^n$, $n = 0, \cdots, t_i - 1$

$$T_{ij} \leqq T_x \leqq T_{ij} + 4 \operatorname{Var} h. \qquad (11)$$

Let J be a subinterval of $[0, 1]$ and

$$A_i^n = \Delta_i^n \times J \subset I_{h(\cdot)}$$

(recall that $h > 1$). We have for $x \in \Delta_{i,j}^n$

(12) $$f^h_{T_x}(x,0)=(f^{t_{ij}}x,0).$$

It follows from (4) that

(13) $$\Delta^n_i=\bigcup_{j=1}^{s_i} f^{t_{ij}}\Delta^n_{ij}.$$

Combining (11), (12) and (13) we obtain

$$\begin{aligned} A^n_i &= \bigcup_{j=1}^{s_i}(\Delta^n_{ij}\times J) \\ &\subset \bigcup_{j=1}^{s_i}\ \bigcup_{0\le t\le 4\operatorname{Var} h} f^h_{T_{ij}+t}(\Delta^n_{ij}\times J) \\ &\subset \bigcup_{j=1}^{s_i}\ \bigcup_{0\le t\le 4\operatorname{Var} h} f^h_{T_{ij}+t}A^n_i. \end{aligned}$$

The second union is infinite but the special structure of the set A^n_i allows us to replace it by a finite union. Namely let δ be the length of J and $K=[4\operatorname{Var} h/\delta]+1$. Obviously

$$\bigcup_{0\le t\le 4\operatorname{Var} h} f^h_{T_{ij}+t}A^n_i\subset\bigcup_{l=0}^{K} f^h_{T_{ij}+l\delta}A^n_i$$

and consequently

(14) $$A^n_i\subset\bigcup_{j=1}^{s_i}\bigcup_{l=0}^{K} f^h_{T_{ij}+l\delta}A^n_i.$$

Now let $A=E\times J$ where E is a set which is measurable with respect to the partition ξ_Δ. Then (14) implies that

(15) $$A\subset\bigcup_{i=1}^{s}\bigcup_{j=1}^{s_i}\bigcup_{l=0}^{K} f^h_{T_{ij}+l\delta}A.$$

Since $h>1$, the number T_{ij} is bigger than t_{ij}. Consequently, if the subinterval Δ is small enough we can arrange that $T_{ij}>N$ for a fixed number N (cf. §1). Furthermore, it follows from (15) that for some i, j, l

(16) $$\lambda_{h(\cdot)}(A\cap f^j_{T_{ij}+l\delta}A)>\frac{1}{(m+1)^2\cdot(K+1)}\lambda_{h(\cdot)}(A).$$

Now let A be an arbitrary set of the form $E\times J$ where $E\subset I$, and

(17) $$\mu(A)<\frac{1}{10(m+1)^2(K+1)}.$$

If the interval Δ is small enough we can approximate A by a set $A_\Delta = E_\Delta \times J$ where E_Δ is measurable with respect to the partition ξ_Δ and

$$\lambda_{h(\cdot)}(A \Delta A_\Delta) < \frac{1}{10}(\mu(A))^2. \tag{18}$$

We have from (16), (17) and (18), where the numbers i, j and l are chosen for the set A_Δ,

$$\begin{aligned}
\lambda_{h(\cdot)}(A \cap f^h_{T_{ij}+l\delta}A) &> \lambda_{h(\cdot)}(A \cap f^h_{T_{ij}+l\delta}A) - 2\lambda_{h(\cdot)}(A \Delta A_\Delta) \\
&> \frac{1}{(m+1)^2(K+1)}\mu(A_\Delta) - \frac{1}{5}(\mu(A))^2 \\
&> \frac{9}{10\cdot(m+1)^2(K+1)}\mu(A) - \frac{1}{5}(\mu(A))^2 \\
&> (9 - \tfrac{1}{5})(\mu(A))^2 > 2(\mu(A))^2.
\end{aligned}$$

Since $T_{ij} + l\delta$ is big enough, if the interval Δ is small enough, this contradicts the mixing property for $\{f^h_t\}$. □

§3. An interesting application of Theorem 2 occurs in the study of billiard flows. Namely, let Q be a polygon in the plane $\mathbf{R}^2$ and assume that all angles of Q are commensurable with π. Then the phase space $Q \times S^1$ of the billiard flow $\{T_t\}$ in Q splits into invariant subsets M_c, $0 \leqq c \leqq \pi/N$ where N is the least common denominator of the numbers n_r where the angles of Q have the form $\alpha_r = \pi m_r/n_r$ (cf. [2]).

Each of the sets M_c has the form $Q \times \Pi_c$ where Π_c is a finite set in S^1 and the flow $\{T_t\}$ restricted to M_c has a natural representation as a special flow over a map defined on $\partial Q \times \Pi_c$. The set $\partial Q \times \Pi_c$ is a union of a finite number of intervals, where length is the natural parameter and the natural invariant measure has a piecewise linear density with respect to this parameter. The roof function is also piecewise linear and, consequently, has bounded variation. Obviously, we can transform $\partial Q \times \Pi_c$ piecewise linearly to the interval $[0,1]$ such that the invariant measure becomes Lebesgue measure and our transformation becomes an I.E.T. The lift of this map transforms the flow $T_t|_{M_c}$ (the restriction of T_t into the invariant set M_c) into some special flow over this I.E.T. with a piecewise linear roof function. Thus, we can apply Theorem 2 and obtain the following.

THEOREM 3. *Let Q be a polygon with angles commensurable with π. The restriction of the billiard flow in Q to any manifold M_c is not mixing.*

References

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